QUADRATIC FORMS, COMPOSITIONS AND TRIALITY OVER $\mathbb{F}_1$

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Abstract. According to Tits, the quadric of dimension 6 over the "field" $\mathbb{F}_1$ with one element is a set of 8 points structured by a permutation of order 2 without fixed points. Subsets that are disjoint from their image under the permutation are the subspaces of the quadric. As in the classical case of hyperbolic quadrics over an arbitrary field, maximal subspaces come in two different types. We define a geometric triality to be a permutation of order 3 of the set consisting of points and maximal subspaces, carrying points to maximal subspaces of one type and maximal subspaces of the other type to points while preserving the incidence relations. We show analogues over $\mathbb{F}_1$ of the one-to-one correspondence between geometric trialities, trialitarian automorphisms of algebraic groups of type D_4, and symmetric composition algebras of dimension 8. Here, the algebraic groups of type D_4 are replaced by their Weyl group, which is the semi-direct product $\mathfrak{S}_2^3 \rtimes \mathfrak{S}_4$, and composition algebras by a certain type of binary operation on a quadric to which a 0 is adjoined. As in the classical case, we show that there are two types of trialities, one related to octonions and the other to Okubo algebras.

1. Introduction

As shown by Tits ([Tit57]) all split simple adjoint algebraic groups can be realized as automorphism groups of geometries. Their Weyl groups can as well be realized as automorphism groups of certain finite geometries, which Tits calls geometries over the "field" $\mathbb{F}_1$ of one element. For example a n-dimensional projective space over $\mathbb{F}_1$ is a set with $n+1$ elements and its automorphism group is the symmetric group on $n+1$ elements $\mathfrak{S}_{n+1}$. The motivation for introducing this "field" which does not exist is as follows. There exist geometries over finite fields with q elements, whose groups of automorphisms are the Chevalley groups over the corresponding fields. If q tends towards 1 one gets geometries having as automorphism groups the Weyl groups. Another introductory paper to projective geometry over $\mathbb{F}_1$ is [Coh04]. Tits's ideas led more than thirty years later to new developments in what is called today $\mathbb{F}_1$-geometry (see [LPn12] for a survey).

In this paper, which is in Tits's original spirit, we classify trialities over $\mathbb{F}_1$. At the end of [Tit57] Tits observes that some important properties of quadrics (over ordinary fields) also hold for quadrics over $\mathbb{F}_1$. An example mentioned by Tits without details is triality. Tits refers to unpublished work of Mlle F. Lenger. We

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were unable to trace her work, but we assume that some of our results are similar to hers.

Classically, triality appears in two different settings: as a geometric property of 6-dimensional quadrics and as trialitarian automorphisms, i.e., outer automorphisms of order 3 of simple adjoint or simple simply connected algebraic groups of type D_4. There are two types of solids on a 6-dimensional quadric, and geometric triality permutes cyclically the points and the two types of solids on the quadric. Following the pioneering work of Élie Cartan ([Car25]) and its beautiful presentation by van der Blij and Springer ([vdBS60]), it was observed by Markus Rost that a specific kind of composition algebras, which Rost called *symmetric compositions*, is well suited to describe triality in both settings (see [KMRT98, Ch. VIII]). Symmetric compositions exist in dimension 2, 4, and 8. As shown in [CKT12] the classifications of 8-dimensional symmetric compositions and of trialities are, in fact, equivalent. One of the aims of this paper is to introduce symmetric compositions over $\mathbb{F}_1$ so that triality over $\mathbb{F}_1$ can be presented in a way parallel to classical triality.

After recalling in Section 2 how algebra and geometry look over $\mathbb{F}_1$, we introduce symmetric compositions over $\mathbb{F}_1$ in Section 3 and prove that they occur only in dimension 2, 4, and 8. In Section 4 we use geometric or combinatorial techniques to describe geometric triality over $\mathbb{F}_1$. Absolute points and hexagons, a tool introduced by Tits in his IHES paper on triality ([Tit59]), play here a fundamental role. We close this section by constructing a bijective correspondence between geometric trialities and 8-dimensional symmetric compositions.
Trialitarian automorphisms over $\mathbb{F}_1$ are outer automorphisms of order 3 of the Weyl group of type D_4, which is the semi-direct product $\mathfrak{S}_2^3 \rtimes \mathfrak{S}_4$. We show in the last part of the paper that the three kind of objects: 8-dimensional symmetric compositions over $\mathbb{F}_1$, geometric trialities over $\mathbb{F}_1$ and trialitarian automorphisms are in bijective correspondence. There are two types of each, which we describe explicitly. One type of 8-dimensional symmetric compositions is closely related to octonions (whose connection with triality in the classical case was already noticed by É. Cartan) and the other case is related to algebras introduced by the theoretical physicist S. Okubo.

2. Geometry and algebra over $\mathbb{F}_1$

Vector spaces and projective spaces. Following Kapranov–Smirnov ([KS95]) we define a *finite-dimensional vector space* $\mathcal{V}$ *over* $\mathbb{F}_1$ as a finite pointed set $\mathcal{V} = \{x_1, \dots, x_n, 0\}$, with n elements $x_1, \dots, x_n$ and a distinguished point 0. The associated *projective space* $\mathbb{P}(\mathcal{V})$ over $\mathbb{F}_1$ is the set $\langle \mathcal{V} \rangle = \mathcal{V} \setminus \{0\}$. The dimension $\dim \mathcal{V}$ of $\mathcal{V}$ is the cardinality $|\langle \mathcal{V} \rangle|$ of $\langle \mathcal{V} \rangle$ and the dimension of $\mathbb{P}(\mathcal{V})$ is equal to $\dim \mathcal{V} - 1$. Thus, as in Tits ([Tit57]), a projective space of dimension $n-1$ over $\mathbb{F}_1$ is a set with n elements. Linear subspaces are subsets containing 0. Any linear subspace $\mathcal{U}$ of $\mathcal{V}$ defines a *linear subvariety* $\langle \mathcal{U} \rangle = \mathcal{U} \setminus \{0\}$ of $\mathbb{P}(\mathcal{V})$. Linear maps of vector spaces over $\mathbb{F}_1$ are maps of pointed sets and the *full linear group* of a vector space $\mathcal{V}$ over $\mathbb{F}_1$ is the permutation group $\mathfrak{S}(\langle \mathcal{V} \rangle)$, which is as the same time the *projective linear group* of $\mathbb{P}(\mathcal{V})$:

$$\mathrm{GL}(\mathcal{V}) = \mathrm{PGL}(\langle \mathcal{V} \rangle) = \mathfrak{S}(\langle \mathcal{V} \rangle).$$

Direct sums $\mathcal{V} \oplus \mathcal{V}'$ of vector spaces are given by disjoint unions where the zero elements are identified.

Quadratic forms. A *(nonsingular) quadratic form* on an even-dimensional vector space $\mathcal{V}$ over $\mathbb{F}_1$ is a bijective self-map of order 2

$$\widetilde{\ }: \mathcal{V} \to \mathcal{V}, \qquad x \mapsto \widetilde{x},$$

without fixed points on $\langle \mathcal{V} \rangle$ and such that $\widetilde{0} = 0$. We call the pair $\mathcal{Q} = (\mathcal{V}, \widetilde{\ })$ an even-dimensional *quadratic space* over $\mathbb{F}_1$. The map $\widetilde{\ }$ will be referred to from now on as the *structure map* of the quadratic space. Orthogonal sums $\mathcal{Q} \perp \mathcal{Q}'$ of quadratic spaces are direct sums of the underlying vector spaces, with the structure map that restricts to the structure map on each summand. A *quadratic subspace* is a linear subspace $\mathcal{U} \subset \mathcal{V}$ that is preserved by the structure map. For any linear subspace $\mathcal{U}$, we set $\mathcal{U}^{\perp} = \{x \in \mathcal{V} \mid \widetilde{x} \notin \mathcal{U}\} \sqcup \{0\}$. In the language of diagram geometry the operator $^{\perp}$ is a *(thin) polarity.* If $\mathcal{U}$ is a nonsingular subspace, then $\mathcal{V} = \mathcal{U} \perp \mathcal{U}^{\perp}$. *Isotropic subspaces* of $\mathcal{V}$ are linear subspaces $\mathcal{U}$ of $\mathcal{V}$ such that $\mathcal{U} \subset \mathcal{U}^{\perp}$, i.e., if $x \in \mathcal{U}$ and $x \neq 0$, then $\widetilde{x} \notin \mathcal{U}$. If $\dim \mathcal{V} = 2n$, isotropic subspaces have dimension at most n. If $\mathcal{U}$ is maximal isotropic, we have $\mathcal{U} = \mathcal{U}^{\perp}$ and for $x \neq 0$ the condition $x \in \mathcal{U}$ is equivalent to $\widetilde{x} \notin \mathcal{U}$.

If $\mathcal{Q} = (\mathcal{V}, \widetilde{\ })$ is a $2n$-dimensional quadratic space, the structure map $\widetilde{\ }$ restricts to a map $\langle \mathcal{V} \rangle \to \langle \mathcal{V} \rangle$ also denoted by $\widetilde{\ }$. Following Tits ([Tit57, p. 287] the pair $\langle \mathcal{Q} \rangle = \big(\langle \mathcal{V} \rangle, \widetilde{\ }\big)$ is a *(hyper)quadric* of dimension $2n-2$ in $\mathbb{P}(\mathcal{V})$. Isotropic subspaces of $\mathcal{Q}$ of dimension $k+1$ define k-dimensional subvarieties of $\langle \mathcal{Q} \rangle$, which we call *$k$-solids*. 0-solids are *points*, 1-solids *lines*, 2-solids *planes*, and $(n-1)$-solids are the *maximal solids*. The structure map $\widetilde{\ }$ extends to a map of k-solids to k-solids still denoted by $\widetilde{\ }$. Observe that $\omega \cap \widetilde{\omega} = \varnothing$ for any solid ω.

We say that two maximal isotropic spaces $\mathcal{U}$ and $\mathcal{U}'$ (resp. two maximal solids ω and ω') are of the *same kind* if $\dim(\mathcal{U} \cap \mathcal{U}')$ has the same parity as $\dim \mathcal{U}$ (resp. if $\dim(\omega \cap \omega')$ has the same parity as $\dim \omega$). Thus maximal isotropic subspaces are of two kinds. Let $C(\langle \mathcal{Q} \rangle)$ be the set of maximal isotropic subspaces of $\mathcal{Q}$ (or the set of maximal solids in $\langle \mathcal{Q} \rangle$). The choice of a decomposition $C(\langle \mathcal{Q} \rangle) = C_1 \sqcup C_2$, where C_1 is the set of subspaces of one fixed kind and C_2 the set of subspaces of the other kind, is an *orientation* ∂ of $\langle \mathcal{Q} \rangle$. More precisely, an orientation is a surjective map

$$\partial \colon C(\langle \mathcal{Q} \rangle) \to \{1, 2\} \tag{2.1}$$

such that $\partial^{-1}(1) = C_1$ is the set of maximal solids of one kind and $\partial^{-1}(2) = C_2$ is the set of maximal solids of the other kind.

Isometries of quadratic spaces $\phi \colon (\mathcal{V}, \widetilde{\ }) \to (\mathcal{V}', \widetilde{\ })$ are linear maps $\phi \colon \mathcal{V} \to \mathcal{V}'$ such that $\widetilde{\phi(x)} = \phi(\widetilde{x})$ for all $x \in \mathcal{V}$. The restrictions of isometries to $\langle \mathcal{V} \rangle$ are the isomorphisms of quadrics. We let $\mathrm{O}(\mathcal{Q})$ $(= \mathrm{PGO}(\langle \mathcal{Q} \rangle))$ denote the group of isometries $\mathcal{Q} \to \mathcal{Q}$ (or the group of isomorphisms $\langle \mathcal{Q} \rangle \to \langle \mathcal{Q} \rangle$) and define $\mathrm{O}^{+}(\mathcal{Q})$ $(= \mathrm{PGO}^{+}(\langle \mathcal{Q} \rangle))$ as the subgroup of isometries (or isomorphisms) that map C_1 to C_1 and C_2 to C_2. The elements of $\mathrm{O}^{+}(\mathcal{Q})$ are the *proper isometries.* Observe that proper isometries respect each of the two orientations.

The group $\mathfrak{S}_n$ acts by permutations on $\mathfrak{S}_2^n$. Viewing $\mathfrak{S}_2^{n-1}$ as a subgroup of $\mathfrak{S}_2^n$ through the exact sequence

$$1 \to \mathfrak{S}_2^{n-1} \to \mathfrak{S}_2^n \xrightarrow{\pi} \mathfrak{S}_2 \to 1, \quad \pi(\epsilon_1, \dots, \epsilon_n) = \epsilon_1 \cdot \ldots \cdot \epsilon_n,$$

we get an action of $\mathfrak{S}_n$ on $\mathfrak{S}_2^{n-1}$.

Lemma 2.2. *For any quadratic space $\mathcal{Q}$ of dimension $2n$, the group $\mathrm{O}(\mathcal{Q}) = \mathrm{PGO}(\langle\, \mathcal{Q}\,\rangle)$ is isomorphic to the semi-direct product $\mathfrak{S}_2^n \rtimes \mathfrak{S}_n$, and $\mathrm{O}^+(\mathcal{Q}) = \mathrm{PGO}^+(\langle\, \mathcal{Q}\,\rangle)$ is isomorphic to $\mathfrak{S}_2^{n-1} \rtimes \mathfrak{S}_n$.*

Proof. For any $x \in \langle\, \mathcal{Q}\,\rangle$, let $\sigma_x \in \mathrm{O}(\mathcal{Q})$ be the reflection

$$\sigma_x(y) = \begin{cases} \widetilde{y} & \text{if } y \in \{x, \widetilde{x}\}, \\ y & \text{if } y \notin \{x, \widetilde{x}\}. \end{cases}$$

Any element of $\mathrm{O}(\mathcal{Q})$ induces a permutation of the set of n pairs $\{x, \widetilde{x}\}$ for $x \in \langle \mathcal{Q} \rangle$. Thus we get a homomorphism $\pi\colon\ \mathrm{O}(\mathcal{Q}) \to \mathfrak{S}_n$ whose kernel is the subgroup $\mathfrak{S}_2^n$ of $\mathrm{O}(\mathcal{Q})$ generated by all reflections σ_x. Moreover the sequence

$$1 \to \mathfrak{S}_2^n \to \mathrm{O}(\mathcal{Q}) \to \mathfrak{S}_n \to 1$$

is split, hence the first claim.
Fixing $x \in \langle\, \mathcal{Q}\,\rangle$, note that every maximal solid ω of $\langle\, \mathcal{Q}\,\rangle$ contains either x or $\widetilde{x}$. If $x \in \omega$, then $\sigma_x(\omega)$ is the solid obtained by substituting $\widetilde{x}$ for x in ω, hence

$$\dim\bigl(\omega \cap \sigma_x(\omega)\bigr) = (\dim \omega) - 1. \tag{2.3}$$

The same equation holds if $\widetilde{x} \in \omega$. Therefore, every reflection switches the two kinds of maximal solids, and a product of reflections is in $\mathrm{O}^+(\mathcal{Q})$ if and only if the number of factors is even. The second claim follows. □

Observe that the center of $\mathrm{O}^+(\mathcal{Q})$ is the group of order 2 generated by the structure map $\widetilde{\ }$.

Lemma 2.4. *The group $\mathrm{PGO}(\langle\, \mathcal{Q}\,\rangle)$ acts transitively on $C(\langle\, \mathcal{Q}\,\rangle)$ and, given an orientation of $\langle\, \mathcal{Q}\,\rangle$, the group $\mathrm{PGO}^+(\langle\, \mathcal{Q}\,\rangle)$ acts transitively on C_1 and on C_2.*

Proof. Let ω, ω' be maximal solids of the quadric $\langle\, \mathcal{Q}\,\rangle$, and let $x_1, \ldots, x_r$ be the points in their intersection, so if $\mathcal{Q} = \{0, x_i, \widetilde{x}_i\}_{i=1}^n$ we may assume

$$\omega = \{x_1, \ldots, x_r, x_{r+1}, \ldots, x_n\} \quad \text{and} \quad \omega' = \{x_1, \ldots, x_r, \widetilde{x_{r+1}}, \ldots, \widetilde{x_n}\}.$$

Using the reflections σ_x defined in the proof of Lemma 2.2, we have

$$\omega' = \sigma_{x_{r+1}} \circ \cdots \circ \sigma_{x_n}(\omega).$$

Note that $\sigma_{x_{r+1}} \circ \cdots \circ \sigma_{x_n} \in \mathrm{PGO}(\langle\, \mathcal{Q}\,\rangle)$, and this element lies in $\mathrm{PGO}^+(\langle\, \mathcal{Q}\,\rangle)$ if $n - r$ is even, which occurs if and only if ω and ω' are maximal solids of the same kind. □

An incidence relation between k-solids, for $k = 0, 1, 2, \ldots, n-1$, is defined by the inclusion. Moreover, two maximal solids of different kinds are called *incident* if the dimension of their intersection has the same parity as their own dimension. With this incidence relation the sets of points, lines, up to maximal solids of $\mathcal{Q}$ define a geometry whose automorphism group is the Weyl group of type D_n (see [Tit57, (3.2)]).

Lemma 2.5. *For any two maximal solids ω, ω' on a quadric $\langle\, \mathcal{Q}\,\rangle$, there exists a finite sequence of maximal solids $\omega_0, \ldots, \omega_s$ of $\langle\, \mathcal{Q}\,\rangle$ such that*

- *$\omega_0 = \omega$, $\omega_s = \omega'$, and*
- *ω_{i-1} and ω_i are incident for all $i = 1, \ldots, s$.*

The integer s is even if and only if ω and ω' are of the same kind.

Proof. By Lemma 2.4, there exist reflections $\sigma_1, \ldots, \sigma_s$ such that $\omega' = \sigma_s \circ \cdots \circ \sigma_1(\omega)$. Set $\omega_i = \sigma_i \circ \cdots \circ \sigma_1(\omega)$, and observe that ω_{i-1} and ω_i are incident by (2.3). □

Example 2.6. Let F be a field, let V be a $2n$-dimensional vector space over F and let $\mathbb{P} = \mathbb{P}(V)$ be the associated projective space of dimension $2n-1$ over F. We designate by $\langle X \rangle$ the subvariety of $\mathbb{P}(V)$ defined by a linear subspace X of V. Let $Q = (V, q)$ be a $2n$-dimensional hyperbolic quadratic space (in the classical sense) over F and let b be the polar of q. Let $\langle Q \rangle$ be the $2n-2$-dimensional quadric in $\mathbb{P}$ defined by the equation $q = 0$. *Lines* (resp. *planes*, ..., $(n-1)$-*solids* on $\langle Q \rangle$ are linear subvarieties defined by 2- (resp. 3-, ..., n-dimensional) totally isotropic subspaces of $\mathbb{P}$. We say again that two maximal solids ω, ω (resp. two maximal isotropic subspaces U and U') are of the *same kind* if $\dim(\omega \cap \omega')$ has the same parity as $\dim \omega$ (resp. $\dim(U \cap U')$ has the same parity as $\dim U$). Let $\mathrm{PGO}(\langle Q \rangle)$ be the subgroup of the group $\mathrm{PGL}(V)$ of collineations of $\mathbb{P}(V)$ mapping $\langle Q \rangle$ to itself. The special projective orthogonal group of $\langle Q \rangle$ is the subgroup $\mathrm{PGO}^+(\langle Q \rangle)$ of $\mathrm{PGO}(\langle Q \rangle)$ of collineations respecting the decomposition of the set of solids into the two types.
Let $\{e_i, f_i\}_{i=1}^n$ be a hyperbolic basis of Q, i.e.,

$$q(e_i) = q(f_j) = 0 \quad \text{and} \quad b(e_i, f_j) = \delta_{ij} \quad \text{for } i,\, j = 1, \ldots, n.$$

The set $\mathcal{Q} = \{e_i, f_i\}_{i=1}^n \cup \{0\}$ with $x \mapsto \widetilde{x}$ defined as $\widetilde{e_i} = f_i$, $\widetilde{f_i} = e_i$, $\widetilde{0} = 0$, is a typical example of a quadratic space over $\mathbb{F}_1$. Elements of $\mathrm{PGO}(\langle \mathcal{Q} \rangle)$ (resp. of $\mathrm{PGO}^+(\langle \mathcal{Q} \rangle)$) are the restrictions of elements of $\mathrm{PGO}(\langle Q \rangle)$ (resp. of $\mathrm{PGO}^+(\langle Q \rangle)$) which map $\mathcal{Q}$ to itself.

Algebras. A finite-dimensional *algebra* $(\mathcal{S}, \star)$ over $\mathbb{F}_1$ is a finite-dimensional $\mathbb{F}_1$-vector space $\mathcal{S}$ together with a map

$$\star \colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}, \quad (x, y) \mapsto x \star y,$$

called the *multiplication*, such that $0 \star x = x \star 0 = 0$ for all $x \in \mathcal{S}$. If $(\mathcal{S}, \star)$ is an algebra over $\mathbb{F}_1$, the *opposite algebra* $(\mathcal{S}, \star^{\mathrm{op}})$, is defined by $x \star^{\mathrm{op}} y = y \star x$.

Example 2.7. Let S be a finite-dimensional algebra (not necessarily associative and not necessarily with unit element) over a field F. If S admits a basis $\{u_i\}$ such that the multiplication table with respect to this basis is monomial, i.e., $u_i \cdot u_j = \lambda_{ijk} u_k$ for some $\lambda_{ijk} \in F$, we deduce from S an algebra $\mathcal{S} = \{v_i\}$ over $\mathbb{F}_1$, with product $\star$ defined by

$$v_i \star v_j = \begin{cases} v_k & \text{if } \lambda_{ijk} \neq 0, \\ 0 & \text{if } \lambda_{ijk} = 0. \end{cases}$$

The philosophy is that an algebra structure that can be defined in terms of multiplication only (i.e., without using the addition) goes over to $\mathbb{F}_1$.

3. Symmetric compositions

Symmetric compositions form a class of algebras that play a fundamental role in triality over arbitrary fields (see [KMRT98, Ch. VIII] and [CKT12] for recent results). In this section, we show that symmetric compositions have analogues over $\mathbb{F}_1$. We first recall the situation over fields.

Symmetric compositions over fields. A *composition* on a finite-dimensional quadratic space (S, n) is a bilinear multiplication $\star$ on S for which the quadratic form is *multiplicative* in the sense that

$$n(x \star y) = n(x)n(y) \quad \text{for all } x,\, y \in S. \tag{3.1}$$

Linearizing this equation yields the following formulas for the polar bilinear form b: for all x, y, $z \in S$,

$$b(x \star y, x \star z) = n(x)b(y, z) \quad \text{and} \quad b(y \star x, z \star x) = b(y, z)n(x) \tag{3.2}$$

and

$$b(x \star y, u \star v) + b(x \star v, u \star y) = b(x, u)b(y, v) \quad \text{for all } x,\, y,\, u,\, v \in S. \tag{3.3}$$

The composition is called *symmetric* if the polar bilinear form b is *associative*; i.e.,

$$b(x \star y, z) = b(x, y \star z) \quad \text{for all } x,\, y,\, z \in S. \tag{3.4}$$

By [KMRT98, (34.1)], we then also have the relations

$$x \star (y \star x) = n(x)y = (x \star y) \star x \quad \text{for all } x,\, y \in S, \tag{3.5}$$

and their linearizations

$$x \star (y \star z) + z \star (y \star x) = b(x, z)y = (x \star y) \star z + (z \star y) \star x \quad \text{for all } x,\, y,\, z \in S. \tag{3.6}$$

Symmetric compositions are completely classified. We refer for example to [KMRT98, §34] for details.

Symmetric compositions over $\mathbb{F}_1$. Let $(\mathcal{S}, \widetilde{\ })$ be a nonzero even-dimensional quadratic space over $\mathbb{F}_1$.

Definition 3.7. A *symmetric composition* on $(\mathcal{S}, \widetilde{\ })$ is an algebra multiplication $\star$ on $\mathcal{S}$ satisfying the following properties for all x, $y \in \mathcal{S}$:

- (SC1) $\widetilde{x \star y} = \widetilde{x} \star \widetilde{y}$.
- (SC2) If x, $y \neq 0$, then $x \star y = 0 \iff x \star \widetilde{y} \neq 0 \iff \widetilde{x} \star y \neq 0 \iff \widetilde{x} \star \widetilde{y} = 0$.
- (SC3) If $x \star y \neq 0$, then $(x \star y) \star \widetilde{x} = y$ and $\widetilde{y} \star (x \star y) = x$.
- (SC4) If $x \star y = 0$, then $(x^{\perp} \star y) \star x = y \star (x \star y^{\perp}) = \{0\}$; i.e., $(u \star y) \star x = y \star (x \star v) = 0$ for all $u \neq \widetilde{x}$ and $v \neq \widetilde{y}$.

Given a symmetric composition, the opposite algebra is also a symmetric composition. We say that two symmetric compositions $\diamond$ and $\star$ on $\mathcal{S}$ are *isomorphic* (resp. *properly isomorphic*) if there is $\phi \in \mathrm{O}(\mathcal{S})$ (resp. $\phi \in \mathrm{O}^{+}(\mathcal{S})$) such that $\phi(x \star y) = \phi(x) \diamond \phi(y)$ for all x, $y \in \mathcal{S}$. An *involution* of $(\mathcal{S}, \star)$ is an isometry ι of order 2 of $\mathcal{S}$ such that $\iota(x \star y) = \iota(y) \star \iota(x)$ for x, $y \in \mathcal{S}$.

Explanation 3.8. The choice of the above rules for symmetric compositions over $\mathbb{F}_1$ can be explained as follows. The idea is to draw consequences of the axioms of classical symmetric compositions with hyperbolic norm for the elements of a hyperbolic basis, ignoring scalar factors and recording only the vanishing or nonvanishing of scalars. Let $\star$ be a classical symmetric composition on a hyperbolic quadratic space (S, n) over a field, let $\{e_i,\, f_i\}_{i=1}^{n}$ be a hyperbolic basis of S, and let $\mathcal{S} = \{e_i,\, f_i\}_{i=1}^{n} \cup \{0\}$. Define on $\mathcal{S}$ the structure map $\widetilde{\ }$ by

$$\widetilde{0} = 0, \quad \widetilde{e_i} = f_i, \quad \widetilde{f_i} = e_i \quad \text{for } i = 1, \ldots, n.$$

Thus, for nonzero x, $y \in \mathcal{S}$, we have

$$b(x,y) = \begin{cases} 1 & \text{if } x = \widetilde{y}, \\ 0 & \text{if } x \neq \widetilde{y}, \end{cases} \qquad \text{and} \quad n(x) = 0.$$

Assuming the composition $\star$ is monomial, as in Example 2.7, we obtain on $\mathcal{S}$ an $\mathbb{F}_1$-algebra structure. We next show that (SC1)–(SC4) hold for this structure.
When x, $y \in \mathcal{S}$ are such that $x \star y \neq 0$, the symmetry condition (3.4) with $z = \widetilde{x \star y}$ yields $b(x, y \star z) \neq 0$, hence

$$\widetilde{x} = y \star (\widetilde{x \star y}). \tag{3.9}$$

Multiplying on the right by y and applying (3.5), we obtain

$$\widetilde{x} \star y = \big(y \star (\widetilde{x \star y})\big) \star y = n(y)\, \widetilde{x \star y} = 0.$$

Similarly, applying (3.4) with $x = \widetilde{y \star z}$ yields

$$(\widetilde{y \star z}) \star y = \widetilde{z} \qquad \text{when } y \star z \neq 0, \tag{3.10}$$

hence, by (3.5),

$$y \star \widetilde{z} = y \star \big((\widetilde{y \star z}) \star y\big) = n(y)\, \widetilde{y \star z} = 0.$$

We have thus shown that if $x \star y \neq 0$, then $\widetilde{x} \star y = x \star \widetilde{y} = 0$, hence $\widetilde{\widetilde{x} \star y} = 0 = x \star \widetilde{y}$.
If $x \star y = 0$, the multiplicativity condition (3.3) yields

$$b(x \star v, u \star y) = b(x,u) b(y,v).$$

If $u = \widetilde{x} \neq 0$ and $v = \widetilde{y} \neq 0$, the right side is nonzero, hence we must have $b(x \star \widetilde{y}, \widetilde{x} \star y) \neq 0$. Therefore, $x \star \widetilde{y} \neq 0$, $\widetilde{x} \star y \neq 0$, and $\widetilde{\widetilde{x} \star y} = x \star \widetilde{y}$. Conditions (SC1) and (SC2) follow, and (SC3) follows from (3.9) and (3.10), in view of (SC1). Finally, (SC4) is a consequence of (3.6).

We next record for later use some immediate consequences of the axioms of a symmetric composition over $\mathbb{F}_1$. We use the following notation: for $x \in \mathcal{S}$,

$$x \star \mathcal{S} = \{x \star y \mid y \in \mathcal{S}\} \quad \text{and} \quad \mathcal{S} \star x = \{y \star x \mid y \in \mathcal{S}\}.$$

Lemma 3.11. *Suppose $\star$ is a symmetric composition on a quadratic $\mathbb{F}_1$-space $(\mathcal{S}, \widetilde{\ })$.*
(1) *Let u, v, $x \in \mathcal{S}$. If $u \star x \neq 0$, then*

$$(u \star x) \star (x \star v) = \begin{cases} 0 & \text{if } v \in (u \star x)^{\perp}, \\ x & \text{if } v = \widetilde{u \star x}. \end{cases}$$

In particular, we have

$$(\mathcal{S} \star x) \star (x \star \mathcal{S}) = \{0, x\}.$$

(2) *For all $x \in \mathcal{S}$ we have $x \star x = 0$ or $\widetilde{x}$.*

Proof. To prove (1), observe that if $v = \widetilde{u \star x} \neq 0$, then (SC3) yields $(u \star x) \star (x \star v) = x$. If $v \in (u \star x)^{\perp}$, we have $(u \star x) \star (x \star v) = 0$ by (SC4).
Now, let $x \in \mathcal{S}$, and assume $x \star x \neq 0$. For $y = x \star x$ we have by (SC3)

$$\widetilde{x} \star y = x = y \star \widetilde{x},$$

hence

$$(y \star \widetilde{x}) \star (\widetilde{x} \star y) = x \star x = y.$$

By (1), the left side is in $\{0, \widetilde{x}\}$, which proves (2). $\square$

Examples 3.12. We claim that there is only one 2-dimensional symmetric composition algebra $\mathcal{S} = \{x, \widetilde{x}, 0\}$ over $\mathbb{F}_1$, given by the multiplication table

$$\begin{array}{c|cc} \star & x & \widetilde{x} \\ \hline x & \widetilde{x} & 0 \\ \widetilde{x} & 0 & x \end{array}$$

To see this, note that if $x \star x \neq \widetilde{x}$, we must have $x \star x = 0$ by Lemma 3.11(2), but then $x \star \widetilde{x} \neq 0$ by (SC2), hence $x \star \widetilde{x} = x$ or $x \star \widetilde{x} = \widetilde{x}$. If $x \star \widetilde{x} = x$, then, by (SC3), $x = x \star (x \star \widetilde{x}) = x \star x = 0$. Similarly the case $x \star \widetilde{x} = \widetilde{x}$ can be excluded.
In dimension 4, the multiplication table

$$\begin{array}{c|cc|cc} \star & x & \widetilde{x} & y & \widetilde{y} \\ \hline x & \widetilde{x} & 0 & y & 0 \\ \widetilde{x} & 0 & x & 0 & \widetilde{y} \\ \hline y & 0 & y & 0 & \widetilde{x} \\ \widetilde{y} & \widetilde{y} & 0 & x & 0 \end{array}$$

defines a symmetric composition. One can verify that this is, up to isomorphism, the unique symmetric composition in dimension 4.

Symmetric compositions in dimension 8**.** Classically, symmetric compositions exist only in dimensions 1, 2, 4 and 8. Over an algebraically closed field, there is one isomorphism class in dimensions 2 and 4, and two classes in dimension 8. These algebras are called *split symmetric compositions*. We describe the two split classes in dimension 8 and show that they lead to symmetric compositions over $\mathbb{F}_1$. We use the representation of the split Cayley algebra $\mathfrak{C}_s$ by Zorn matrices.

We let $\bullet$ denote the usual scalar product on $F^3 = F \times F \times F$, and $\times$ the vector product: for $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in F^3$,

$$a \bullet b = a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{and} \qquad a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

The Zorn algebra (see [Zor30, p. 144]) is a representation of the split Cayley algebra as the set of matrices

$$\mathfrak{Z} = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \;\middle|\; \alpha, \beta \in F,\ a, b \in F^3 \right\} \tag{3.13}$$

with the product

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \cdot \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma + a \bullet d & \alpha c + \delta a - b \times d \\ \gamma b + \beta d + a \times c & \beta\delta + b \bullet c \end{pmatrix},$$

the norm

$$n \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} = \alpha\beta - a \bullet b,$$

and the conjugation

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \mapsto \overline{\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}} = \begin{pmatrix} \beta & -a \\ -b & \alpha \end{pmatrix},$$

which is such that $\xi \cdot \overline{\xi} = \overline{\xi} \cdot \xi = n(\xi)$ for all $\xi \in \mathfrak{Z}$.

The new product

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} * \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \overline{\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}} \cdot \overline{\begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix}} = \begin{pmatrix} \beta\delta + a \bullet d & -\beta c - \gamma a - b \times d \\ -\delta b - \alpha d + a \times c & \alpha\gamma + b \bullet c \end{pmatrix}$$

defines on $\mathfrak{Z}$ the structure of a symmetric composition algebra with a hyperbolic norm (see [KMRT98]). We call $(\mathfrak{Z}, n, *)$ the *para-Zorn algebra*. Its automorphism group is isomorphic to the automorphism group of the split Cayley algebra.

Let (u_1, u_2, u_3) be the standard basis of F^3. The set $\{e_i, f_i\}_{i=1}^4$ of Zorn matrices given by

$$\begin{aligned} e_i &= \begin{pmatrix} 0 & u_i \\ 0 & 0 \end{pmatrix}, \ i = 1,2,3, \quad \text{and} \quad e_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ f_i &= \begin{pmatrix} 0 & 0 \\ u_i & 0 \end{pmatrix}, \ i = 1,2,3, \quad \text{and} \quad f_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{3.14}$$

is a hyperbolic basis for the quadratic space $(\mathfrak{Z}, n)$. We have

$$\overline{e_i} = -e_i, \ \overline{f_i} = -f_i \quad \text{for } i = 1,\ 2,\ 3, \text{ and} \quad \overline{e_4} = f_4.$$

The multiplication table of the para-Zorn algebra is

(3.15)

$*$	e_1	f_1	e_2	f_2	e_3	f_3	e_4	f_4
e_1	0	e_4	f_3	0	$-f_2$	0	$-e_1$	0
f_1	f_4	0	0	$-e_3$	0	e_2	0	$-f_1$
e_2	$-f_3$	0	0	e_4	f_1	0	$-e_2$	0
f_2	0	e_3	f_4	0	0	$-e_1$	0	$-f_2$
e_3	f_2	0	$-f_1$	0	0	e_4	$-e_3$	0
f_3	0	$-e_2$	0	e_1	f_4	0	0	$-f_3$
e_4	0	$-f_1$	0	$-f_2$	0	$-f_3$	f_4	0
f_4	$-e_1$	0	$-e_2$	0	$-e_3$	0	0	e_4

The second example of a split 8-dimensional symmetric composition is obtained by another twist of the multiplication of Zorn matrices. Consider a Zorn matrix as above,

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}, \quad \text{where } \alpha,\, \beta \in F \text{ and } a = (a_1, a_2, a_3),\ b = (b_1, b_2, b_3) \in F^3.$$

Let $\varphi\colon a \mapsto a^\varphi$ be the cyclic permutation $(a_1, a_2, a_3) \mapsto (a_2, a_3, a_1)$ and define

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}^\theta = \begin{pmatrix} \alpha & a^\varphi \\ b^\varphi & \beta \end{pmatrix}.$$

The map θ is an automorphism of order 3 of the Zorn algebra $\mathfrak{Z}$. The algebra with the new multiplication given by

$$x \star y = \overline{x}^\theta \cdot \overline{y}^{\theta^{-1}}$$

is another symmetric composition, known as the *pseudo-octonion algebra* or *split Okubo algebra*. Using the basis (3.14), we get the multiplication table:

(3.16)

$\star$	e_1	f_1	e_2	f_2	e_3	f_3	e_4	f_4
e_1	f_1	0	$-f_3$	0	0	e_4	$-e_2$	0
f_1	0	$-e_1$	0	e_3	f_4	0	0	$-f_2$
e_2	0	e_4	f_2	0	$-f_1$	0	$-e_3$	0
f_2	f_4	0	0	$-e_2$	0	e_1	0	$-f_3$
e_3	$-f_2$	0	0	e_4	f_3	0	$-e_1$	0
f_3	0	e_2	f_4	0	0	$-e_3$	0	$-f_1$
e_4	0	$-f_3$	0	$-f_1$	0	$-f_2$	f_4	0
f_4	$-e_3$	0	$-e_1$	0	$-e_2$	0	0	e_4

Remark. Ignoring signs in the two multiplication tables given above, we obtain algebras over $\mathbb{F}_1$, according to Example 2.7 and 3.8. We call them the *para-Zorn algebra*, resp. the *pseudo-octonion algebra* over $\mathbb{F}_1$. As we shall see later (see Corollary 4.26), every 8-dimensional symmetric composition over $\mathbb{F}_1$ is isomorphic to a para-Zorn algebra or a pseudo-octonion algebra. Moreover symmetric compositions over $\mathbb{F}_1$ occur only in dimensions 2, 4, and 8, see Corollary 3.21.

Remark. The conjugation of the split Cayley algebra induces an involution of the para-Zorn algebra. As we shall see later, the pseudo-octonion algebra also admits involutions, however the conjugation of the para-Zorn algebra commutes with the automorphisms of the algebra, in contrast to the involutions of the pseudo-octonion algebra (see Proposition 4.19 and Theorem 4.25).

Symmetric compositions and maximal solids. Let $\star$ be a symmetric composition on a quadratic space $(\mathcal{S}, \tilde{\ })$ of dimension $2n$ over $\mathbb{F}_1$. In this subsection, we show that the maximal isotropic subspaces of $\mathcal{S}$ can be described by means of the composition $\star$. As a result, we prove that $\dim \mathcal{S} = 2$, 4, or 8.
For $x \in \mathcal{S}$, let ℓ_x, r_x be the linear maps $\mathcal{S} \to \mathcal{S}$ defined by

$$\ell_x \colon y \mapsto x \star y \quad \text{and} \quad r_x \colon y \mapsto y \star x, \quad \text{for } y \in \mathcal{S},$$

so

$$\begin{aligned} \operatorname{Im} \ell_x &= x \star \mathcal{S}, & \operatorname{Ker} \ell_x &= \{y \in \mathcal{S} \mid x \star y = 0\}, \\ \operatorname{Im} r_x &= \mathcal{S} \star x, & \operatorname{Ker} r_x &= \{y \in \mathcal{S} \mid y \star x = 0\}. \end{aligned}$$

Lemma 3.17. *For every nonzero $x \in \mathcal{S}$, the spaces $\operatorname{Im} \ell_x$, $\operatorname{Ker} r_x$, $\operatorname{Im} r_x$, and $\operatorname{Ker} \ell_x$ are maximal isotropic. Moreover,*

$$\operatorname{Im} \ell_x = \operatorname{Ker} r_x, \qquad \operatorname{Im} r_x = \operatorname{Ker} \ell_x,$$

and

$$\mathcal{S} = (x \star \mathcal{S}) \oplus (\widetilde{x} \star \mathcal{S}) = (\mathcal{S} \star x) \oplus (\mathcal{S} \star \widetilde{x}).$$

Proof. If $z = x \star y \neq 0$ for some $y \in \mathcal{S}$, then by (SC3) we have $z \star \widetilde{x} = y \neq 0$, hence $z \star x = 0$ by (SC2). Thus $\operatorname{Im} \ell_x \subset \operatorname{Ker} r_x$. For the reverse inclusion, observe that if $z \star x = 0$, then $z \star \widetilde{x} \neq 0$ by (SC2), hence $z = x \star (z \star \widetilde{x})$ by (SC3), which proves $z \in \operatorname{Im} \ell_x$. The equality $\operatorname{Im} r_x = \operatorname{Ker} \ell_x$ is proved by similar arguments.
Axiom (SC2) shows that $(\operatorname{Ker} \ell_x) \cap (\operatorname{Ker} \ell_{\widetilde{x}}) = \{0\}$ and $(\operatorname{Ker} \ell_x) \cup (\operatorname{Ker} \ell_{\widetilde{x}}) = \mathcal{S}$, hence

$$\mathcal{S} = (\operatorname{Ker} \ell_x) \oplus (\operatorname{Ker} \ell_{\widetilde{x}}) = (\mathcal{S} \star x) \oplus (\mathcal{S} \star \widetilde{x}). \tag{3.18}$$

On the other hand, we have $\operatorname{Ker}\ell_{\widetilde{x}} = \widetilde{\operatorname{Ker}\ell_x}$ by (SC1), hence

$$(\operatorname{Ker}\ell_x) \cap \widetilde{\operatorname{Ker}\ell_x} = \{0\}.$$

It follows that $\operatorname{Ker}\ell_x$ is an isotropic subspace, and (3.18) shows that $\dim \operatorname{Ker}\ell_x = n$. Therefore, $\operatorname{Ker}\ell_x$ is a maximal isotropic subspace. Likewise, $\operatorname{Ker} r_x$ is a maximal isotropic subspace. □

Lemma 3.17 shows that for all nonzero $x \in \mathcal{S}$, the sets $\langle x \star \mathcal{S}\rangle$ and $\langle \mathcal{S} \star x\rangle$ are maximal solids of the quadric $\langle \mathcal{S}\rangle$.

Lemma 3.19. *Let ω be a maximal solid and let y be a point in $\langle \mathcal{S}\rangle$. If the maximal solids ω and $\langle \mathcal{S} \star y\rangle$ are incident, there exists $x \in \mathcal{S}$ such that $\omega = \langle x \star \mathcal{S}\rangle$ and $x \star y = 0$. Similarly, if ω and $\langle y \star \mathcal{S}\rangle$ are incident, then there exists $z \in \mathcal{S}$ such that $\omega = \langle \mathcal{S} \star z\rangle$ and $y \star z = 0$.*

Proof. Let $\mathcal{U} \subset \mathcal{S}$ be the maximal isotropic subspace such that $\omega = \langle \mathcal{U}\rangle$. Assuming ω and $\langle \mathcal{S} \star y\rangle$ are incident, we have $n-1$ points in the intersection: let

$$\omega \cap \langle \mathcal{S} \star y\rangle = \{u_1, \dots, u_{n-1}\}$$

and let $v_i \in \mathcal{S}$ be such that $u_i = v_i \star y$ for $i = 1, \dots, n-1$. Since $\dim(\mathcal{S} \star y) = n$, there exists one element in $\langle \mathcal{S} \star y\rangle$ that does not lie in ω. We write this element as $\widetilde{x} \star y$ for some nonzero $x \in \mathcal{S}$, and show that $\mathcal{U} = x \star \mathcal{S}$. Note that $x \star y = 0$ since $\widetilde{x} \star y \neq 0$.
Since $\mathcal{U}$ is maximal isotropic and does not contain $\widetilde{x} \star y$, it must contain $\widetilde{\widetilde{x} \star y} = x \star \widetilde{y}$. To show that $\mathcal{U} = x \star \mathcal{S}$, it now suffices to prove $u_i \in x \star \mathcal{S}$ for $i = 1, \dots, n-1$, or, equivalently by Lemma 3.17, that $u_i \star x = 0$ for $i = 1, \dots, n-1$. We have $\widetilde{x} \neq v_i$, hence $v_i \in x^\perp$ for $i = 1, \dots, n-1$. Moreover, $x \star y = 0$ by (SC2) since $\widetilde{x} \star y \neq 0$. Therefore, it follows from (SC4) that $(v_i \star y) \star x = 0$. We have thus proved the first claim. The proof of the last claim is similar. □

Proposition 3.20. *Let $\star$ be a symmetric composition on a quadratic space $(\mathcal{S}, \widetilde{\ })$ over $\mathbb{F}_1$. For any given nonzero $x \in \mathcal{S}$, all the maximal isotropic subspaces of the same kind as $x \star \mathcal{S}$ are of the form $y \star \mathcal{S}$, and all the maximal isotropic subspaces of the opposite kind are of the form $\mathcal{S} \star y$ for some $y \in \mathcal{S}$. In particular, every maximal isotropic subspace of $\mathcal{S}$ is of the form $y \star \mathcal{S}$ or $\mathcal{S} \star y$ for some nonzero $y \in \mathcal{S}$.*

Note that a maximal isotropic subspace may be simultaneously of the form $y \star \mathcal{S}$ and of the form $\mathcal{S} \star z$: this occurs for the 2-dimensional composition algebra, see Examples 3.12.

Proof. Let $\mathcal{U} \subset \mathcal{S}$ be a maximal isotropic subspace and let $\omega = \langle \mathcal{U}\rangle$. By Lemma 2.5, there exists a sequence of maximal solids $\omega_0, \dots, \omega_s$ such that $\omega_0 = \langle x \star \mathcal{S}\rangle$, $\omega_s = \omega$, and ω_{i-1} and ω_i are incident for $i = 1, \dots, s$. Lemma 3.19 shows that there exist $y_1, \dots, y_s \in \mathcal{S}$ such that

$$\omega_i = \begin{cases} \langle \mathcal{S} \star y_i\rangle & \text{if } i \text{ is odd,} \\ \langle y_i \star \mathcal{S}\rangle & \text{if } i \text{ is even.} \end{cases}$$

Moreover, by Lemma 2.5, s is even if and only if ω and $\langle x \star \mathcal{S}\rangle$ are of the same kind. Thus

$$\mathcal{U} = \begin{cases} \mathcal{S} \star y_s & \text{if } \mathcal{U} \text{ and } x \star \mathcal{S} \text{ are of opposite kinds,} \\ y_s \star \mathcal{S} & \text{if } \mathcal{U} \text{ and } x \star \mathcal{S} \text{ are of the same kind.} \end{cases}$$

□

Let $C(\mathcal{S}) = C\big(\langle\, \mathcal{S}\,\rangle\big) \sqcup \{0\}$. The structure map $\tilde{}$ extends obviously to a structure map (also denoted $\tilde{}$) on $C(\mathcal{S})$, so that $C(\mathcal{S})$ is a quadratic space over $\mathbb{F}_1$ and the linear maps

$$\gamma_1,\, \gamma_2\colon\, \mathcal{S} \mapsto C(\mathcal{S}), \quad \gamma_1(x) = x\star\mathcal{S}, \quad \gamma_2(y) = \mathcal{S}\star y$$

extend to a map of quadratic spaces

$$\gamma = \gamma_1 \perp \gamma_2\colon\, \mathcal{S}\perp\mathcal{S} \to C(\mathcal{S}).$$

We have $\dim C(\mathcal{S}) = 2^{\frac{1}{2}\dim\mathcal{S}}$.

Corollary 3.21. *Let $\star$ be a symmetric composition on a quadratic space $(\mathcal{S}, \tilde{})$ over $\mathbb{F}_1$.*

(1) *The map $\gamma\colon\, \mathcal{S}\perp\mathcal{S}\to C(\mathcal{S})$ is surjective.*
(2) $\dim\mathcal{S} = 2$*,* 4*, or* 8*.*
(3) *If $\mathcal{S}$ has dimension 8, the map $\gamma\colon\, \mathcal{S}\perp\mathcal{S}\to C(\mathcal{S})$ is bijective. In particular, if $x\star\mathcal{S} = y\star\mathcal{S}$ or $\mathcal{S}\star x = \mathcal{S}\star y$ for some x, $y\in\mathcal{S}$, then $x = y$.*

Proof. (1) readily follows from Proposition 3.20. Let $\dim\mathcal{S} = 2n$, so the set $\langle\, C(\mathcal{S})\,\rangle$ of maximal isotropic subspaces of $\mathcal{S}$ has 2^n elements. In view of (1) there are at most $4n$ maximal isotropic subspaces of $\mathcal{S}$. Thus we have $2^n \le 4n$, which implies $n\le 4$. If $n = 3$, two maximal isotropic subspaces are of opposite kinds if and only if their intersection has even dimension. Thus for all nonzero $x\in\mathcal{S}$, the subspaces $x\star\mathcal{S}$ and $\widetilde{x}\star\mathcal{S}$ are of opposite kinds. By Proposition 3.20, it follows that $x\star\mathcal{S} = \mathcal{S}\star y$ for some $y\in\mathcal{S}$. Therefore, the maps γ_1 and γ_2 are surjective. This is a contradiction since $\dim\mathcal{S} = 6$ and $\dim C(\mathcal{S}) = 8$, so $\dim\mathcal{S} = 6$ is excluded, and we have proved (2). Finally, (3) follows from the fact that $2^n = 4n$ if $n = 4$. □

4. Geometric triality

Let Z be a 6-dimensional quadric over $\mathbb{F}_1$, with structure map $\tilde{}$. From now on we call the maximal solids of Z simply solids, so that the objects of the associated geometry are points, lines, planes, and solids of two kinds on Z. As in Section 2, we extend the incidence relation given by inclusion, by saying that two solids of different kinds are incident if their intersection is a plane.

Classical triality in projective geometry permutes points and the two kinds of solids on a 6-dimensional quadric over a field F (see for example [Stu13], [Car38], [Wei38], [Che54], [Tit59] or [vdBS60]). We define a *geometric triality* on Z as a pair (τ, ∂), where ∂ is an orientation $C(Z) = C_1\sqcup C_2$ of Z and τ is a map

$$\tau\colon\, Z\sqcup C_1\sqcup C_2 \to Z\sqcup C_1\sqcup C_2$$

with the following properties:

(GT1) τ commutes with the structure map $\tilde{}\colon x\mapsto\widetilde{x}$;
(GT2) τ preserves the incidence relations;
(GT3) $\tau(Z) = C_1$, $\tau(C_1) = C_2$, and $\tau(C_2) = Z$;
(GT4) $\tau^3 = I$.

Note that the intersection of two solids ω, ω' of the same kind is a line, unless $\omega = \omega'$ or $\omega = \widetilde{\omega}'$. Therefore, for any line $\{x, y\}$ in Z, the intersection $\tau(x) \cap \tau(y)$ is a line. We extend the definition of τ to the set L of lines by setting

$$\tau\{x, y\} = \tau(x) \cap \tau(y).$$

Thus extended, the map τ still preserves the incidence relations. It will be clear from Theorems 4.7 and 4.10 that τ is determined by its action on L.

Remark. Although it is determined by τ, the orientation ∂ is part of the definition of a triality. If $\widehat{\partial}$ is the opposite orientation, i.e., the composition of ∂ with the transposition of 1 and 2, then every triality (τ, ∂) yields another triality $(\tau^2, \widehat{\partial})$. It is convenient to call ∂ the *orientation* of the triality (τ, ∂).

Coordinates. We may identify the various elements of the geometry of the quadric Z with vectors in $\mathbb{R}^4$, as follows. Let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the standard basis of $\mathbb{R}^4$. We set

$$Z = \{\pm\xi_1, \pm\xi_2, \pm\xi_3, \pm\xi_4\}$$

and define $\widetilde{z} = -z$ for all $z \in Z$. Next, we identify each line $\{x, y\}$ with $x + y \in \mathbb{R}^4$ and each solid $\{u, v, x, y\} \in C(Z)$ with $\frac{1}{2}(u + v + x + y) \in \mathbb{R}^4$. Thus we may fix an orientation by setting

$$C_1 = \big\{\pm\tfrac{1}{2}(\xi_1 + \xi_2 + \xi_3 + \xi_4), \pm\tfrac{1}{2}(\xi_1 + \xi_2 - \xi_3 - \xi_4), \\ \pm\tfrac{1}{2}(\xi_1 - \xi_2 - \xi_3 + \xi_4), \pm\tfrac{1}{2}(\xi_1 - \xi_2 + \xi_3 - \xi_4)\big\}$$

and

$$C_2 = \big\{\pm\tfrac{1}{2}(\xi_1 - \xi_2 - \xi_3 - \xi_4), \pm\tfrac{1}{2}(\xi_1 + \xi_2 + \xi_3 - \xi_4), \\ \pm\tfrac{1}{2}(\xi_1 + \xi_2 - \xi_3 + \xi_4), \pm\tfrac{1}{2}(\xi_1 - \xi_2 + \xi_3 + \xi_4)\big\}.$$

Incidence between points, lines, and solids occurs if and only if the usual scalar product of the corresponding vectors u, v satisfies $u \cdot v > 0$, and the intersection of two solids ω, ω' of the same type is the line $\omega + \omega'$.

The groups $\mathrm{PGO}(Z) = \mathfrak{S}_2^4 \rtimes \mathfrak{S}_4$ and $\mathrm{PGO}^+(Z) = \mathfrak{S}_2^3 \rtimes \mathfrak{S}_4$ (see Lemma 2.2) are identified with subgroups of $\mathrm{O}_4(\mathbb{R})$ as follows. Let P be the subgroup of $\mathrm{O}_4(\mathbb{R})$ generated by all the permutations of the standard basis of $\mathbb{R}$. Let D be the subgroup of $\mathrm{O}_4(\mathbb{R})$ consisting of all diagonal matrices $\mathrm{Diag}(\nu_1, \nu_2, \nu_3, \nu_4)$ with $\nu_i = \pm 1$ and let D^+ be the subgroup of D consisting of matrices $\mathrm{Diag}(\nu_1, \nu_2, \nu_3, \nu_4)$ with the supplementary condition $\prod_i \nu_i = 1$. Then $\mathrm{PGO}(Z)$ is the subgroup of $\mathrm{O}_4(\mathbb{R})$ generated by P and D and $\mathrm{PGO}^+(Z)$ is the subgroup of $\mathrm{O}_4(\mathbb{R})$ generated by P and D^+. The action of $\mathrm{PGO}(Z)$ on Z, C_1 and C_2 is given by the restriction of the natural action of $\mathrm{O}_4(\mathbb{R})$ on $\mathbb{R}^4$.

Proposition 4.1. *Let $Z = \{\pm\xi_1, \dots, \pm\xi_4\} \subset \mathbb{R}^4$ and let ∂ be the orientation for which C_1 and C_2 are as above. There is a one-to-one correspondence between trialities (τ, ∂) and orthogonal matrices $T \in \mathrm{O}_4(\mathbb{R})$ such that $T^3 = 1$ and $TZ = C_1$, which associates to (τ, ∂) the matrix of τ in the basis $(\xi_1, \dots, \xi_4)$.*

Proof. For every triality (τ, ∂), the matrix T of τ satisfies $TZ = C_1$ and $T^3 = 1$ since $\tau(Z) = C_1$ and $\tau^3 = I$. It is also orthogonal since any two vectors in C_1 are orthogonal unless they are equal or opposite. Conversely, given a matrix $T \in \mathrm{O}_4(\mathbb{R})$ such that $T^3 = 1$ and $TZ = C_1$, we define a bijective map $\tau\colon Z \to C_1$ by mapping

ξ_i to the i-th column of T and $-\xi_i$ to the opposite of the i-th column of T, for $i = 1, \ldots, 4$. Then $\tau(C_1) = \tau^2(Z)$, so we need to show that $\tau^2(Z) = C_2$. Since $T^3 = 1$ and T is orthogonal, T^2 is the transpose T^t. Since $TZ = C_1$, all the entries of T are $\pm\frac{1}{2}$, hence the same holds for T^2. The vectors in $\mathbb{R}^4$ whose coordinates are all $\pm\frac{1}{2}$ form the set $C_1 \cup C_2$, so $T^2Z \subset C_1 \cup C_2$. But $(TZ) \cap (T^2Z) = \varnothing$ since $Z \cap (TZ) = Z \cap C_1 = \varnothing$. Therefore, $T^2Z = C_2$. Since $T^3 = 1$, we have $\tau(C_2) = Z$, hence τ satisfies (GT3) and (GT4). It also satisfies (GT1) since multiplication by T is a linear map, and (GT2) because T is orthogonal and incidence is equivalent to positivity of the scalar product. □

Corollary 4.2. *A geometric triality (τ, ∂) on Z is uniquely determined by its restriction $\tau|_Z \colon Z \to C_1$.*

Trialities with absolute points. From now on, geometrical or combinatorial methods will be used. Absolute points play a fundamental role.

Definition 4.3. An *absolute point* of a triality (τ, ∂) on Z is a point $z \in Z$ such that $z \in \tau(z)$, i.e., the point z and the solid $\tau(z)$ are incident. Then $\tau(z)$ and $\tau^2(z)$ are incident, and $\tau^2(z)$ and z are incident, so the condition defining an absolute point can be rephrased as

$$|\tau(z) \cap \tau^2(z)| = 3, \qquad \text{or} \qquad z \in \tau^2(z).$$

For any absolute point z of a triality τ, we let

$$\pi(z) = \tau(z) \cap \tau^2(z).$$

This is a plane in Z.

Lemma 4.4. *If $\{x, y\}$ is a line of Z fixed under some geometric triality (τ, ∂), then x and y are absolute points and $\{x, y\} = \pi(x) \cap \pi(y)$.*

Proof. Since $\{x, y\}$ is fixed under τ we have

$$\{x, y\} = \tau\{x, y\} = \tau(x) \cap \tau(y) \qquad \text{and} \qquad \{x, y\} = \tau^2\{x, y\} = \tau^2(x) \cap \tau^2(y),$$

hence

$$\{x, y\} = \tau(x) \cap \tau(y) \cap \tau^2(x) \cap \tau^2(y). \tag{4.5}$$

Thus $|\tau(x) \cap \tau^2(x)| \geq 2$, which implies $\tau(x)$ and $\tau^2(x)$ are incident and x is absolute. Similarly, y is absolute, and (4.5) shows that $\{x, y\} = \pi(x) \cap \pi(y)$. □

Definition 4.6. An *hexagon* in a 6-dimensional quadric Z over $\mathbb{F}_1$ is a pair $H = (V, E)$ consisting of a set $V \subset Z$ of six points stable under $\tilde{}$ and a set E of six lines between points of V such that the graph with vertex set V and edge set E is a circuit.

Theorem 4.7. *Suppose (τ, ∂) is a triality on Z for which there exists an absolute point. Then the pair (V, E) where V is the set of absolute points of Z and E is the set of lines fixed under τ is an hexagon. Moreover, for every hexagon (V, E) in Z and any orientation ∂ there is a unique geometric triality (τ, ∂) on Z such that V is the set of absolute points of τ and E is the set of fixed lines under τ.*

Proof. Fix an absolute point a and let

$$\pi(a) = \{a, b, c\}.$$

Claim 1: The lines $\{a, b\}$ and $\{a, c\}$ are fixed under τ.

To prove the claim, observe that by applying τ to the incidence relation $b \in \tau^2(a)$ we obtain $a \in \tau(b)$, hence

$$a \in \tau(a) \cap \tau(b) = \tau\{a, b\}.$$

Similarly, since $b \in \tau(a)$ we have $a \in \tau^2(b)$, hence

$$a \in \tau^2(a) \cap \tau^2(b) = \tau^2\{a, b\}.$$

Applying τ^2, we obtain $\tau\{a, b\} \subset \tau^2(a)$, hence $\tau\{a, b\} \subset \pi(a)$. Thus $\tau\{a, b\}$ is a line containing a in the plane $\pi(a)$, which means that τ permutes the lines through a in $\pi(a)$. Since there are only two such lines and τ has order 3, the action of τ on the lines through a in $\pi(a)$ is trivial, and Claim 1 is proved.

Now, let

$$\tau(a) = \{a, b, c, d\}.$$

Claim 2: The points b and c are absolute, and d is not absolute.
Claim 1 and Lemma 4.4 readily show that b and c are absolute. Now, suppose d is absolute, hence $d \in \tau^2(d)$. Since $d \in \tau(a)$, we also have $a \in \tau^2(d)$, hence $\{a, d\} \subset \tau^2(d)$ and therefore applying τ we see that $d \in \tau\{a, d\}$. But we have $\{a, d\} \subset \tau(a)$, hence $\tau\{a, d\} \subset \tau^2(a)$. Therefore,

$$d \in \tau\{a, d\} \subset \tau^2(a),$$

and it follows that $d \in \tau(a) \cap \tau^2(a) = \pi(a)$, a contradiction.

From Claim 2 we derive that $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ also are absolute points, and $\widetilde{d}$ is not an absolute point. We have $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c} \notin \{a, b, c\}$ since $\{a, b, c\}$ is a plane, hence

$$Z = \{a, b, c, d, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}\}$$

and the set of absolute points is

$$V = \{a, b, c, \widetilde{a}, \widetilde{b}, \widetilde{c}\}.$$

Claim 3: $\pi(b) = \{a, b, \widetilde{c}\}$ and $\pi(c) = \{a, \widetilde{b}, c\}$.
By Claim 2, all the points in $\pi(b)$ and $\pi(c)$ are absolute. Claim 1 and Lemma 4.4 show that $\pi(a) \cap \pi(b) = \{a, b\}$, hence

$$\{a, b\} \subset \pi(b) \subset V \setminus \{c\}.$$

The third point of $\pi(b)$ cannot be $\widetilde{a}$ nor $\widetilde{b}$ since $\pi(b)$ is a plane. Thus the only possibility is $\pi(b) = \{a, b, \widetilde{c}\}$. The argument for $\pi(c)$ is similar.

From Claim 3 it follows that

$$\pi(\widetilde{b}) = \{\widetilde{a}, \widetilde{b}, c\} \quad \text{and} \quad \pi(\widetilde{c}) = \{\widetilde{a}, b, \widetilde{c}\}.$$

We also have $\pi(\widetilde{a}) = \{\widetilde{a}, \widetilde{b}, \widetilde{c}\}$, hence Claim 1 shows that the lines $\{a, b\}$, $\{a, c\}$, $\{b, \widetilde{c}\}$, $\{\widetilde{b}, c\}$, $\{\widetilde{a}, \widetilde{b}\}$, $\{\widetilde{a}, \widetilde{c}\}$ are fixed under τ. To complete the proof of the first part of Theorem 4.7, it remains to see that the other lines between the points in V are not fixed. Suppose for instance that $\{b, c\}$ is fixed. Then Lemma 4.4 shows that $\pi(b) \cap \pi(c) = \{b, c\}$, contradicting Claim 3. Similarly, from the determination of $\pi(x)$ for all $x \in V$ it readily follows that the only fixed lines are those determined above; they are the edge set of an hexagon with vertex set V.
To prove the second part, we show that the geometric triality can be uniquely reconstructed from the hexagon of absolute points and fixed lines. Suppose there is a geometric triality (τ, ∂) for which the hexagon of absolute points and fixed lines

is as above. The neighbors of the vertex a in the hexagon yield $\pi(a) = \tau(a) \cap \tau^2(a)$, hence we have
$$\{\tau(a), \tau^2(a)\} = \{\{a, b, c, d\}, \{a, b, c, \widetilde{d}\}\}.$$
Of the two solids on the right side, one is in C_1 and the other in C_2, hence $\tau(a)$ and $\tau^2(a)$ are uniquely determined. Similarly, $\tau(x)$ and $\tau^2(x)$ are uniquely determined for all $x \in V$, and it only remains to determine $\tau(d)$, $\tau(\widetilde{d})$, and $\tau^2(d)$, $\tau^2(\widetilde{d})$. Of the solids in C_1, there are just two that are not of the form $\tau(x)$ for $x \in V$. The one that contains $\widetilde{d}$ must be $\tau(d)$, and the one that contains d must be $\tau(\widetilde{d})$; thus $\tau(d)$ and $\tau(\widetilde{d})$ are uniquely determined and, similarly, $\tau^2(d)$ and $\tau^2(\widetilde{d})$ are uniquely determined.
To complete the proof, we still have to show that, given an orientation ∂, for each hexagon in Z there exists a geometric triality (τ, ∂) with the given hexagon as the pair of absolute points and fixed lines. The uniqueness proof above is constructive; it thus suffices to check that the maps $\tau\colon Z \to C_1$ and $\tau^2\colon Z \to C_2$ defined above yield a triality, which can be done by direct computations. An alternative approach is to use coordinates: suppose
$$Z = \{a, b, c, d, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}\},$$
the given hexagon is
$$\Big\{\{a, b, c, \widetilde{a}, \widetilde{b}, \widetilde{c}\},\ \{\{a, b\},\ \{b, \widetilde{c}\},\ \{\widetilde{c}, \widetilde{a}\},\ \{\widetilde{a}, \widetilde{b}\},\ \{\widetilde{b}, c\}\ \{c, a\}\}\Big\},$$
and the orientation ∂ is such that $\{a, b, c, d\} \in C_1$. We identify Z with a subset of $\mathbb{R}^4$ by mapping a, $\widetilde{b}$, $\widetilde{c}$, d to the elements ξ_1, ξ_2, ξ_3, ξ_4 of the standard basis. The map $\tau\colon Z \to C_1$ constructed above is given by
$$\begin{aligned}
\tau(a) &= \{a, b, c, d\}, & \tau(\widetilde{a}) &= \{\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}\},\\
\tau(b) &= \{a, b, \widetilde{c}, \widetilde{d}\}, & \tau(\widetilde{b}) &= \{\widetilde{a}, \widetilde{b}, c, d\},\\
\tau(c) &= \{a, \widetilde{b}, c, \widetilde{d}\}, & \tau(\widetilde{c}) &= \{\widetilde{a}, b, \widetilde{c}, d\},\\
\tau(d) &= \{\widetilde{a}, b, c, \widetilde{d}\}, & \tau(\widetilde{d}) &= \{a, \widetilde{b}, \widetilde{c}, d\},
\end{aligned}$$
or, after the identification, by the linear map $\mathbb{R}^4 \to \mathbb{R}^4$ with matrix
$$T_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}. \tag{4.8}$$
It is readily verified that this matrix is orthogonal with cube 1, mapping Z to C_1, hence it is a triality. □

Corollary 4.9. *Let ∂ be a fixed orientation of Z. There are 16 trialities (τ, ∂) with absolute points on Z. All these trialities are conjugate under* $\mathrm{PGO}^+(Z)$.

Proof. There are 16 hexagons in Z, and they are all permuted by $\mathrm{PGO}^+(Z)$. □

Trialities without absolute points. We now turn to trialities without absolute points.

Theorem 4.10. *Let (τ, ∂) be a geometric triality on Z without absolute points. There are four hexagons (V_1, E_1), …, (V_4, E_4) with disjoint edge sets such that each edge set E_i is preserved under τ and $E_1 \sqcup E_2 \sqcup E_3 \sqcup E_4$ is the set of all lines* *in Z. Any one of these hexagons determines the triality uniquely if the order in which the edges are permuted is given. More precisely, given an orientation ∂ of Z, an hexagon (V,E) in Z and an orientation of the circuit of edges of E, there is a unique triality (τ,∂) on Z without absolute points that permutes the edges in E in the prescribed direction.*

Proof. Note that it is already clear from Lemma 4.4 that τ permutes the lines of Z without leaving any of them fixed.

Claim 1: For any $a \in Z$, we have $\tau(a) \cap \tau^2(a) = \{\widetilde{a}\}$.
Indeed, $\tau(a)$ and $\tau^2(a)$ each contains a or $\widetilde{a}$ since they are solids, but they cannot contain a, lest a be an absolute point for τ. Moreover, $|\tau(a) \cap \tau^2(a)| = 1$ since otherwise $\tau(a)$ and $\tau^2(a)$ are incident and a is absolute.

Claim 2: For any line $\{a,b\}$ in Z, the line $\tau\{a,b\}$ intersects $\{\widetilde{a},\widetilde{b}\}$ in one point.
If $\tau\{a,b\} = \{\widetilde{a},\widetilde{b}\}$, then $\widetilde{a} \in \tau(b)$ and $\widetilde{b} \in \tau(a)$. This last incidence relation implies $\widetilde{a} \in \tau^2(b)$, hence $\widetilde{a} \in \tau(b) \cap \tau^2(b)$. This is impossible in view of Claim 1.
We set $\{c,\widetilde{d}\} = \tau\{a,b\} = \tau(a) \cap \tau(b)$. If $\{c,\widetilde{d}\}$ is disjoint from $\{\widetilde{a},\widetilde{b}\}$, then since $\widetilde{a} \in \tau(a)$ and $\widetilde{b} \in \tau(b)$ we derive that $\widetilde{a} \notin \tau(b)$ and $\widetilde{b} \notin \tau(a)$, hence

$$\tau(a) = \{\widetilde{a},b,c,\widetilde{d}\} \qquad \text{and} \qquad \tau(b) = \{a,\widetilde{b},c,\widetilde{d}\}.$$

From Claim 1 it follows that

$$\tau^2(a) = \{\widetilde{a},\widetilde{b},\widetilde{c},d\} \qquad \text{and} \qquad \tau^2(b) = \{\widetilde{a},\widetilde{b},\widetilde{c},d\}.$$

Thus, $\tau^2(a) = \tau^2(b)$, a contradiction. Claim 2 is thus proved.

Now, let $\{a,b\}$ be an arbitrary line in Z. Interchanging a and b if necessary, we may assume by Claim 2 that $\widetilde{a} \in \tau\{a,b\}$. Let $\tau\{a,b\} = \{c,\widetilde{a}\}$ for some $c \in Z \setminus \{a,\widetilde{a},b,\widetilde{b}\}$. Claim 2 also shows that $\tau\{c,\widetilde{a}\}$ contains $\widetilde{c}$ or a. In the latter case, we have

$$a \in \tau\{c,\widetilde{a}\} = \tau^2\{a,b\} = \tau^2(a) \cap \tau^2(b),$$

hence a is an absolute point, a contradiction. Therefore, $\tau\{c,\widetilde{a}\} = \{z,\widetilde{c}\}$ for some $z \in Z$. Repeating the same argument we see that $\tau\{z,\widetilde{c}\}$ contains $\widetilde{z}$. But

$$\tau\{z,\widetilde{c}\} = \tau^2\{c,\widetilde{a}\} = \tau^3\{a,b\},$$

so $\widetilde{z} = a$ or b. If $\widetilde{z} = a$ we have $\tau\{c,\widetilde{a}\} = \{\widetilde{c},\widetilde{a}\}$ hence $\widetilde{a} \in \tau(\widetilde{a})$ and $\widetilde{a}$ is an absolute point. Since there is no such points, we must have $\widetilde{z} = b$, hence

$$\tau\{a,b\} = \{c,\widetilde{a}\}, \quad \tau\{c,\widetilde{a}\} = \{\widetilde{b},\widetilde{c}\}, \quad \text{and} \quad \tau\{\widetilde{b},\widetilde{c}\} = \{a,b\}. \tag{4.11}$$

Applying $\widetilde{\ }$, we obtain

$$\tau\{\widetilde{a},\widetilde{b}\} = \{\widetilde{c},a\}, \quad \tau\{\widetilde{c},a\} = \{b,c\}, \quad \text{and} \quad \tau\{b,c\} = \{\widetilde{a},\widetilde{b}\}. \tag{4.12}$$

Thus the set

$$E_1 = \big\{\{a,b\},\{b,c\},\{c,\widetilde{a}\},\{\widetilde{a},\widetilde{b}\},\{\widetilde{b},\widetilde{c}\},\{\widetilde{c},a\}\big\}$$

is stable under the action of τ. It is the edge set of the hexagon $H_1 = (V_1,E_1)$, where

$$V_1 = \{a,b,c,\widetilde{a},\widetilde{b},\widetilde{c}\}.$$

Starting with lines that are not in E_1, we obtain three more hexagons whose edge sets are stable under τ:

$$\begin{aligned}
H_2 &= \Big\{\{a,c,\widetilde{d},\widetilde{a},\widetilde{c},d\},\ \big\{\{a,c\},\ \{c,\widetilde{d}\},\ \{\widetilde{d},\widetilde{a}\},\ \{\widetilde{a},\widetilde{c}\},\ \{\widetilde{c},d\},\ \{d,a\}\big\}\Big\},\\
H_3 &= \Big\{\{a,\widetilde{d},b,\widetilde{a},d,\widetilde{b}\},\ \big\{\{a,\widetilde{d}\},\ \{\widetilde{d},b\},\ \{b,\widetilde{a}\},\ \{\widetilde{a},d\},\ \{d,\widetilde{b}\},\ \{\widetilde{b},a\}\big\}\Big\},\\
H_4 &= \Big\{\{b,d,c,\widetilde{b},\widetilde{d},\widetilde{c}\},\ \big\{\{b,d\},\ \{d,c\},\ \{c,\widetilde{b}\},\ \{\widetilde{b},\widetilde{d}\},\ \{\widetilde{d},\widetilde{c}\},\ \{\widetilde{c},b\}\big\}\Big\}.
\end{aligned}$$

To complete the proof, we show that for any orientation ∂ of Z, any given hexagon (V,E) and any orientation on the circuit of edges E, there is a unique geometric triality (τ,∂) without absolute point that permutes the edges in E in the prescribed direction. Let $Z=\{a,b,c,d,\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}\}$ and consider in Z the hexagon $H_1=(V_1,E_1)$ above. Suppose the orientation ∂ is such that $\{a,b,c,d\}\in C_1$. We seek a triality (τ,∂) without absolute point that permutes the edges as in (4.11) and (4.12). Since $\tau\{a,b\}=\{c,\widetilde{a}\}$ we know that $\widetilde{a},\ c\in\tau(b)$. On the other hand, $b\notin\tau(b)$ since b is not an absolute point, hence $\widetilde{b}\in\tau(b)$. Thus $\tau(b)$ is either $\{\widetilde{a},\widetilde{b},c,d\}$ or $\{\widetilde{a},\widetilde{b},c,\widetilde{d}\}$. For the orientation ∂, the first of these solids is in C_1 and the second in C_2, so

$$\tau(b)=\{\widetilde{a},\widetilde{b},c,d\}.$$

Similarly, from $\tau\{\widetilde{c},a\}=\{b,c\}$ we derive

$$\tau(a)=\{\widetilde{a},b,c,\widetilde{d}\}$$

and from $\tau\{b,c\}=\{\widetilde{a},\widetilde{b}\}$ we derive

$$\tau(c)=\{\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}\}.$$

The solids $\tau(d)$ and $\tau(\widetilde{d})$ are uniquely determined by the conditions $\widetilde{d}\in\tau(d)$, $d\in\tau(\widetilde{d})$, and

$$\{\tau(d),\tau(\widetilde{d})\}=C_1\setminus\{\tau(a),\tau(\widetilde{a}),\tau(b),\tau(\widetilde{b}),\tau(c),\tau(\widetilde{c})\},$$

so that

$$\tau(d)=\{a,\widetilde{b},c,\widetilde{d}\}.$$

The map $\tau^2\colon Z\to C_2$ is determined by τ in view of Claim 1. To see that τ is indeed a geometric triality, we use coordinates, identifying Z with $\{\pm\xi_1,\dots,\pm\xi_4\}\subset\mathbb{R}^4$ as follows:

$$a=\xi_1,\qquad b=-\xi_2,\qquad c=-\xi_3,\qquad d=\xi_4.$$

Then τ is given by the matrix

$$\frac{1}{2}\begin{pmatrix}-1&1&1&1\\-1&-1&-1&1\\-1&1&-1&-1\\-1&-1&1&-1\end{pmatrix}. \tag{4.13}$$

Computation shows that the corresponding linear map is an orthogonal transformation of $\mathbb{R}^4$ with cube 1 that maps Z to C_1, hence τ is a triality. □

Proposition 4.14. *Let (τ,∂) be a triality without absolute points and with hexagons $\{H_1,\,H_2,\,H_3,\,H_4\}$. For any proper isometry $\phi\in\mathrm{PGO}^+(Z)$ the set of hexagons of the triality $(\phi\circ\tau\circ\phi^{-1},\partial)$ is $\{\phi(H_1),\,\phi(H_2),\,\phi(H_3),\,\phi(H_4)\}$.*

Proof. The edge set of each hexagon is an orbit under the action of the group generated by τ and the structure map $\widetilde{\ }$. The claim follows from the fact that the orbits of $\phi\circ\tau\circ\phi^{-1}$ are the images under ϕ of the orbits of τ. □

Corollary 4.15. *Let ∂ be a fixed orientation on Z. There are 8 geometric trialities (τ, ∂) on Z without absolute points. These trialities are conjugate under the group* $\mathrm{PGO}^+(Z)$.

Proof. There are 4 hexagons in Z containing a given line as an edge, and each of these hexagons can be oriented in two different ways. The 8 oriented hexagons are permuted under the action of $\mathrm{PGO}^+(Z)$. The last claim follows from Proposition 4.14. □

Automorphisms. Let Z be a 6-dimensional quadric and let (τ, ∂) be a triality on Z. The proper isometries $\phi \in \mathrm{PGO}^+(Z)$ that conjugate (τ, ∂) to itself are called *automorphisms* of (τ, ∂). We let $\mathrm{Aut}(\tau, \partial)$ denote the group of automorphisms of (τ, ∂). Obviously the structure map $\tilde{\ }$ is contained in $\mathrm{Aut}(\tau, \partial)$.

Theorem 4.16. *If (τ, ∂) admits absolute points, then* $\mathrm{Aut}(\tau, \partial)$ *is isomorphic to the dihedral group $\mathfrak{D}_{12} = \mathfrak{S}_2 \times \mathfrak{S}_3$.*
If (τ, ∂) does not admit absolute points, then $\mathrm{Aut}(\tau, \partial)$ *is isomorphic to the double cover $\widetilde{\mathfrak{A}}_4$ ($\simeq \mathrm{SL}_2(\mathbb{F}_3)$) of the alternating group $\mathfrak{A}_4$.*

Proof. Assume that (τ, ∂) admits absolute points. In view of Theorem 4.7 the triality (τ, ∂) is uniquely determined by its hexagon of absolute points and fixed lines. Thus the group of automorphisms of (τ, ∂) is isomorphic to the group $\mathfrak{D}_{12}$ of automorphisms of the hexagon. Observe that an automorphism of the hexagon extends to an automorphism of the quadric in such a way that the action on the pair of non absolute points has to be such that the extension is a proper isometry.
Assume now that the triality (τ, ∂) does not admit absolute points. We set $Z = \{a, b, c, d, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}\}$ and assume τ is as in the proof of Theorem 4.10. It follows from Proposition 4.14 that the automorphism ϕ induces a permutation of the hexagons. Thus we have a group homomorphism $\psi\colon \mathrm{Aut}(\tau, \partial) \to \mathfrak{S}_4$. The claim now follows from the following lemma:

Lemma 4.17. *The image of ψ is the group $\mathfrak{A}_4$ and the sequence*

$$1 \to \{I, \tilde{\ }\} \to \mathrm{Aut}(\tau, \partial) \xrightarrow{\psi} \mathfrak{A}_4 \to 1$$

is non-split exact.

Proof. Consider the following transformation $\phi \in \mathrm{PGO}^+(Z)$:

$$\phi\colon a \mapsto b \mapsto c \mapsto \widetilde{a} \mapsto \widetilde{b} \mapsto \widetilde{c} \mapsto a, \qquad d \leftrightarrow \widetilde{d}. \tag{4.18}$$

It preserves the hexagon H_1 and permutes its edges in the same direction as τ, hence $\phi \in \mathrm{Aut}(\tau, \partial)$. Inspection shows that ϕ permutes H_2, H_3, and H_4 cyclically. Similar transformations can be defined to preserve any of the hexagons and to permute the others cyclically, hence the image of ψ contains $\mathfrak{A}_4$. On the other hand, any automorphism of (τ, ∂) that fixes H_1 is a power of ϕ, hence the image of ψ does not contain any transposition fixing H_1, so the image of ψ is $\mathfrak{A}_4$.
Now, assume $\theta \in \operatorname{Ker} \psi$. Since it fixes H_1, it must preserve the pair $\{d, \widetilde{d}\}$. Similarly, it must preserve each pair $\{a, \widetilde{a}\}$, $\{b, \widetilde{b}\}$, $\{c, \widetilde{c}\}$. If it is the identity on $\{a, \widetilde{a}\}$, then it must also be the identity on $\{b, \widetilde{b}\}$ and on $\{c, \widetilde{c}\}$ since it fixes H_1, hence it must be the identity since it is a proper isometry. Similarly, we get $\theta = I$ if θ leaves any element of Z fixed, hence $\operatorname{Ker} \psi = \{I, \tilde{\ }\}$. Finally the fact that the 6-cycle (4.18) maps to a 3-cycle in $\mathfrak{A}_4$ shows that the sequence is not split. □

Involutions. Given a geometric triality (τ, ∂) on a 6-dimensional quadric Z over $\mathbb{F}_1$, let $G_\tau \subset \mathrm{PGO}(Z)$ be the group of automorphisms of Z that conjugate to itself the group $\{I, \tau, \tau^2\}$. An element $\phi \in G_\tau$ satisfies $\phi \circ \tau \circ \phi^{-1} = \tau^2$ if and only if it exchanges C_1 and C_2, hence

$$G_\tau \cap \mathrm{PGO}^+(Z) = \mathrm{Aut}(\tau, \partial).$$

An *involution* of (τ, ∂) is an element $\phi \in G_\tau \setminus \mathrm{Aut}(\tau, \partial)$ of order 2.

Proposition 4.19. *If the geometric triality (τ, ∂) has absolute points, then the map $\gamma \in \mathrm{PGO}(Z)$ that leaves all the absolute points fixed and exchanges the two non-absolute points is an involution, and $G_\tau = \mathrm{Aut}(\tau, \partial) \times \{I, \gamma\} \simeq \mathfrak{D}_{12} \times \mathfrak{S}_2$.*
If the geometric triality (τ, ∂) has no absolute points, then G_τ is isomorphic to the double cover $\widetilde{\mathfrak{S}}_4$ ($\simeq \mathrm{GL}_2(\mathbb{F}_3)$) of the symmetric group $\mathfrak{S}_4$ characterized by the property that transpositions of $\mathfrak{S}_4$ lift to elements of order 2, while products of two disjoint transpositions lift to elements of order 4. In particular, G_τ contains involutions.

Proof. In the first case, it is readily verified that γ is an involution. Every automorphism of (τ, ∂) preserves the absolute points, hence commutes with γ. Therefore, G_τ is the direct product of $\mathrm{Aut}(\tau, \partial)$ and $\{I, \gamma\}$.
Now, suppose (τ, ∂) has no absolute points. Set $Z = \{a, b, c, d, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}\}$ and assume (τ, ∂) is as in the proof of Theorem 4.10. The map ϕ that leaves a, d (hence also $\widetilde{a}$ and $\widetilde{d}$) fixed and exchanges b and $\widetilde{c}$ (hence also $\widetilde{b}$ and c) preserves the hexagon H_1 and reverses the orientation of its circuit of edges, hence it conjugates τ into τ^2. It is thus an involution of (τ, ∂). Note that ϕ also preserves the hexagon H_4 and exchanges H_2 and H_3. Since every even permutation of $\{H_1, H_2, H_3, H_4\}$ can be realized as an action of an automorphism of (τ, ∂), by Lemma 4.17, it follows that the action of G_τ on the set of hexagons yields a surjective map $G_\tau \to \mathfrak{S}_4$. As observed in the proof of Lemma 4.17, the kernel of this map is the center $\{I, \widetilde{\ }\}$ of $\mathrm{PGO}(Z)$. □

Theorem 4.20. *There are four proper isomorphism classes of geometric trialities under the group $\mathrm{PGO}^+(Z)$ and there are two isomorphism classes of geometric trialities under the group $\mathrm{PGO}(Z)$.*

Proof. The first claim follows from Corollaries 4.9 and 4.15, the second from the existence of involutions for geometric trialities. □

Symmetric compositions and geometric trialities. In this subsection, we establish a one-to-one correspondence between symmetric compositions on a quadratic space $(\mathcal{S}, \widetilde{\ })$ of dimension 8 over $\mathbb{F}_1$ and geometric trialities on the corresponding quadric $Z = \langle \mathcal{S} \rangle$. Let $\star$ be a symmetric composition on $(\mathcal{S}, \widetilde{\ })$. The bijection γ of Corollary 3.21 allows us to define an orientation $\partial_\star$ of the quadric Z by setting

$$C_1 = \{\langle x \star \mathcal{S} \rangle \mid x \in \mathcal{S}\}, \qquad C_2 = \{\langle \mathcal{S} \star y \rangle \mid y \in \mathcal{S}\}. \tag{4.21}$$

To describe the corresponding geometry on Z, it is useful to describe all possible intersections of maximal solids of opposite kinds.

Proposition 4.22. *For any nonzero x, $y \in \mathcal{S}$, the following conditions are equivalent:*

(a) $|\langle x \star \mathcal{S} \rangle \cap \langle \mathcal{S} \star y \rangle| = 3$;

(b) $x \star y = 0$.

Similarly, the following conditions are equivalent:

(a') $|\langle\, x \star \mathcal{S}\,\rangle \cap \langle\, \mathcal{S} \star y\,\rangle| = 1$;

(b') $x \star y \neq 0$.

When (a') and (b') hold, we have $\langle\, x \star \mathcal{S}\,\rangle \cap \langle\, \mathcal{S} \star y\,\rangle = \{x \star y\}$ since clearly $x \star y$ lies in $x \star \mathcal{S}$ and $\mathcal{S} \star y$.

Proof. Suppose (b) holds. We then have $(z \star y) \star x = 0$ for all $z \neq \widetilde{x}$ by (SC4). There are four elements z such that $z \star y \neq 0$ and one of them is $\widetilde{x}$. Thus for the three others we have $(z \star y) \star x = 0$, which means $z \star y \in \operatorname{Ker} r_x = x \star \mathcal{S}$. Thus,

$$|\langle\, x^{\perp} \star y\,\rangle| = 3 \quad \text{and} \quad x^{\perp} \star \mathcal{S} \subset (x \star \mathcal{S}) \cap (\mathcal{S} \star y).$$

Since $\langle\, x \star \mathcal{S}\,\rangle$ and $\langle\, \mathcal{S} \star y\,\rangle$ are solids of different kinds, $|\langle\, x \star \mathcal{S}\,\rangle \cap \langle\, \mathcal{S} \star y\,\rangle|$ must be odd. If $x \star y = 0$ we have $(z \star y) \star x = 0$ for all $z \neq \widetilde{x}$. Therefore, we have (a). Conversely, (a) implies (b) by Lemma 3.19. Now, (a') is equivalent to the negation of (a) since $\langle\, x \star \mathcal{S}\,\rangle$ and $\langle\, \mathcal{S} \star y\,\rangle$ are solids of different kinds, and (b') is the negation of (b), so (a') and (b') are equivalent. $\square$

We next define a correspondence between geometric trialities and symmetric compositions.

Proposition 4.23. *Let $(\mathcal{S}, \tilde{\ })$ be an 8-dimensional quadratic space over $\mathbb{F}_1$.*

(1) *For any symmetric composition $\star$ on $(\mathcal{S}, \tilde{\ })$, the orientation (4.21) and the map*
$$\tau_\star \colon x \mapsto \langle\, x \star \mathcal{S}\,\rangle \mapsto \langle\, \mathcal{S} \star x\,\rangle \mapsto x \quad \text{for } x \in \langle\, S\,\rangle$$
define a geometric triality $(\tau_\star, \partial_\star)$ on $\langle\, \mathcal{S}\,\rangle$.

(2) *Conversely, given a geometric triality (τ, ∂), the multiplication defined by*
$$x \star_\tau 0 = 0 \star_\tau x = 0 \qquad \text{for } x \in \mathcal{S}$$
and, for nonzero x, $y \in \mathcal{S}$,
$$x \star_\tau y = \begin{cases} z & \text{if } \tau(x) \cap \tau^2(y) = \{z\}, \\ 0 & \text{if } |\tau(x) \cap \tau^2(y)| = 3 \end{cases}$$
is a symmetric composition on $(\mathcal{S}, \tilde{\ })$.

Proof. (1) By definition, the map $\tau_\star$ satisfies (GT3) and (GT4). Moreover, it satisfies (GT1) because (SC1) holds for $\star$. To prove (1), it remains to show that (GT2) holds, i.e., that $\tau_\star$ preserves the incidence relations. For nonzero x, $y \in \mathcal{S}$, we have to see that

$$x \in y \star \mathcal{S} \iff |\langle\, x \star \mathcal{S}\,\rangle \cap \langle\, \mathcal{S} \star y\,\rangle| = 3 \iff \mathcal{S} \star x \ni y.$$

By Lemma 3.17 and Proposition 4.22, these conditions are all equivalent to $x \star y = 0$.
(2) We show that $\tau_\star$ satisfies (SC1)–(SC4). Axiom (SC1) readily follows from (GT1). Axiom (SC2) translates into the following condition on τ: for x, $y \in \langle\, \mathcal{S}\,\rangle$,

$$|\tau(x) \cap \tau^2(y)| = 3 \iff |\tau(x) \cap \tau^2(\widetilde{y})| = 1.$$

This equivalence holds because $\tau^2(\widetilde{y}) = \widetilde{\tau^2(y)}$. Likewise, (SC3) is equivalent to the following condition on τ: for x, y, $z \in \langle\, \mathcal{S}\,\rangle$,

$$\tau(x) \cap \tau^2(y) = \{z\} \;\Rightarrow\; \tau(z) \cap \tau^2(\widetilde{x}) = \{y\} \text{ and } \tau(\widetilde{y}) \cap \tau^2(z) = \{x\}. \tag{4.24}$$

Assume $\tau(x)\cap\tau^2(y)=\{z\}$. Since $z\in\tau(x)$ and τ preserves the incidence relations, we have $|\tau(z)\cap\tau^2(x)|=3$, hence $|\tau(z)\cap\tau^2(\widetilde{x})|=1$. Similarly, since $z\in\tau^2(y)$ we have $y\in\tau(z)$, and since $|\tau(x)\cap\tau^2(y)|=1$ we have $y\notin\tau^2(x)$, hence $y\in\tau^2(\widetilde{x})$. Therefore, $\tau(z)\cap\tau^2(\widetilde{x})=\{y\}$. Applying the same argument after a cyclic permutation of x, y, and z, we obtain

$$\tau(z)\cap\tau^2(\widetilde{x})=\{y\}\ \Rightarrow\ \tau(y)\cap\tau^2(\widetilde{z})=\{\widetilde{x}\}.$$

Therefore, (4.24) holds. It only remains to prove (SC4), which translates to the following statement: for x, y, u, $v\in\langle\,\mathcal{S}\,\rangle$ with $u\neq\widetilde{x}$ and $v\neq\widetilde{y}$, if $|\tau(x)\cap\tau^2(y)|=3$, then

- either $|\tau(u)\cap\tau^2(y)|=3$, or
- $\tau(u)\cap\tau^2(y)=\{z\}$ for some $z\in\langle\,\mathcal{S}\,\rangle$, and $|\tau(z)\cap\tau^2(x)|=3$,

and, likewise,

- either $|\tau(x)\cap\tau^2(v)|=3$, or
- $\tau(x)\cap\tau^2(v)=\{z\}$ for some $z\in\langle\,\mathcal{S}\,\rangle$, and $|\tau(y)\cap\tau^2(z)|=3$.

Assume $\tau^2(y)=\{z_1,\,z_2,\,z_3,\,z_4\}$ and $\tau(x)\cap\tau^2(y)=\{z_1,\,z_2,\,z_3\}$. Since $u\neq\widetilde{x}$, we have $\tau(u)\cap\tau^2(y)\neq\{z_4\}$, so either $|\tau(u)\cap\tau^2(y)|=3$, or $\tau(u)\cap\tau^2(y)=\{z_i\}$ for some $i\in\{1,2,3\}$. Then $z_i\in\tau(x)$, so $|\tau(z_i)\cap\tau^2(x)|=3$ since τ preserves the incidence relation. Likewise, if $\tau(x)=\{z_1,\,z_2,\,z_3,\,z_5\}$ and $v\neq\widetilde{y}$, then $\tau(x)\cap\tau^2(v)\neq\{z_5\}$, so either $|\tau(x)\cap\tau^2(v)|=3$, or $\tau(x)\cap\tau^2(v)=\{z_i\}$ for some $i\in\{1,2,3\}$. Then $z_i\in\tau^2(y)$, so $|\tau(y)\cap\tau^2(z_i)|=3$. □

Theorem 4.25. *Let $(\mathcal{S},\,\widetilde{}\,)$ be an 8-dimensional quadratic space over $\mathbb{F}_1$. The maps $\star\mapsto(\tau_\star,\partial_\star)$ and $(\tau,\partial)\mapsto\star_\tau$ are inverse one-to-one correspondences between symmetric compositions on $(\mathcal{S},\,\widetilde{}\,)$ and geometric trialities on $\langle\,\mathcal{S}\,\rangle$. Isomorphic (resp. properly isomorphic) symmetric compositions correspond to isomorphic (resp. properly isomorphic) geometric trialities.*

Proof. Let $\star$ be a symmetric composition. By definition, we have for nonzero x, $y\in\mathcal{S}$

$$x\star_{\tau_\star}y=\begin{cases}z & \text{if } \langle\,x\star\mathcal{S}\,\rangle\cap\langle\,\mathcal{S}\star y\,\rangle=\{z\},\\ 0 & \text{if } |\langle\,x\star\mathcal{S}\,\rangle\cap\langle\,\mathcal{S}\star y\,\rangle|=3.\end{cases}$$

Proposition 4.22 shows that $x\star y=0$ if $|\langle\,x\star\mathcal{S}\,\rangle\cap\langle\,\mathcal{S}\star y\,\rangle|=3$ and that $x\star y\neq0$ if $|\langle\,x\star\mathcal{S}\,\rangle\cap\langle\,\mathcal{S}\star y\,\rangle|=1$. Since $x\star y\in\langle\,x\star\mathcal{S}\,\rangle\cap\langle\,\mathcal{S}\star y\,\rangle$, we must have $\{x\star y\}=\langle\,x\star\mathcal{S}\,\rangle\cap\langle\,\mathcal{S}\star y\,\rangle$ in the latter case, so $\star_{\tau_\star}=\star$.
Starting with a geometric triality (τ,∂), we have $\langle\,x\star_\tau\mathcal{S}\,\rangle\subset\tau(x)$ and $\langle\,\mathcal{S}\star_\tau x\,\rangle\subset\tau^2(x)$ for all $x\in\langle\,\mathcal{S}\,\rangle$, so in fact $\langle\,x\star_\tau\mathcal{S}\,\rangle=\tau(x)$ and $\langle\,\mathcal{S}\star_\tau x\,\rangle=\tau^2(x)$ since $\langle\,x\star_\tau\mathcal{S}\,\rangle$ and $\langle\,\mathcal{S}\star_\tau x\,\rangle$ are solids. Therefore, $(\tau_{\star_\tau},\partial_{\star_\tau})=(\tau,\partial)$. We have thus shown that the correspondences are inverse of each other. The last statement is clear. □

In view of the classification of geometric trialities in Theorem 4.20, we readily derive from the preceding theorem the classification of symmetric compositions in dimension 8:

Corollary 4.26. *(1) There are four proper isomorphism classes of 8-dimensional symmetric compositions over $\mathbb{F}_1$, given by the para-Zorn algebra, the pseudo-octonion algebra and their opposites.*
(2) There are two isomorphism classes of 8-dimensional symmetric compositions over $\mathbb{F}_1$, given by the para-Zorn algebra and the pseudo-octonion algebra.

It is also clear from the correspondence in Theorem 4.25 that automorphisms of symmetric compositions are automorphisms of the corresponding geometric trialities. Therefore, Theorem 4.16 readily yields:

Corollary 4.27. *The group of automorphisms of the para-Zorn algebra over $\mathbb{F}_1$ is a dihedral group $\mathfrak{D}_{12}$. The group of automorphisms of the pseudo-octonion algebra over $\mathbb{F}_1$ is isomorphic to the double cover $\widetilde{\mathfrak{A}}_4$.*

Note that, according to Tits [Tit57], $\mathfrak{D}_{12}$ is the exceptional group of type G_2 over $\mathbb{F}_1$. The description of groups of type G_2 as automorphism groups of para-Zorn algebras thus also holds over $\mathbb{F}_1$.

5. Trialitarian automorphisms

It is well known that the Weyl group $\mathfrak{S}_2^3 \rtimes \mathfrak{S}_4$ of type D_4 admits outer automorphisms of order 3 ("trialitarian automorphisms"). Symmetric compositions over $\mathbb{F}_1$ or geometric trialities can be used to construct such automorphisms. We view $\mathfrak{S}_2^3 \rtimes \mathfrak{S}_4$ as the group $\mathrm{O}^+(\mathcal{S})$ for a quadratic space $\mathcal{S}$ or as the group $\mathrm{PGO}^+(Z)$ of a 6-dimensional quadric Z. Following Tits this group is the projective orthogonal group $\mathrm{PGO}_8^+(\mathbb{F}_1)$. We need the following fact.

Lemma 5.1. *If α, β are trialitarian automorphisms of $\mathrm{PGO}_8^+(\mathbb{F}_1)$, then $\alpha \circ \beta^{-1}$ or $\alpha \circ \beta^{-2}$ is an inner automorphism.*

Proof. The claim is well known. A proof is for example in [Ban69] or [Fra01], see also [FH03]. □

Geometric trialities and trialitarian automorphisms. The relation between geometric trialities and trialitarian automorphisms is straightforward: let (τ, ∂) be a geometric triality on a 6-dimensional quadric Z over $\mathbb{F}_1$. Assume the orientation ∂ is given by $C(Z) = C_1 \sqcup C_2$. For any element $g \in \mathrm{PGO}^+(Z)$, we write $C_1(g)$ for the permutation induced by g on C_1, and set

$$\rho_\tau(g) = \tau|_Z^{-1} \circ C_1(g) \circ \tau|_Z.$$

Proposition 5.2. *The map ρ_τ is a trialitarian automorphism of $\mathrm{PGO}^+(Z)$.*

Proof. It is clear from the definition that ρ_τ is an automorphism of $\mathrm{PGO}^+(Z)$. To prove that ρ_τ is an outer automorphism and that $\rho_\tau^3 = I$, we use coordinates to identify Z with the set of vectors in the standard basis of $\mathbb{R}^4$ and their opposites:

$$Z = \{\pm\xi_1, \pm\xi_2, \pm\xi_3, \pm\xi_4\}.$$

As seen in §4, we may also identify C_1 with a set of vectors in $\mathbb{R}^4$, and $\mathrm{PGO}^+(Z)$ embeds in $\mathrm{O}_4(\mathbb{R})$: we have homomorphisms

$$\mu\colon \mathrm{PGO}^+(Z) \hookrightarrow \mathrm{O}_4(\mathbb{R}) \quad \text{and} \quad \mu'\colon \mathrm{PGO}(C_1) \hookrightarrow \mathrm{O}_4(\mathbb{R}),$$

which assign to each isomorphism of Z or of C_1 the matrix of the induced orthogonal transformation of $\mathbb{R}^4$ in the standard basis. For $g \in \mathrm{PGO}^+(Z)$ and $\varepsilon_1, \ldots, \varepsilon_4 \in \{\pm 1\}$ such that $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$, we have

$$C_1(g)\big(\tfrac{1}{2}(\varepsilon_1\xi_1 + \varepsilon_2\xi_2 + \varepsilon_3\xi_3 + \varepsilon_4\xi_4)\big) = \tfrac{1}{2}\big(\varepsilon_1 g(\xi_1) + \varepsilon_2 g(\xi_2) + \varepsilon_3 g(\xi_3) + \varepsilon_4 g(\xi_4)\big).$$

Therefore, the following diagram commutes:

$$\begin{tikzcd} \mathrm{PGO}^+(Z) \arrow[r, "\mu"] \arrow[d, "C_1"'] & \mathrm{O}_4(\mathbb{R}) \arrow[d, equal] \\ \mathrm{PGO}(C_1) \arrow[r, "\mu'"] & \mathrm{O}_4(\mathbb{R}). \end{tikzcd}$$

Writing T for the matrix of τ in the standard basis of $\mathbb{R}^4$, as in Proposition 4.1, we thus have

$$\mu\bigl(\rho_\tau(g)\bigr) = T^{-1}\mu(g)T \qquad \text{for } g \in \mathrm{PGO}^+(Z). \tag{5.3}$$

Since $T^3 = 1$, it follows that $\rho_\tau^3 = I$, and since $TZ = C_1$ we have $T \notin \mu\bigl(\mathrm{PGO}^+(Z)\bigr)$, so ρ_τ is an outer automorphism. □

Lemma 5.4. *Let (τ, ∂) be a geometric triality on Z and let $f \in \mathrm{PGO}^+(Z)$. The map $\tau\circ f\colon Z \to C_1$ extends to a triality $(\tau\circ f, \partial)$ on Z if and only if $\rho_\tau^2(f)\circ\rho_\tau(f)\circ f = I$. When this condition holds, we have $\rho_{\tau\circ f} = \mathrm{Int}(f^{-1}) \circ \rho_\tau$.*

Proof. Using the same notation as in the proof of Proposition 5.2, we see that $T\mu(f)$ is the matrix representing the map $\tau \circ f$. By Proposition 4.1, it follows that this map extends to a geometric triality $(\tau \circ f, \partial)$ if and only if $\bigl(T\mu(f)\bigr)^3 = 1$. This condition holds if and only if $\rho_\tau^2(f) \circ \rho_\tau(f) \circ f = I$. Assuming it holds, we have for $g \in \mathrm{PGO}^+(Z)$

$$\mu\bigl(\rho_{\tau\circ f}(g)\bigr) = \mu(f)^{-1}T^{-1}\mu(g)T\mu(f) = \mu\bigl(f^{-1}\rho_\tau(g)f\bigr),$$

which proves the last claim. □

Theorem 5.5. *For any 6-dimensional quadric Z over $\mathbb{F}_1$, the map $(\tau, \partial) \mapsto \rho_\tau$ is a bijection between geometric trialities on Z and trialitarian automorphisms of $\mathrm{PGO}^+(Z)$.*

Proof. Suppose (τ, ∂) and (τ', ∂') are geometric trialities such that $\rho_\tau = \rho_{\tau'}$. Using the same notation as in the proof of Proposition 5.2 and letting T (resp. T') denote the matrix of τ (resp. τ'), it follows from (5.3) that $T'T^{-1}$ centralizes $\mu\bigl(\mathrm{PGO}^+(Z)\bigr)$, hence $T' = \pm T$. Since ${T'}^3 = T^3 = 1$, we must have $T' = T$, hence $(\tau, \partial) = (\tau', \partial')$. Now, suppose ρ is an arbitrary trialitarian automorphism of $\mathrm{PGO}^+(Z)$, and let (τ, ∂) be a geometric triality on Z. By Lemma 5.1, we have

$$\rho = \mathrm{Int}(f^{-1}) \circ \rho_\tau \qquad \text{or} \qquad \rho = \mathrm{Int}(f^{-1}) \circ \rho_\tau^2 \qquad \text{for some } f \in \mathrm{PGO}^+(Z).$$

In the latter case, we substitute $(\tau^2, \widehat{\partial})$ for (τ, ∂) (where $\widehat{\partial}$ is the opposite orientation of ∂). Since $\rho_{\tau^2} = \rho_\tau^2$, we may thus consider only the first case. Since $\rho^3 = I$, we have $\rho_\tau^2(f) \circ \rho_\tau(f) \circ f = I$, hence Lemma 5.4 shows that $(\tau \circ f, \partial)$ is a geometric triality such that $\rho_{\tau\circ f} = \rho$. Therefore, the map $(\tau, \partial) \mapsto \rho_\tau$ is onto. □

Corollary 5.6. *In the Weyl group of type D_4, the subgroup fixed under a trialitarian automorphism is isomorphic to either the dihedral group $\mathfrak{D}_{12}$ or the double covering $\widetilde{\mathfrak{A}}_4$, depending on whether the trialitarian automorphism corresponds under the bijection of Theorem 5.5 to a geometric triality with absolute points or without absolute points.*

Proof. Let (τ, ∂) be a geometric triality. By definition, the subgroup fixed under ρ_τ is the group of automorphisms of (τ, ∂). Therefore, the corollary follows from Theorem 4.16. □

Corollary 5.6 was verified in [KT10] using the software program Magma.

Symmetric compositions and trialitarian automorphisms. Since symmetric compositions are in bijection with geometric trialities and geometric trialities are in bijection with trialitarian automorphisms of $\mathrm{PGO}_8^+(\mathbb{F}_1)$, symmetric compositions and trialitarian automorphisms are in bijection. We describe such a bijection directly.

Proposition 5.7. *Let $\star$ be a symmetric composition on an 8-dimensional quadratic space $(\mathcal{S}, \widetilde{\ })$ over $\mathbb{F}_1$.*

(1) *For every $f \in \mathrm{O}^+(\mathcal{S})$, there are unique elements $f_1,\ f_2 \in \mathrm{O}^+(\mathcal{S})$ such that*

$$f(x \star \mathcal{S}) = f_1(x) \star \mathcal{S} \quad \text{and} \quad f(\mathcal{S} \star x) = \mathcal{S} \star f_2(x) \quad \text{for all } x \in \mathcal{S}. \tag{5.8}$$

These maps satisfy the following identities for all $x,\ y \in \mathcal{S}$:

$$\begin{aligned} f(x \star y) &= f_1(x) \star f_2(y) \\ f_1(x \star y) &= f_2(x) \star f(y) \\ f_2(x \star y) &= f(x) \star f_1(y) \end{aligned}$$

(2) *For every $f \in \mathrm{O}(\mathcal{S}) \setminus \mathrm{O}^+(\mathcal{S})$, there are unique elements $f_1,\ f_2 \in \mathrm{O}(\mathcal{S})$ such that*

$$f(x \star \mathcal{S}) = \mathcal{S} \star f_2(x) \quad \text{and} \quad f(\mathcal{S} \star x) = f_1(x) \star \mathcal{S} \quad \text{for all } x \in \mathcal{S}.$$

These maps satisfy the following identities for all $x,\ y \in \mathcal{S}$:

$$\begin{aligned} f(x \star y) &= f_1(y) \star f_2(x) \\ f_1(x \star y) &= f_2(y) \star f(x) \\ f_2(x \star y) &= f(y) \star f_1(x) \end{aligned}$$

Proof. Suppose $f \in \mathrm{O}^+(\mathcal{S})$. Then f maps maximal isotropic subspaces of one kind to maximal isotropic subspaces of the same kind, and since by Corollary 3.21 every maximal isotropic subspace has a unique representation in the form $x \star \mathcal{S}$ or $\mathcal{S} \star x$, there are bijective maps $f_1,\ f_2 \colon \mathcal{S} \to \mathcal{S}$ defined by the property (5.8) (and by $f_1(0) = f_2(0) = 0$). We have $f_1 \in \mathrm{O}(\mathcal{S})$ since

$$f_1(\widetilde{x}) \star \mathcal{S} = f(\widetilde{x} \star \mathcal{S}) = f\widetilde{(x \star \mathcal{S})} = \widetilde{f_1(x)} \star \mathcal{S}\,.$$

Similarly, $f_2 \in \mathrm{O}(\mathcal{S})$.

To prove the identities in (1), we may assume x and y are nonzero. If $x \star y \neq 0$, Proposition 4.22 implies that

$$\{x \star y\} = \langle\, x \star \mathcal{S} \,\rangle \cap \langle\, \mathcal{S} \star y \,\rangle,$$

so that

$$\{f(x \star y)\} = \langle\, f(x \star \mathcal{S}) \,\rangle \cap \langle\, f(\mathcal{S} \star y) \,\rangle = \langle\, f_1(x) \star \mathcal{S} \,\rangle \cap \langle\, \mathcal{S} \star f_2(y) \,\rangle\,.$$

By Proposition 4.22 again, it follows that $f_1(x) \star f_2(y) = f(x \star y)$. We thus get the first identity of (1) when $x \star y \neq 0$. For the second, assuming $x \star y \neq 0$, we deduce from $(x \star y) \star \widetilde{x} = y$ that

$$f\big((x \star y) \star \widetilde{x}\big) = f_1(x \star y) \star f_2(\widetilde{x}) = f(y). \tag{5.9}$$

Since f is an isometry, we have $f(\widetilde{x}) = \widetilde{f(x)}$. Multiplying on the left (5.9) by $f_2(x)$ gives the second formula when $x \star y \neq 0$. The proof of the third formula is similar. If $x \star y = 0$, then $x \star \widetilde{y} \neq 0$ by (SC2), and the preceding arguments yield $f_1(x) \star f_2(\widetilde{y}) \neq 0$, $f_2(x) \star f(\widetilde{y}) \neq 0$, and $f(x) \star f_1(\widetilde{y}) \neq 0$. Since f, f_1, and f_2 are isometries, they commute with $\widetilde{\ }$, hence

$$f_1(x) \star f_2(y) = f_2(x) \star f(y) = f(x) \star f_1(y) = 0.$$

The identities in (1) thus hold for all x, $y \in \mathcal{S}$.
From the second identity it follows that for all x, $y \in \mathcal{S}$

$$f_1(x \star \mathcal{S}) = f_2(x) \star \mathcal{S} \quad \text{and} \quad f_1(\mathcal{S} \star y) = \mathcal{S} \star f(y),$$

hence f_1 preserves the types of maximal isotropic subspaces. Therefore, $f_1 \in \mathrm{O}^+(\mathcal{S})$. Likewise, the third identity shows that $f_2 \in \mathrm{O}^+(\mathcal{S})$. The proof of (1) is thus complete. The proof of (2) is similar. □

Given a symmetric composition $\star$ on $(\mathcal{S}, \widetilde{\ })$, we use Proposition 5.7 to define a map $\rho_\star \colon \mathrm{O}^+(\mathcal{S}) \to \mathrm{O}^+(\mathcal{S})$ by

$$\rho_\star(f) = f_1 \qquad \text{for } f \in \mathrm{O}^+(\mathcal{S}).$$

Proposition 5.10. *Let $(\tau_\star, \partial_\star)$ be the geometric triality corresponding to the symmetric composition $\star$ (see Proposition 4.23), and let $\rho_{\tau_\star}$ be the associated trialitarian automorphism. We have $\rho_\star = \rho_{\tau_\star}$.*

Proof. Let $f \in \mathrm{O}^+(\mathcal{S}) = \mathrm{PGO}^+(\langle\, \mathcal{S}\,\rangle)$. By definition of $\rho_{\tau_\star}$, we have for $x \in \langle\, \mathcal{S}\,\rangle$

$$\rho_{\tau_\star}(f)(x) = \tau_\star^{-1}\bigl(C_1(f)(\tau_\star(x))\bigr) = \tau_\star^{-1}\,\langle\, f(x \star \mathcal{S})\,\rangle = \tau_\star^{-1}\,\langle\, f_1(x) \star \mathcal{S}\,\rangle = f_1(x).$$

□

It readily follows from Proposition 5.10 that $\rho_\star$ is a trialitarian automorphism. Combining Theorems 4.25 and 5.5, we obtain:

Theorem 5.11. *The map $\star \mapsto \rho_\star$ defines a bijection between the set of symmetric compositions over the quadratic space $(\mathcal{S}, \widetilde{\ })$ and the set of trialitarian automorphisms of $\mathrm{PGO}_8^+(\mathbb{F}_1)$.*

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