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Collocated Output-Feedback Stabilization of a 2×2 Quasilinear Hyperbolic System using Backstepping

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Abstract— **We consider the problem of output feedback stabilization for a quasilinear 2×2 system of first-order hyperbolic PDEs with boundary actuation and measurement. We design an output feedback control law, with actuation and measurement on only one end of the domain, and prove local H^2 exponential stability of the closed-loop system. The proof of stability is based on the construction of a strict Lyapunov function which includes the observer states. The feedback law and output injection gains are found using the backstepping method for 2×2 system of first-order hyperbolic linear PDEs, developed by the authors in a previous work, which is briefly reviewed.**

I. Introduction

In this paper we are concerned with the problem of output feedback boundary stabilization for a 2×2 system of first-order hyperbolic *quasilinear* PDEs. We consider actuation in only one of the boundaries, and measurements in the same end. The quasilinear case is of interest since several relevant physical systems are described by 2×2 systems of first-order hyperbolic quasilinear PDEs, such as open channels, transmission lines, gas flow pipelines or road traffic models.

This problem has been previously considered for 2×2 systems [8] and even $n \times n$ systems [13], using the explicit evolution of the Riemann invariants along the characteristics. More recently, an approach using control Lyapunov functions has been developed, for 2×2 systems [3] and $n \times n$ systems [4]. These results use only static output feedback. However these results do not deal with the same class of systems considered in this work (which includes an extra term in the equations); with this term, it has been shown in [1], [2] that there are examples (even in the linear case) for which there are no control Lyapunov functions of the form $\int_0^1 z^T Q(x) z dx$ (see the next section for notation) which would allow the computation of a static output feedback law to stabilize the system (even if feedback is allowed on both sides of the boundary).

Several other authors have also studied this problem. For instance, the linear case has been analyzed in [24] (using a Lyapunov approach) and in [14] (using a spectral approach). The nonlinear case has been considered by [6] and [9] using a Lyapunov approach, and in [16], [17], and [7] using a Riemann invariants approach.

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The basis of the design used in this paper, which needs actuation on only one end, is the backstepping method [10]; initially developed for parabolic equations, it has been used for first-order hyperbolic equations [12], delay systems [11], second-order hyperbolic equations [18], fluid flows [20], nonlinear PDEs [21] and even PDE adaptive designs [19].

Our design builds upon the recently developed backstepping method for 2×2 system of first-order hyperbolic linear PDEs [22] and uses Lyapunov functions to prove stability (see [15] for a study of Lyapunov functions for infinite-dimensional systems). There, a linear output feedback law is designed; we show that this linear law makes the closed-loop quasilinear system locally exponentially stable in the H^2 sense. The feedback law and the output injection gains are obtained as solutions of two well-posed 4 x 4 systems of first-order hyperbolic linear PDEs. The proof of stability is based on [4]; we construct a strict Lyapunov function, locally equivalent to the H^2 norm, and written in coordinates defined by the (invertible) backstepping transformation. The argument follows similar lines to the one presented [23], however there are several non-trivial modifications. First, we have additional states due to the observer which are nonlinearly coupled with the plant. We use as variables the observer estimates and observer error, and apply two different backstepping transformations on these states (a control transformation on the observer estimate and an observer transformation on the observer error), resulting on a nonlinearly coupled 4x4 quasilinear system of hyperbolic PDEs. In [23] local H^2 stability was proved by using Lyapunov functions for the first and second time derivative of the state; here, it is needed to use a (non-diagonal) Lyapunov function for the mixed space-time second derivative.

The paper is organized as follows. In Section II we formulate the problem. In Section III we briefly review the linear backstepping design from [22]. In Section IV we present our main result, which is proven in Section V. We finish in Section VI with some concluding remarks.

II. Problem Statement

Consider the system

$$z_t+\Lambda(z,x)z_x+f(z,x)=0, \quad x\in[0,1],\, t\in[0,+\infty), \quad (1)$$

where $z : [0,1]\times[0,\infty) \to \mathbb{R}^2$, $\Lambda : \mathbb{R}^2\times[0,1] \to \mathcal{M}_{2,2}(\mathbb{R})$, $f : \mathbb{R}^2\times[0,1]\to\mathbb{R}^2$, where $\mathcal{M}_{2,2}(\mathbb{R})$ denotes the set of 2×2 real matrices. We consider that $\Lambda(z,x)$ is twice continuously differentiable with respect to z and x. Without loss of generality we also consider $\Lambda(0,x)$ a diagonal matrix with nonzero eigenvalues $\Lambda_1(x)$ and $\Lambda_2(x)$ which are, respectively, positive and negative, i.e.,

$$\Lambda(0,x)=\operatorname{diag}(\Lambda_1(x),\Lambda_2(x)),\Lambda_1(x)>0,\Lambda_2(x)<0, \quad (2)$$

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where $\operatorname{diag}(\Lambda_1, \Lambda_2)$ denotes the diagonal matrix with Λ_1 in the first position of the diagonal and Λ_2 in the second.

Also assume that $f(0, x) = 0$ (so that there is an equilibrium at the origin) and that f is twice continuously differentiable with respect to z. Denote

$$\frac{\partial f}{\partial z}(0, x) = \begin{bmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{bmatrix}, \tag{3}$$

and assume that $f_{ij} \in \mathcal{C}^1([0,1])$.

Denoting $z = [z_1 \ z_2]^T$, we study classical solutions of the system under the following boundary conditions

$$z_1(0, t) = q z_2(0, t),\ z_2(1, t) = U_c(t), \tag{4}$$

for $q \neq 0$, and $U_c(t)$ the actuation. Our task is to find a feedback law for $U_c(t)$ to make the origin of (1),(4) locally exponentially stable, using only measurements of $z_1(1, t)$.

Remark 1: The case with $f = 0$ in (1) was addressed in [3] and [4] by using control Lyapunov functions to design a static output feedback law; this approach has been shown to fail ([1], [2]) for some cases with $f \neq 0$, at least for a Lypunov function of the form $\int_0^1 z^T Q(x) z dx$.

Remark 2: The case $q = 0$ in (4) can be considered by modifying the design as in [22]. Also, nonlinear boundary conditions can be addressed by slightly modifying the proof. We don't include either case for the sake of clarity.

III. Output feedback stabilization of 2×2 hyperbolic linear systems

To stabilize (1)–(4), we use the design presented in [22]. For the reader's sake, the result is reviewed next.

Consider the following linear system

$$w_t = \Sigma(x) w_x + C(x) w, \tag{5}$$

where

$$\Sigma(x) = \begin{pmatrix} -\epsilon_1(x) & 0 \\ 0 & \epsilon_2(x) \end{pmatrix}, C(x) = \begin{pmatrix} 0 & c_1(x) \\ c_2(x) & 0 \end{pmatrix}, \tag{6}$$

where $\epsilon_1(x), \epsilon_2(x) > 0$ and c_1, c_2 are $\mathcal{C}^2([0,1])$ functions and with boundary conditions

$$u(0, t) = q v(0, t),\ v(1, t) = U_c(t), \tag{7}$$

where $w = [u \ v]^T$. Only $u(1, t)$ is measurable.

Then, selecting $U_c(t)$ as

$$U_c = \int_0^1 K^{vu}(1, \xi) \hat{u}(\xi, t) d\xi + \int_0^1 K^{vv}(1, \xi) \hat{v}(\xi, t) d\xi, \tag{8}$$

where $\hat{u}$ and $\hat{v}$ are computed from

$$\hat{w}_t = \Sigma(x)\hat{w}_x + C(x)\hat{w} - \epsilon_1(x) p(x)(u(1, t) - \hat{u}(1, t)), \tag{9}$$

where $\hat{w} = [\hat{u} \ \hat{v}]^T$, $p(x) = [P^{uu}(x, 1) \ P^{vu}(x, 1)]^T$ and with boundary conditions

$$\hat{u}(0, t) = q\hat{v}(0, t),\ \hat{v}(1, t) = U_c(t), \tag{10}$$

it can be shown that the origin of system (5)–(7) is exponentially stable, where the kernels K^{vu} and K^{vv} are solution of the following kernel equations:

$$\begin{aligned}
\epsilon_1(x) K^{uu}_x + \epsilon_1(\xi) K^{uu}_\xi &= -\epsilon_1'(\xi) K^{uu} - c_2(\xi) K^{uv}, &(11)\\
\epsilon_1(x) K^{uv}_x - \epsilon_2(\xi) K^{uv}_\xi &= \epsilon_2'(\xi) K^{uv} - c_1(\xi) K^{uu}, &(12)\\
\epsilon_2(x) K^{vu}_x - \epsilon_1(\xi) K^{vu}_\xi &= \epsilon_1'(\xi) K^{vu} + c_2(\xi) K^{vv}, &(13)\\
\epsilon_2(x) K^{vv}_x + \epsilon_2(\xi) K^{vv}_\xi &= -\epsilon_2'(\xi) K^{vv} + c_1(\xi) K^{vu}, &(14)
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
K^{uu}(x, 0) &= \frac{\epsilon_2(0)}{q\epsilon_1(0)} K^{uv}(x, 0), &(15)\\
K^{uv}(x, x) &= \frac{c_1(x)}{\epsilon_1(x) + \epsilon_2(x)}, &(16)\\
K^{vu}(x, x) &= -\frac{c_2(x)}{\epsilon_1(x) + \epsilon_2(x)}, &(17)\\
K^{vv}(x, 0) &= \frac{q\epsilon_1(0)}{\epsilon_2(0)} K^{vu}(x, 0). &(18)
\end{aligned}$$

Similarly, the kernels P^{uu} and P^{vu} are solution of the following kernel equations:

$$\begin{aligned}
\epsilon_1(x) P^{uu}_x + \epsilon_1(\xi) P^{uu}_\xi &= -\epsilon_1'(\xi) P^{uu} - c_1(x) P^{vu}, &(19)\\
\epsilon_1(x) P^{uv}_x - \epsilon_2(\xi) P^{uv}_\xi &= \epsilon_2'(\xi) P^{uv} - c_1(x) P^{vv}, &(20)\\
\epsilon_2(x) P^{vu}_x - \epsilon_1(\xi) P^{vu}_\xi &= \epsilon_1'(\xi) P^{vu} + c_2(x) P^{uu}, &(21)\\
\epsilon_2(x) P^{vv}_x + \epsilon_2(\xi) P^{vv}_\xi &= -\epsilon_2'(\xi) P^{vv} + c_2(x) P^{uv}, &(22)
\end{aligned}$$

with boundary conditions:

$$\begin{aligned}
P^{uu}(0, \xi) &= q P^{vu}(0, \xi), &(23)\\
P^{uv}(x, x) &= \frac{c_1(x)}{\epsilon_1(x) + \epsilon_2(x)}, &(24)\\
P^{vu}(x, x) &= -\frac{c_2(x)}{\epsilon_1(x) + \epsilon_2(x)}, &(25)\\
P^{vv}(0, \xi) &= \frac{1}{q} P^{uv}(0, \xi), &(26)
\end{aligned}$$

Both kernel equations evolve in the triangular domain $\mathcal{T} = \{(x, \xi) : 0 \leq \xi \leq x \leq 1\}$. In [22] it is shown that if the coefficients ϵ_1, ϵ_2 are $\mathcal{C}^2([0,1])$, and if c_1 and c_2 are $\mathcal{C}^2([0,1])$, then the kernels belong to $\mathcal{C}^2(\mathcal{T})$.

Define $\|w(\cdot, t)\|_{L^2} = \sqrt{\int_0^1 (u^2(\xi, t) + v^2(\xi, t)) d\xi}$. The following result holds.

Theorem 1: Consider systems (5)–(7) and (9)–(10) with control law (8) and initial conditions $w_0, \hat{w}_0 \in L^2([0,1])$. Then, there exists $\lambda > 0$ and $c > 0$ such that

$$\|w(\cdot, t)\|_{L^2} + \|\hat{w}(\cdot, t)\|_{L^2} \leq c\, \mathrm{e}^{-\lambda t} \left(\|w_0\|_{L^2} + \|\hat{w}_0\|_{L^2} \right). \tag{27}$$

The proof of the theorem will be of use later in Section V.

Proof: Define the observer estimate error $\tilde{w} = w - \hat{w}$ and denote

$$K = \begin{pmatrix} K^{uu} & K^{uv} \\ K^{vu} & K^{vv} \end{pmatrix}, P = \begin{pmatrix} P^{uu} & P^{uv} \\ P^{vu} & P^{vv} \end{pmatrix}. \tag{28}$$

Defining new "target" variables $\hat{\gamma} = \begin{bmatrix} \hat{\alpha} & \hat{\beta} \end{bmatrix}^T$ and $\tilde{\gamma} =$

$\begin{bmatrix}\tilde{\alpha} & \tilde{\beta}\end{bmatrix}^T$ by using the following transformations

$$\hat{\gamma}(x,t) = \hat{w}(x,t) - \int_0^x K(x,\xi)\hat{w}(\xi,t)d\xi, \quad (29)$$

$$\tilde{w}(x,t) = \tilde{\gamma}(x,t) - \int_x^1 P(x,\xi)\tilde{\gamma}(\xi,t)d\xi, \quad (30)$$

it can be proven that, if the kernels verify (11)–(26), then $\hat{\gamma}$ and $\tilde{\gamma}$ verify the following equations:

$$\tilde{\gamma}_t = \Sigma(x)\tilde{\gamma}_x, \quad (31)$$

$$\hat{\gamma}_t = \Sigma(x)\hat{\gamma}_x - \bar{p}(x)\tilde{\alpha}(1,t), \quad (32)$$

where $\bar{p}(x) = \epsilon_1(x)p(x) - \int_0^x K(x,\xi)\epsilon_1(\xi)p(\xi)d\xi$, with boundary conditions

$$\tilde{\alpha}(0,t) = q\tilde{\beta}(0,t), \hat{\alpha}(0,t) = q\hat{\beta}(0,t), \tilde{\beta}(1,t) = \hat{\beta}(1,t) = 0 \quad (33)$$

which have an exponentially stable equilibrium at the origin, as we show next. Define

$$D(x) = \begin{bmatrix} A\frac{e^{-\mu x}}{\epsilon_1(x)} & 0 \\ 0 & B\frac{e^{\mu x}}{\epsilon_2(x)} \end{bmatrix}, \quad (34)$$

where $B = q^2A + \lambda_2$, $A = \lambda_2 e^{\mu}$, and $\mu = \lambda_1\bar{\epsilon}$, where $\bar{\epsilon} = \max_{x\in[0,1]}\left\{\frac{1}{\epsilon_1(x)}, \frac{1}{\epsilon_2(x)}\right\}$, with $\lambda_1, \lambda_2 > 0$ as large as desired. Select, for $c_1 > 0$, the following Lyapunov function

$$U = \int_0^1 \tilde{\gamma}^T(x,t)D(x)\tilde{\gamma}(x,t)dx + c_1\int_0^1 \hat{\gamma}^T(x,t)D(x)\hat{\gamma}(x,t)dx. \quad (35)$$

Computing $\dot{U}$ and integrating by parts, we obtain

$$\dot{U} \leq -\lambda_1 U - \lambda_2\left(\tilde{\alpha}^2(1,t) + \tilde{\beta}^2(0,t)\right) - c_1\lambda_2\left(\hat{\alpha}^2(1,t) + \hat{\beta}^2(0,t)\right) + 2c_1\left|\tilde{\alpha}(1,t)\int_0^1 \bar{p}^T(x)D(x)\hat{\gamma}(x,t)dx\right|, \quad (36)$$

Using Young's inequality and since D is diagonal and positive, the last term of (36) can be dominated as follows:

$$2c_1\left|\tilde{\alpha}(1,t)\int_0^1 \bar{p}^T(x)D(x)\hat{\gamma}(x,t)dx\right| \leq \frac{c_1}{c_2}\int_0^1 \hat{\gamma}(x,t)^T D(x)\hat{\gamma}(x,t)dx + c_1c_2\tilde{\alpha}^2(1,t)\left|\int_0^1 \bar{p}^T(x)D(x)\bar{p}(x)dx\right|, \quad (37)$$

for any $c_2 > 0$, and choosing $c_2 = \frac{2c_1}{\lambda_1}$ and c_1 such that

$$c_1 = \sqrt{\frac{\lambda_1\lambda_2}{4\left|\int_0^1 \bar{p}^T(x)D(x)\bar{p}(x)dx\right|}}, \quad (38)$$

we obtain that

$$\dot{U} \leq -\frac{\lambda_1}{2}U - \frac{\lambda_2}{2}\left(\tilde{\alpha}^2(1,t) + \tilde{\beta}^2(0,t)\right) - c_1\lambda_2\left(\hat{\alpha}^2(1,t) + \hat{\beta}^2(0,t)\right). \quad (39)$$

This shows L^2 exponential stability of the origin for the $\tilde{\gamma}$-$\hat{\gamma}$ system. To extend the result to the original system, an inverse transformation to (29)–(30) is defined as follows

$$\hat{w}(x,t) = \hat{\gamma}(x,t) + \int_0^x L\hat{\gamma}(\xi,t)d\xi, \quad (40)$$

$$\tilde{\gamma}(x,t) = \tilde{w}(x,t) + \int_x^1 R\tilde{w}(\xi,t)d\xi, \quad (41)$$

where the inverse kernel matrices L and R (whose kernels are solutions of a system of equations analogous to (11)–(18)) can be shown to exist. The theorem then follows by using the inverse and direct transformations to relate the L^2 norms of $\tilde{\gamma}$-$\hat{\gamma}$ and $\tilde{w}$-$\hat{w}$ and then using that $w = \tilde{w} + \hat{w}$. ■

IV. Main Result: Application of Backstepping to the Nonlinear System

We wish to show that the linear design (8)–(10) works locally for the nonlinear system (including the nonlinear terms in the observer). Thus, we design the following nonlinear observer:

$$\hat{z}_t + \Lambda(\hat{z},x)\hat{z}_x + f(\hat{z},x) + r(x)\left(z_1(1,t) - \hat{z}_1(1,t)\right) = 0, \quad (42)$$

with boundary conditions

$$\hat{z}_1(0,t) = q\hat{z}_2(0,t),\ \hat{z}_2(1,t) = U_c(t). \quad (43)$$

Now, we write our quasilinear system in a form equivalent (up to linear terms) to (5) and (9). Define

$$\varphi_1(x) = \exp\left(\int_0^x \frac{f_{11}(s)}{\Lambda_1(s)}ds\right), \quad (44)$$

$$\varphi_2(x) = \exp\left(-\int_0^x \frac{f_{22}(s)}{\Lambda_2(s)}ds\right). \quad (45)$$

We define new state variables w and $\hat{w}$ from z and $\hat{z}$ using the following transformation:

$$w = \begin{bmatrix} \varphi_1(x) & 0 \\ 0 & \varphi_2(x) \end{bmatrix} z = \Phi(x)z,\ \hat{w} = \Phi(x)\hat{z}, \quad (46)$$

so that

$$z = \begin{bmatrix} \frac{1}{\varphi_1(x)} & 0 \\ 0 & \frac{1}{\varphi_2(x)} \end{bmatrix} w = \Phi^{-1}(x)w,\ \hat{z} = \Phi^{-1}(x)\hat{w}. \quad (47)$$

It follows that w-$\hat{w}$ verify the following equations:

$$0 = w_t + \bar{\Lambda}(w,x)w_x + \bar{f}(w,x), \quad (48)$$

$$0 = \hat{w}_t + \bar{\Lambda}(\hat{w},x)\hat{w}_x + \bar{f}(\hat{w},x) + p(x)(u(1,t) - \hat{u}(1,t)), \quad (49)$$

where $p(x) = \frac{1}{\phi(1)}\Phi(x)r(x)$ and

$$\bar{\Lambda}(w,x) = \Phi(x)\Lambda(\Phi^{-1}(x)w,x)\Phi^{-1}(x), \quad (50)$$

$$\bar{f}(w,x) = \Phi(x)f(\Phi^{-1}(x)w,x) + \bar{\Lambda}(w,x)\begin{bmatrix} -\frac{f_{11}(x)}{\Lambda_1(x)} & 0 \\ 0 & \frac{f_{22}(x)}{\Lambda_2(x)} \end{bmatrix} w. \quad (51)$$

It is evident that $\bar{\Lambda}(0,x) = \Phi(x)\Lambda(0,x)\Phi^{-1}(x) = \Lambda(0,x)$ and that $\bar{f}(0,x) = 0$. Also,

$$C(x) = -\left.\frac{\partial \bar{f}(w,x)}{\partial w}\right|_{w=0} = \begin{bmatrix} 0 & -f_{12} \\ -f_{21} & 0 \end{bmatrix}. \quad (52)$$

Thus, it is possible to write (48)–(49) as two linear systems plus nonlinear terms:

$$\begin{aligned} w_t &= (\Sigma(x) - \Lambda_{NL}(w,x))\, w_x + C(x)w \\ &\quad - f_{NL}(w,x), \qquad (53)\\ \hat{w}_t &= (\Sigma(x) - \Lambda_{NL}(\hat{w},x))\, \hat{w}_x + C(x)\hat{w} \\ &\quad - f_{NL}(\hat{w},x) - p(x)(u(1,t)-\hat{u}(1,t)), \qquad (54)\end{aligned}$$

where $\Sigma(x) = -\Lambda(0,x)$ and

$$\begin{aligned} \Lambda_{NL}(w,x) &= \bar{\Lambda}(w,x) + \Sigma(x), \qquad (55)\\ f_{NL}(w,x) &= \bar{f}(w,x) + C(x)w. \qquad (56)\end{aligned}$$

It is clear that the nonlinear terms verify $\Lambda_{NL}(0,x) = 0$, $f_{NL}(0,x) = \frac{\partial f_{NL}}{\partial w}(0,x) = 0$.

Computing the boundary conditions of (53) by combining (4) with the transformation (46), one obtains

$$u(0,t) = qv(0,t), \hat{u}(0,t) = q\hat{v}(0,t), v(1,t) = \hat{v}(1,t) = \bar{U}_c(t), \qquad (57)$$

where $\bar{U}_c(t) = U_c(t)/\varphi_2(1)$.

Notice that the linear part of (53)-(54) is identical to (5), the boundary conditions are the same, and the coefficients $C(x)$ and $\Sigma(x)$ verify the assumptions of Section III. Thus we consider using the output feedback law:

$$\bar{U}_c = \int_0^1 K^{vu}(1,\xi)\hat{u}(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)\hat{v}(\xi,t)d\xi, \qquad (58)$$

with $p(x) = \epsilon_1(x)\left[P^{uu}(x,1)\ P^{vu}(x,1)\right]^T$. This implies the following control law and output injection gains for the system in z-$\hat{z}$ variables:

$$\begin{aligned} U_c &= \varphi_2(1)\int_0^1 K^{vu}(1,\xi)\frac{\hat{z}_1(\xi,t)}{\varphi_1(\xi)}d\xi \\ &\quad + \varphi_2(1)\int_0^1 K^{vv}(1,\xi)\frac{\hat{z}_2(\xi,t)}{\varphi_2(\xi)}d\xi, \qquad (59)\\ r(x) &= \phi(1)\epsilon_1(x)\Phi^{-1}(x)\left[P^{uu}(x,1)\ P^{vu}(x,1)\right]^T, \qquad (60)\end{aligned}$$

where the kernels are computed from (11)–(26) using the coefficients $C(x)$ and $\Sigma(x)$ obtained from Λ and f.

Denoting:

$$q_0 = \begin{bmatrix} 1 \\ -q \end{bmatrix}, q_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, k(x) = \begin{bmatrix} \frac{\varphi_2(1)K^{vu}(1,x)}{\varphi_1(x)} \\ \frac{\varphi_2(1)K^{vv}(1,x)}{\varphi_2(x)} \end{bmatrix}, \qquad (61)$$

the boundary conditions of the closed loop system can be written as:

$$q_0^T z(0,t) = 0,\ q_1^T z(1,t) = \int_0^1 k^T(\xi)\hat{z}(\xi,t)d\xi. \qquad (62)$$

Define the norms $\|z(\cdot,t)\|_{H_1} = \|z(\cdot,t)\|_{L^2} + \|z_x(\cdot,t)\|_{L^2}$ and $\|z(\cdot,t)\|_{H_2} = \|z(\cdot,t)\|_{H^1} + \|z_{xx}(\cdot,t)\|_{L^2}$. We now state the main result of the paper.

Theorem 2: Consider the z-system (1) with boundary conditions (62) and initial conditions $z_0 = [z_{0_1}\ \ z_{0_2}]^T \in H^2([0,1])$, where the kernels K^{vu} and K^{vv} are obtained from (11)–(18), and the $\hat{z}$-system (42) with boundary conditions (43) and initial conditions $\hat{z}_0 = [\hat{z}_{0_1}\ \ \hat{z}_{0_2}]^T \in H^2([0,1])$, with r defined as in (60) where the kernels P^{uv} and P^{uu} are obtained from (19)–(26). Then, if the following compatibility conditions are verified for z_0

$$q_0^T z_0(0) = 0, \qquad (63)$$

$$q_1^T z_0(1) = \int_0^1 k^T(\xi)\hat{z}_0(\xi)d\xi, \qquad (64)$$

$$q_0^T\Lambda(z_0(0),0)z_0'(0) = -q_0^T f(z_0(0),0), \qquad (65)$$

$$\begin{aligned} q_1^T\Lambda(z_0(1),1)z_0'(1) &= \int_0^1 k^T(\xi)\left(\Lambda(\hat{z}_0(\xi),\xi)\hat{z}_0'(\xi)\right. \\ &\quad \left. + f(\hat{z}_0(\xi),\xi)\right)d\xi - q_1^T f(z_0(1),1), \qquad (66)\end{aligned}$$

and similar conditions for $\hat{z}_0$ (substituting z_0 for $\hat{z}_0$ in (63)–(66)), there exists $\delta > 0$, $\lambda > 0$ and $c > 0$ such that if $\|z_0\|_{H^2} + \|\hat{z}_0\|_{H^2} \leq \delta$ then:

$$\|z(\cdot,t)\|_{H^2} + \|\hat{z}(\cdot,t)\|_{H^2} \leq c\,\mathrm{e}^{-\lambda t}\left(\|z_0\|_{H^2} + \|\hat{z}_0\|_{H^2}\right). \qquad (67)$$

V. Proof of Theorem 2

We first establish some notation. For an $\mathbb{R}^2$ vector $\gamma(x)$ with components $\alpha(x)$ and $\beta(x)$ denote $|\gamma(x)| = |\alpha(x)| + |\beta(x)|$, and

$$\|\gamma\|_\infty = \sup_{x\in[0,1]}|\gamma(x)|,\ \|\gamma\|_{L^1} = \int_0^1 |\gamma(\xi)|d\xi. \qquad (68)$$

For a 2×2 matrix M, denote:

$$|M| = \max\{|Mv|; \gamma\in\mathbb{R}^2, |\gamma| = 1\}. \qquad (69)$$

For the kernel matrices K and L denote

$$\|K\|_\infty = \sup_{(x,\xi)\in\mathcal{T}}|K(x,\xi)|. \qquad (70)$$

For $\gamma \in H^2([0,1])$, recall the following well-known inequalities:

$$\begin{aligned} \|\gamma\|_{L^1} &\leq C_1\|\gamma\|_{L^2} \leq C_2\|\gamma\|_\infty, \qquad (71)\\ \|\gamma\|_\infty &\leq C_3\left[\|\gamma\|_{L^2} + \|\gamma_x\|_{L^2}\right] \leq C_4\|\gamma\|_{H^1}, \qquad (72)\\ \|\gamma_x\|_\infty &\leq C_5\left[\|\gamma_x\|_{L^2} + \|\gamma_{xx}\|_{L^2}\right] \leq C_6\|\gamma\|_{H^2}. \qquad (73)\end{aligned}$$

In what follows we denote $|\gamma| = |\gamma(x,t)|$ and $\|\gamma\| = \|\gamma(\cdot,t)\|$ to simplify our notation.

To prove Theorem 2, we first define the observer error $\tilde{w} = w - \hat{w}$. The equation verified by $\tilde{w}$ is:

$$\begin{aligned} \tilde{w}_t &= \Sigma(x)\tilde{w}_x + C(x)\tilde{w} - \Lambda_{NL}(\tilde{w}+\hat{w},x)\tilde{w}_x \\ &\quad - \left(\Lambda_{NL}(\tilde{w}+\hat{w},x) - \Lambda_{NL}(\hat{w},x)\right)\hat{w}_x \\ &\quad - f_{NL}(\tilde{w}+\hat{w},x) + f_{NL}(\hat{w},x) + p(x)\tilde{u}(1,t), \qquad (74)\end{aligned}$$

with boundary conditions $\tilde{u}(0,t) = q\tilde{v}(0,t)$, $\tilde{v}(1,t) = 0$. Now, we apply the (invertible) backstepping transformation (29) to the nonlinear system (54), and using the inverse transformation (40) the transformed system can be expressed fully in terms of $\hat{\gamma}$ and $\tilde{\gamma}(1,t)$ as:

$$\hat{\gamma}_t = \Sigma(x)\hat{\gamma}_x + F_2[\hat{\gamma},\hat{\gamma}_x] + \bar{p}(x)\tilde{\alpha}(1,t). \qquad (75)$$

Similarly, applying the inverse observer backstepping transformation (41) to the nonlinear system (74) and using transformations (30) and (40) the transformed system can be expressed fully in terms of $\tilde{\gamma}$ and $\hat{\gamma}$ as:

$$\tilde{\gamma}_t = \Sigma(x)\tilde{\gamma}_x + G_2[\tilde{\gamma},\hat{\gamma},\tilde{\gamma}_x,\hat{\gamma}_x]. \qquad (76)$$

The definitions and properties of the functionals F_2 and G_2 (and similar functionals that will appear from here on) is skipped for lack of space and will appear in a forthcoming journal publication.

The boundary conditions are

$$\hat{\alpha}(0,t) = q\hat{\beta}(0,t), \tilde{\alpha}(0,t) = q\tilde{\beta}(0,t), \hat{\beta}(1,t) = \tilde{\beta}(1,t) = 0. \quad (77)$$

Differentiating twice with respect to x in the direct and inverse transformations (29)–(30) and (40)–(41), it can be shown that the H^2 norms of $\hat{\gamma}$ and $\tilde{\gamma}$ are equivalent to those of $\hat{w}$ and $\tilde{w}$. Thus, if we show H^2 local stability of the origin for (75) and (76), the same holds for w-$\hat{w}$ and thus for z-$\hat{z}$.

We proceed by analyzing (using a Lyapunov function) the growth of $\|\hat{\gamma}\|_{L^2} + \|\tilde{\gamma}\|_{L^2}$, $\|\hat{\gamma}_t\|_{L^2} + \|\tilde{\gamma}_t\|_{L^2}$ and $\|\hat{\gamma}_{tx}\|_{L^2} + \|\tilde{\gamma}_{tx}\|_{L^2}$. The relations of these norms with $\|\hat{\gamma}\|_{H^2} + \|\tilde{\gamma}\|_{H^2}$ norm is given in the following two lemmas:

Lemma 1: There exists $\delta > 0$ such that for $\|\hat{\gamma}\|_\infty + \|\tilde{\gamma}\|_\infty < \delta$ then the norm defined by $\|\hat{\gamma}_t\|_{L^2} + \|\tilde{\gamma}_t\|_{L^2} + \|\hat{\gamma}\|_{L^2} + \|\tilde{\gamma}\|_{L^2}$ is equivalent to $\|\hat{\gamma}\|_{H^1} + \|\tilde{\gamma}\|_{H^1}$.

Proof: The lemma is proven directly bounding the norms (for small $\|\hat{\gamma}\|_\infty + \|\tilde{\gamma}\|_\infty$) in (75)–(76). ■

Lemma 2: There exists $\delta > 0$ such that for $\|\hat{\gamma}\|_\infty + \|\tilde{\gamma}\|_\infty + \|\hat{\gamma}_t\|_\infty + \|\tilde{\gamma}_t\|_\infty < \delta$ then the norm defined by $\|\hat{\gamma}_{tx}\|_{L^2} + \|\tilde{\gamma}_{tx}\|_{L^2} + \|\hat{\gamma}_t\|_{L^2} + \|\tilde{\gamma}_t\|_{L^2} + \|\hat{\gamma}\|_{L^2} + \|\tilde{\gamma}\|_{L^2}$ is equivalent to $\|\hat{\gamma}\|_{H^2} + \|\tilde{\gamma}\|_{H^2}$.

Proof: Taking an x-derivative in (75)–(76) one finds a equation that relates $\hat{\gamma}_{tx}$ and $\tilde{\gamma}_{tx}$ with $\hat{\gamma}_{xx}$ and $\tilde{\gamma}_{xx}$. Bounding these variables and using Lemma 1, the result is shown. ■

A. Analyzing the growth of $\|\hat{\gamma}\|_{L^2} + \|\tilde{\gamma}\|_{L^2}$

Define U as in (36). Proceeding analogously to (36)–(39), and using the properties of the functionals F_2 and G_2, there exists δ_1, such that for $\|\hat{\gamma}\|_\infty + \|\tilde{\gamma}\|_\infty < \delta_1$, we get:

$$\begin{aligned} \dot{U} \quad \leq \quad & -\frac{\lambda_1}{2}U - \frac{\lambda_2}{2}\left(\tilde{\alpha}^2(1,t) + \tilde{\beta}^2(0,t)\right) \\ & -c_1\lambda_2\left(\hat{\alpha}^2(1,t) + \hat{\beta}^2(0,t)\right) \\ & +K_1\left(\|\hat{\gamma}_x\|_\infty + \|\tilde{\gamma}_x\|_\infty\right)\left(\|\hat{\gamma}\|^2_{L^2} + \|\tilde{\gamma}\|^2_{L^2}\right) \\ & +K_2\left(\|\hat{\gamma}\|_\infty + \|\tilde{\gamma}\|_\infty\right)\left(\|\hat{\gamma}\|^2_{L^2} + \|\tilde{\gamma}\|^2_{L^2}\right), \quad (78) \end{aligned}$$

and using inequality (72) and noting $\left(\|\hat{\gamma}\|^2_{L^2} + \|\tilde{\gamma}\|^2_{L^2}\right) \leq K_4 U^{1/2}$, we obtain the following theorem:

Theorem 3: There exists δ_1 such that if $\|\hat{\gamma}\|_\infty + \|\tilde{\gamma}\|_\infty < \delta_1$ then

$$\dot{U} \quad \leq \quad -\lambda_3 U + K_1 U^{3/2} + K_2\left(\|\hat{\gamma}_x\|_\infty + \|\tilde{\gamma}_x\|_\infty\right)U, \quad (79)$$

where λ_3, K_1 and K_2 are positive constants.

B. Analyzing the growth of $\|\hat{\gamma}_t\|_{L^2} + \|\tilde{\gamma}_t\|_{L^2}$

Define $\hat{\eta} = \hat{\gamma}_t$ and $\tilde{\eta} = \tilde{\gamma}_t$. Taking partial derivative in t in both (75) and (76) we obtain equations for $\hat{\eta}$ and $\tilde{\eta}$:

$$\hat{\eta}_t = \left(\Sigma(x) - F_1[\hat{\gamma}]\right)\hat{\eta}_x + F_3[\hat{\gamma}, \hat{\gamma}_x, \hat{\eta}] + \bar{p}(x)\tilde{\eta}_1(1,t), \quad (80)$$

$$\begin{aligned} \tilde{\eta}_t = & \left(\Sigma(x) - G_1[\hat{\gamma}, \tilde{\gamma}]\right)\tilde{\eta}_x + \left(F_1[\hat{\gamma}] - G_1[\hat{\gamma}, \tilde{\gamma}]\right)\hat{\eta}_x \\ & +G_3[\hat{\gamma}, \hat{\gamma}_x, \hat{\eta}, \tilde{\gamma}, \tilde{\gamma}_x, \tilde{\eta}]. \quad (81) \end{aligned}$$

The boundary conditions are

$$\begin{aligned} \hat{\eta}_1(0,t) \quad &= \quad q\hat{\eta}_2(0,t),\ \tilde{\eta}_1(0,t) = q\tilde{\eta}_2(0,t), \quad (82) \\ \hat{\eta}_2(1,t) \quad &= \quad \tilde{\eta}_2(1,t) = 0. \quad (83) \end{aligned}$$

Consider the following Lyapunov function

$$\begin{aligned} V \quad = \quad & \int_0^1 \tilde{\eta}^T(x,t)D(x)\tilde{\eta}(x,t)dx \\ & +c_1\int_0^1 \hat{\eta}^T(x,t)D(x)\hat{\eta}(x,t)dx. \quad (84) \end{aligned}$$

with c_1 as in (35). Proceeding analogously to (36)–(39), and using the properties of the functionals, there exists δ_2, such that for $\|\hat{\gamma}\|_\infty + \|\tilde{\gamma}\|_\infty < \delta_2$, we get:

$$\begin{aligned} \dot{V} \quad \leq \quad & -\frac{\lambda_1}{2}V - \frac{\lambda_2}{2}\left(\tilde{\eta}_1^2(1,t) + \tilde{\eta}_2^2(0,t)\right) \\ & -c_1\lambda_2\left(\hat{\eta}_1^2(1,t) + \hat{\eta}_2^2(0,t)\right) \\ & +K_1\left(\|\hat{\eta}_x\|_\infty + \|\tilde{\eta}_x\|_\infty\right)(V+U) \\ & +K_2\left(\|\hat{\gamma}\|_\infty + \|\tilde{\gamma}\|_\infty\right)\left(\|\hat{\gamma}\|^2_{L^2} + \|\tilde{\gamma}\|^2_{L^2}\right), \quad (85) \end{aligned}$$

Theorem 4: There exists δ_2 such that if $\|\hat{\gamma}\|_\infty + \|\tilde{\gamma}\|_\infty < \delta_2$, then

$$\begin{aligned} \dot{V} \ \leq \ & -\lambda_2 V - \lambda_3\left(\hat{\eta}_1^2(1,t) + \hat{\eta}_2^2(0,t) + \tilde{\eta}_1^2(1,t) + \tilde{\eta}_2^2(0,t)\right) \\ & +K_3\left(\|\hat{\eta}_x\|_\infty + \|\tilde{\eta}_x\|_\infty\right)(V+U) \quad (86) \end{aligned}$$

for $\lambda_2, \lambda_3, K_3$ positive constants.

C. Analyzing the growth of $\|\hat{\gamma}_{tx}\|_{L^2} + \|\tilde{\gamma}_{tx}\|_{L^2}$

This step contains the main differences of this paper with [23]. Define $\hat{\theta} = \hat{\eta}_x$ and $\tilde{\theta} = \tilde{\eta}_x$. Taking a partial derivative in x in (80)–(81) we obtain equations for $\hat{\theta}$ and $\tilde{\theta}$:

$$\begin{aligned} \hat{\theta}_t = & \left(\Sigma(x) - F_1[\hat{\gamma}]\right)\hat{\theta}_x + \Sigma'(x)\hat{\theta} \\ & +F_4[\hat{\gamma}, \hat{\gamma}_x, \hat{\eta}, \hat{\eta}_x, \hat{\theta}] + \bar{p}'(x)\tilde{\eta}_1(1,t), \quad (87) \\ \tilde{\theta}_t = & \left(\Sigma(x) - G_1[\hat{\gamma}, \tilde{\gamma}]\right)\tilde{\theta}_x + \left(F_1[\hat{\gamma}] - G_1[\hat{\gamma}, \tilde{\gamma}]\right)\hat{\theta}_x \\ & +\Sigma'(x)\tilde{\theta} + G_4[\hat{\gamma}, \hat{\gamma}_x, \hat{\eta}, \hat{\eta}_x, \hat{\theta}, \tilde{\gamma}, \tilde{\gamma}_x, \tilde{\eta}, \tilde{\eta}_x, \tilde{\theta}]. \quad (88) \end{aligned}$$

The boundary conditions for $\hat{\theta}$ and $\tilde{\theta}$ are

$$\begin{aligned} \hat{\theta}_1(0,t) \quad &= \quad \bar{q}_{01}[\hat{\gamma}]\hat{\theta}_2(0,t) + \bar{q}_1[\hat{\gamma}]\tilde{\eta}_1(1,t) \\ & \quad\quad +Q_1[\hat{\gamma}, \hat{\gamma}_x, \hat{\eta}, \hat{\theta}], \quad (89) \\ \hat{\theta}_2(1,t) \quad &= \quad \bar{q}_2[\hat{\gamma}]\tilde{\eta}_1(1,t) + Q_2[\hat{\gamma}, \hat{\gamma}_x, \hat{\eta}, \hat{\theta}], \quad (90) \\ \tilde{\theta}_1(0,t) \quad &= \quad \bar{q}_{02}[\hat{\gamma}, \tilde{\gamma}]\tilde{\theta}_2(0,t) \\ & \quad\quad +Q_3[\hat{\gamma}, \hat{\gamma}_x, \hat{\eta}, \hat{\theta}, \tilde{\gamma}, \tilde{\gamma}_x, \tilde{\eta}, \tilde{\theta}], \quad (91) \\ \tilde{\theta}_2(1,t) \quad &= \quad Q_4[\hat{\gamma}, \hat{\gamma}_x, \hat{\eta}, \hat{\theta}, \tilde{\gamma}, \tilde{\gamma}_x, \tilde{\eta}, \tilde{\theta}], \quad (92) \end{aligned}$$

where the linear part of the functionals appearing in (89)–(92) is $\bar{q}_{01}[0] = \bar{q}_{02}[0,0] = -\frac{q\epsilon_2(0)}{\epsilon_1(0)}$, $\bar{q}_2[0] = -\frac{\bar{p}_2(1)}{\epsilon_2(1)}$ and $\bar{q}_1[0] = -\frac{\bar{p}_2(0)\epsilon_1(0)+q\bar{p}_1(0)\epsilon_2(0)}{\epsilon_2(0)\epsilon_1(0)}$ and zero for all Q_i's.

To find a Lyapunov function for ($\hat{\theta}$,$\tilde{\theta}$), we use the next lemma (whose proof is given in [23]):

Lemma 3: There exists $\delta > 0$ such that, for $\|\hat{\gamma}\|_\infty < \delta$, there exists a symmetric matrix $R[\hat{\gamma}] > 0$ verifying:

$$R[\hat{\gamma}]\left(\Sigma(x) - F_1[\hat{\gamma}]\right) - \left(\Sigma(x) - F_1[\hat{\gamma}]\right)^T R[\hat{\gamma}] = 0, \quad (93)$$

and the following bounds:

$$\begin{aligned} R[\hat{\gamma}](x) \quad &\leq \quad c_1 + c_2\|\hat{\gamma}\|_\infty, \quad (94) \\ \left|\left(\left(R[\hat{\gamma}] - \bar{D}(x)\right)\Sigma(x)\right)_x\right| \quad &\leq \quad c_2\|\hat{\gamma}\|_\infty\left(1 + \|\hat{\gamma}_x\|_\infty\right), \quad (95) \\ \left|\left(R[\hat{\gamma}]\right)_t\right| \quad &\leq \quad c_3\left(|\hat{\eta}| + \|\hat{\eta}\|_{L^1}\right), \quad (96) \end{aligned}$$

where $\bar{D} = D\Sigma^2$ and c_1, c_2, c_3 are positive constants.

An analogous lemma produces a symmetric matrix $S[\hat{\gamma},\tilde{\gamma}] > 0$ verifying the identity:

$$S[\hat{\gamma},\tilde{\gamma}]\left(\Sigma(x)-G_1[\hat{\gamma},\tilde{\gamma}]\right)-\left(\Sigma(x)-G_1[\hat{\gamma},\tilde{\gamma}]\right)^T S[\hat{\gamma},\tilde{\gamma}]=0, \tag{97}$$

with analogous properties to (94)–(96).

Using $R[\hat{\gamma}]$ and $S[\hat{\gamma}+\tilde{\gamma}]$, we define the following (non-diagonal) Lyapunov function:

$$\begin{aligned} W=&\int_0^1\left(\hat{\theta}(x,t)+\tilde{\theta}(x,t)\right)^T S[\hat{\gamma}+\tilde{\gamma}]\left(\hat{\theta}(x,t)+\tilde{\theta}(x,t)\right)dx \\ &+\int_0^1 \hat{\theta}^T(x,t)R[\gamma]\hat{\theta}(x,t)dx. \end{aligned} \tag{98}$$

Computing the time derivative of W and choosing $\|\hat{\gamma}\|_\infty+\|\hat{\eta}\|_\infty+\|\tilde{\gamma}\|_\infty+\|\tilde{\eta}\|_\infty$ small enough, we find:

$$\begin{aligned} \dot{W} \quad \leq \quad & -\lambda_4 W-\lambda_5\left(\tilde{\theta}_1^2(1,t)+\tilde{\theta}_2^2(0,t)\right) \\ & -\lambda_5\left(\hat{\theta}_1^2(1,t)+\hat{\theta}_2^2(0,t)\right)+K_1\tilde{\eta}_1^2(1,t) \\ & +K_2WV^{1/2}+K_3VW^{1/2}+K_4W^{3/2}, \end{aligned} \tag{99}$$

Thus, we finally obtain the following theorem:

Theorem 5: There exists δ_3 such that if $\|\hat{\gamma}\|_\infty+\|\hat{\eta}\|_\infty+\|\tilde{\gamma}\|_\infty+\|\tilde{\eta}\|_\infty<\delta_3$ then

$$\begin{aligned} \dot{W} \quad \leq \quad & -\lambda_5 W+K_1\tilde{\eta}_1^2(1,t)+K_2WV^{1/2} \\ & +K_3VW^{1/2}+K_4W^{3/2}, \end{aligned} \tag{100}$$

where $\lambda_5, K_1, K_2, K_3, K_4$ are positive constants.

D. Proof of H^2 stability of ($\hat{\gamma}$,$\tilde{\gamma}$)

Defining $S=U+V+\kappa W$, combining Theorems 3, 4, and 5, and selecting κ to cancel the $\eta_1^2(1,t)$ terms in W, there exists δ such that if $\|\hat{\gamma}\|_\infty+\|\hat{\eta}\|_\infty+\|\tilde{\gamma}\|_\infty+\|\tilde{\eta}\|_\infty<\delta$

$$\dot{S}\leq -\lambda S+CS^{3/2}, \tag{101}$$

for $\lambda, C>0$. Following [3] and noting $\|\hat{\gamma}\|_\infty+\|\hat{\eta}\|_\infty+\|\tilde{\gamma}\|_\infty+\|\tilde{\eta}\|_\infty\leq C_2 S$, then for sufficiently small $S(0)$, it follows that $S(t)\to 0$ exponentially.

Given that S is equivalent to $\|\hat{\gamma}\|_{H^2}+\|\tilde{\gamma}\|_{H^2}$ when $\|\hat{\gamma}\|_\infty+\|\hat{\eta}\|_\infty+\|\tilde{\gamma}\|_\infty+\|\tilde{\eta}\|_\infty$ is sufficiently small, we obtain that if the initial conditions of ($\hat{\gamma}$,$\tilde{\gamma}$) verify compatibility conditions analogous to (63)–(66), there exists $\delta>0$ and $c>0$ such that if $\|\hat{\gamma}_0\|_{H^2}+\|\tilde{\gamma}_0\|_{H^2}\leq\delta$, then:

$$\|\hat{\gamma}\|_{H^2}+\|\tilde{\gamma}\|_{H^2}\leq c\,\mathrm{e}^{-\lambda t}\left(\|\hat{\gamma}_0\|_{H^2}+\|\tilde{\gamma}_0\|_{H^2}\right). \tag{102}$$

This proves Theorem 2.

VI. Concluding remarks

We have solved the problem of output feedback boundary stabilization for a 2×2 system of first-order hyperbolic quasilinear PDEs with actuation and measurement on only one boundary. We have then shown H^2 local exponential stability of the state and of the observer error.

It would be of interest to extend the method to $n\times n$ systems. For instance, a 3×3 first-order hyperbolic system of interest is the Saint-Venant-Exner system, which models open channels with a moving sediment bed [5]. While extending the Lyapunov analysis to $n\times n$ systems has been done [4], it remains an open problem to extend backstepping to such systems, even in the linear case. In general, the method would need n^2 kernels resulting in a $n^2\times n^2$ system of coupled first-order hyperbolic equations, whose well-posedness depends critically on the exact choice of the transformation and target system; these are still to be defined.

References

[1] G. Bastin and J.-M. Coron, "Further Results on boundary feedback stabilisation of 2×2 hyperbolic systems over a bounded interval," *Proceedings of 8th NOLCOS*, pp. 1081–1085, 2010.

[2] G. Bastin and J.-M. Coron, "On boundary feedback stabilization of non-uniform linear 2×2 hyperbolic systems over a bounded interval," accepted for publication in *Systems and Control Letters*, 2011.

[3] J.-M. Coron, B. d'Andrea-Novel and G. Bastin, "A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws," *IEEE Trans. on Automatic Control*, vol. 52, pp. 2–11, 2006.

[4] J.-M. Coron, G. Bastin and B. d'Andrea-Novel, "Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems," *SIAM Journal of Control and Optimization*, vol. 47, pp. 1460–1498, 2008.

[5] A. Diagne, G. Bastin and J.-M. Coron, "Lyapunov exponential stability of linear hyperbolic systems of balance laws," to appear in *Proceedings of the 18th IFAC World Congress*, 2011.

[6] M. Dick, M. Gugat and G. Leugering, "Classical solutions and feedback stabilisation for the gas flow in a sequence of pipes," *Networks and heterogeneous media*, vol. 5, pp. 691–709, 2010.

[7] V. Dos Santos and C. Prieur, "Boundary control of open channels with numerical and experimental validations," *IEEE Trans. Control Syst. Tech.*, vol. 16, pp. 1252–1264, 2008.

[8] J-M. Greenberg and T-t. Li, "The effect of boundary damping for the quasilinear wave equations," *Journal of Differential Equations*, vol. 52, pp. 66–75, 1984.

[9] M. Gugat and M. Herty, "Existence of classical solutions and feedback stabilisation for the flow in gas networks," *ESAIM Control Optimisation and Calculus of Variations*, vol. 17, pp. 28–51, 2011.

[10] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs*, SIAM, 2008

[11] M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Birkhauser, 2009.

[12] M. Krstic and A. Smyshlyaev, "Backstepping boundary control for first order hyperbolic PDEs and application to systems with actuator and sensor delays," *Syst. Contr. Lett.*, vol. 57, pp. 750–758, 2008.

[13] T-t. Li, *Global Classical Solutions for Quasilinear Hyperbolic Systems*, Wiley, 1994.

[14] X. Litrico and V. Fromion, "Boundary control of hyperbolic conservation laws using a frequency domain approach," *Automatica*, vol. 45, pp. 647–656, 2009.

[15] Z.-H. Luo, B.-Z. Guo, O. Morgul, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, Springer, 1999.

[16] C. Prieur, "Control of systems of conservation laws with boundary errors," *Networks and Heterogeneous Media*, vol. 4, pp. 393–407, 2009.

[17] C. Prieur, J. Winkin and G. Bastin, "Robust boundary control of systems of conservation laws," *Mathematics of Control, Signals, and Systems*, vol. 20, pp. 173–197, 2008.

[18] A. Smyshlyaev, E. Cerpa and M. Krstic, "Boundary stabilization of a 1-D wave equation with in-domain antidamping," *SIAM Journal of Control and Optimization*, vol. 48, pp. 4014–4031, 2010.

[19] A. Smyshlyaev and M. Krstic, *Adaptive Control of Parabolic PDEs*, Princeton University Press, 2010.

[20] R. Vazquez and M. Krstic, *Control of Turbulent and Magnetohydrodynamic Channel Flow*, Birkhauser, 2008.

[21] R. Vazquez and M. Krstic, "Control of 1-D parabolic PDEs with Volterra nonlinearities — Part I: Design," *Automatica*, vol. 44, pp. 2778–2790, 2008.

[22] R. Vazquez, M. Krstic and J.-M. Coron, "Backstepping Boundary Stabilization and State Estimation of a 2×2 Linear Hyperbolic System," *50th IEEE CDC and ECC*, Orlando, 2011.

[23] R. Vazquez, J.-M. Coron, M. Krstic and G. Bastin, "Local Exponential H^2 Stabilization of a 2×2 Quasilinear Hyperbolic System Using Backstepping," *50th IEEE CDC and ECC*, Orlando, 2011.

[24] C.Z. Xu and G. Sallet, "Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems," *ESAIM Control Optimisation and Calculus of Variations*, vol. 7, pp. 421–442, 2002.