CHAPTER 3

Trusses

3.1 Kinematic model of a rod.

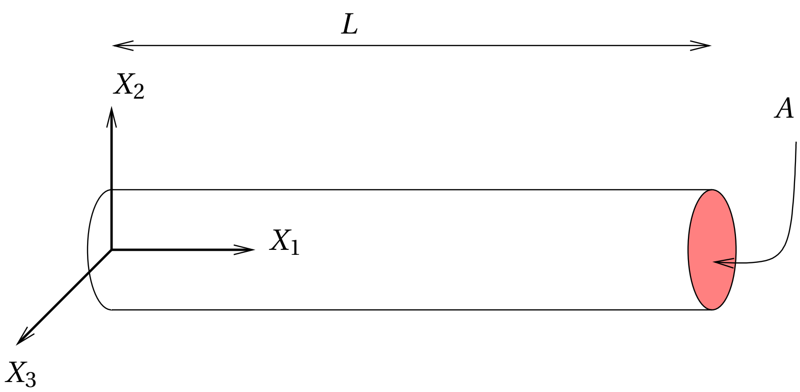

Figure 3.1: A rod with its local frame.

We are going to develop structural elements starting from the simplest one. The kinematic behavior of each element will be defined first using its local or natural frame $\mathbf{X} = \{X_1, X_2, X_3\}$ as $\mathbf{U} = \{U_1, U_2, U_3\}$ with

$$U_1 = U_1(X_1, Y_1, Z_1),$$
$$U_2 = U_2(X_1, Y_1, Z_1),$$
$$U_3 = U_3(X_1, Y_1, Z_1).$$

The rod is the simplest structural element (see Figure 3.1): we use X_1 for its principal axis. Rods only accomodate axis loads (traction or compression). A rod can only have displacements along its principal axis:

$$U_1 = U_1(X_1)$$
$$U_2 = 0$$
$$U_3 = 0$$

Deformations can be written as

$$\epsilon = \begin{Bmatrix} \frac{dU_1}{dX_1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The energy of deformation of the bar is written as

$$U = \frac{1}{2}\int_{\Omega} E\left(\frac{dU_1}{dX_1}\right)^2 d\mathbf{X}. \tag{3.1}$$

Assuming that the bar has a constant section $A = \int dX_2 dX_3$ and that Young modulus does not depend on $\mathbf{X}$, formula (3.1) simplifies to

$$U = \frac{EA}{2}\int_0^L \left(\frac{dU_1}{dX_1}\right)^2 dX_1 = \frac{EA}{2}\int_0^L (U_1')^2\, dX_1. \tag{3.2}$$

Let us now use results of §2 to establish solutions for simple configurations. Assume first that our rod is fixed at its origin $X_1 = 0$ and that a force $F[N]$ is applied to it at its end $X_1 = L$. We have

$$\pi = \frac{EA}{2}\int_0^L (U_1')^2\, dX_1 - FU_1(L).$$

We have

$$\delta\pi = EA\int_0^L U_1'\delta U_1'\, dX_1 - F\delta U_1(L) = 0 \ \ \forall \delta U_1.$$

Integration by parts gives

$$-EA\int_0^L U_1''\delta U_1\, dX_1 + EAU_1'\delta U_1|_0^L - F\delta U_1(L) = 0 \ \ \forall \delta U_1.$$

Regrouping terms and taking into account that $\delta U_1(0) = 0$, we have

$$EA\int_0^L U_1''\delta U_1\, dX_1 + (F - EAU_1')\delta U_1(L) = 0 \ \ \forall \delta U_1.$$

First, we choose a family of δU_1 that are zero at both ends. This means that

$$\int_0^L U_1''\delta U_1\, dX_1 = 0 \ \ \forall \delta U_1.$$

The fundamental lemma of the calculus of variation ensure us that

$$U_1'' = 0 \quad \rightarrow \quad U_1 = \alpha + \beta X_1$$

with α and β two constants. We have first to choose $\alpha = 0$ because U_1 is kinematically admissible so that $U_1(0) = 0$.

Now, let us consider the other variations that are non zero on $X_1 = L$. We have $EAU_1' = F$ for $X_1 = L$. This means that $EA\beta = F$. The solution is then

$$U_1 = \frac{F}{EA} X_1.$$

Let's now assume that the rod is submitted to its own weight. Assume that the gravity acts on the positive X_1 direction and that the rod is made of a material of density ρ. Gravity is a volume force of amplitude ρg. We have

$$\pi(U_1) = \frac{EA}{2} \int_0^L \left(U_1'\right)^2 dX_1 - A \int_0^L \rho g U_1 \, dX_1.$$

We then find

$$\delta\pi = A\left(\int_0^L \left[EU_1'\delta U_1' - \rho g \delta U_1\right] dX_1\right) = 0 \ \ \forall \delta U_1.$$

Integration by parts give

$$\int_0^L (-EU_1'' - \rho g)\delta U_1 \, dX_1 + EU_1'\delta U_1(L) = 0 \ \ \forall \delta U_1.$$

We finally get

$$U_1(X_1) = -\frac{\rho g}{2E} X_1^2 + \alpha X_1 + \beta.$$

Condition $U_1(0) = 0$ leads to $\beta = 0$. Condition $U_1'(L) = 0$ leads to

$$-\frac{\rho g L}{E} + \alpha = 0.$$

Finally, we get

$$\boxed{U_1(X_1) = \frac{\rho g}{2E}\left(2LX_1 - X_1^2\right).} \tag{3.3}$$

It is indeed possible to compute the normal effort in every section X_1 of the rod as

$$N = A\sigma_{11} = EA\epsilon_{11} = EA\frac{\mathrm{d}U_1}{\mathrm{d}X_1} = \rho g A(L - X_1).$$

3.2 The Ritz method

The Ritz method is a direct method to find an approximate solution for boundary value problems. Let's come back to the problem of a rod submitted to its own weight.

Exact solution of the problem is given in (3.3). Assume that we look for an approximate solution of the following form

$$U_1^r(X_1) = \alpha + \beta X_1. \tag{3.4}$$

We'd like to use the principle of total potential energy. For that, we need U_1^r to be compatible: constant α has then to be equal to zero for having $U_1^r(0) = 0$. This last form $U_1^r = \beta X_1$ is of course an approximation of the exact solution. Taking into account the approximation, the total potential energy of the system can be written as

$$\pi(U_1^r) = \frac{EA}{2}\int_0^L \beta^2 dX_1 - A\int_0^L \rho g \beta X_1 dX_1 = \frac{EAL\beta^2}{2} + \rho g A\beta\frac{L^2}{2}.$$

In this case, π is a function of one parameter β: it is not a functional anymore. Its minimum can be computed by zeroing the derivative of π with respect to β. After simple calculations, we find

$$\frac{\mathrm{d}\pi}{\mathrm{d}\beta} = 0 \quad \rightarrow \quad \beta = \frac{\rho g L}{2E}$$

and

$$U_1^r(X_1) = \frac{\rho g L}{2E} X_1. \tag{3.5}$$

Now let's look at the difference between the exact solution (3.3) and its linear approximation (3.5). It is interresting to see that (3.5) is exact at both end of the rod. This fact is not a stroke of luck.

Let us come back to the problem of elasticity. Assume $\mathbf{u}^r$ to be an approximate solution that has been found by the method of Ritz among a family of parametrized solutions $\mathbf{u}^* \in \mathscr{U}^* \subset \mathscr{U}$. We have

$$\pi(\mathbf{u}^r) \leq \pi(\mathbf{u}^*) \;\; , \quad \forall \mathbf{u}^*$$

or, if we assume no volume forces,

$$\frac{1}{2}\int_\Omega \epsilon(\mathbf{u}^r) : \mathbf{C} : \epsilon(\mathbf{u}^r)\, dv - \int_{\Gamma_N} \mathbf{F}\cdot\mathbf{u}^r\, ds - \frac{1}{2}\int_\Omega \epsilon(\mathbf{u}^*) : \mathbf{C} : \epsilon(\mathbf{u}^*)\, dv + \int_{\Gamma_N} \mathbf{F}\cdot\mathbf{u}^*\, ds \leq 0.$$

This can be re-written as

$$\frac{1}{2}\int_\Omega \epsilon(\mathbf{u}^r) : \mathbf{C} : \epsilon(\mathbf{u}^r)\, dv - \frac{1}{2}\int_\Omega \epsilon(\mathbf{u}^*) : \mathbf{C} : \epsilon(\mathbf{u}^*)\, dv - \int_{\Gamma_N} \mathbf{F}\cdot(\mathbf{u}^r - \mathbf{u}^*)\, ds \leq 0. \tag{3.6}$$

Assume that $\mathbf{u} \in \mathscr{U}$ is the exact solution: it verifies the principle of virtual work

$$\int_\Omega \epsilon(\mathbf{u}) : \mathbf{C} : \epsilon(\delta\mathbf{u})\, dv - \int_{\Gamma_N} \mathbf{F}\cdot(\delta\mathbf{u}) ds = 0 \quad \forall \delta\mathbf{u} \in \mathscr{U}_0.$$

It should be clear at that point that $\delta\mathbf{u} = \mathbf{u}^r - \mathbf{u}^* \in \mathscr{U}_0$ in (3.6) is a variation. Then,

$$\int_{\Gamma_N} \mathbf{F}\cdot(\mathbf{u}^r - \mathbf{u}^*)\, ds = \int_\Omega \epsilon(\mathbf{u}) : \mathbf{C} : \epsilon(\mathbf{u}^r - \mathbf{u}^*)\, dv.$$

We have then

$$\frac{1}{2}\int_{\Omega} \epsilon(\mathbf{u}^r) : \mathbf{C} : \epsilon(\mathbf{u}^r)\, dv - \frac{1}{2}\int_{\Omega} \epsilon(\mathbf{u}^*) : \mathbf{C} : \epsilon(\mathbf{u}^*)\, dv - \int_{\Omega} \epsilon(\mathbf{u}) : \mathbf{C} : \epsilon(\mathbf{u}^r - \mathbf{u}^*)\, dv \le 0. \tag{3.7}$$

Strains ϵ are linear operators. Equation (3.7) can be re-written as

$$\boxed{\underbrace{\frac{1}{2}\int_{\Omega} \epsilon(\mathbf{u}^r - \mathbf{u}) : \mathbf{C} : \epsilon(\mathbf{u}^r - \mathbf{u})\, dv}_{U(\mathbf{u}^r - \mathbf{u})} - \underbrace{\frac{1}{2}\int_{\Omega} \epsilon(\mathbf{u}^* - \mathbf{u}) : \mathbf{C} : \epsilon(\mathbf{u}^* - \mathbf{u})\, dv}_{U(\mathbf{u}^* - \mathbf{u})} \le 0.} \tag{3.8}$$

Equation (3.7) has a very interresting and useful interpretation. It says

$$U(\mathbf{u}^r - \mathbf{u}) \le U(\mathbf{u}^* - \mathbf{u})$$

which means that the difference in term of energy between the exact solution $\mathbf{u}$ and its Ritz approximation $\mathbf{u}^r$ is smaller than the difference between $\mathbf{u}$ and any other approximation $\mathbf{u}^*$. This means that the solution obtained by Ritz method is *the best in terms of energy*. It is an orthogonal projection of the exact solution onto the space of approximation where distances are measured in terms of energy (see Figure 3.2). It actually also means that, whenever the space of approximation contains the exact solution, Ritz method will find it.

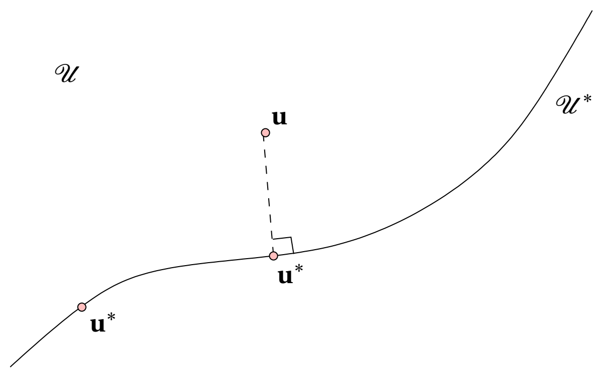

Figure 3.2: Illustration of the orthogonality property (3.8).

3.3 Elastic energy of a truss structure

Equation (3.2) allows to compute the elastic energy of one single rod. In (3.4), we have then assumed that the axial displacement $U_1(X_1)$ of the rod was linear. This is indeed a very common assumption: if the rod is loaded on its two ends, this is indeed the exact solution. When volume forces are involved such as gravity, then this is an approximate solution. The form (3.4) is not the most convenient: coefficients

α and β do not have a clear significations. One more interresting way of expressing the displacement along the rod is to use

$$U_1(X_1) = U_1^1\left(1 - \frac{X_1}{L}\right) + U_1^2 \frac{X_1}{L}. \tag{3.9}$$

Here, coefficients U_1^1 and U_1^2 have a clear meaning. For $X_1 = 0$, we get $U_1(0) = U_1^1$ which means that U_1^1 is the displacement of the left end of the rod. Similarly, for $X_1 = L$, we get $U_1(L) = U_1^2$ which means that U_1^2 is the displacement of the right end of the rod. In the finite element method, the unknowns are usually the displacements of the nodes of the structure. This allows in fact to compute the elastic energy of a whole truss structure.

Consider (3.9). The elastic energy (3.2) of this rod can be expressed as

$$U = \frac{EA}{2}\int_0^L (U_1')^2\, dX_1 = \frac{EA}{2L}(U_1^2 - U_1^1)^2 = \frac{1}{2}\left(\begin{array}{c} U_1^1 \\ U_1^2 \end{array}\right)^T \left[\begin{array}{cc} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{array}\right]\left(\begin{array}{c} U_1^1 \\ U_1^2 \end{array}\right)$$

Writing the elastic energy in matrix form will be very useful in what follows.

A truss consists of straight rods connected at joints or nodes. In a truss, loads are applied to nodes only. A planar truss is a truss for which all members are in a given plane and where all the loads are applied in the same plane. We consider planar trusses first.

The kinematic behavior of a planar truss can be described by its nodal displacements. Assume a truss with N nodes and B bars (or rods). Node i has coordinates $\mathbf{x}^i = \{x_1^i, x_2^i\}$ and its displacement is noted $\mathbf{u}^i = \{u_1^i, u_2^i\}$. Note here that positions and displacements of the nodes are given using the same system of coordinates (we use lower case letters for coordinates in the common euclidian frame). Bar j connects node b_1^j and b_2^j. It has a length L^j, a section A^j and it is made of a material of Young modulus E^j We want now to compute the elastic energy U^j of bar j as a function of global displacements $\mathbf{u}^{b_1^j}$ and $\mathbf{u}^{b_2^j}$. Assume a bar that has its local axis X_1 inclined with an angle θ with the global x_1 axis (see Figure 3.3). Any vector has two sets of coordinates, one expressed in the global frame $\mathbf{v} = \{v_1, v_2\}$ and one in the local frame $\mathbf{V} = \{V_1, V_2\}$. We have

$$\mathbf{V} = \left(\begin{array}{c} V_1 \\ V_2 \end{array}\right) = \left[\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array}\right]\left(\begin{array}{c} v_1 \\ v_2 \end{array}\right) = [\mathbf{T}]\mathbf{v}.$$

and

$$\mathbf{v} = \left(\begin{array}{c} v_1 \\ v_2 \end{array}\right) = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]\left(\begin{array}{c} V_1 \\ V_2 \end{array}\right) = [\mathbf{T}]^T\mathbf{V}.$$

The elastic energy of rod j can be expressed as

$$U^j = \frac{1}{2}\left(\begin{array}{c} U_1^{b_1^j} \\ U_2^{b_1^j} \\ U_1^{b_2^j} \\ U_2^{b_2^j} \end{array}\right)^T \left[\begin{array}{cccc} \frac{E^jA^j}{L^j} & 0 & -\frac{E^jA^j}{L^j} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{E^jA^j}{L^j} & 0 & \frac{E^jA^j}{L^j} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\left(\begin{array}{c} U_1^{b_1^j} \\ U_2^{b_1^j} \\ U_1^{b_2^j} \\ U_2^{b_2^j} \end{array}\right) = \frac{1}{2}(\mathbf{U}^j)^T[\mathbf{K}^j](\mathbf{U}^j)$$

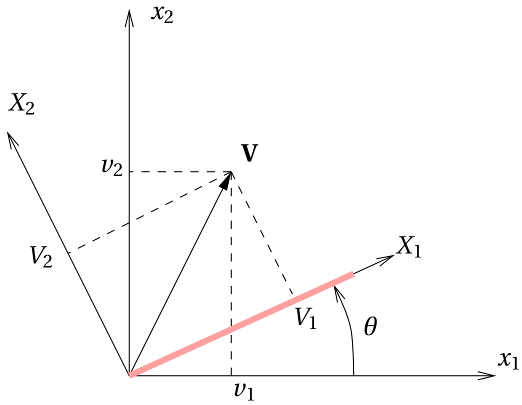

Figure 3.3: Local and global frames

where $[\mathbf{K}^j]$ is the local stiffness matrix of bar j and $(\mathbf{U}^j)$ its local displacement vector. The zeros in the local stifness matrix indicate that no energy is stored for any displacement along the local X_2 axis. We use then the transformation formula $(\mathbf{U}^j) = [\mathbf{T}^j](\mathbf{u}^j)$ to express the energy as a function of global displacements $(\mathbf{u}^j)$

$$U^j = \frac{1}{2}(\mathbf{U}^j)^T[\mathbf{K}^j](\mathbf{U}^j) = \frac{1}{2}(\mathbf{u}^j)^T[\mathbf{T}^j]^T[\mathbf{K}^j][\mathbf{T}^j](\mathbf{u}^j) = \frac{1}{2}(\mathbf{u}^j)^T[\mathbf{k}^j](\mathbf{u}^j).$$

Here,

$$[\mathbf{k}^j] = [\mathbf{T}^j]^T[\mathbf{K}^j][\mathbf{T}^j] \tag{3.10}$$

is the global stifness matrix of bar j. Energy is an extensive quantity and the energy of a given truss is computed as

$$U = \sum_{j=1}^{B} U^j.$$

3.4 Global equilibrium of a truss

3.4.1 Stiffness Matrix

Let's now define the global displacement vector

$$(\mathbf{u}) = \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \\ \vdots \\ u_1^N \\ u_2^N \end{pmatrix}$$

as a vector of size $2N$ that contains all nodal displacements of the truss. Node i of the truss correspond to entries $2i-1$ and $2i$, entry $2i-1$ corresponding to the displacement along x_1 and entry $2i$ corresponding to the displacement along x_2.

Our aim now is to express the elastic energy of the truss as the quadratic form

$$U = \frac{1}{2}(\mathbf{u})^T[\mathbf{k}](\mathbf{u})$$

where $[\mathbf{k}]$ is the global stifness matrix of the truss of size $2N \times 2N$. The energy of bar

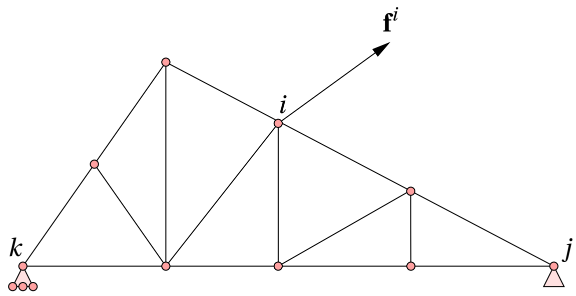

Figure 3.4: A truss with loads and fixations.

j can also be written as the quadratic form

$$U^j = \frac{1}{2}(\mathbf{u})^T[\mathbf{k}_e^j](\mathbf{u})$$

with $[\mathbf{k}_e^j]$ the extended stiffness matrix i.e. a $2N \times 2N$ matrix that is essentially filled up with zeros and where the only 16 non zero entries are the ones of $[\mathbf{k}^j]$. The indices of the 4 degrees of freedom of bar j are

$$\iota = \{\iota_1, \iota_2, \iota_3, \iota_4\} = \{2b_1^j - 1, 2b_1^j, 2b_2^j - 1, 2b_2^j\}$$

and the 16 non zero entries are

$$[\mathbf{k}_e^j]_{\iota_i, \iota_j} = [\mathbf{k}^j]_{i,j} \quad , \quad i, j = 1, \dots, 4.$$

The elastic energy of a truss is therefore

$$U = \sum_{j=1}^{B} U^j = \frac{1}{2}\sum_{j=1}^{B}(\mathbf{u})^T[\mathbf{k}_e^j](\mathbf{u}) = \frac{1}{2}(\mathbf{u})^T\left[\sum_{j=1}^{B}[\mathbf{k}_e^j]\right](\mathbf{u}) = \frac{1}{2}(\mathbf{u})^T[\mathbf{k}](\mathbf{u})$$

with $[\mathbf{k}] = \sum_{j=1}^{B}[\mathbf{k}_e^j]$ i.e. the global stiffness matrix $[\mathbf{k}]$ is simply “assembled” as the sum of extended stiffness matrices $[\mathbf{k}_e^j]$.

3.4.2 Force Vector

A truss is loaded at its vertices. If $\mathbf{f}^i = \{f_1^i, f_2^i\}$ (see Figure 3.4) is the external load at vertex i, we define the global force vector as

$$(\mathbf{f}) = \begin{pmatrix} f_1^1 \\ f_2^1 \\ f_1^2 \\ f_2^2 \\ \vdots \\ f_1^N \\ f_2^N \end{pmatrix}.$$

The work of external loads can the be expressed as

$$W_{ext} = (\mathbf{u})^T(\mathbf{f})$$

and the total potential energy of the truss is

$$\boxed{\pi = \frac{1}{2}(\mathbf{u})^T[\mathbf{k}](\mathbf{u}) - (\mathbf{u})^T(\mathbf{f}).} \quad (3.11)$$

3.4.3 Supports

The natural thing to do now could be to minimize π with respect to $(\mathbf{u})$. Yet, the princple of the minimum of total potential energy only applies to displacements that are admissible i.e. that verify *a priori* the essential boundary conditions i.e. the displacements that are compatible with the supports. On the truss of Figure 3.4, displacement of node j along x_2 should be equal to zero: $(\mathbf{u})_{2j} = 0$. Similarly, the displacement vector of node k should be equal to zero: $(\mathbf{u})_{2k-1} = (\mathbf{u})_{2k} = 0$.

Lagrange multipliers

The most elegant way of dealing with supports is to use constrained optimization. Assume that the supports of the truss consist in M linear constraints

$$[\mathbf{c}](\mathbf{u}) = (\bar{\mathbf{u}}) \quad (3.12)$$

where $[\mathbf{c}]$ is a $M \times 2N$ matrix of full rank and where $(\bar{\mathbf{u}})$ is the right hand side i.e. a vector of size M.

In mathematical optimization, *the method of Lagrange multipliers* is a strategy for finding the local maxima and minima of a function subject to equality constraints.

For instance, consider the optimization problem maximize $f(x, y)$ subject to $g(x, y) = 0$. We introduce a new variable λ called a *Lagrange multiplier* and study the Lagrangian defined by

$$\mathscr{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y).$$

We then solve

$$\nabla_{x,y,\lambda}\mathscr{L}(x,y,\lambda) = 0.$$

This is the method of Lagrange multipliers. Note that $\nabla_{\lambda}\mathscr{L}(x,y,\lambda) = 0$ of course implies $g(x,y) = 0$.

We apply the method of Lagrange multipliers to our truss problem. Let us define $(\boldsymbol{\lambda})$ to be a M-vector that contains the Lagrange multipliers. We define the Lagrangian

$$\mathscr{L}((\mathbf{u}),(\boldsymbol{\lambda})) = \frac{1}{2}(\mathbf{u})^T[\mathbf{k}](\mathbf{u}) - (\mathbf{u})^T(\mathbf{f}) + [[\mathbf{c}](\mathbf{u}) - (\bar{\mathbf{u}})]^T(\boldsymbol{\lambda}). \tag{3.13}$$

We then look for a stationary point of $\mathscr{L}$:

$$\frac{\partial\mathscr{L}}{\partial(\mathbf{u})} = (\mathbf{0}) \quad \rightarrow \quad [\mathbf{k}](\mathbf{u}) - (\mathbf{f}) + [\mathbf{c}]^T(\boldsymbol{\lambda}) = (\mathbf{0}),$$

$$\frac{\partial\mathscr{L}}{\partial(\boldsymbol{\lambda})} = (\mathbf{0}) \quad \rightarrow \quad [\mathbf{c}](\mathbf{u}) - (\bar{\mathbf{u}}) = (\mathbf{0}).$$

The following system of $2N + M$ equations allows to find $(\mathbf{u})$ and $(\boldsymbol{\lambda})$ in one linear system solve:

$$\boxed{\begin{bmatrix} [\mathbf{k}] & [\mathbf{c}]^T \\ [\mathbf{c}] & [\mathbf{0}] \end{bmatrix} \begin{pmatrix} (\mathbf{u}) \\ (\boldsymbol{\lambda}) \end{pmatrix} = \begin{pmatrix} (\mathbf{f}) \\ (\bar{\mathbf{u}}) \end{pmatrix}.} \tag{3.14}$$

Lagrange multipliers usually have an interresting interpretation. Here $[\mathbf{c}]^T(\boldsymbol{\lambda})$ are *reaction forces* that are associated to the constraints. In the simplest case of the simply supported node j of Figure 3.4, the Lagrange multiplier relative to constraint $u_{2j} = 0$ is the vertical reaction force at node j. The formulation (3.14) has the great advantage to provide both displacements and reaction forces!

3.5 Example

Finite elements are not meant to be solved on paper: they should be programmed and solved on a computer. Nevertheless, let's solve one simple truss unsing finite elements and Lagrange multipliers in order to illustrate the method.

It is indeed easy to compute analytically the stiffness matrix (3.10) of bar j with its characteristics A^j, E^j and L^j:

$$[\mathbf{k}^j] = [\mathbf{T}^j]^T[\mathbf{K}^j][\mathbf{T}^j] =$$

$$\frac{E^jA^j}{L^j}\begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) & -\cos^2(\theta) & -\sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) & -\sin(\theta)\cos(\theta) & -\sin^2(\theta) \\ -\cos^2(\theta) & -\sin(\theta)\cos(\theta) & \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & -\sin^2(\theta) & \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{bmatrix}.$$

Consider the truss depicted on Figure 3.5. Assume A and E constant for both bars. The truss has 2 bars and 3 nodes. The stiffness matrix of bar 1 ($b_1^1 = 2$, $b_2^1 = 3$, $\theta =$

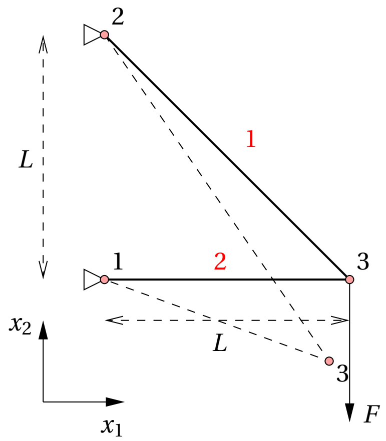

Figure 3.5: A simple truss.

$-\pi/4$) is

$$[\mathbf{k}^1] = \frac{EA}{2\sqrt{2}L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The stiffness matrix of bar 2 ($b_1^2 = 1$, $b_2^2 = 3$, $\theta = 0$) is

$$[\mathbf{k}^2] = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The stiffness matrix of the truss is assembled as

$$[\mathbf{k}] = \underbrace{\frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{[\mathbf{k}_e^1]} + \underbrace{\frac{EA}{2\sqrt{2}L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}}_{[\mathbf{k}_e^2]}.$$

We see here that it is mandatory to impose the value of $(\mathbf{u})_2$ because this degree of freedom is not associated to any stiffness: all elements of row 2 of $[\mathbf{k}]$ are identically equal to 0. In fact, at least 3 constraints have to be imposed to a truss in order to avoid global translation (2 kinematic modes) and rotation (1 kinematic mode). In the case of the truss of Figure 3.5, we find

$$[\mathbf{c}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

and $(\bar{\mathbf{u}}) = [\mathbf{0}]$. The force vector is then computed as

$$(\mathbf{f})^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -F \end{bmatrix}.$$

At that point, solving the problem on paper is cumbersome: we have here a system of 10 equations with 10 unknowns!

In the case of simple constraints such as zeroing a displacement, one way to simplify the resolution on paper is to remove lines and columns of the stiffness matrix that correspond to the displacements that are null. We have then:

$$[\mathbf{k}] = \frac{EA}{2\sqrt{2}L} \begin{bmatrix} 1+2\sqrt{2} & -1 \\ -1 & 1 \end{bmatrix}.$$

Here, the constraints have been implicitely taken into account and finding the static equilibrium of the truss consist in solving

$$(\mathbf{u}) = [\mathbf{k}]^{-1}(\mathbf{f}).$$

We have

$$[\mathbf{k}]^{-1} = \frac{L}{EA} \begin{bmatrix} 1 & 1 \\ 1 & 1+2\sqrt{2} \end{bmatrix}.$$

and the solution is

$$u_1^3 = -\frac{FL}{EA} \quad , \quad u_2^3 = -\frac{FL}{EA}(1+2\sqrt{2}).$$