CHAPTER 5

Frames

5.1 The Bernoulli-Euler model for beams

The kinematic behavior of a beam involves in-plane flexure. In its local coordinates, the Bernoulli-Euler for a beam writes:

$$
\begin{aligned}
U_1 &= U_1(X_1) - X_2 \frac{dU_2}{dX_1} \\
U_2 &= U_2(X_1) \\
U_3 &= 0.
\end{aligned}
\tag{5.1}
$$

Here, U_2 represents the displacement of the neutral axis of the beam along X_2 (see Figure 5.1).

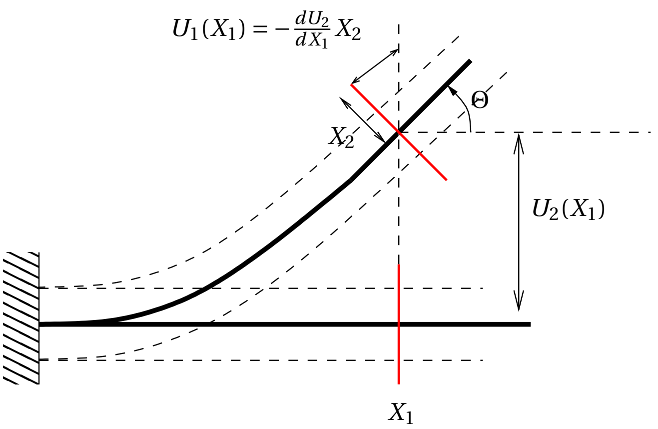

Figure 5.1: The Bernoulli-Euler model for a beam (pure bending).

This model can be seen as the sum of two model: (i) a bar model (3.1) that allows energy to be stored in traction/compression and (ii) a pure flexure model that allows to store flexural energy only. Assume the bar to be of length L, with a bending stiffness EI and a axial stiffness EA. The energy of deformation of the bar is written as

$$U = \frac{1}{2}\int_0^L EA\left(\frac{dU_1}{dX_1}\right)^2 \, dX_1 + \frac{1}{2}\int_0^L EI\left(\frac{d^2U_2}{dX_1^2}\right)^2 \, dX_1. \tag{5.2}$$

The energy of deformation has to be finite. This means that U_1 has to have its first derivative square-integrable. It is possible to prove that this condition is verified for every $U_1 \in C^0$, the space of continuous functions. The flexure term involves the square of the second derivative of U_2. Functions that have their second derivative square-integrable should not only be continuous. They should also have their first derivative continuous. We have to choose $U_2 \in C^1$, the space of continuously derivable functions.

5.2 Finite Elements for Beams

The extra difficulty that arises here comes from that fact that U_2 should be C^1. The most obvious fashion of ensuring that a discretization is C^0 is to use degrees of freedom that are displacements at nodes. For getting a C^1 discretization, one has simply to add degrees of freedom that are the first derivatives of U_2 at nodes. A beam element has therefore 6 degrees of freedom:

- U_1^1 and U_1^2, the two nodal displacements along X_1.
- U_2^1 and U_2^2, the two nodal displacements along X_2.
- Θ_3^1 and Θ_3^2, the two nodal derivatives of U_2 with respect to X_1. Those are noted Θ_3 because theu correspond to rotations around X_3.

Displacement $U_2(X_1)$ depends on 4 parameters U_2^1, U_2^2, Θ_3^1 and Θ_3^2. Cubic polynomials $N_j(X_1)$, $j = 1,\ldots,4$ are used to discretize U_2 as:

$$\begin{aligned} U_2(X_1) &= N_1(X_1)U_2^1 + N_2(X_1)\Theta_3^1 + N_3(X_1)U_2^2 + N_4(X_1)\Theta_3^2 \\ &= \sum_{i=1}^{4} \tilde{U}_i N_i(X_1) \end{aligned} \tag{5.3}$$

with $\tilde{\mathbf{U}} = \{U_2^1, \Theta_3^1, U_2^2, \Theta_3^2\}$. It is easy to computes functions N_j. The sixteen constants that are required to describe the 4 cubic polynomials are selected in order to satisfy the following 16 relation

$$\begin{array}{rcl} (U_2)(0) = U_2^1 & \to & N_1(0) = 1,\ N_2(0) = 0,\ N_3(0) = 0,\ N_4(0) = 0, \\ (U_2')(0) = \Theta_3^1 & \to & N_1'(0) = 0,\ N_2'(0) = 1,\ N_3'(L) = 0,\ N_4'(L) = 0, \\ (U_2)(L) = U_2^2 & \to & N_1(L) = 0,\ N_2(L) = 0,\ N_3(L) = 1,\ N_4(L) = 0, \\ (U_2')(L) = \Theta_3^2 & \to & N_1'(L) = 0,\ N_2'(L) = 0,\ N_3'(L) = 0,\ N_4'(L) = 1. \end{array}$$

For example, we can assume

$$N_1(X_1) = A + BX_1 + CX_1^2 + DX_1^3.$$

We have $N1_(0) = 1 \rightarrow A = 1$. Then, $N_1'(0) = 0 \rightarrow B = 0$, $N_1(L) = 0 \rightarrow 1 + CL^2 + DL^3 = 0$ and $N_1'(L) = 0 \rightarrow 2CL + 3DL^2 = 0$. It is easy to determine B and C using

$$\begin{bmatrix} L^2 & L^3 \\ 2L & 3L^2 \end{bmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

We find $C = -3/L^2$ and $D = 2/L^3$. Then, posing $t = \left(\frac{X_1}{L}\right)$, we find

$$N_1(t) = 1 - 3t^2 + 2t^3.$$

After computations, we find

$$\begin{aligned} N_1(t) &= 1 - 3t^2 + 2t^3 \\ N_2(t) &= Lt(t-1)^2 \\ N_3(t) &= t^2(-2t+3) \\ N_4(t) &= -L(1-t)t^2. \end{aligned} \tag{5.4}$$

Polynomials N_i of (5.4) are called *Hermite polynomials*. They are represented in Figure 5.2 Let us now write the U (see (5.2)) using approximation (5.3) for the transver-

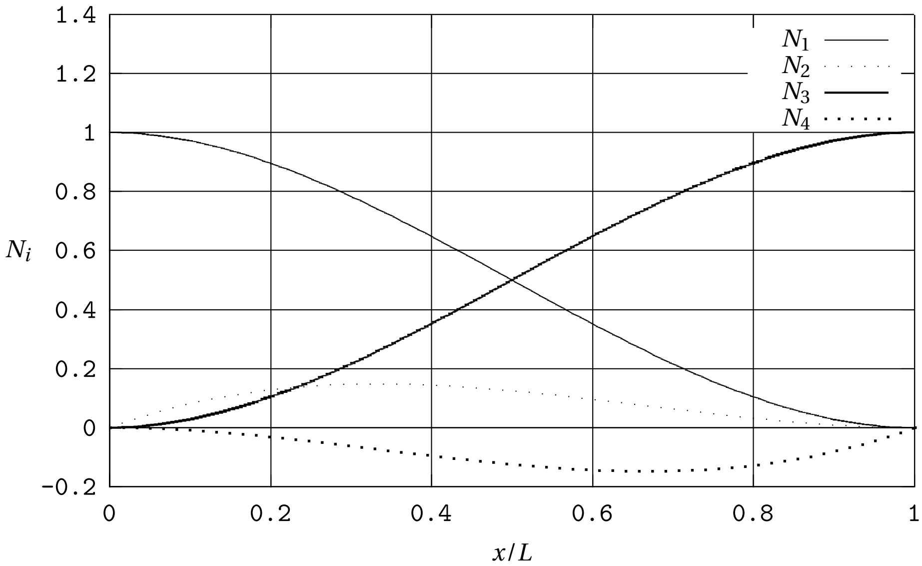

Figure 5.2: Cubic Hermite polynomials.

sal displacement U_2 and using (3.9) for U_1:

$$\begin{aligned} U &= \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\tilde{U}_i\tilde{U}_j\int_0^L EI\frac{d^2N_i}{dX_1^2}\frac{d^2N_j}{dX_1^2}\,dX_1 + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}U_1^iU_1^j\int_0^L EA\frac{dM_i}{dX_1}\frac{dM_j}{dX_1}\,dX_1 \\ &= \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\tilde{U}_i\tilde{U}_jK_{ij}^f + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}U_1^iU_1^jK_{ij}^a. \end{aligned} \tag{5.5}$$

Assuming EA and EI to be constant along the bar, we have the expressions

$$[\mathbf{K}^f] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{5.6}$$

and

$$[\mathbf{K}^a] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{5.7}$$

Grouping the two terms, we get

$$\begin{aligned} U &= \frac{1}{2}\begin{pmatrix} U_1^1 \\ U_2^1 \\ \Theta_3^1 \\ U_1^2 \\ U_2^2 \\ \Theta_3^2 \end{pmatrix}^T \begin{bmatrix} K_{11}^a & 0 & 0 & K_{12}^a & 0 & 0 \\ 0 & K_{11}^f & K_{12}^f & 0 & K_{13}^f & K_{14}^f \\ 0 & K_{21}^f & K_{22}^f & 0 & K_{23}^f & K_{24}^f \\ K_{21}^a & 0 & 0 & K_{22}^a & 0 & 0 \\ 0 & K_{31}^f & K_{32}^f & 0 & K_{33}^f & K_{34}^f \\ 0 & K_{41}^f & K_{42}^f & 0 & K_{43}^f & K_{44}^f \end{bmatrix} \begin{pmatrix} U_1^1 \\ U_2^1 \\ \Theta_3^1 \\ U_1^2 \\ U_2^2 \\ \Theta_3^2 \end{pmatrix} \\ &= \frac{1}{2}(\mathbf{U})^T[\mathbf{K}](\mathbf{U}). \end{aligned} \tag{5.8}$$

We have now to give an expression of the energy in global coordinates i.e. using the 6 global degrees of freedom of the beam. Assume that the bar is inclined of an angle α with respect to the X_1 axis, we can write

$$(\mathbf{u}) = [\mathbf{T}](\mathbf{U})$$

with

$$[\mathbf{T}] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.9}$$

We have then

$$U = \frac{1}{2}(\mathbf{U})^T[\mathbf{K}](\mathbf{U}) = \frac{1}{2}(\mathbf{u})^T[\mathbf{T}]^T[\mathbf{K}][\mathbf{T}](\mathbf{u}) = \frac{1}{2}(\mathbf{u})^T[\mathbf{k}](\mathbf{u}).$$

The analytical computation of $[\mathbf{k}] = [\mathbf{T}]^T[\mathbf{K}][\mathbf{T}]$ is not given explicitely. This stiffness matrix is usually computed numerically.

5.3 Solving planar trusses with finite elements

The resolution of a planar frame uses the same kind of process as the one we described in Chapters 3 and 4. Here, three degrees of freedom $\mathbf{u}^j = \{u_1^j, u_2^j, \theta_3^j\}$ have to be defined for each node j of the frame. Each bar is associated to a 6×6 stiffness matrix. Forces and moments can be applied to joints of the frame in the same fashion as it was done for trusses. Yet, in the case of frames, continuous distributions of forces $\tau(X_1)[N/m]$ can be applied, for taking into account proper weights of elements or for taking into accound wind forces.

Let us try to solve the problem of a Cantilever beam as it is depicted on Figure 5.3.

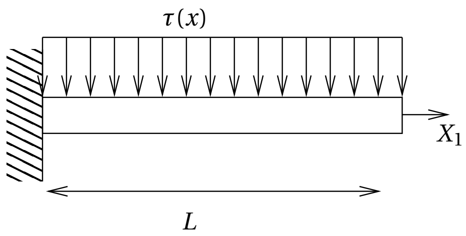

Figure 5.3: A Cantilever Beam.

Its exact solution can easily be found:

$$U_2(X_1) = -\frac{\tau X_1^2(6L^2 - 4LX_1 + X_1^2)}{24EI}, \quad M(X_1) = EIU_2'' = -\frac{\tau(L^2 - 2LX_1 + X_1^2)}{2}, \quad T(X_1) = M'(X_1) = \tau(X_1 - L).$$

Now let's try to solve this problem using one single finite element. The frame has two nodes as depicted in Figure 5.4.

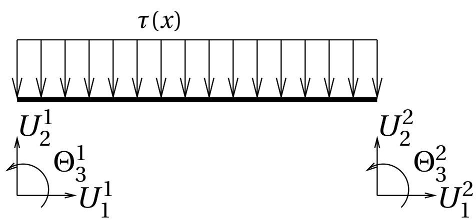

Figure 5.4: A Cantilever Beam with 6 degrees of freedom.

The work of load $\tau(X_1)$ is computed as

$$W_{ext} = \int_0^L \tau(X_1)U_2(X_1)dX_1 = \sum_{i=1}^4 \tilde{U}_i \int_0^L \tau(X_1)N_i(X_1)dX_1 = \sum_{i=1}^4 \tilde{U}_i\tilde{F}_i$$

with $(\tilde{\mathbf{U}}) = \{U_2^1, \Theta_3^1, U_2^2, \Theta_3^2\}$ and $(\tilde{\mathbf{F}}) = \{T_2^1, M_3^1, T_2^2, M_3^2\}$ a nodal force vector that is a projection of τ onto the shape functions N_i of the beam. This process can be seen as replacing a continuous distribution of forces $\tau(X_1)$ by nodal forces: two vertical loads T_2^1 and T_2^2 associated to nodes 1 and 2 and two moments of force M_3^1 and M_3^2 associated to nodes 1 and 2 as well. Solving the Cantilever problem with one finite element consist in solving a simplified problem where the continuous load have been replaced by "equivalent" nodal forces and moments. In this case, the finite element method will find the exact solution of the simplified problem, but not to the initial problem with continuous loads. Assume τ to be constant. We have

$$(\tilde{\mathbf{F}}) = \left\{-\tau L/2, \tau L^2/12, -\tau L/2, -\tau L^2/12\right\}.$$

The solution to the simplified problem is solved using finite elements. We have $U_2^1 = \Theta_3^1 = 0$ and

$$\frac{EI}{L^3}\left[\begin{array}{cc} 12 & -6L \\ -6L & 4L^2 \end{array}\right]\left(\begin{array}{c} U_2^2 \\ \Theta_3^2 \end{array}\right) = \left(\begin{array}{c} -\tau L/2 \\ -\tau L^2/12 \end{array}\right)$$

or

$$\left[\begin{array}{cc} 12 & -6L \\ -6L & 4L^2 \end{array}\right]\left(\begin{array}{c} U_2^2 \\ \Theta_3^2 \end{array}\right) = -\frac{\tau L^4}{12EI}\left(\begin{array}{c} 6 \\ L \end{array}\right).$$

We find

$$U_2^2 = -\frac{5\tau L^4}{24EI} \quad \text{and} \quad \Theta_3^2 = -\frac{\tau L^3}{3EI}.$$

The finite element solution finally writes

$$U_2(t) = -\frac{5\tau L^4}{24EI} t^2 (-2t+3) + \frac{\tau L^4}{3EI} (1-t)\, t^2.$$

For $X_2 = L$ we of course get $U_2(1) = U_2^2 = -\frac{5\tau L^4}{24EI}$ which is different from the exact value $-\frac{3\tau L^4}{24EI}$.

The best way of improving the results is to split the beam in multiple finite elements. Figure 5.5 shows the simplified finite element problem arising from a 2 element discretization. Here, we see that moments at interior nodes fall down to zero because both left and right beams provide the node opposite contributions. Here, the problem has 6 degrees of freedom (degress of freedom for node 1 are all zero) and it is not possible to solve it by hand.

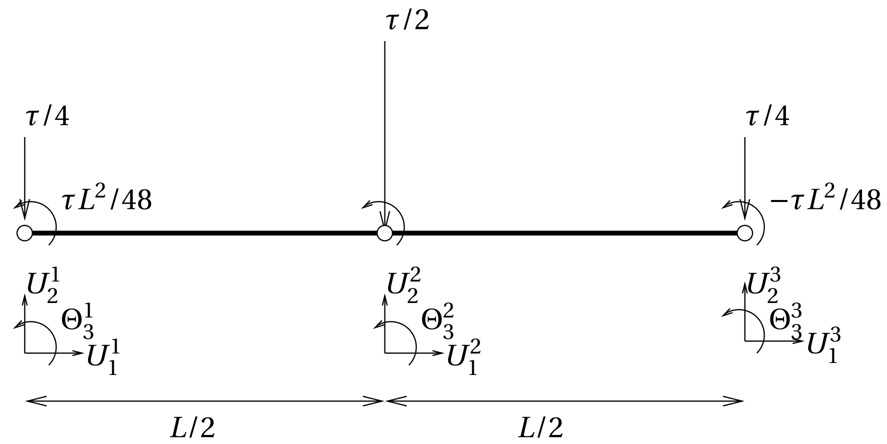

Figure 5.5: Discretization using 2 finite elements