CONJUGACY CLASSES OF TRIALITARIAN AUTOMORPHISMS AND SYMMETRIC COMPOSITIONS

V. CHERNOUSOV, M.-A. KNUS, AND J.-P. TIGNOL

Dedicated with great friendship to Eva Bayer on the occasion of her 60^{th} *birthday*

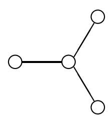
Abstract. The trialitarian automorphisms considered in this paper are the outer automorphisms of order 3 of adjoint classical groups of type D_4 over arbitrary fields. A one-to-one correspondence is established between their conjugacy classes and similarity classes of symmetric compositions on 8-dimensional quadratic spaces. Using the known classification of symmetric compositions, we distinguish two conjugacy classes of trialitarian automorphisms over algebraically closed fields. For type I, the group of fixed points is connected and simple of type G_2, whereas, if the characteristic of the field is different from 3, it is adjoint of type A_2 for trialitarian automorphisms of type II.

1. Introduction

Among simple algebraic groups of classical type only the simple adjoint algebraic groups $G = \mathbf{PGO}^+(n)$ and the simple simply connected algebraic groups $G = \mathbf{Spin}(n)$, where n is the norm of an octonion algebra, admit outer automorphisms of order 3, known as trialitarian automorphisms, see [KMRT98, (42.7)] or [Jac64]. These groups are of type D_4 and there is a split exact sequence of algebraic groups

$$1 \to \mathbf{Int}(G) \to \mathbf{Aut}(G) \to \mathfrak{S}_3 \to 1 \tag{1.1}$$

where the permutation group of three elements $\mathfrak{S}_3$ is viewed as the group of automorphisms of the Dynkin diagram of type D_4:



In view of the exact sequence (1.1), all the trialitarian automorphisms of G can be obtained from a fixed one ρ by composing ρ or ρ^{-1} with inner automorphisms. They are not necessarily conjugate to ρ, however. Our goal is to classify trialitarian automorphisms defined over an arbitrary field F up to conjugation in the group $\mathbf{Aut}(G)(F)$ of F-automorphisms of G. We achieve this goal by relating trialitarian automorphisms to symmetric compositions and using the known classification of composition algebras.

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We take as guiding principle the analogous description of outer automorphisms of order 2 of $\mathbf{PGL}_n$ for $n \geq 3$, which could be termed *dualitarian automorphisms*: they have the form $f \mapsto \sigma(f)^{-1}$, where σ is the adjoint involution of a nonsingular symmetric or skew-symmetric bilinear form. Dualitarian automorphisms on $\mathbf{PGL}_n$ are thus in one-to-one correspondence with nonsingular symmetric or skew-symmetric bilinear forms on an n-dimensional vector space up to scalar multiples, and dualitarian automorphisms are conjugate if and only if the corresponding bilinear forms are similar. There are two types of dualitarian automorphisms, distinguished by the type of their groups of fixed points, which can be either symplectic or (in characteristic different from 2) orthogonal. In characteristic 2, the non-symplectic case leads to group schemes that are not smooth.

Likewise, we set up a one-to-one correspondence between trialitarian automorphisms of $\mathbf{PGO}^+(n)$, for n a 3-fold Pfister quadratic form, and symmetric compositions up to scalar multiples on the underlying vector space of n, and use it to define a bijection between conjugacy classes of trialitarian automorphisms and similarity classes of symmetric compositions. We isolate two types of symmetric compositions. Type I is related to octonion algebras; the fixed subgroups are connected simple of type G_2. When the characteristic is not 3, type II is related to central simple algebras of degree 3; the fixed subgroups are adjoint of type ${}^1\mathrm{A}_2$ or ${}^2\mathrm{A}_2$. In characteristic 3, the fixed subgroups under trialitarian automorphisms of type II are not smooth.

Triality for simple Lie groups first appears in the paper [Car25] of É. Cartan, who already noticed that octonions can be used to explicitly define trialitarian automorphisms.[1] The observation that symmetric compositions are particularly well suited for that purpose is due to M. Rost.

As far as we know a complete classification of trialitarian automorphisms of simple groups of type D_4 had only been obtained over $\mathbb{C}$ (in relation with Lie algebras, see [WG68]), over finite fields, in [GL83, (9.1)], and in [GLS98, Theorem (4.7.1)] for the algebraic closure of a finite field of characteristic different from 3. A summary of known results for Lie algebras can be found in [Knu09]. Like for duality, there is also a projective geometric version of triality, which is in fact older than Cartan's triality and goes back to Study [Stu13]. A classification of geometric trialities, as well as of their groups of automorphisms, was done by Tits in [Tit59].

If not explicitly mentioned F denotes throughout the paper an arbitrary field.

2. Similarities of quadratic spaces

A quadratic space over F is a finite-dimensional vector space V over F with a quadratic form $q\colon V \to F$. We always assume that q is nonsingular, in the sense that the *polar bilinear form* b_q defined by

$$b_q(x,y) = q(x+y) - q(x) - q(y) \qquad \text{for } x,\, y \in V$$

has radical $\{0\}$. We also assume throughout that $\dim V$ is *even*. Let ad_q denote the involution on $\mathrm{End}_F V$ such that

$$b_q\big(f(x), y\big) = b_q\big(x, \mathrm{ad}_q(f)(y)\big) \qquad \text{for all } f \in \mathrm{End}_F V \text{ and } x,\, y \in V.$$

This involution is said to be *adjoint* to q. Let $\mathbf{GO}(q)$ be the F-algebraic group of similarities of (V,q), whose group of rational points $\mathbf{GO}(q)(F)$ consists of linear

[1] We refer to [KMRT98, pp. 510–511] and [SV00, §3.8] for historical comments on triality.

maps $f\colon V\to V$ for which there exists a scalar $\mu(f)\in F^\times$, called the *multiplier* of f, such that

$$q\big(f(x)\big)=\mu(f)q(x)\qquad\text{for all } x\in V.$$

The center of $\mathbf{GO}(q)$ is the multiplicative group $\mathbf{G}_{\mathrm{m}}$, whose rational points are viewed as homotheties. Let $\mathbf{PGO}(q)$ be the F-algebraic group of automorphisms of $\operatorname{End}_F V$ that commute with the adjoint involution ad_q. This group is identified with the quotient $\mathbf{GO}(q)/\mathbf{G}_{\mathrm{m}}$, acting on $\operatorname{End}_F V$ by inner automorphisms: for $f\in\mathbf{GO}(q)(F)$, we let $[f]$ be the image of f in $\mathbf{GO}(q)(F)/\mathbf{G}_{\mathrm{m}}(F)$ and identify $[f]$ with

$$\operatorname{Int}[f]\colon \operatorname{End}_F V\to\operatorname{End}_F V,\qquad \phi\mapsto f\phi f^{-1},$$

see [KMRT98, §23]. For simplicity, we write

$$\mathrm{GO}(q)=\mathbf{GO}(q)(F)\qquad\text{and}\qquad \mathrm{PGO}(q)=\mathbf{PGO}(q)(F)=\mathrm{GO}(q)/F^\times.$$

Let $C(V,q)$ be the Clifford algebra of the quadratic space (V,q) and let $C_0(V,q)$ be the even Clifford algebra. We let σ be the canonical involution of $C(V,q)$, such that $\sigma(x)=x$ for $x\in V$, and use the same notation for its restriction to $C_0(V,q)$. Every similarity $f\in\mathrm{GO}(q)$ induces an automorphism $C_0(f)$ of $(C_0(V,q),\sigma)$ such that

$$C_0(f)(xy)=\mu(f)^{-1}f(x)f(y)\qquad\text{for } x,\ y\in V, \tag{2.1}$$

see [KMRT98, (13.1)]. This automorphism depends only on the image $[f]=fF^\times$ of f in $\mathrm{PGO}(q)$, and we shall use the notation $C_0[f]$ for $C_0(f)$. The similarity f is *proper* if $C_0[f]$ fixes the center of $C_0(V,q)$ and *improper* if it induces a nontrivial automorphism of the center of $C_0(V,q)$ (see [KMRT98, (13.2)]). Proper similarities define an algebraic subgroup $\mathbf{GO}^+(q)$ in $\mathbf{GO}(q)$, and we let $\mathbf{PGO}^+(q)=\mathbf{GO}^+(q)/\mathbf{G}_{\mathrm{m}}$, a subgroup of $\mathbf{PGO}(q)$. The groups $\mathbf{GO}^+(q)$ and $\mathbf{PGO}^+(q)$ are the connected components of the identity in $\mathbf{GO}(q)$ and $\mathbf{PGO}(q)$ respectively, see [KMRT98, §23.B]. Conjugation by an improper similarity is an outer automorphism of $\mathbf{PGO}^+(q)$, since the induced automorphism on the center of $C_0(V,q)$ is nontrivial. As pointed out in the introduction, more outer automorphisms can be defined when the form q is the norm of an octonion algebra, i.e., a 3-fold Pfister form. In this case, we call the quadratic space a *3-fold Pfister quadratic space.*

3. Symmetric compositions

Let (S,n) be a quadratic space of dimension 8 over F.

Definition 3.1. A *symmetric composition* on (S,n) is an F-bilinear map

$$\star\colon S\times S\to S,\qquad (x,y)\mapsto x\star y\quad\text{for } x,\ y\in S$$

subject to the following conditions:

(1) there exists $\lambda_\star\in F^\times$, called the *multiplier* of the symmetric composition $\star$, such that
$$n(x\star y)=\lambda_\star n(x)n(y)\qquad\text{for all } x,\ y\in S;$$

(2) for all $x,\ y,\ z\in S$,
$$b_n(x\star y,z)=b_n(x,y\star z).$$

A symmetric composition with multiplier $\lambda_\star=1$ is called *normalized.*

This definition of symmetric compositions is not identical to the one given in [KMRT98, §34], where (S, n) can *a priori* be a nonsingular quadratic space of arbitrary finite dimension (but in fact $\dim S = 1$, 2, 4 or 8 by a theorem of Hurwitz, see [KMRT98, (33.28)]), and all the symmetric compositions are normalized.

The following lemma is an immediate consequence of the definition of a symmetric composition:

Lemma 3.2. *Let $\star$ be a symmetric composition on (S, n) with multiplier $\lambda_\star$.*

(1) *For any scalar $\nu \in F^\times$, $\star$ is a symmetric composition on the quadratic space $(S, \nu n)$ with multiplier $\nu^{-1}\lambda_\star$.*
(2) *For any scalar $\nu \in F^\times$, the bilinear map $\nu\cdot\star\colon (x, y) \mapsto \nu x\star y$ is a symmetric composition on the quadratic space (S, n) with multiplier $\nu^2\lambda_\star$.*

Lemma 3.3. *Under condition (1), condition (2) of the definition of a symmetric composition is equivalent to*

$$x \star (y \star x) = \lambda_\star n(x)y = (x \star y) \star x \qquad \textit{for all } x,\ y \in S.$$

Proof. The claim follows by applying [KMRT98, (34.1)] to the (normalized) composition $\star$ on $(S, \lambda_\star n)$. □

Since the symmetric compositions do not change when the quadratic form n is scaled, we may and will always assume without loss of generality that n represents 1. It is then a 3-fold Pfister form, by [KMRT98, (33.18), (33.29)].

Example 3.4. Let $(\mathbb{O}, n)$ be an octonion algebra with norm n, multiplication $(x, y) \mapsto x \cdot y$, identity 1, and conjugation $x \mapsto \overline{x}$. The multiplication

$$x \star y = \overline{x} \cdot \overline{y}$$

defines a normalized symmetric composition on $(\mathbb{O}, n)$, called the *para-octonion composition* (see for example [KMRT98, §34.A]). Observe that $\overline{x \star y} = \overline{y} \star \overline{x}$ for all x, $y \in \mathbb{O}$.

More examples—and a complete classification of symmetric compositions—are given in Section 9.

Definitions 3.5. Let (S, n) be a 3-fold Pfister quadratic space and let $\star$ and $\diamond$ be symmetric compositions on (S, n). A *similarity* $f : \star \to \diamond$ is an element $f \in \mathrm{GO}(n)$ such that

$$f(x \star y) = f(x) \diamond f(y) \qquad \text{for all } x,\ y \in S.$$

The multipliers of f, $\star$ and $\diamond$ are then related by $\lambda_\star = \lambda_\diamond\mu(f)$. Similarities with multiplier $\mu(f) = 1$ are called *isomorphisms*. In particular, similarities between symmetric compositions with the same multiplier are isomorphisms.

The *opposite* of the symmetric composition $\star$ on (S, n) is the symmetric composition $\star^{\mathrm{op}}$ on (S, n) defined by

$$x \star^{\mathrm{op}} y = y \star x \qquad \text{for } x,\ y \in S.$$

The multiplier of $\star^{\mathrm{op}}$ is the same as the multiplier of $\star$.

Symmetric compositions $\star$ and $\diamond$ are said to be *similar* if there is a similarity $\star \to \diamond$; they are said to be *antisimilar* if $\star^{\mathrm{op}}$ and $\diamond$ are similar.

Proposition 3.6. *Let n be a 3-fold quadratic Pfister form on a vector space S, and let $\star$ be a symmetric composition on (S, n). There exists up to isomorphism a unique normalized composition $\diamond$ on (S, n) similar to $\star$.*

Proof. Let $u \in S$ be such that $n(u) = 1$. Consider the maps $\ell^\star_u$, $r^\star_u\colon S \to S$ defined by

$$\ell^\star_u(x) = u \star x \qquad \text{and} \qquad r^\star_u(x) = x \star u \qquad \text{for } x \in S.$$

Define a new multiplication $\diamond$ on S by

$$x \diamond y = \lambda_\star^{-2}\ell^\star_u\bigl(r^\star_u(x) \star r^\star_u(y)\bigr) \qquad \text{for } x,\, y \in S.$$

Condition (1) for a symmetric composition yields

$$n\bigl(\ell^\star_u(x)\bigr) = \lambda_\star n(x) = n\bigl(r^\star_u(x)\bigr) \qquad \text{for all } x \in S.$$

It is then easy to check that $\diamond$ is a normalized symmetric composition on (S, n). Moreover, by Lemma 3.3, we have

$$\ell^\star_u \circ r^\star_u = r^\star_u \circ \ell^\star_u = \lambda_\star \operatorname{Id}_S .$$

From the definition of $\diamond$, it follows that

$$\lambda_\star^{-1} r^\star_u(x \diamond y) = \bigl(\lambda_\star^{-1} r^\star_u(x)\bigr) \star \bigl(\lambda_\star^{-1} r^\star_u(y)\bigr) \qquad \text{for all } x,\, y \in S,$$

hence $\lambda_\star^{-1} r^\star_u \colon \diamond \to \star$ is a similarity. If $\diamond'$ is another normalized composition similar to $\star$, then $\diamond$ and $\diamond'$ are similar, hence isomorphic since they have the same multiplier. □

4. From symmetric compositions to trialitarian automorphisms

Throughout this section, we fix a 3-fold Pfister quadratic space (S, n) and a symmetric composition $\star$ on (S, n) with multiplier $\lambda_\star$. We show how to associate to this composition a trialitarian automorphism $\rho_\star$ of $\mathbf{PGO}^+(n)$ defined over F, i.e., an outer automorphism of order 3 in $\mathbf{Aut}\bigl(\mathbf{PGO}^+(n)\bigr)(F)$.

For each element $x \in S$ we define linear maps $\ell^\star_x$, $r^\star_x\colon S \to S$ by

$$\ell^\star_x(y) = x \star y \qquad \text{and} \qquad r^\star_x(y) = y \star x \qquad \text{for } y \in S.$$

Consider $\begin{pmatrix} 0 & \lambda_\star^{-1} r^\star_x \\ \ell^\star_x & 0 \end{pmatrix} \in M_2(\operatorname{End} S) = \operatorname{End}_F(S \oplus S)$. By Lemma 3.3, we have

$$\begin{pmatrix} 0 & \lambda_\star^{-1} r^\star_x \\ \ell^\star_x & 0 \end{pmatrix}^2 = n(x) \cdot \operatorname{Id}_{S \oplus S} .$$

Therefore, by the universal property of Clifford algebras, the map

$$x \mapsto \begin{pmatrix} 0 & \lambda_\star^{-1} r^\star_x \\ \ell^\star_x & 0 \end{pmatrix}, \quad x \in S$$

induces an F-algebra homomorphism

$$\alpha^\star \colon C(S, n) \to \operatorname{End}_F(S \oplus S).$$

Proposition 4.1. *The map $\alpha^\star$ is an isomorphism of F-algebras*

$$\alpha^\star \colon C(S, n) \xrightarrow{\sim} \operatorname{End}_F(S \oplus S).$$

It restricts to an isomorphism of F-algebras with involution

$$\alpha^\star_0 \colon \bigl(C_0(S, n), \sigma\bigr) \xrightarrow{\sim} (\operatorname{End}_F S, \operatorname{ad}_n) \times (\operatorname{End}_F S, \operatorname{ad}_n)$$

such that for x, $y \in S$

$$\alpha^\star_0(x \cdot y) = (\lambda_\star^{-1} r^\star_x \ell^\star_y, \lambda_\star^{-1} \ell^\star_x r^\star_y).$$

Proof. This is shown in [KMRT98, (35.1)]. For completeness, we reproduce the easy argument. The map $\alpha^\star$ is injective since the Clifford algebra $C(S,n)$ is simple. It restricts to an F-algebra embedding $C_0(S,n)\hookrightarrow(\operatorname{End}S)\times(\operatorname{End}S)$. Since $\dim S=8$, dimension count shows that this embedding is an isomorphism. Using Lemma 3.3, it is easy to check that the involution σ corresponds to $\operatorname{ad}_n\times\operatorname{ad}_n$ under $\alpha_0^\star$. □

For f, g, $h\in\operatorname{GO}(n)$, define the F-algebra automorphism $\Theta(g,h)$ of $(\operatorname{End}_F S)\times(\operatorname{End}_F S)$ by

$$\Theta(g,h)\colon(\varphi,\psi)\mapsto(g\psi g^{-1},h\varphi h^{-1})$$

and consider the following diagrams:

$D_\star^+(f,g,h)$

$$\begin{tikzcd} C_0(S,n) \arrow[r, "\alpha_0^\star"] \arrow[d, "C_0[f]"'] & (\operatorname{End}_F S)\times(\operatorname{End}_F S) \arrow[d, "\operatorname{Int}[g]\times\operatorname{Int}[h]"] \\ C_0(S,n) \arrow[r, "\alpha_0^\star"] & (\operatorname{End}_F S)\times(\operatorname{End}_F S) \end{tikzcd}$$

and

$D_\star^-(f,g,h)$

$$\begin{tikzcd} C_0(S,n) \arrow[r, "\alpha_0^\star"] \arrow[d, "C_0[f]"'] & (\operatorname{End}_F S)\times(\operatorname{End}_F S) \arrow[d, "\Theta(g,h)"] \\ C_0(S,n) \arrow[r, "\alpha_0^\star"] & (\operatorname{End}_F S)\times(\operatorname{End}_F S). \end{tikzcd}$$

Lemma 4.2. *Let $\star$ be a symmetric composition on (S,n). For f, g, $h\in\operatorname{GO}(n)$, the following statements are equivalent:*

(a^+) *there exists a scalar $\lambda\in F^\times$ such that*
$$\lambda f(x\star y)=g(x)\star h(y)\qquad \textit{for all } x,\ y\in S;$$
(b^+) *there exists a scalar $\mu\in F^\times$ such that*
$$\mu g(x\star y)=h(x)\star f(y)\qquad \textit{for all } x,\ y\in S;$$
(c^+) *there exists a scalar $\nu\in F^\times$ such that*
$$\nu h(x\star y)=f(x)\star g(y)\qquad \textit{for all } x,\ y\in S;$$
(d^+) *the diagram $D_\star^+(f,g,h)$ commutes;*
(e^+) *the diagram $D_\star^+(g,h,f)$ commutes;*
(f^+) *the diagram $D_\star^+(h,f,g)$ commutes.*

When they hold, the scalars λ, μ, ν and the multipliers of f, g, h are related by
$$\lambda\mu=\mu(h),\quad \mu\nu=\mu(f),\quad \lambda\nu=\mu(g).$$
Moreover, the similarities f, g, and h are all proper in this case.

Likewise, for f, g, $h\in\operatorname{GO}(n)$ the following statements are equivalent:

(a^-) *there exists a scalar $\lambda\in F^\times$ such that*
$$\lambda f(x\star y)=h(y)\star g(x)\qquad \textit{for all } x,\ y\in S;$$
(b^-) *there exists a scalar $\mu\in F^\times$ such that*
$$\mu g(x\star y)=f(y)\star h(x)\qquad \textit{for all } x,\ y\in S;$$
(c^-) *there exists a scalar $\nu\in F^\times$ such that*
$$\nu h(x\star y)=g(y)\star f(x)\qquad \textit{for all } x,\ y\in S;$$

(d^-) *the diagram* $D^-_\star(f,h,g)$ *commutes;*
(e^-) *the diagram* $D^-_\star(g,f,h)$ *commutes;*
(f^-) *the diagram* $D^-_\star(h,g,f)$ *commutes.*

When they hold, the scalars λ, μ, ν *and the multipliers of* f, g, h *are related by*

$$\lambda\mu=\mu(h),\quad \mu\nu=\mu(f),\quad \lambda\mu=\mu(g).$$

Moreover, the similarities f, g, *and* h *are all improper in this case.*

Proof. This is essentially proved in [KMRT98, (35.4)]. We give a proof for the reader's convenience.

(a^+)⇒(b^+) Multiplying each side of (a^+) on the left by $h(y)$ and using Lemma 3.3, we obtain

$$\lambda h(y)\star f(x\star y)=\lambda_\star n\big(h(y)\big)g(x)\qquad\text{for } x,\,y\in S.$$

If y is anisotropic, the map $r^\star_y$ is a bijection whose inverse is $\lambda_\star^{-1}n(y)^{-1}\ell^\star_y$. Letting $X=y$ and $Y=x\star y$, we derive from the preceding equation

$$\lambda h(X)\star f(Y)=\mu(h)g(X\star Y)\qquad\text{for } X,\,Y\in S \text{ with } Y \text{ anisotropic}.$$

Since generic vectors in S are anisotropic, (b^+) follows with $\mu=\mu(h)\lambda^{-1}$. Similar arguments yield (b^+)⇒(c^+) with $\nu=\mu(f)\mu^{-1}$ and (c^+)⇒(a^+) with $\lambda=\mu(g)\nu^{-1}$.

Now, assume (a^+), (b^+), and (c^+) hold. From (b^+) and (c^+), and from $\mu\nu=\mu(f)$, we readily derive

$$g\big((y\star z)\star x\big)=\mu^{-1}h(y\star z)\star f(x)=\mu(f)^{-1}\big(f(y)\star g(z)\big)\star f(x)$$

and

$$h\big(x\star(z\star y)\big)=\nu^{-1}f(x)\star g(z\star y)=\mu(f)^{-1}f(x)\star\big(h(z)\star f(y)\big)$$

for all x, y, $z\in S$. These equations mean that diagram $D^+_\star(f,g,h)$ commutes, hence (d^+) holds. Similar computations show that (e^+) and (f^+) hold.

Now, assume (d^+) holds. The map

$$x\mapsto\begin{pmatrix}0 & \lambda_\star^{-1}r^\star_{f(x)}\\ \mu(f)^{-1}\ell^\star_{f(x)} & 0\end{pmatrix}$$

also yields by the universal property of Clifford algebras an F-algebra isomorphism $\beta\colon C(S,n)\xrightarrow{\sim}\operatorname{End}_F(S\oplus S)$. The automorphism $\beta\circ(\alpha^\star)^{-1}$ of $\operatorname{End}_F(S\oplus S)$ is inner by the Skolem–Noether theorem, and it preserves $(\operatorname{End}_F S)\times(\operatorname{End}_F S)$ diagonally embedded, hence

$$\beta\circ(\alpha^\star)^{-1}=\operatorname{Int}\begin{pmatrix}\varphi&0\\0&\psi\end{pmatrix}\qquad\text{for some invertible } \varphi,\,\psi\in\operatorname{End}_F S.$$

It follows that for $x\in S$

$$\varphi\circ r^\star_x\circ\psi^{-1}=r^\star_{f(x)}\qquad\text{and}\qquad \psi\circ\ell^\star_x\circ\varphi^{-1}=\mu(f)^{-1}\ell^\star_{f(x)},$$

which means that for x, $y\in S$

$$\varphi(y\star x)=\psi(y)\star f(x)\qquad\text{and}\qquad \psi(x\star y)=\mu(f)^{-1}f(x)\star\varphi(y).\tag{4.3}$$

For x, $y\in S$ we have

$$\beta(x\cdot y)=(\lambda_\star^{-1}\mu(f)^{-1}r^\star_{f(x)}\ell^\star_{f(x)},\lambda_\star^{-1}\mu(f)^{-1}\ell^\star_{f(x)}r^\star_{f(x)})=\alpha_0^\star\circ C_0[f](x\cdot y),$$

hence (d^+) yields

$$\operatorname{Int}\begin{pmatrix}\varphi&0\\0&\psi\end{pmatrix}=\operatorname{Int}\begin{pmatrix}g&0\\0&h\end{pmatrix}.$$

It follows that $[\varphi] = [g]$ and $[\psi] = [h]$, hence (4.3) yields (c^+) and (b^+). Similarly, $(e^+)\Rightarrow(a^+)$, (c^+), and $(f^+)\Rightarrow(a^+)$, (b^+). Thus, all the statements (a^+)–(f^+) are equivalent. When they hold, (d^+) shows that $C_0[f]$ is the identity on the center of $C_0(S, n)$, hence f is proper. Similarly, (e^+) shows that g is proper and (f^+) that h is proper.

The proof of the second part is similar. Details are left to the reader. □

Corollary 4.4. *Let $\star$ be a symmetric composition on (S, n). Every automorphism $\star \to \star$ is a proper isometry.*

Proof. For any automorphism f, condition (a^+) of Lemma 4.2 holds with $\lambda = 1$ and $g = h = f$. □

Theorem 4.5. *Let $\star$ be a symmetric composition on (S, n).*

(1) *For any $f \in \mathrm{GO}^+(n)$, there exist g, $h \in \mathrm{GO}^+(n)$ such that the equivalent conditions (a^+)–(f^+) in Lemma 4.2 hold.*
(2) *For any improper similarity $f \in \mathrm{GO}(n)$, there exist improper similarities g, $h \in \mathrm{GO}(n)$ such that the equivalent conditions (a^-)–(f^-) in Lemma 4.2 hold.*

In each case, the similarities g and h are uniquely determined up to multiplication by scalars, so $[g]$ and $[h] \in \mathrm{PGO}(n)$ are uniquely determined.

Proof. (1) The automorphism $C_0[f]$ of $C_0(S, n)$ preserves the canonical involution σ and restricts to the identity on the center. Therefore, we may find g, $h \in \mathrm{GO}(n)$ such that the diagram $D^+_\star(f, h, g)$ commutes. The elements $[g]$, $[h] \in \mathrm{PGO}(n)$ are uniquely determined by this condition (or by any other in the list (a^+)–(f^+) from Lemma 4.2), and Lemma 4.2 shows that g and h are proper similarities.

(2) If f is improper, then $C_0[f]$ restricts to the nontrivial automorphism of the center of $C_0(S, n)$, hence it fits in a diagram $D^-_\star(f, h, g)$ for some similarities g, $h \in \mathrm{GO}(n)$, which are improper by Lemma 4.2. □

Note that scaling g and/or h we may change the scalars λ, μ, ν in conditions (a^+)–(c^+) or (a^-)–(c^-). We may for instance choose g and h so that $\lambda = 1$, or, as in [KMRT98, (35.4)], so that $\lambda = \mu(f)^{-1}$, $\mu = \mu(g)^{-1}$, and $\nu = \mu(h)^{-1}$. In this case, $\mu(f)\mu(g)\mu(h) = 1$.

In view of Theorem 4.5, we may define a map

$$\rho_\star\colon\ \mathrm{PGO}^+(n) \to \mathrm{PGO}^+(n)$$

by carrying any $[f] \in \mathrm{PGO}^+(n)$ to the unique $[g] \in \mathrm{PGO}^+(n)$ such that the diagram $D^+_\star(f, g, h)$ commutes for some $h \in \mathrm{GO}^+(n)$. Since $\rho_\star$ is defined in a functorial way, it actually defines a map of F-algebraic groups

$$\rho_\star\colon\ \mathbf{PGO}^+(n) \to \mathbf{PGO}^+(n).$$

Theorem 4.6. *The map $\rho_\star$ is an outer automorphism of order* 3 *of* $\mathbf{PGO}^+(n)$ *and $\rho_\star^{-1} = \rho_{\star^{op}}$.*

Proof. Since commutation of $D^+_\star(f, g, h)$ implies that $D^+_\star(g, h, f)$ and $D^+_\star(h, f, g)$ commute, we have

$$\rho_\star[g] = [h] \qquad \text{and} \qquad \rho_\star[h] = [f],$$

hence $\rho_\star^3 = \mathrm{Id}$. Now, let f_1, $f_2 \in \mathrm{GO}^+(n)$ and assume g_1, g_2, h_1, $h_2 \in \mathrm{GO}^+(n)$ are such that $D_\star^+(f_1, g_1, h_1)$ and $D_\star^+(f_2, g_2, h_2)$ commute. By Lemma 4.2 we may find scalars λ_1, $\lambda_2 \in F^\times$ such that for all x, $y \in S$

$$\lambda_1 f_1(x \star y) = g_1(x) \star h_1(y) \qquad \text{and} \qquad \lambda_2 f_2(x \star y) = g_2(x) \star h_2(y),$$

hence

$$\lambda_1\lambda_2 f_1 f_2(x \star y) = \lambda_1 f_1\big(g_2(x) \star h_2(y)\big) = g_1 g_2(x) \star h_1 h_2(y).$$

Therefore, $\rho_\star[f_1 f_2] = [g_1 g_2]$, showing that $\rho_\star$ is an automorphism. We refer to [KMRT98, (35.6)] for the fact that $\rho_\star$ is an outer algebraic group automorphism. (That it is outer also follows from the fact that $\rho_\star$ lifts to an automorphism of $\mathbf{Spin}(n)$ that is not the identity on the center, see (8.2)). □

The definition of $\rho_\star$ above depends on the isomorphism $\alpha_0^\star$ through the diagram $D_\star^+(f, g, h)$. We next show that, conversely, the automorphism $\rho_\star$ determines $\alpha_0^\star$.

Proposition 4.7. *Let $\star$ and $\diamond$ be symmetric compositions on (S, n). We have $\rho_\star = \rho_\diamond$ if and only if $\alpha_0^\star = \alpha_0^\diamond$.*

Proof. If $\alpha_0^\star = \alpha_0^\diamond$, then the diagrams $D_\star^+(f, g, h)$ and $D_\diamond^+(f, g, h)$ coincide for all f, g, $h \in \mathrm{GO}^+(n)$, hence $\rho_\star = \rho_\diamond$. Conversely, suppose that $\rho_\star = \rho_\diamond$. Then $\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}$ is an automorphism of $(\operatorname{End} S) \times (\operatorname{End} S)$ that commutes with the involution $\mathrm{ad}_n \times \mathrm{ad}_n$ and that makes the following diagram commute for any $f \in \mathrm{GO}^+(n)$

(4.8)
$$\begin{array}{ccc}
(\operatorname{End} S) \times (\operatorname{End} S) & \xrightarrow{\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}} & (\operatorname{End} S) \times (\operatorname{End} S) \\
{\scriptstyle \mathrm{Int}(\rho_\star[f]) \times \mathrm{Int}(\rho_\star^2[f])}\downarrow & & \downarrow{\scriptstyle \mathrm{Int}(\rho_\star[f]) \times \mathrm{Int}(\rho_\star^2[f])} \\
(\operatorname{End} S) \times (\operatorname{End} S) & \xrightarrow{\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}} & (\operatorname{End} S) \times (\operatorname{End} S).
\end{array}$$

Suppose first that $\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}$ exchanges the two factors, so there exist automorphisms θ_1, θ_2 of $(\operatorname{End} S, \mathrm{ad}_n)$ such that

$$\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}(a, b) = \big(\theta_1(b), \theta_2(a)\big) \qquad \text{for all } a,\ b \in \operatorname{End} S.$$

Automorphisms of $(\operatorname{End} S, \mathrm{ad}_n)$ are inner automorphisms induced by similarities of (S, n), so we may find s_1, $s_2 \in \mathrm{GO}(n)$ such that $\theta_i = \mathrm{Int}[s_i]$ for $i = 1, 2$. By commutativity of diagram (4.8), we have

$$\rho_\star[f] \cdot [s_1] = [s_1] \cdot \rho_\star^2[f] \quad \text{and} \quad \rho_\star^2[f] \cdot [s_2] = [s_2] \cdot \rho_\star[f] \qquad \text{for all } f \in \mathrm{GO}^+(n).$$

Thus, $\mathrm{Int}[s_1] \circ \rho_\star^2 = \rho_\star$, hence $\mathrm{Int}[s_1] = \rho_\star^2$ and therefore $\rho_\star = \mathrm{Int}[s_1^2]$. Since the square of any similarity is a proper similarity, it follows that $\rho_\star$ is an inner automorphism of $\mathbf{PGO}^+(n)$, a contradiction. Therefore, $\alpha_0^\diamond \circ (\alpha_0^\star)^{-1}$ preserves the two factors, and we have

$$\alpha_0^\diamond \circ (\alpha_0^\star)^{-1} = \mathrm{Int}[s_1] \times \mathrm{Int}[s_2]$$

for some s_1, $s_2 \in \mathrm{GO}(n)$. Commutativity of diagram (4.8) yields

$$\rho_\star[f] \cdot [s_1] = [s_1] \cdot \rho_\star[f] \quad \text{and} \quad \rho_\star^2[f] \cdot [s_2] = [s_2] \cdot \rho_\star^2[f] \qquad \text{for all } f \in \mathrm{GO}^+(n).$$

Therefore, $[s_1]$ and $[s_2]$ centralize $\mathrm{PGO}^+(n)$. It follows that s_1 and s_2 are proper similarities, since conjugation by an improper similarity is an outer automorphism of $\mathrm{PGO}^+(n)$. Since the center of $\mathrm{PGO}^+(n)$ is trivial, it follows that $[s_1] = [s_2] = 1$, hence $\alpha_0^\diamond \circ (\alpha_0^\star)^{-1} = \mathrm{Id}$. □

5. The one-to-one correspondence

As in the preceding section, we fix a 3-fold Pfister quadratic space (S,n) over F. Our goal is to show that the map $\star \mapsto \rho_\star$ defines a one-to-one correspondence between symmetric compositions on (S,n) up to a scalar factor and trialitarian automorphisms of $\mathbf{PGO}^+(n)$ defined over F.

Recall that given a symmetric composition $\star$ on (S,n), linear operators $\ell_x^\star$ and $r_x^\star$ on S are defined for any $x \in S$ by

$$\ell_x^\star(y) = x \star y \qquad \text{and} \qquad r_x^\star(y) = y \star x \qquad \text{for } y \in S.$$

If x is anisotropic, condition (1) in the definition of symmetric compositions shows that $\ell_x^\star$ and $r_x^\star$ are similarities of (S,n) with multiplier $\lambda_\star n(x)$. We shall see in Corollary 7.4 below that $\ell_x^\star$ and $r_x^\star$ are improper similarities for every anisotropic vector $x \in S$. At this stage, we can at least prove:

Lemma 5.1. *The following conditions are equivalent:*

(a) *there exists an anisotropic vector $x \in S$ such that $\ell_x^\star$ is an improper similarity;*
(b) *there exists an anisotropic vector $x \in S$ such that $r_x^\star$ is an improper similarity;*
(c) *for every anisotropic vector $x \in S$, the similarity $\ell_x^\star$ is improper;*
(d) *for every anisotropic vector $x \in S$, the similarity $r_x^\star$ is improper.*

Proof. By Lemma 3.3 we have for all $x \in S$

$$r_x^\star \circ \ell_x^\star = \lambda_\star n(x) \cdot \mathrm{Id}_S,$$

and $\lambda_\star n(x) \cdot \mathrm{Id}_S$ is a proper similarity. Thus $r_x^\star$ and $\ell_x^\star$ are either both proper or improper, proving (a) $\iff$ (b) and (c) $\iff$ (d). Since (c) $\Rightarrow$ (a) is clear, it only remains to show (a) $\Rightarrow$ (c). For this, we use a homotopy argument. Let x, $y \in S$ be anisotropic vectors, and let t be an indeterminate over F. Consider the vector $u(t) = x(1-t) + yt \in S(t) = S_{F(t)}$. It gives rise to a rational morphism

$$\phi\colon \mathbb{A}^1 \dashrightarrow \mathbf{GO}(S,n),$$

induced by $t \mapsto \ell_{u(t)}^\star$, of the affine line over F to $\mathbf{GO}(S,n)$. Note that ϕ is defined at a if the vector $u(a) = x(1-a) + ya$ is anisotropic. Since $\mathrm{GO}(S,n)$ contains an improper similarity g of S, the variety of $\mathbf{GO}(S,n)$ is the disjoint union of two irreducible subvarieties $\mathbf{GO}^+(S,n)$ and $g\,\mathbf{GO}^+(S,n)$. Since $\mathbb{A}^1$ is irreducible the image of ϕ is contained either in $\mathbf{GO}^+(S,n)$ or in $g\,\mathbf{GO}^+(S,n)$. It follows immediately that the similarities $\phi(0) = x$ and $\phi(1) = y$ are both proper or both improper. $\square$

Lemma 5.2. *Let f, $g \in \mathrm{GO}(n)$ and let $\star$ be a symmetric composition on (S,n). The following conditions are equivalent:*

(a) *the map $\diamond\colon S \times S \to S$ defined by*

$$x \diamond y = f(x) \star g(y) \qquad \text{for } x,\ y \in S$$

is a symmetric composition on (S,n);
(b) *for all x, $y \in S$,*

$$f(x) \star g(y) = \mu(g) g^{-1}\big(x \star f(y)\big);$$

(c) *f and g are proper similarities such that*

$$\rho_\star^2[f]\cdot\rho_\star[f]\cdot[f]=1 \quad \textit{and} \quad [g]=\rho_\star^2[f]^{-1} \quad \textit{in } \mathrm{PGO}^+(n).$$

When they hold, the multiplier of $\diamond$ is $\lambda_\diamond=\mu(f)\mu(g)\lambda_\star$ and

$$\rho_\diamond=\mathrm{Int}[f^{-1}]\circ\rho_\star=\mathrm{Int}[g^{-1}]\circ\rho_\star\circ\mathrm{Int}[f].$$

Moreover, assuming (a)–(c) *hold, the following conditions on $[h]\in\mathrm{PGO}^+(n)$ are equivalent:*

(i) $\rho_\diamond=\mathrm{Int}[h^{-1}]\circ\rho_\star\circ\mathrm{Int}[h]$,
(ii) $[f]=\rho_\star[h]^{-1}\cdot[h]=[h]\cdot\rho_\diamond[h]^{-1}$.

Proof. We first show (a) $\Longleftrightarrow$ (b). Since f and g are similarities and $\star$ is a symmetric composition, we have

$$n\big(f(x)\star g(y)\big)=\lambda_\star\mu(f)\mu(g)n(x)n(y) \qquad \text{for all } x,\, y\in S, \tag{5.3}$$

hence the map $\diamond$ satisfies condition (1) in the definition of symmetric compositions. Therefore, (a) is equivalent to

$$b_n(f(x)\star g(y),z)=b_n\big(x,f(y)\star g(z)\big) \qquad \text{for all } x,\, y,\, z\in S.$$

Since $\star$ is a symmetric composition and g is a similarity, we may rewrite the right side as

$$b_n\big(x\star f(y),g(z)\big)=\mu(g)b_n\big(g^{-1}\big(x\star f(y)\big),z\big).$$

Therefore, (a) is equivalent to

$$b_n(f(x)\star g(y),z)=\mu(g)b_n\big(g^{-1}\big(x\star f(y)\big),z\big) \qquad \text{for all } x,\, y,\, z\in S.$$

Since n is nonsingular, this condition is also equivalent to (b).

Now, assume (b) holds. Fixing an anisotropic vector $x\in S$ and considering each side of the equation in (b) as a function of y, we have

$$\ell^\star_{f(x)}\circ g=\mu(g)g^{-1}\circ\ell^\star_x\circ f.$$

By Lemma 5.1, the similarities $\ell^\star_{f(x)}\circ g$ and $\mu(g)g^{-1}\circ\ell^\star_x$ are both proper or both improper, hence this equation shows that f is proper. Likewise, fixing an anisotropic vector y and considering each side of the equation in (b) as a function of x, we have

$$r^\star_{g(y)}\circ f=\mu(g)g^{-1}\circ r^\star_{f(y)}.$$

Since by Lemma 5.1 the similarities $r^\star_{g(y)}$ and $r^\star_{f(y)}$ are either both proper or both improper, this equation shows that g is proper because f is proper.

By Lemma 4.2, condition (b) is equivalent to the commutativity of diagram $D^+_\star(g^{-1},f,gf^{-1})$, hence to

$$\rho_\star[g^{-1}]=[f] \quad \text{and} \quad \rho_\star^2[g^{-1}]=[gf^{-1}]. \tag{5.4}$$

Since $\rho_\star^3=\mathrm{Id}$, it follows that $[g]=\rho_\star^2[f]^{-1}$ and $\rho_\star^2[f]\cdot\rho_\star[f]\cdot[f]=1$. We have thus proved (b) $\Rightarrow$ (c). Conversely, we readily deduce (5.4) from (c), hence the diagram $D^+_\star(g^{-1},f,gf^{-1})$ commutes. By Lemma 4.2, we may find λ, μ, $\nu\in F^\star$ such that for all x, $y\in S$

$$\lambda\, g^{-1}(x\star y)=f(x)\star gf^{-1}(y), \tag{5.5}$$

$$\mu\, f(x\star y)=gf^{-1}(x)\star g^{-1}(y), \tag{5.6}$$

$$\nu\, gf^{-1}(x\star y)=g^{-1}(x)\star f(y), \tag{5.7}$$

and

$$\lambda\mu = \mu(gf^{-1}), \quad \mu\nu = \mu(g^{-1}), \quad \lambda\nu = \mu(f).$$

These last equations yield $\lambda^2\mu\nu = \mu(g)$. To obtain (b) from (5.5), it suffices to prove $\lambda = \mu(g)$, wich amounts to $\lambda\mu\nu = 1$. For this, observe that

$$\lambda\mu\nu\, x \star y = \lambda\mu\nu\, gf^{-1} \circ f \circ g^{-1}(x \star y) \qquad \text{for all } x,\, y \in S.$$

Compute the right side using successively (5.5), (5.6), and (5.7):

$$\begin{aligned}\lambda\mu\nu\, gf^{-1} \circ f \circ g^{-1}(x \star y) &= \mu\nu\, gf^{-1} \circ f\bigl(f(x) \star gf^{-1}(y)\bigr)\\ &= \nu\, gf^{-1}\bigl(g(x) \star f^{-1}(y)\bigr)\\ &= x \star y.\end{aligned}$$

Thus, $\lambda\mu\nu\, x \star y = x \star y$ for all x, $y \in S$, hence $\lambda\mu\nu = 1$ and it follows that (c)⇒(b).

Now, assume (a), (b), and (c) hold. The equation $\lambda_\diamond = \mu(f)\mu(g)\lambda_\star$ easily follows from (5.3). For $\varphi \in \mathrm{GO}^+(n)$ Theorem 4.5 yields φ', $\varphi'' \in \mathrm{GO}^+(n)$ such that

$$\varphi(x \star y) = \varphi'(x) \star \varphi''(y) \qquad \text{for all } x,\, y \in S,$$

so $\rho_\star[\varphi] = [\varphi']$. Then

$$\varphi(x \diamond y) = \varphi\bigl(f(x) \star g(y)\bigr) = \varphi' f(x) \star \varphi'' g(y) = f^{-1}\varphi' f(x) \diamond g^{-1}\varphi'' g(y),$$

hence $\rho_\diamond[\varphi] = [f^{-1}\varphi' f]$. It follows that $\rho_\diamond = \mathrm{Int}[f^{-1}] \circ \rho_\star$. From (c) it is easily derived that $[f]^{-1} = [g^{-1}] \cdot \rho_\star[f]$, hence we also have $\rho_\diamond = \mathrm{Int}[g^{-1}] \circ \rho_\star \circ \mathrm{Int}[f]$.

Finally, (i) holds if and only if $\rho_\diamond = \mathrm{Int}\bigl([h]^{-1} \cdot \rho_\star[h]\bigr) \circ \rho_\star$. This equation is equivalent to (ii) since $\rho_\diamond = \mathrm{Int}[f^{-1}] \circ \rho_\star$ and the center of $\mathbf{PGO}^+(n)$ is trivial. □

Theorem 5.8. *The assignment $\star \mapsto \rho_\star$ defines a one-to-one correspondence between symmetric compositions on (S, n) up to scalars and trialitarian automorphisms of $\mathbf{PGO}^+(n)$ defined over F.*

Proof. We first show that the map is onto. Let τ be a trialitarian automorphism of $\mathbf{PGO}^+(n)$ over F and let $\star$ be a symmetric composition on (S, n), so $\rho_\star$ also is a trialitarian automorphism. In view of the exact sequence (1.1), we have either $\tau \equiv \rho_\star$ or $\tau \equiv \rho_\star^{-1} \mod \mathrm{Int}\bigl(\mathrm{PGO}^+(n)\bigr)$. Substituting $\star^{\mathrm{op}}$ for $\star$ if necessary, we may assume $\tau \equiv \rho_\star \mod \mathrm{Int}\bigl(\mathrm{PGO}^+(n)\bigr)$, hence there exists $f \in \mathrm{GO}^+(n)$ such that

$$\tau = \mathrm{Int}[f^{-1}] \circ \rho_\star.$$

Since $\tau^3 = \rho_\star^3 = \mathrm{Id}$, we must have $[f]^{-1} \cdot \rho_\star[f]^{-1} \cdot \rho_\star^2[f]^{-1} = 1$. Let $g \in \mathrm{GO}^+(n)$ be such that $[g] = \rho_\star^2[f]^{-1}$. Lemma 5.2 then shows that $\tau = \rho_\diamond$ for the symmetric composition $\diamond$ defined by

$$x \diamond y = f(x) \star g(y) \qquad \text{for } x,\, y \in S.$$

Now, let $\star$ and $\diamond$ be two symmetric compositions on (S, n). Suppose $\diamond$ is a scalar multiple of $\star$, say there exists $\mu \in F^\times$ such that $x \diamond y = \mu x \star y$ for all x, $y \in S$. Then $\lambda_\diamond = \mu^2\lambda_\star$, and for all $x \in S$ we have $\ell_x^\diamond = \mu\ell_x^\star$ and $r_x^\diamond = \mu r_x^\star$. Therefore, $\alpha_0^\diamond = \alpha_0^\star$, hence $\rho_\diamond = \rho_\star$ by Proposition 4.7. Thus, symmetric compositions that are scalar multiples of each other define the same trialitarian automorphism, and it only remains to show the converse: if $\star$ and $\diamond$ are symmetric compositions such that $\rho_\star = \rho_\diamond$, then $\star$ and $\diamond$ are multiples of each other. By Proposition 4.7, the hypothesis $\rho_\star = \rho_\diamond$ implies $\alpha_0^\star = \alpha_0^\diamond$, hence

$$\lambda_\star^{-1}\ell_x^\star r_y^\star = \lambda_\diamond^{-1}\ell_x^\diamond r_y^\diamond \qquad \text{for all } x,\, y \in S.$$

Fix an anisotropic vector $x \in S$ and let $\varphi = (\ell_x^\diamond)^{-1}\ell_x^\star$, so

$$\lambda_\diamond\lambda_\star^{-1}\varphi r_y^\star = r_y^\diamond \qquad \text{for all } y \in S.$$

This equation means that

$$z \diamond y = \lambda_\diamond\lambda_\star^{-1}\varphi(z \star y) \qquad \text{for all } y,\, z \in S. \tag{5.9}$$

The map φ is a similarity since $\ell_x^\diamond$ and $\ell_x^\star$ are similarities. Its multiplier is

$$\mu(\varphi) = \mu(\ell_x^\diamond)^{-1}\mu(\ell_x^\star) = \lambda_\diamond^{-1}\lambda_\star.$$

Suppose first φ is improper. Theorem 4.5 then yields improper similarities φ', φ'' such that

$$\varphi(z \star y) = \mu(\varphi)\varphi'(y) \star \varphi''(z) \qquad \text{for all } y,\, z \in S,$$

hence

$$z \diamond y = \varphi''(z) \star^{\mathrm{op}} \varphi'(y) \qquad \text{for all } y,\, z \in S.$$

Lemma 5.2 yields a contradiction since φ' and φ'' are improper. Therefore, φ is proper and Theorem 4.5 yields φ', $\varphi'' \in \mathrm{GO}^+(n)$ such that

$$\varphi(z \star y) = \mu(\varphi)\varphi'(z) \star \varphi''(y) \qquad \text{for all } y,\, z \in S,$$

hence

$$z \diamond y = \varphi'(z) \star \varphi''(y) \qquad \text{for all } y,\, z \in S.$$

By Lemma 5.2 it follows that $\rho_\diamond = \mathrm{Int}[\varphi'^{-1}] \circ \rho_\star$. Since by hypothesis $\rho_\star = \rho_\diamond$, this equation implies $[\varphi'] = 1$. But $[\varphi'] = \rho_\star[\varphi]$, so also $[\varphi] = 1$. Equation (5.9) then shows that $\diamond$ and $\star$ are scalar multiples of each other. $\square$

6. Classification of conjugacy classes of trialitarian automorphisms

We next show that under the one-to-one correspondence of Theorem 5.8 similarity of symmetric compositions corresponds to conjugacy of trialitarian automorphisms.

Proposition 6.1. *Let $h\colon \diamond \to \star$ be a similarity of symmetric compositions on (S, n). Then*

$$\rho_\star = \mathrm{Int}[h] \circ \rho_\diamond \circ \mathrm{Int}[h]^{-1}. \tag{6.2}$$

Conversely, if (6.2) holds for some $h \in \mathrm{GO}(n)$, then some scalar multiple of h is a similarity $\diamond \to \star$.

Proof. Suppose first $h\colon \diamond \to \star$ is a similarity of symmetric compositions, so

$$h(x \diamond y) = h(x) \star h(y) \qquad \text{for all } x,\, y \in S. \tag{6.3}$$

Let $\varphi \in \mathrm{GO}^+(n)$. Theorem 4.5 yields φ', $\varphi'' \in \mathrm{GO}^+(n)$ such that

$$\varphi(x \star y) = \varphi'(x) \star \varphi''(y) \qquad \text{for all } x,\, y \in S,$$

and $\rho_\star[\varphi] = [\varphi']$ by definition. Applying φ to each side of (6.3), we find

$$\varphi h(x \diamond y) = \varphi' h(x) \star \varphi'' h(y).$$

The right side is

$$hh^{-1}\varphi' h(x) \star hh^{-1}\varphi'' h(y) = h\big(h^{-1}\varphi' h(x) \diamond h^{-1}\varphi'' h(y)\big),$$

hence

$$h^{-1}\varphi h(x \diamond y) = h^{-1}\varphi' h(x) \diamond h^{-1}\varphi'' h(y) \qquad \text{for all } x,\, y \in S.$$

Therefore, $\rho_\diamond\big([h^{-1}\varphi h]\big) = [h^{-1}] \cdot \rho_\star[\varphi] \cdot [h]$, proving (6.2).

Conversely, assume (6.2) holds, and set

$$x \rhd y = h^{-1}\big(h(x) \star h(y)\big) \qquad \text{for } x,\, y \in S,$$

so $\rhd$ is a symmetric composition similar to $\star$ under h. The arguments above show that

$$\rho_{\rhd} = \mathrm{Int}[h^{-1}] \circ \rho_\star \circ \mathrm{Int}[h] = \rho_\diamond,$$

hence Theorem 5.8 implies that $\diamond$ is a multiple of $\rhd$. If $\lambda \in F^\times$ is such that $x \rhd y = \lambda\, x \diamond y$ for all x, $y \in S$, then $\lambda\, h(x \diamond y) = h(x) \star h(y)$, hence

$$\lambda^{-1} h(x \diamond y) = \lambda^{-1} h(x) \star \lambda^{-1} h(y) \qquad \text{for all } x,\, y \in S.$$

This equation shows that $\lambda^{-1}h\colon \diamond \to \star$ is a similarity. □

For the following statement, recall from Proposition 3.6 that every symmetric composition is similar to a normalized symmetric composition, which is unique up to isomorphism.

Theorem 6.4. *Let $\star$ and $\diamond$ be symmetric compositions on (S, n), and let $\overline{\star}$, $\overline{\diamond}$ be normalized symmetric compositions similar to $\star$ and to $\diamond$ respectively. The following conditions are equivalent:*

(a) *$\star$ and $\diamond$ are similar;*
(b) *$\overline{\star}$ and $\overline{\diamond}$ are isomorphic;*
(c) *$\rho_\star$ and $\rho_\diamond$ are conjugate in $\mathrm{Aut}\big(\mathbf{PGO}^+(n)\big)$ over F;*
(d) *$\rho_{\overline{\star}}$ and $\rho_{\overline{\diamond}}$ are conjugate in $\mathrm{Aut}\big(\mathbf{PGO}^+(n)\big)$ over F.*

Proof. It is clear from the definition of $\overline{\star}$ and $\overline{\diamond}$ that (a) $\iff$ (b). Since $\star$ and $\overline{\star}$ are similar, Proposition 6.1 shows that $\rho_\star$ and $\rho_{\overline{\star}}$ are conjugate in $\mathrm{Aut}\big(\mathbf{PGO}^+(n)\big)$ over F. Similarly, $\rho_\diamond$ and $\rho_{\overline{\diamond}}$ are conjugate, hence it is clear that (c) $\iff$ (d). Proposition 6.1 also yields (a)$\Rightarrow$(c), so it only remains to show (c)$\Rightarrow$(a).

Suppose ϕ is an automorphism of $\mathbf{PGO}^+(n)$ defined over F such that

$$\rho_\star = \phi \circ \rho_\diamond \circ \phi^{-1}.$$

Let $f \in \mathrm{GO}(n)$ be an improper similitude. The restriction of $\mathrm{Int}[f]$ to $\mathbf{PGO}^+(n)$ is an outer automorphism whose square is inner, hence each coset of $\mathrm{Aut}\big(\mathbf{PGO}^+(n)\big)$ modulo $\mathrm{Int}\big(\mathbf{PGO}^+(n)\big)$ is represented over F by an element from the set

$$\{\mathrm{Id},\ \rho_\diamond,\ \rho_\diamond^2,\ \mathrm{Int}[f],\ \mathrm{Int}[f]\circ\rho_\diamond,\ \mathrm{Int}[f]\circ\rho_\diamond^2\}.$$

Since $\phi \circ \rho_\diamond \circ \phi^{-1}$ does not change when ϕ is multiplied on the right by $\rho_\diamond$ or $\rho_\diamond^{-1}$, we may assume the coset of ϕ is represented by Id or by $\mathrm{Int}[f]$, hence $\phi = \mathrm{Int}[h]$ for some $h \in \mathrm{GO}(n)$. It follows from Proposition 6.1 that $\star$ and $\diamond$ are similar. □

Theorem 6.4 shows that the correspondence $\star \mapsto \rho_\star$ induces a one-to-one correspondence between similarity classes of symmetric compositions, or isomorphism classes of normalized symmetric compositions, and conjugacy classes of trialitarian automorphisms over F in $\mathrm{Aut}\big(\mathbf{PGO}^+(n)\big)$.

Corollary 6.5. (1) *Two symmetric compositions $\star$ and $\diamond$ on (S, n) are similar or antisimilar if and only if the subgroups generated by $\rho_\star$ and $\rho_\diamond$ in $\mathrm{Aut}\big(\mathbf{PGO}^+(n)\big)$ are conjugate.*

(2) *Two normalized symmetric compositions $\star$ and $\diamond$ on (S, n) are isomorphic or anti-isomorphic if and only if the subgroups generated by $\rho_\star$ and $\rho_\diamond$ in $\mathrm{Aut}\big(\mathbf{PGO}^+(n)\big)$ are conjugate.*

Proof. The subgroups generated by $\rho_\star$ and $\rho_\diamond$ are conjugate if and only if $\rho_\star$ is conjugate to $\rho_\diamond$ or to $\rho_\diamond^{-1}$. Since $\rho_\diamond^{-1} = \rho_{\diamond^{\mathrm{op}}}$ by Theorem 4.6, the corollary follows from Theorem 6.4. □

To complete this section, we discuss fixed points of trialitarian automorphisms, which will be used to classify trialitarian automorphisms in Section 9. For any symmetric composition $\star$ on (S,n), we let $\mathbf{Aut}(\star)$ be the F-algebraic group of automorphisms of $\star$, whose group of F-rational points $\mathrm{Aut}(\star)$ consists of the isomorphisms $f\colon \star \to \star$. Corollary 4.4 shows that $\mathbf{Aut}(\star) \subset \mathbf{GO}^+(n)$.

Theorem 6.6. *Let $\star$ be a symmetric composition on (S,n). The canonical map $\phi\colon\ \mathbf{GO}^+(n) \to \mathbf{PGO}^+(n)$ induces an isomorphism from $\mathbf{Aut}(\star)$ to the subgroup $\mathbf{PGO}^+(n)^{\rho_\star}$ of $\mathbf{PGO}^+(n)$ fixed under the trialitarian automorphism $\rho_\star$.*

Proof. If $f \in \mathrm{Aut}(\star)$, then $\rho_\star[f] = [f]$, hence ϕ gives rise to a canonical map
$$\psi\colon\ \mathbf{Aut}(\star) \to \mathbf{PGO}^+(n)^{\rho_\star}.$$
To prove that ψ is an isomorphism it suffices to show that $\operatorname{Ker}\psi = 1$ and that ψ is a quotient map (see [Wat79, Corollary 15.4]). The kernel of ψ represents the functor which takes a commutative F-algebra A to $\operatorname{Ker}[\mathbf{Aut}(\star)(A) \to \mathbf{PGO}^+(n)^{\rho_\star}(A)]$. Let a be in this kernel. Note that $\operatorname{Ker}\psi \subset \operatorname{Ker}\phi = \mathbf{G}_{\mathrm{m}}$. Hence a is a homothety $r\,\mathrm{Id}_{S_A}$ for some $r \in A^\times$. But such a map is in $\mathbf{Aut}(\star)(A)$ if and only if $r = 1$. Thus $\operatorname{Ker}\phi$ represents the trivial functor and so $\operatorname{Ker}\phi = 1$. To see that ψ is a quotient map we need to show that for every commutative F-algebra A and every $g \in \mathbf{PGO}^+(n)^{\rho_\star}(A)$ there exists a faithfully flat extension $A \to B$ and an element $f \in \mathbf{Aut}(\star)(B)$ such that $\psi(f) = g$ (see [Wat79, Theorem 15.5]). Since ϕ is a quotient map we can find an F-algebra B and an element $f_0 \in \mathbf{GO}^+(n)(B)$ such that $\phi(f_0) = g$. Since g is fixed by $\rho_\star$ there are scalars α, $\beta \in B^\times$ such that
$$f_0(x \star y) = \alpha f_0(x) \star \beta f_0(y) \qquad \text{for all } x,\ y \in S_B.$$
Let $f = \alpha\beta f_0$. Then f is an automorphism of $\star$ and $\psi(f) = \phi(f_0) = g$. □

7. The twist of a symmetric composition

As in the preceding sections, we fix a 3-fold Pfister quadratic space (S,n) over F. If $\star$ and $\diamond$ are symmetric compositions on (S,n), then the trialitarian automorphisms $\rho_\star$ and $\rho_\diamond$ are related by inner automorphisms of $\mathbf{PGO}^+(n)$: in view of the exact sequence (1.1), we have either
$$\rho_\star \equiv \rho_\diamond \bmod \operatorname{Int}\bigl(\mathbf{PGO}^+(n)\bigr) \quad \text{or} \quad \rho_\star \equiv \rho_\diamond^{-1} \bmod \operatorname{Int}\bigl(\mathbf{PGO}^+(n)\bigr).$$
We use this observation to relate the symmetric compositions $\star$ and $\diamond$.

Proposition 7.1. *Let $\star$ and $\diamond$ be two symmetric compositions on (S,n). If $\rho_\star \equiv \rho_\diamond \bmod \operatorname{Int}\bigl(\mathbf{PGO}^+(n)\bigr)$, then there exist f, $g \in \mathrm{GO}^+(n)$ such that*
$$x \diamond y = f(x) \star g(y) \qquad \text{for all } x,\ y \in S. \tag{7.2}$$
If $\rho_\star \equiv \rho_\diamond^{-1} \bmod \operatorname{Int}\bigl(\mathbf{PGO}^+(n)\bigr)$, then there exist f, $g \in \mathrm{GO}^+(n)$ such that
$$x \diamond y = g(y) \star f(x) \qquad \text{for all } x,\ y \in S.$$

Proof. Suppose first $\rho_\diamond = \operatorname{Int}[f^{-1}] \circ \rho_\star$ for some $f \in \mathrm{GO}^+(n)$. Since $\rho_\diamond^3 = \mathrm{Id}$, we have $\rho_\star^2[f] \cdot \rho_\star[f] \cdot [f] = 1$. Let $g \in \mathrm{GO}^+(n)$ be such that $[g] = \rho_\star^2[f^{-1}]$, and define

$$x \rhd y = f(x) \star g(y) \qquad \text{for } x,\, y \in S.$$

By Lemma 5.2, $\rhd$ is a symmetric composition on (S, n), and

$$\rho_\rhd = \operatorname{Int}[f^{-1}] \circ \rho_\star = \rho_\diamond.$$

Therefore, Theorem 5.8 shows that $\rhd$ is a scalar multiple of $\diamond$. Scaling g, we may assume (7.2) holds.

If $\rho_\star \equiv \rho_\diamond^{-1} \bmod \operatorname{Int}\bigl(\mathbf{PGO}^+(n)\bigr)$, then $\rho_{\star^{\mathrm{op}}} \equiv \rho_\diamond \bmod \operatorname{Int}\bigl(\mathbf{PGO}^+(n)\bigr)$ since $\rho_{\star^{\mathrm{op}}} = \rho_\star^{-1}$ by Theorem 4.6. The first part of the proof yields $f,\, g \in \mathrm{GO}^+(n)$ such that

$$x \diamond y = f(x) \star^{\mathrm{op}} g(y) = g(y) \star f(x) \qquad \text{for } x,\, y \in S.$$

□

When the symmetric composition $\diamond$ is given by (7.2), then we must have $[g] = \rho_\star^2[f^{-1}]$, by Lemma 5.2. The map g is therefore uniquely determined by f up to a scalar factor, and we say $\diamond$ is a *twist* of $\star$ through the similarity f. Thus, by Proposition 7.1, given a symmetric composition $\star$ on (S, n), every symmetric composition on (S, n) is a twist of $\star$ or $\star^{\mathrm{op}}$.

Twisting has its origin in Petersson [Pet69], where the following special case is considered: suppose $f \colon \star \to \star$ is a similarity (i.e., f is an automorphism of $\star$), and $f^3 = \mathrm{Id}_S$. Since f is an automorphism, we have $f \in \mathrm{GO}^+(n)$ and $\rho_\star[f] = [f]$: see Theorem 6.6. Since $f^3 = \mathrm{Id}_S$, we have $\rho_\star^2[f] \cdot \rho_\star[f] \cdot [f] = 1$, hence we may choose $g = f^{-1}$ in the discussion above. By Lemma 5.2, the product

$$x \star_f y = f(x) \star f^{-1}(y) \qquad \text{for } x,\, y \in S \tag{7.3}$$

defines a symmetric composition.

The idea of twisting was further developed in [Eld00].

Corollary 7.4. *For every symmetric composition $\diamond$ on (S, n) and every anisotropic vector $x \in S$, the similarities $\ell_x^\diamond$ and $r_x^\diamond$ are improper.*

Proof. Let $\star$ be a para-octonion composition on (S, n) (see Example 3.4). The similarities $\ell_1^\star$ and $r_1^\star$, where 1 is the identity of the octonion algebra, coincide with the conjugation map, which is an improper isometry. Therefore, by Lemma 5.1, the similarities $\ell_x^\star$ and $r_x^\star$ are improper for every anisotropic vector $x \in S$. Propositiom 7.1 shows that there exist $f,\, g \in \mathrm{GO}^+(n)$ such that either

$$x \diamond y = f(x) \star g(y) \quad \text{for all } x,\, y \in S \quad \text{or} \quad x \diamond y = g(y) \star f(x) \quad \text{for all } x,\, y \in S.$$

Therefore, for every anisotropic vector $x \in S$ we have either $\ell_x^\diamond = \ell_{f(x)}^\star \circ g$ and $r_x^\diamond = r_{g(x)}^\star \circ f$, or $\ell_x^\diamond = r_{f(x)}^\star \circ g$ and $r_x^\diamond = \ell_{g(x)}^\star \circ f$. The similarities $\ell_x^\diamond$ and $r_x^\diamond$ are therefore improper. □

8. Spin groups and symmetric compositions

Let $\star$ be a normalized symmetric composition on a 3-fold Pfister quadratic space (S, n). We briefly point out in this section how the results of the preceding sections can be modified to apply to the group $\mathbf{Spin}(n)$ instead of $\mathbf{PGO}^+(n)$.

Recall that the F-rational points of the F-algebraic group $\mathbf{Spin}(n)$ are given by

$$\mathrm{Spin}(n) = \{c \in C_0(S, n)^\times \mid cSc^{-1} \subset S \quad \text{and} \quad c\sigma(c) = 1\}$$

(see for example [KMRT98, 35.C.]). The isomorphism $\alpha_0^\star$ introduced in Proposition 4.1 can be used to give a convenient description of $\mathrm{Spin}(n)$: for $c \in \mathrm{Spin}(n)$, define $f,\ f_1,\ f_2 \in \mathrm{End}_F S$ as follows:

$$f(x) = cxc^{-1} \text{ for } x \in S, \qquad \alpha_0^\star(c) = (f_1, f_2).$$

The map f is a proper isometry. Similarly, since $c\sigma(c) = 1$ and $\alpha_0^\star$ is an isomorphism of algebras with involution, f_1 and f_2 are isometries. We have $C_0[f] = \mathrm{Int}(c)|_{C_0(S,n)}$, hence the diagram $D_\star^+(f, f_1, f_2)$ commutes. Therefore, f_1 and f_2 are proper isometries. Let $\mathrm{O}^+(n)$ be the group of proper isometries of (S, n). The following result is shown in [KMRT98, (35.7)] (assuming $\mathrm{char}\, F \neq 2$):

Proposition 8.1. *The map $c \mapsto (f, f_1, f_2)_\star$ defines an isomorphism*

$$\mathrm{Spin}(n) \xrightarrow{\sim} \{(f, f_1, f_2)_\star \mid f_i \in \mathrm{O}^+(n)(F),\ f(x \star y) = f_1(x) \star f_2(y),\ x, y \in S\}.$$

Moreover any of the three relations

$$\begin{array}{rcl} f(x \star y) & = & f_1(x) \star f_2(y) \\ f_1(x \star y) & = & f_2(x) \star f(y) \\ f_2(x \star y) & = & f(x) \star f_1(y) \end{array}$$

implies the two others.

Observe that the representation of elements of $\mathrm{Spin}(n)$ as triples of elements of $\mathrm{O}^+(n)$ depends on the choice of the composition $\star$, hence the notation $(f, f_1, f_2)_\star$.

We have an obvious trialitarian automorphism $\widehat{\rho}_\star$ of $\mathbf{Spin}(n)$ defined over F by

$$\widehat{\rho}_\star \colon (f, f_1, f_2)_\star \mapsto (f_1, f_2, f)_\star.$$

The center $\mathbf{Z}$ of $\mathbf{Spin}(n)$ is the scheme $\boldsymbol{\mu}_2^2$ with $\boldsymbol{\mu}_2(F) = \pm 1$. Viewing $\mathbf{Z}(F)$ as kernel of the multiplication map

$$\boldsymbol{\mu}_2^3(F) \to \boldsymbol{\mu}_2(F), \quad (\varepsilon_1, \varepsilon_2, \varepsilon_3) \mapsto \varepsilon_1\varepsilon_2\varepsilon_3,$$

the group scheme $\mathbf{Z}$ admits a natural $\mathfrak{S}_3$-action. The exact sequence of schemes

$$1 \to \mathbf{Z} \to \mathbf{Spin}(n) \to \mathbf{PGO}^+(n) \to 1, \tag{8.2}$$

where the morphism $\mathbf{Spin}(n) \to \mathbf{PGO}^+(n)$ is induced by $(f, f_1, f_2)_\star \mapsto [f]$, is $\mathfrak{S}_3$-equivariant (see [KMRT98, (35.13)]). Thus the trialitarian action on $\mathbf{Spin}(n)$ is a lift of the trialitarian action on $\mathbf{PGO}^+(n)$.

Most of the results on trialitarian actions on $\mathbf{PGO}^+(n)$ hold for $\mathbf{Spin}(n)$. Moreover, in a similar way we have a natural action of $\mathfrak{S}_3$ on $\mathbf{Spin}(n)$. Let $\mathbf{H}$ be the semidirect product of $\mathbf{Spin}(n)$ and $\mathfrak{S}_3$.

Theorem 8.3. *Two normalized symmetric compositions $\star$ and $\diamond$ on (S, n) are isomorphic if and only if the trialitarian automorphisms $\widehat{\rho}_\star$ and $\widehat{\rho}_\diamond$ are conjugate over F in $\mathbf{H}$.*

One of the main steps in the proof for $\mathbf{PGO}^+(n)$ was to show that any trialitarian automorphism is induced by a symmetric composition. We describe this step for $\mathbf{Spin}(n)$.

Proposition 8.4. *Every trialitarian automorphism of $\mathbf{Spin}(n)$ over F in $\mathbf{H}$ has the form $\widehat{\rho}_\diamond$ for some normalized symmetric composition $\diamond$.*

Proof. We show how to modify the proof of the corresponding statement in Theorem 5.8 for $\mathbf{PGO}^+(n)$. Let τ be a trialitarian automorphism of $\mathbf{Spin}(n)$ over F, let $\star$ be a symmetric composition on (S, n) and let $\widehat{\rho}_\star$ be the associated trialitarian automorphism. In view of the exact sequence (1.1), we have either $\tau \equiv \widehat{\rho}_\star$ or $\tau \equiv \widehat{\rho}_\star^{-1} \bmod \mathrm{Spin}(n)$. Substituting $\star^{\mathrm{op}}$ for $\star$ if necessary, we may assume $\tau \equiv \widehat{\rho}_\star \bmod \mathrm{Spin}(n)$, hence there exists $(h, h_1, h_2)_\star \in \mathrm{Spin}(n)$ such that

$$\tau = \mathrm{Int}\big((h, h_1, h_2)_\star^{-1}\big) \circ \widehat{\rho}_\star,$$

which means that for all $(f, f_1, f_2)_\star \in \mathrm{Spin}(n)$,

$$\tau\big((f, f_1, f_2)_\star\big) = (h^{-1} f_1 h, h_1^{-1} f_2 h_1, h_2^{-1} f h_2)_\star.$$

It follows from $\tau^3 = \widehat{\rho}_\star^3 = \mathrm{Id}$ that $h_2 h_1 h = \mathrm{Id}_S$. Let

$$x \diamond y = h(x) \star h_2^{-1}(y) \quad \text{for } x,\, y \in S.$$

One deduces easily from $h_2 h_1 h = \mathrm{Id}_S$ that $(x \diamond y) \diamond x = x \diamond (y \diamond x) = n(x) y$ for $x,\, y \in S$. Thus $\diamond$ is a normalized symmetric composition on (S, n). For $(f, f_1, f_2)_\star \in \mathrm{Spin}(n)$ and $x,\, y \in S$ we have

$$\begin{aligned} f(x \diamond y) &= f\big(h(x) \star h_2^{-1}(y)\big) \\ &= f_1 h(x) \star f_2 h_2^{-1}(y) \\ &= h^{-1} f_1 h(x) \diamond h_2 f_2 h_2^{-1}(y) \end{aligned} \tag{8.5}$$

Let

$$\mathrm{Spin}(n) = \{(f, f_1', f_2')_\diamond \mid f,\, f_1',\, f_2' \in \mathrm{O}^+(n),\, f(x \diamond y) = f_1'(x) \star f_2'(y),\ x, y \in S\}$$

be the presentation of $\mathrm{Spin}(n)$ using the symmetric composition $\diamond$. It follows from (8.5) that

$$f_1' = h^{-1} f_1 h \quad \text{and} \quad f_2' = h_2^{-1} f_2 h_2,$$

so that the passage from the presentation of $\mathrm{Spin}(n)$ using $\star$ to the presentation of $\mathrm{Spin}(n)$ using $\diamond$ is described as

$$(f, f_1, f_2)_\star \mapsto (f, h^{-1} f_1 h, h_2 f_2 h_2^{-1})_\diamond. \tag{8.6}$$

Since $\widehat{\rho}_\diamond(f, f_1', f_2')_\diamond = (f_1', f_2', f)_\diamond$, we get, using (8.6),

$$\widehat{\rho}_\diamond\big((f, f_1, f_2)_\star\big) = (h^{-1} f_1 h, h h_2 f_2 h_2^{-1} h^{-1}, h_2^{-1} f h_2)_\star$$

so that the relation $h_2 h_1 h = \mathrm{Id}_S$ implies $\tau = \widehat{\rho}_\diamond$. □

9. Classification of symmetric compositions

It follows from Theorem 6.4 that the classification of symmetric compositions up to similarity, the classification of *normalized* symmetric compositions up to isomorphism, and the classification of trialitarian automorphisms up to conjugation are essentially equivalent. In this section we recall the classification of normalized symmetric compositions of dimension 8 over arbitrary fields.

We first consider symmetric compositions over algebraically closed fields. The norm form n is then hyperbolic. Octonion algebras over such fields are split and it is convenient to choose as a model the Zorn algebra, which is defined in arbitrary characteristic.

The Zorn algebra. We denote by $\bullet$ the usual scalar product on $F^3 = F \times F \times F$, and by $\times$ the vector product: for $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in F^3$,

$$a \bullet b = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \text{and} \qquad a \times b = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

The Zorn algebra is the set of matrices

$$\mathfrak{Z} = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \middle| \, \alpha, \beta \in F, \ a, b \in F^3 \right\}$$

with the product

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \cdot \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma + a \bullet d & \alpha c + \delta a - b \times d \\ \gamma b + \beta d + a \times c & \beta\delta + b \bullet c \end{pmatrix},$$

the norm

$$n \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} = \alpha\beta - a \bullet b,$$

and the conjugation

$$\overline{\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}} = \begin{pmatrix} \beta & -a \\ -b & \alpha \end{pmatrix},$$

which is such that $\xi \cdot \overline{\xi} = \overline{\xi} \cdot \xi = n(\xi)$ for all $\xi \in \mathfrak{Z}$ (see [Zor30, p. 144]).

In view of Example 3.4 the product

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} * \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \overline{\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}} \cdot \overline{\begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix}} = \begin{pmatrix} \beta\delta + a \bullet d & -\beta c - \gamma a - b \times d \\ -\delta b - \alpha d + a \times c & \alpha\gamma + b \bullet c \end{pmatrix},$$

defines a symmetric composition on the quadratic space $(\mathfrak{Z}, n)$. We call $\star$ the *para-Zorn composition.*

We use the technique of Section 7 to twist the para-Zorn composition. Let $p\colon F^3 \to F^3$ be the map $(x_1, x_2, x_3) \mapsto (x_3, x_1, x_2)$, and let

$$\pi\colon \mathfrak{Z} \to \mathfrak{Z}, \qquad \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} = \begin{pmatrix} \alpha & p(a) \\ p(b) & \beta \end{pmatrix}.$$

The map π is an automorphism of $\star$ of order 3, so we may consider the twisted composition $\star_\pi$ as in (7.3), which we call the *split Petersson symmetric composition.* (The algebra $(\mathfrak{Z}, \star_\pi)$ is also known as the *split pseudo-octonion algebra.*)

Theorem 9.1. *Over an algebraically closed field, there are exactly two symmetric compositions up to isomorphism: the para-Zorn composition and the split Petersson composition.*

Proof. The claim is a result of Petersson [Pet69, Satz 2.7] if the field has characteristic different from 2 and 3, and is due to Elduque-Pérez [EP96] in arbitrary characteristic. □

Over arbitrary fields we consider two kinds of symmetric compositions, which we call type I and type II. Symmetric compositions of *type* I are forms of the para-Zorn composition, i.e., $\star$ is of *type* I if and only if $\star$ is isomorphic to the para-Zorn composition after scalar extension to an algebraic closure. Similarly, we say that $\star$ is of *type* II if it is a form of the split Petersson composition. Thus, Theorem 9.1 shows that every symmetric composition is either of type I or of type II.

Type I. The classification of symmetric compositions of type I is particularly simple.

Theorem 9.2. *Let (S, n) be a 3-fold Pfister quadratic space over an arbitrary field F. Up to isomorphism, there is a unique normalized symmetric composition of type I on (S, n), given by the para-octonion composition.*

Proof. Forms of the split para-octonion algebra are para-octonion algebras (see for example [KMRT98, §34 A]), hence symmetric compositions of type I are para-octonion compositions as defined in Example 3.4. Isomorphisms of octonion algebras are isomorphisms of the corresponding para-octonion algebras, and octonion algebras with isometric norms are isomorphic (see for example [KMRT98, (33.19)]). □

The automorphism group of a symmetric composition of type I is the automorphism group of the corresponding octonion algebra; it is a simple algebraic group of type G_2. Accordingly, symmetric compositions of type I are also called *symmetric compositions of type* G_2.

Type II. The classification of symmetric compositions of type II has a completely different flavor in characteristic 3. We first discuss the case where the characteristic is different from 3, and distinguish two subcases, depending on whether the base field contains a primitive cube root of unity or not.

Suppose first F is a field of characteristic different from 3 containing a primitive cube root of unity ω. Let A be a central simple F-algebra of degree 3. For the reduced characteristic polynomial of $a \in A$, we use the notation

$$X^3 - \mathrm{Trd}(a)X^2 + \mathrm{Srd}(a)X - \mathrm{Nrd}(a)\,1,$$

so Trd is the reduced trace map on A, Srd is the reduced quadratic trace map, and Nrd is the reduced norm. Let $A^0 \subset A$ be the kernel of Trd. We define a multiplication $\star$ on A^0 by

$$x \star y = \frac{yx - \omega xy}{1 - \omega} - \tfrac{1}{3}\,\mathrm{Trd}(xy)\,1 \tag{9.3}$$

and a quadratic form n by

$$n(x) = -\tfrac{1}{3}\,\mathrm{Srd}(x). \tag{9.4}$$

The following result can be found for instance in [KMRT98, (34.19), (34.25)]:

Proposition 9.5. *The quadratic space (A^0, n) is hyperbolic, and $\star$ is a normalized symmetric composition on (A^0, n).*

When the algebra A is split, the composition $\star$ is isomorphic to the split Petersson composition. Symmetric compositions as in Proposition 9.5 are called *Okubo compositions.* The automorphism group of an Okubo composition associated to a central simple algebra A is $\mathbf{PGL}_1(A)$. Therefore, these compositions are also called *symmetric compositions of type* ${}^1\mathrm{A}_2$.

Suppose next F is a field of characteristic different from 3 that does not contain a primitive cube root of unity. Let ω be a primitive cube root of unity in some separable closure of F, and let $K = F(\omega)$, a separable quadratic extension of F. Let B be a central simple K-algebra of degree 3 with a unitary involution τ leaving F fixed. We let $\mathrm{Sym}(B, \tau)^0$ denote the F-vector space of τ-symmetric elements of reduced trace zero. Formula (9.3) defines a multiplication on $\mathrm{Sym}(B,\tau)^0$, and the form n of (9.4) is a quadratic form on $\mathrm{Sym}(B,\tau)^0$.

The following result is proved in [KMRT98, (34.35)]:

Proposition 9.6. *The quadratic space* $(\mathrm{Sym}(B,\tau)^0, n)$ *is a* 3*-fold Pfister quadratic space that becomes hyperbolic over* K*, and* $\star$ *is a normalized symmetric composition on this space.*

Normalized symmetric compositions as in Proposition 9.6 are also called Okubo compositions. The automorphism group of an Okubo composition associated to a unitary involution τ on a central simple K-algebra B is $\mathbf{PGU}(B,\tau)$. Therefore, these compositions are also called *symmetric compositions of type* ${}^2\mathrm{A}_2$.

Theorem 9.7. *Let* (S,n) *be a* 3*-fold Pfister quadratic space over a field* F *of characteristic different from* 3*. Let* ω *be a primitive cube root of unity in a separable closure of* F*.*

(1) *If* $\omega \in F$*, then every normalized symmetric composition of type II on* (S,n) *is isomorphic to the Okubo composition associated to some central simple* F*-algebra of degree* 3*, uniquely determined up to isomorphism. Such compositions exist if and only if* n *is hyperbolic.*
(2) *If* $\omega \notin F$*, let* $K = F(\omega)$*. Every normalized symmetric composition of type II on* (S,n) *is isomorphic to the Okubo composition associated to some central simple* K*-algebra of degree* 3 *with unitary involution, uniquely determined up to isomorphism. Such compositions exist if and only if* n *is split by* K*.*

Proof. See Elduque-Myung [EM93, p. 2487] or [KMRT98, (34.37)]. □

Now, suppose the characteristic of F is 3. Normalized symmetric compositions of type II over F are extensively discussed in [EP96], [Eld97], [Eld99] and [Eld00]. They can be viewed as a specialization of symmetric compositions of type II over fields containing a primitive cube root of unity. To introduce them we first consider symbol algebras of degree 3. Any central simple algebra A of degree 3 over a field containing a cubic primitive root of unity ω is a symbol algebra, i.e., A has generators x, y such that $x^3 = a$, $y^3 = b$ and $yx = \omega\, xy$ for some a, $b \in F^\times$. We consider the Okubo composition $\star$ associated with A. The following basis of A^0:

$$\begin{array}{llll} e_1 = x & e_2 = y & e_3 = \omega^2\, xy & e_4 = \omega\, xy^{-1} \\ f_1 = x^{-1} & f_2 = y^{-1} & f_3 = \omega^2\, x^{-1}y^{-1} & f_4 = \omega\, x^{-1}y \end{array} \tag{9.8}$$

is hyperbolic for the form n and the multiplication table with respect to this basis of the product $\star$ is given by

	e_1	f_1	e_2	f_2	e_3	f_3	e_4	f_4
e_1	af_1	0	0	$-e_4$	0	$-f_2$	$-af_3$	0
f_1	0	$a^{-1}e_1$	$-f_4$	0	$-e_2$	0	0	$-a^{-1}e_3$
e_2	$-e_3$	0	bf_2	0	$-be_4$	0	$-e_1$	0
f_2	0	$-f_3$	0	$b^{-1}e_2$	0	$-b^{-1}f_4$	0	$-f_1$
e_3	$-af_4$	0	0	$-e_1$	abf_3	0	0	$-bf_2$
f_3	0	$-a^{-1}e_4$	$-f_1$	0	0	$a^{-1}b^{-1}e_3$	$-b^{-1}e_2$	0
e_4	0	$-f_2$	0	$-b^{-1}e_3$	$-af_1$	0	$a^{-1}bf_4$	0
f_4	$-e_2$	0	$-bf_3$	0	0	$-a^{-1}e_1$	0	$ab^{-1}e_4$

This multiplication table does not involve ω, hence specialising gives a symmetric composition $\diamond_{a,b}$ over arbitrary fields, also of characteristic 3.

Proposition 9.9. *1) The symmetric composition $\diamond_{1,1}$ is isomorphic to the split Petersson composition.*
2) Let $\overline{F}$ be an algebraic closure of F and let $c,\ d \in \overline{F}$ be such that $c^3 = a$ and $d^3 = b$. The symmetric compositions $\diamond_{a,b}$ and $\diamond_{1,1}$ are isomorphic over $F(c,d)$. Thus the symmetric composition $\diamond_{a,b}$ splits over $F(c,d)$.

Proof. 1) and 2) follow by comparing multiplication tables. □

Theorem 9.10. *Let (S,n) be a 3-fold Pfister quadratic space over a field F of characteristic 3. Every normalized symmetric composition of type II on (S,n) is isomorphic to a symmetric composition of the form $\diamond_{a,b}$ for some $a,\ b \in F^\times$.*

Proof. The claim follows from [Eld99, p. 291] and a comparison of multiplication tables. □

Conditions for isomorphism $\diamond_{a,b} \xrightarrow{\sim} \diamond_{a',b'}$ are discussed in [Eld99].

The automorphism groups (as group schemes) of the compositions $\diamond_{a,b}$ are not smooth in characteristic 3 and are, as far as we know, not studied in the literature. Their groups of rational points are described in [Eld99].

Observe that symmetric compositions of type II in characteristic 3 are not necessarily split over separably closed fields, in contrast to symmetric compositions of type I and to symmetric compositions of type II in characteristic different from 3.

References

[Car25] Élie Cartan. Le principe de dualité et la théorie des groupes simples et semi-simples. *Bull. Sci. Math*, 49:361–374, 1925.

[Eld97] Alberto Elduque. Symmetric composition algebras. *J. Algebra*, 196:283–300, 1997.

[Eld99] Alberto Elduque. Okubo algebras in characteristic 3 and their automorphisms. *Communications in Algebra*, 27:3009–3030, 1999.

[Eld00] A. Elduque. A note on triality and symmetric compositions. *Abh. Math. Sem. Univ. Hamburg*, 70:125–135, 2000.

[EM93] Alberto Elduque and Hyo Chul Myung. On flexible composition algebras. *Comm. Algebra*, 21(7):2481–2505, 1993.

[EP96] Alberto Elduque and José María Pérez. Composition algebras with associative bilinear form. *Comm. Algebra*, 24(3):1091–1116, 1996.

[Fau88] John R. Faulkner. Finding octonion algebras in associative algebras. *Proc. Amer. Math. Soc.*, 104(4):1027–1030, 1988.

[GL83] Daniel Gorenstein and Richard Lyons. The local structure of finite groups of characteristic 2 type. *Mem. Amer. Math. Soc.*, 42(276):vii+731, 1983.

[GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon. *The classification of the finite simple groups. Number 3. Part I. Chapter A*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[Jac64] Nathan Jacobson. Triality and Lie algebras of type D_4. *Rend. Circ. Mat. Palermo (2)*, 13:129–153, 1964.

[KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The Book of Involutions*. Number 44 in American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, R.I., 1998. With a preface in French by J. Tits.

[Knu09] Max-Albert Knus. Trialitarian automorphisms of Lie algebras. *Transformation Groups*, 14(1):361–386, 2009.

[Oku78] Susumu Okubo. Pseudo-quaternion and pseudo-octonion algebras. *Hadronic J.*, 1(4):1250–1278, 1978.

[Pet69] Holger P. Petersson. Eine Identität fünften Grades, der gewisse Isotope von Kompositions-Algebren genügen. *Math. Z.*, 109:217–238, 1969.

[Stu13] Ernst Study. Grundlagen und Ziele der analytischen Kinematik. *Sitzungsber. Berl. Math. Ges.*, 12:36–60, 1913.

[SV00] Tonny A. Springer and Ferdinand D. Veldkamp. *Octonions, Jordan Algebras and Exceptional Groups.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

[Tit59] Jacques Tits. Sur la trialité et certains groupes qui s'en déduisent. *Publ. Math. IHES*, 2:14–60, 1959.

[Wat79] William C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1979.

[WG68] Joseph A. Wolf and Alfred Gray. Homogeneous spaces defined by Lie group automorphisms. I. *J. Differential Geometry*, 2:77–114, 1968.

[Zor30] Max Zorn. Theorie der alternativen Ringe. *Abh. Math. Seit. Hamb. Univ.*, 8:123–147, 1930.

Department of Mathematical and Statistical Sciences, University of Alberta, 632 Central Academic Building, Edmonton, AB T6G 2G1, Canada
E-mail address: `chernous@math.ualberta.ca`

Department Mathematik, ETH Zentrum, CH-8092 Zürich, Switzerland
E-mail address: `knus@math.ethz.ch`

ICTEAM Institute, Université catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium, and Zukunftskolleg, Universität Konstanz, D-78457 Konstanz, Germany
E-mail address: `jean-pierre.tignol@uclouvain.be`