# CHAPTER 1

## Calculus of variations

In the context of calculus, the derivative $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$ of a function $f(x) \in C^2(\mathbb{R})$ is a well defined concept. If $f'(x) = 0$ for $x = x_0$, then $f$ attains either a maximum or a minumum at that point. This means that, for a given neighborhood $|x - x_0| < \delta$, then either $f(x) < f(x_0)$ ($f$ is maximum, $f''(x_0) < 0$) or $f(x) > f(x_0)$ ($f$ is minimum, $f''(x_0) > 0$).

We will see in forthcoming section that finding the static equilibrium of an elastic structure can is a problem that can be written in the following fashion. Consider a deformable structure $\Omega \in \mathbb{R}^3$ and a displacement field

$$\mathbf{u}(\mathbf{x}) = \{u_1(x_1, x_2, x_3), u_2(x_1, x_2, x_3), u_3(x_1, x_2, x_3)\}$$

that is admissible. We will be more clear about what is an admissible displacement in the next chapter but for now, we assume that a displacement field $\mathbf{u}$ is admissible if it belongs to a vector valued function space $U$ that contains all possible admissible displacements.

The static equilibrium of a structure $\Omega$ that is loaded by some volume forces $\mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \Omega$ and by surface loads $\mathbf{F}(\mathbf{x})$, $\mathbf{x} \in \Gamma_F$ consist in finding the $\mathbf{u} \in U$ that minimizes the potential energy

$$\pi(\mathbf{u}) = \frac{1}{2}\int_\Omega \sigma_{ij}\epsilon_{ij}\, dv - \int_\Omega f_i u_i\, dv - \int_{\Gamma_F} F_i u_i\, ds. \tag{1.1}$$

where

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) = u_{(i,j)} \quad , \quad \sigma_{ij} = \lambda \epsilon_{mm}\delta_{ij} + 2\mu\epsilon_{ij},$$

with Lamé coefficients

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \quad \mu = \frac{E}{2(1+\nu)},$$

and with $E$ and $\nu$ being the Young modulus and the Poisson ratio.

The potential energy $\pi$ that is defined in (1.1) is not a function in the classical sense. A function $f(x)$ transforms reals into reals: $f: \mathbb{R} \to \mathbb{R}$. The argument $\mathbf{u}$ of $\pi(\mathbf{u})$ is not a real but a function. Such an object that takes a function as argument and returns reals is called a *functional*. How could we possibly find the extremum of a functional? We could try to mimic the definition of the derivative of a function:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}. \tag{1.2}$$

Here is the problem: how could we possibly define $d\mathbf{u}$ when $\mathbf{u}$ is a function?

## 1.1 The brachistochrone curve

At this point, it is instructive to present a very famous problem that has lead to the development of the calculus of variations. Most of the theory that is developed here is due to the king of mathematicians: Leonhard Euler (1707-1783). In 1696, Johann Bernoulli (1667 - 1747) that was Euler's Ph.D. advisor introduced the following problem[1]. Assume that we have to design the most exciting roller-coaster ever. The ride starts at a point $\mathbf{x}^1(x_1, y_1)$ where the car is launched with a zero speed. The car then goes down following a planar curve $y(x)$ and ends at point $\mathbf{x}^2(x_2, y_2)$. For sake of simplicity, and without loss of generality, we choose $\mathbf{x}^1 = \{0, 0\}$ and we assume the acceleration of gravity to be oriented along the positive $y$'s (see Figure 1.1).

We assume now that the most exciting ride will be the one that minimizes the time for going from $\mathbf{x}^1$ to $\mathbf{x}^2$. The curve that minimizes the ride time is called the *brachistochrone curve*.

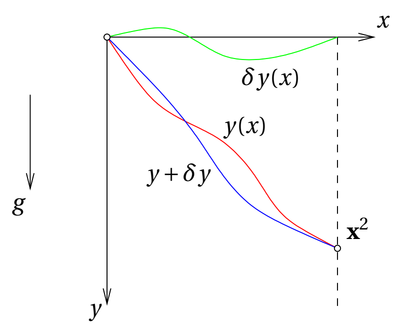


Figure 1.1: Brachistochrone curve.

Classical mechanics allow to write

$$\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t}.$$

[1] The statement has been rephrased

The time $T$ required to go from $\mathbf{x}^1$ to $\mathbf{x}^2$ is computed as follows:

$$T = \int_{\mathbf{x}^1}^{\mathbf{x}^2} \frac{dl}{V} = \int_{\mathbf{x}^1}^{\mathbf{x}^2} \frac{\sqrt{dx^2 + dy^2}}{V} = \int_0^{x_2} \frac{\sqrt{1 + y'^2}}{V} dx$$

where $V$ is the velocity of the car. We have assumed that the car was initially at rest. If $m$ is the mass of the car and if $g$ is the acceleration of gravity, we have

$$mgy = \frac{mV^2}{2} \quad \rightarrow \quad V = \sqrt{2gy}.$$

Time $T$ is finally computed as:

$$\boxed{T(y) = \int_0^{x_2} \sqrt{\frac{1 + y'^2}{2gy}} dx.} \tag{1.3}$$

In (1.3), $T$ is a functional: its argument is the shape of the curve $y(x)$ i.e. a function of $x$. qui est une fonctionnelle dont l'argument est la "forme" du rail, i.e. $y(x)$. Now comes the question of admissibility. Ca we choose any $y(x)$? Of course not: $y(x)$ must pass through $\mathbf{x}^1$ and $\mathbf{x}^2$. This is a the admissibility condition and we write $y \in U$ with

$$U = \{y(x) \mid y(x_1) = y_1, \;\; y(x_2) = y_2\}.$$

We now introduce the concept of variation. In order to find the minimum of $T$ among all $y$'s, we have to define the equivalent of a $dx$ in the definition of the derivative (1.2). We have to define a perturbation $\delta y(x)$ that verifies

$$y + \delta y \in U.$$

In other words, the perturbed function $y + \delta y$ should still be admissible in order to be able to evaluate $T(y + \delta y)$. The condition

$$\delta y(x_1) = \delta y(x_2) = 0$$

allows to write

$$y(x_1) + \delta y(x_1) = y_1 \quad \text{and} y(x_2) + \delta y(x_2) = y_2$$

which means that $y + \delta y$ is admissible and we define $\delta u \in U_0$ with the function space of variations

$$U_0 = \{\delta y(x) \mid \delta y(x_1) = \delta y(x_2) = 0\}$$

There is indeed another admissibility condition. Functional $T(y)$ should be "computable" i.e. $y(x) > 0$. If $y < 0$ for any $x \in [0, x_2]$, then the car would never finish the ride. Here, we do not ask $\delta y$ to be positive because $\delta y$ is assumed to be an infinitesimal variation of $y$, as it is the case for the standard derivative.

## 1.2 A fundamental result

**Proposition 1.2.1** *If a continuous function $f(x)$ in $[a,b]$ and if*

$$\int_a^b f(x)\, w(x)\, \mathrm{d}x = 0$$

*for every continuous function $w(x)$ such that $w(a) = w(b) = 0$, then $f(x) = 0$ for all $x$ in $[a,b]$.*

**Proof** Suppose that $f(x)$ is non zero, say positive at some point in $[a,b]$. Then $f$ is positive for some interval $[x_1, x_2]$ contained in $[a,b]$. If we set

$$w(x) = (x - x_1)(x_2 - x)$$

for $x \in [x_1, x_2]$ and $w(x) = 0$ otherwise, then $w(x)$ satisfies the conditions of the hypothesis. However

$$\int_a^b f(x)\, w(x)\, \mathrm{d}x = \int_{x_1}^{x_2} f(x)\, (x - x_1)(x_2 - x)\, \mathrm{d}x > 0$$

since the integrand is positive (except at $x_1$ and $x_2$ where it is null). This contradiction proves the result.

Note here that the condition $w(a) = w(b) = 0$ is not necessary: what it is shown here is that proposition 1.2.1 holds even for functions $w(x)$ that vanish at the boundary. Result 1.2.1 is called the *fundemental lemma of the calculus of variations*. This result has many extensions. See [1], Chapter 1, §3 for more details.

## 1.3 Euler-Lagrange equations

Finding the extrema (minima or maxima) of a functional $T(y)$ is similar to finding the extrema of functions $f(x)$. The first variation of a functional $T$ is defined as

$$\delta T(x, y, y') = T(x, y + \delta y, y' + \delta y') - T(x, y, y').$$

Any variation $\delta y$ can be written as the difference of two admissible functions $y_a$ and $y_b$. We have then

$$(\delta y)' = (y_a - y_b)' = (y'_a - y'_b) = \delta y'.$$

A development in Taylor series of $T$ around $y$ and $y'$ gives

$$T(x, y + \delta y, y' + \delta y') = T(x, y, y') + \left( \frac{\partial T}{\partial y} \delta y + \frac{\partial T}{\partial y'} \delta y' \right) + \mathcal{O}(\delta^2)$$

where $\mathcal{O}(\delta^2)$ refers to terms containing $\delta y^2$, $\delta y'^2$, $\delta y^3$... Neglecting those terms, we have

$$\boxed{\delta T(x, y, y') = \frac{\partial T}{\partial y} \delta y + \frac{\partial T}{\partial y'} \delta y'.}$$

$T$ is extremal if and only if its first variation $\delta T$ is equal to zero for all variations $\delta y$. There is a well known version of this result when $T$ has the form

$$T = \int_{x_1}^{x_2} F(x, y, y') dx$$

as it is for the Brachistochrone curve. Using integration by parts, we obtain

$$\begin{aligned} \delta T &= \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx \\ &= \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right) \delta y dx + \frac{\partial F}{\partial y'} \delta y \bigg|_{x_1}^{x_2} \\ &= \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right) \delta y dx = 0 \quad \forall \delta y \in U_0. \end{aligned}$$

According to the fundamental lemma of calculus of variations 1.2.1, the part of the integrand in parentheses is zero, i.e.

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0} \tag{1.4}$$

Equation (1.4) is called the Euler-Lagrange equation. It is an ordinary differential equation, generally non linear, which can be solved to obtain the extremal function $y(x)$.

If $F$ does not depend on $x$ explicitely, equation (1.4) can be simplified. We have

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y'' + \frac{\partial F}{\partial x},$$

where the last term drops out because $F$ does not depend on $x$ explicitely. Rearranging this yields

$$y' \frac{\partial F}{\partial y} = \frac{\mathrm{d}F}{\mathrm{d}x} - \frac{\partial F}{\partial y'} y''.$$

We then substitue $y' \frac{\partial F}{\partial y}$ into (1.4) to get

$$\frac{\mathrm{d}F}{\mathrm{d}x} - \frac{\partial F}{\partial y'} y'' - y' \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y'} = 0.$$

The last term can be expanded as

$$y' \frac{d}{dx} \frac{\partial F}{\partial y'} = \frac{\mathrm{d}}{\mathrm{d}x} \left( \frac{\partial F}{\partial y'} y' \right) - \frac{\partial F}{\partial y'} y'',$$

and equation (1.4) can finally be written as

$$\frac{\mathrm{d}}{\mathrm{d}x} \left( F - y' \frac{\partial F}{\partial y'} \right) = 0.$$

that can be integrated to find the so-called Beltrami formula

$$\boxed{F - y'\frac{\partial F}{\partial y'} = C.} \tag{1.5}$$

Let us come back to the Brachistochrone curve. In that case (see Equation (1.3)),

$$F = \sqrt{\frac{1+y'^2}{2gy}}$$

and Beltrami's formula (1.5) can be used to find the ordinary differential equation of the Brachistochrone curve:

$$\left[1+(y')^2\right]\ y = \frac{1}{2gC^2} = D \tag{1.6}$$

with $D > 0$. The solution of (1.6) is not obvious. Let us do the following change of variables:

$$y' = \tan t.$$

We have

$$1 + y'^2 = 1 + \tan^2 t = \frac{1}{\cos^2 t}.$$

Then

$$y = D\cos^2 t = \frac{D}{2}(1+\cos 2t).$$

Finding $x$ is then rather simple. We have $y' = \tan t$. We can also derive $y$ explicitely as

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = 2D\sin t\cos t\ \frac{\mathrm{d}t}{\mathrm{d}x}.$$

We have finally

$$dx = 2D\ \cos^2 t\ dt \quad \rightarrow \quad x = D\left(t + \frac{1}{2}\sin 2t\right) + c$$

with an integration constant $c$. Posing $R = D/2$ and $u = 2t$, we have finally the parametric equations of the Brachistochrone curve

$$\begin{aligned} x &= R(u + \sin u) + c \\ y &= R(1+\cos u). \end{aligned} \tag{1.7}$$

Those equations correspond to a cycloid. A cycloid is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage. We still have 2 constants $R$ and $c$ to fix. Point $y = 0$ correspond to $u = \pi$. We need $x(\pi) = 0$ which means that $c = -R\pi$. Then, we want the point $\mathbf{x} = \{x_2, y_2\}$ to belong to the curve, which allows to fix $R$.

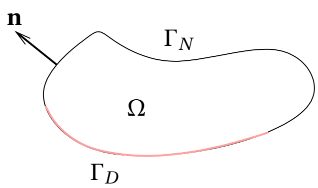


Figure 1.2: A domain $\Omega$ and its boundary $\Gamma = \Gamma_D \cup \Gamma_N$.

## 1.4 The Laplace problem

Consider a domain $\Omega \subset \mathbb{R}^3$ with its boundary $\Gamma = \Gamma_D \cup \Gamma_N$ of normal $\mathbf{n}$ (see Figure 1.2). The Laplace problem consist in finding $u(\mathbf{x}) \in H^1(\Omega)$ solution of

$$\begin{aligned} \nabla \cdot (\kappa \nabla u) &= 0 \text{ on } \Omega \\ u &= \bar{u} \text{ on } \Gamma_D \\ \kappa \nabla u \cdot \mathbf{n} &= \bar{q} \text{ on } \Gamma_N \end{aligned} \tag{1.8}$$

with $\kappa > 0$ function of $\mathbf{x}$. Problem (1.8) correspond to many practical situations: steady state thermal equilibrium, incompressible and irrotational fluids or membrane equilibrium. We now take the problem the other way around. Starting from differential form (1.8), we'd like to build a variational formulation of the Laplace problem.

First use the fundamental lemma of calculus of variations 1.2.1 to claim that

$$\int_\Omega [\nabla \cdot (\kappa \nabla u)] \; w \, dv = 0 \;\; \forall w$$

is equivalent to $\nabla^2 u = 0$ in $\Omega$. Integration by parts in $\mathbb{R}^3$

$$\int_\Omega \nabla \cdot \mathbf{a} b \, dv = -\int_\Omega \mathbf{a} \cdot \nabla b \, dv + \int_\Gamma \mathbf{a} \cdot \mathbf{n} b \, ds.$$

leads to

$$-\int_\Omega \kappa \nabla u \cdot \nabla w \, dv + \int_\Gamma \kappa \nabla u \cdot \mathbf{n} \; w \, ds = 0 \;\; \forall w$$

We then write

$$-\int_\Omega \kappa \nabla u \cdot \nabla w \, dv + \int_{\Gamma_D} \kappa \nabla u \cdot \mathbf{n} \; w \, ds + \int_{\Gamma_N} \kappa \nabla u \cdot \mathbf{n} \; w \, ds = 0 \;\; \forall w.$$

If we choose $w = \delta u$ i.e. if $w$ is a variation, then $w|_{\Gamma_D} = 0$ and we obtain

$$-\int_\Omega \kappa \nabla u \cdot \nabla \delta u \, dv + \int_{\Gamma_N} \kappa \nabla u \cdot \mathbf{n} \, \delta u \, ds = 0 \;\; \forall \delta u.$$

Let us define the space of admissible $u$'s to be

$$U = \{u \in H^1(\Omega) \;\; , u|_{\Gamma_D} = \bar{u}\}$$

and the space of variations

$$U_0 = \{\delta u \in H^1(\Omega) \ , u|_{\Gamma_D} = 0\}.$$

The Laplace problem (1.8) is equivalent to find $u \in U$ that verifies

$$\boxed{\int_\Omega \kappa \nabla u \cdot \nabla \delta u \, dv - \int_{\Gamma_N} q \, \delta u \, ds = 0 \ \ \forall \delta u \in U_0.} \tag{1.9}$$

Here, the signs of the terms of the equation has been changed for clarity. Now, equation (1.9) is not written as $\delta T = 0$ with $T$ a functional. It is indeed possible to do so. We define the following functional that is called the potential energy

$$\Pi(u) = \frac{1}{2} \int_\Omega \kappa \, (\nabla u)^2 \; dv - \int_{\Gamma_N} q \; u \; ds. \tag{1.10}$$

Obviously, $\delta\Pi = 0$ is equivalent to (1.9).

Equation (1.9) is a variational or weak formulation of the Laplace problem. It is called weak in opposition to the strong form (1.8) that have solutions for $u$'s that are twice differentiable ($u \in C^2(\Omega)$). Solutions of the weak form (1.9) only require that $\int_\Omega (\nabla u)^2 \, dv < \infty$. The space of functions that have their first derivatives square integrable is called $H^1(\Omega)$. This space is larger than $C^2(\Omega)$ which means that even though every strong solution is a weak solution, there may exist weak solutions that do not correspond to strong solutions: this is the case when $\kappa$ is discontinuous i.e. when the domain is composed of two distinct "materials". This is one of the reasons why variational forms are prefered to strong forms for solving PDEs on computers.

A last interresting question is to know if $u$ corresponds to a maximum or a minimum of $\Pi$. For that, we use a very similar technique as the one we use in calculus: we look at second order derivatives. We have

$$\delta(\delta\Pi(u)) = \frac{1}{2} \int_\Omega \kappa \, (\nabla \delta u)^2 \; dv \geq 0 \tag{1.11}$$

and the solution correspond to the minimum of the potential energy.

## 1.5 A different formulation of the Laplace problem

## 1.6 Problems

### 1.6.1 The Brachistochrone curve with friction

Write the problem of the brachistochrone curve taking into account coulombian friction. First compute the velocity of the car. Then write the functional $T(x, y, y')$. Find the corresponding ODE and solve it.

### 1.6.2 Geodesics

Another well known minimization problem is the construction of geodesics on a curved surface, meaning the curves of minimal length. Given two points **a** and **b** lying on a surface $S \subset \mathbb{R}^3$, we seek the curve $C \subset S$ that joins them and has the minimal possible length.

Assume the surface to have the simple form

$$z = F(x, y) \quad \rightarrow \quad \mathbf{a} = \{a, \alpha, F(a, \alpha)\} \quad \text{and} \quad \mathbf{b} = \{b, \beta, F(b, \beta\}.$$

Assume that $C$ is written in the $(x, y)$ plane as $y = u(x)$. We have of course $z = F(x, u(x))$. Compute the length $L(u)$ of $C$ as a function of $u$.

# Bibliography


[1] Izrail Moiseevich Gelfand, Izrail Moiseevitch Gelfand, Sergei Vasilevich Fomin, and Richard A Silverman. *Calculus of variations.* Courier Corporation, 2000.