CHAPTER 2

Variational principles in mechanics

2.1 Linear Elasticity

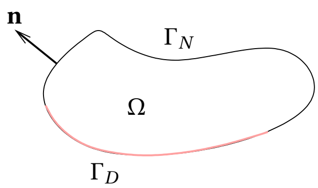

Figure 2.1: A domain Ω and its boundary $\Gamma = \Gamma_D \cup \Gamma_N$.

Consider a domain $\Omega \subset \mathbb{R}^3$ with its boundary $\Gamma = \Gamma_D \cup \Gamma_N$ of normal $\mathbf{n}$ (see Figure 2.1).

The problem of *linear elasticity* consist in finding the deformations ϵ and the internal stresses σ (N/m^2) on every $\mathbf{x}$ of Ω when it is submitted to external loads. We usually distinguish volume loads $\mathbf{f}$ (N/m^3) and surface loads $\mathbf{F}$ (N/m^2). The strong form of the problem consist in finding $\epsilon_{ij}(\mathbf{x})$ and $\sigma_{ij}(\mathbf{x})$ solution of the following equations:

$$\partial_j \sigma_{ij} + f_i = 0 \text{ on } \Omega \tag{2.1}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) = u_{(i,j)} \tag{2.2}$$

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \tag{2.3}$$

$$u_i = U_i \text{ on } \Gamma_D \tag{2.4}$$

$$\sigma_{ij}\, n_j = F_i \text{ on } \Gamma_N \tag{2.5}$$

The three equations (2.1) (one per direction) are expressing the equilibrium of internal stresses with the external volume loads. Equations (2.5) are expressing surface equilibrium on Γ_N of normal stresses with the external surface loads $\mathbf{F}$. Equations (2.1) and (2.5) both express equilibrium of forces. Those are the equations of static. When a stress field σ verifies (2.1) and (2.5), it is said *statically admissible*. We introduce the space of statically admissible stresses $\mathscr{S}$ as:

$$\mathscr{S} = \left\{\sigma(\mathbf{x}) \mid \partial_j \sigma_{ij} + f_i = 0 \ \forall \mathbf{x} \in \Omega, \ \sigma_{ij} \ n_j = F_i \ \forall \mathbf{x} \in \Gamma_F\right\}.$$

It is now possible to introduce the space variations of σ as

$$\mathscr{S}_0 = \left\{\delta\sigma(\mathbf{x}) \mid \partial_j \delta\sigma_{ij} = 0 \ \forall \mathbf{x} \in \Omega, \ \delta\sigma_{ij} \ n_j = 0 \ \forall \mathbf{x} \in \Gamma_F\right\}$$

Clearly, if $\sigma \in \mathscr{S}$ and $\delta\sigma \in \mathscr{S}_0$, then $\sigma + \delta\sigma \in \mathscr{S}$ and $\delta\sigma$ is a variation.

In continium mecahnics, we imagine Ω to be composed of a set of infinitesimal volumes or material points. Each volume is assumed to be connected to its neighbors without any gaps or overlaps. Certain mathematical conditions have to be satisfied to ensure that gaps/overlaps do not develop when a continuum body is deformed. A body that deforms without developing any gaps/overlaps is called a compatible body. Equations (2.2) and (2.4) are compatibility equations. Equations (2.2) say that, whenever ϵ is expressed as $\epsilon_{ij} = u_{(i,j)}$ where $\mathbf{u}$ is a displacement field, then no gaps/overlaps can appear in the continuum. Then, $\mathbf{u}$ should also be compatible with external kinematical constraints or supports. Equations (2.4) express external compatibility conditions. The space kinematically admissible displacement fields is defiend as

$$\mathscr{U} = \{\mathbf{u}(\mathbf{x}) \mid \mathbf{u} \in U(\Omega), \ u_i = U_i \ \forall \mathbf{x} \in \Gamma_D\}.$$

Here, $U(\Omega)$ a vector-valued function space that will be described in more details later. For now, let's assume that U is a space of functions that are sufficiently smooth such that no gaps/overlaps can appear in the continuum. We then introduce the space of variations:

$$\mathscr{U}_0 = \{\delta\mathbf{u}(\mathbf{x}) \mid \delta\mathbf{u} \in U(\Omega), \ \delta u_i = 0 \ \forall \mathbf{x} \in \Gamma_D\}.$$

Clearly, if $\mathbf{u} \in \mathscr{U}$ and $\delta\mathbf{u} \in \mathscr{U}_0$, then $\mathbf{u} + \delta\mathbf{u} \in \mathscr{U}_0$ and $\delta\mathbf{u}$ is a variation.

Finally, equations (2.3) are the linear and elastic constitutive equations. In the isotropic case, they simplify as

$$\sigma_{ij} = \lambda \epsilon_{mm} \delta_{ij} + 2\mu \epsilon_{ij} \tag{2.6}$$

with Lamé coefficients

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \quad \mu = \frac{E}{2(1+\nu)},$$

where E and ν are the Young modulus and the Poisson ratio. Constitutive law (2.17) can be inverted as

$$\epsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}.$$

The elastic energy that is stored in Ω due to its deformation can be written as

$$U(\mathbf{u}) = \frac{1}{2} \int_\Omega \sigma_{ij} \epsilon_{ij} dv = \frac{1}{2} \int_\Omega \epsilon_{ij} \epsilon_{kl} c_{ijkl} dv.$$

It is a functional, as defined in §1. The energy of a system has to be finite. This is indeed the weakest physical principle that defines how smooth a displacement field should be. Deformations ϵ being derivatives of $\mathbf{u}$, then space U is defined as the space of vector-valued fuctions that have their first derivative square integrable on Ω:

$$U = \left\{ \mathbf{u}\{u_1, u_2, u_3\} \mid \int_\Omega (\partial_j u_i)^2 dv < \infty \right\}$$

2.2 The Principle of Virtual Work

Assume $\mathbf{u} \in \mathscr{U}(\Omega)$ a compatible displacement. A variational form of (2.1) can be written as

$$\int_\Omega \left(\partial_j \sigma_{ij} + f_i\right) \delta u_i \, dv = 0 \quad \forall \delta u_i \in \mathscr{U}_0. \tag{2.7}$$

Equations (2.7) implies (2.1) thanks to the fundamental lemma of the calculus of variations 1.2.1. Integration by parts formula

$$\int_\Omega \nabla \cdot \mathbf{f} \, g dv = -\int_\Omega \mathbf{f} \cdot \nabla g dv + \int_\Gamma \mathbf{f} \cdot \mathbf{n} g ds,$$

applied to (2.7) gives

$$\int_\Omega \left(-\sigma_{ij} \, \partial_j \delta u_i + f_i \, \delta u_i \right) dv + \int_\Gamma \sigma_{ij} \, n_j \, \delta u_i \, ds = 0 \quad \forall \delta u_i \in \mathscr{U}_0. \tag{2.8}$$

Let us decompose the surface integral in two parts:

$$\int_\Gamma \sigma_{ij} \, n_j \, \delta u_i \, ds = \underbrace{\int_{\Gamma_D} \sigma_{ij} \, n_j \, \delta u_i \, ds}_{\delta u_i|_{\Gamma_D} = 0} + \underbrace{\int_{\Gamma_N} \sigma_{ij} \, n_j \, \delta u_i \, ds}_{\sigma_{ij} \, n_j|_{\Gamma_N} = F_i}. \tag{2.9}$$

We now take into account that

$$\sigma_{ij} \, \partial_j \delta u_i = \sigma_{ij} \, \delta u_{(i,j)},$$

to obtain the following variational form: find $\mathbf{u} \in \mathscr{U}$ that verifies

$$\boxed{\int_\Omega \left(-\sigma_{ij} \, \delta \epsilon_{ij} + f_i \, \delta u_i\right) \, dv + \int_{\Gamma_F} F_i \, \delta u_i \, ds = 0 \quad \forall \delta u_i \in \mathscr{U}_0.} \tag{2.10}$$

Form (2.10) is called the *principle of virtual work*. Every solution $\mathbf{u}$ that verifies the string form (2.1)-(2.5) is a solution of (2.10). Yet, some weaker solutions do verify (2.10) while not being sufficiently smooth to verify (2.1)-(2.5). Variational formulation (2.10) is the most general form of the equations of elasticity.

2.3 The Principle of Complementary Virtual Work

There exist another variational principle that is actually the dual of (2.10). Assume $\sigma \in \mathscr{S}$ to be a statically admissible stress field and let's write compatibility condition (2.2) in the following variational form :

$$\int_\Omega \epsilon_{ij} \delta\sigma_{ij} dv = \int_\Omega \frac{1}{2} \left(\partial_i u_j + \partial_j u_i\right) \delta\sigma_{ij} \quad \forall \delta\sigma_{ij} \in \mathscr{S}_0. \tag{2.11}$$

We the use the symmetry of σ to write

$$\partial_i u_j \delta\sigma_{ij} = \partial_i u_j \delta\sigma_{ji} = \partial_j u_i \delta\sigma_{ij}$$

which, combined to (2.11) leads to

$$\int_\Omega \epsilon_{ij} \delta\sigma_{ij} dv = \int_\Omega \partial_j u_i \delta\sigma_{ij} \quad \forall \delta\sigma_{ij} \in \mathscr{S}_0. \tag{2.12}$$

The right hand side of (2.12) can be integrated by parts to give

$$\int_\Omega \epsilon_{ij} \delta\sigma_{ij} dv = \int_\Gamma u_i \delta\sigma_{ij} n_j ds - \underbrace{\int_\Omega u_i \partial_j \delta\sigma_{ij} dv}_{=0 \text{ car } \delta\sigma \in \mathscr{S}_0} \quad \forall \delta\sigma_{ij} \in \mathscr{S}_0. \tag{2.13}$$

Let's decompose the surface integral of (2.13) in two parts:

$$\int_\Gamma u_i \delta\sigma_{ij} n_j ds = \underbrace{\int_{\Gamma_U} \delta\sigma_{ij}\; n_j\; u_i\; ds}_{u_i|_{\Gamma_U} = U_i} + \underbrace{\int_{\Gamma_F} \delta\sigma_{ij}\; n_j\; u_i\; ds}_{\delta\sigma_{ij}\; n_j|_{\Gamma_F} = 0}. \tag{2.14}$$

The following formulation is called the principle of complementary virtual work: find $\sigma \in \mathscr{S}$ solution of

$$\boxed{\int_\Omega \epsilon_{ij} \delta\sigma_{ij} dv = \int_{\Gamma_U} U_i \delta\sigma_{ij} n_j ds \quad \forall \delta\sigma_{ij} \in \mathscr{S}_0.} \tag{2.15}$$

Formulation (2.15) is way less useful in practice than (2.10). It actually requires to start with stresses that are statically admissible *a priori*. This requires to find a function space for σ that is in equilibrim with the external forces and this equilibrium involves a partial differential equation that is to be verified *a priori*! Very few numerical methods use (2.15) in practice.

2.4 The Principle of Total Potental Energy

There are cases when formulation (2.10) is equivalent to the minimization of a functional. Assume a linear elastic material. The principle of virtual work can be written as

$$\int_\Omega \left(\epsilon_{ij} c_{ijkl} \sigma_{kl} + f_i\right)\; \delta u_i\; dv = 0 \quad \forall \delta u_i \in \mathscr{U}_0. \tag{2.16}$$

Let us recall the definition of the elastic energy.

$$U = \frac{1}{2}\int_\Omega c_{ijkl}\epsilon_{kl}\epsilon_{ij} dv$$

We define the functional of total potential energy as the difference between the elastic energy U and the work of external loads **f** and **F**:

$$\pi = U - W_{ext} = \frac{1}{2}\int_\Omega c_{ijkl}\epsilon_{kl}\epsilon_{ij} dv - \int_\Omega f_i u_i\, dv - \int_{\Gamma_N} F_i u_i\, ds,$$

We have:

$$\delta\pi = \int_\Omega c_{ijkl}\epsilon_{kl}\delta\epsilon_{ij}\, dv - \int_\Omega f_i\delta u_i\, dv - \int_{\Gamma_N} F_i\delta u_i\, ds$$

which is indeed formulation (2.10). We have that

$$\delta\pi = 0 \quad , \quad \forall \delta u_i \in \mathscr{U}_0$$

is equivalent to the principle of virtual work (2.10).

It is easy to see that static equilibrium correspond to the minimum of π. For that, we compute

$$\delta^2\pi = \delta(\delta\pi) = \int_\Omega c_{ijkl}\delta\epsilon_{ij}\delta\epsilon_{kl} dv = 2U(\delta\epsilon).$$

Elastic energy being positive, the equilibrium correspond to the minimum of the total potential energy.

2.5 The Principle of Total Complementary Potental Energy

Let's again consider the linear elastic case. The total complementary potental energy is the following functional:

$$\pi^*(\sigma) = U - W^*_{ext} = \frac{1}{2}\int_\Omega d_{ijkl}\sigma_{kl}\sigma_{ij} dv - \int_{\Gamma_D} \sigma_{ij} n_j U_i\, ds.$$

The variation of π^* is

$$\delta^{(1)}\pi^* = \int_\Omega d_{ijkl}\sigma_{kl}\delta\sigma_{ij} dv - \int_{\Gamma_U} \delta\sigma_{ij} n_j U_i\, ds.$$

We have then

$$\delta\pi^* = 0 \quad , \quad \forall \delta_{sigma} \in \mathscr{S}_0$$

is equivalent to the principle of complementary virtual work. Again, this extremum correspond to a minimum.

2.6 Matrix notations

Strains ϵ, stresses σ or Hooke's law (2.3) can be written in index notations. Matrix notations can also be used, especially if we restrict our interest to standard euclidian coordinates. In this case, we can write Hooke's law in the following form:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{xz} \\ 2\epsilon_{yz} \end{pmatrix} \tag{2.17}$$

or in compact form

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon}$$

with **C** that is called Hooke's rigidity matrix. The inverse relation is written

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{xz} \\ 2\epsilon_{yz} \end{pmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} \tag{2.18}$$

or in compact form

$$\boldsymbol{\epsilon} = \mathbf{C}^{-1}\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\sigma}$$

where **D** is called Hooke's matrix of flexibility or complicance. The principle of virtual work can be written as: find $\mathbf{u} \in \mathscr{U}$ solution of

$$\boxed{\int_\Omega \boldsymbol{\sigma}^T(\mathbf{u})\, \epsilon(\delta\mathbf{u})\, dv = \int_\Omega \mathbf{f}\, \delta\mathbf{u}\, dv + \int_{\Gamma_N} \mathbf{F}\, \delta\mathbf{u}\, ds \quad \forall \delta\mathbf{u} \in \mathscr{U}_0.} \tag{2.19}$$

The principle of complementary virtual work can be written as: find $\boldsymbol{\sigma} \in \mathscr{S}$ solution of

$$\boxed{\int_\Omega \boldsymbol{\epsilon}^T\, \delta\boldsymbol{\sigma}\, dv = \int_{\Gamma_D} (\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{u}\, ds \quad \forall \delta\boldsymbol{\sigma} \in \mathscr{S}_0.} \tag{2.20}$$

2.7 The hypercircle of Prager and Synge

Assume $\boldsymbol{\epsilon}_{ex}, \boldsymbol{\sigma}_{ex}$ to be the exact solution of a given problem of elasticity. Let $\boldsymbol{\sigma} \in \mathscr{S}$ be a statically admissible stress and $\mathbf{u} \in \mathscr{U}$ be a compatible displacement. We write now the following functional

$$E^2(\epsilon(\mathbf{u}), \boldsymbol{\sigma}) = \frac{1}{2}\int_\Omega (\boldsymbol{\sigma} - \mathbf{C}\boldsymbol{\epsilon})^T\, \mathbf{D}\, (\boldsymbol{\sigma} - \mathbf{C}\boldsymbol{\epsilon})\, dv. \tag{2.21}$$

which is called the error in constitutive relation. Functional E of (2.21) is non negative and is equal to zero only if $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon}$ i.e. if the constitutive law is fullfilled. We then add $\boldsymbol{\sigma}_{ex} - \mathbf{C}\boldsymbol{\epsilon}_{ex}$ in (2.21) to obtain

$$\begin{aligned} E^2(\boldsymbol{\epsilon}(\mathbf{u}), \boldsymbol{\sigma}) &= \frac{1}{2}\int_\Omega (\boldsymbol{\sigma} - \mathbf{C}\boldsymbol{\epsilon} + \boldsymbol{\sigma}_{ex} - \mathbf{C}\boldsymbol{\epsilon}_{ex})^T \mathbf{D} (\boldsymbol{\sigma} - \mathbf{C}\boldsymbol{\epsilon} + \boldsymbol{\sigma}_{ex} - \mathbf{C}\boldsymbol{\epsilon}_{ex})\, dv \\ &= E^2(\boldsymbol{\epsilon}_{ex}, \boldsymbol{\sigma}) + E^2(\boldsymbol{\epsilon}, \boldsymbol{\sigma}_{ex}) + \int_\Omega (\boldsymbol{\sigma} - \boldsymbol{\sigma}_{ex})^T (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_{ex})\, dv \end{aligned} \tag{2.22}$$

Let us expand the last term of (2.22) as

$$\begin{aligned} &\int_\Omega (\boldsymbol{\sigma} - \boldsymbol{\sigma}_{ex})^T (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_{ex})\, dv \overset{\mathbf{u},\mathbf{u}_{ex} \in \mathscr{U}}{=} \int_\Omega (\boldsymbol{\sigma} - \boldsymbol{\sigma}_{ex})^T 1/2(\nabla + \nabla^T)(\mathbf{u} - \mathbf{u}_{ex})\, dv = \\ &-\int_\Omega \underbrace{\nabla\, (\boldsymbol{\sigma} - \boldsymbol{\sigma}_{ex})^T}_{=0 \text{ car } \boldsymbol{\sigma}, \boldsymbol{\sigma}_{ex} \in \mathscr{S}} (\mathbf{u} - \mathbf{u}_{ex})\, dv + \int_\Gamma (\boldsymbol{\sigma} - \boldsymbol{\sigma}_{ex})\, \mathbf{n}\, (\mathbf{u} - \mathbf{u}_{ex})\, ds = \\ &\int_{\Gamma_F} \underbrace{(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{ex})\, \mathbf{n}}_{=0 \text{ car } \boldsymbol{\sigma}, \boldsymbol{\sigma}_{ex} \in \mathscr{S}} (\mathbf{u} - \mathbf{u}_{ex})\, ds + \int_{\Gamma_U} (\boldsymbol{\sigma} - \boldsymbol{\sigma}_{ex})\, \mathbf{n} \underbrace{(\mathbf{u} - \mathbf{u}_{ex})}_{=0 \text{ car } \mathbf{u}, \mathbf{u}_{ex} \in \mathscr{U}} ds \\ &= 0. \end{aligned} \tag{2.23}$$

We should regognize now that $\boldsymbol{\sigma} - \boldsymbol{\sigma}_{ex} \in \mathscr{S}_0$ and that $\mathbf{u} - \mathbf{u}_{ex} \in \mathscr{S}_0$: relation (2.23) expresses then the orthogonality of $\delta\mathbf{u}$'s and $\delta\boldsymbol{\sigma}$'s. The hypercircle theorem of Prager and Synge writes then

$$E^2(\boldsymbol{\epsilon}(\mathbf{u}), \boldsymbol{\sigma}) = E^2(\boldsymbol{\epsilon}_{ex}, \boldsymbol{\sigma}) + E^2(\boldsymbol{\epsilon}, \boldsymbol{\sigma}_{ex}). \tag{2.24}$$

It says that the square of the distance (measured in term of E) between two admissible fields $\boldsymbol{\epsilon}(\mathbf{u})$ and $\boldsymbol{\sigma}$ is equal to the sum of the square of the distances between $\boldsymbol{\epsilon}_{ex}$ and $\boldsymbol{\sigma}$ and between $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}_{ex}$). This has a graphical interpretation (see Figure 2.2).

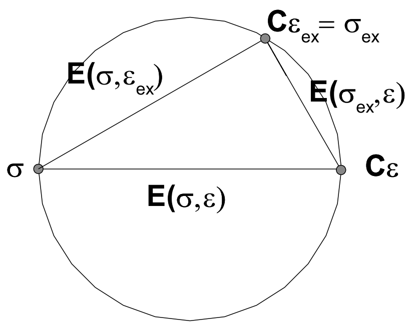

Figure 2.2: Hypercircle of Prager and Synge