Community detection in complex networks

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Outline

Network: definition
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Why networks?
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  Some basic notions
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  Spectral partitioning
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   Modularity maximization
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Detecting groups of vertices with the same behavior...
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Detecting groups of vertices with the same behavior...
A network (or graph) is a set of points joined by lines.

Example

A point is called a vertex.
A line is called an edge.
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Why networks?

Networks allow to model systems with interacting agents.

The structure of the network is fundamental for the understanding of the underlying system.
Examples

- *Technological networks*
  - *Internet*


- *Telephone networks*
- *Transportation networks*
• Social networks
  - Friendship network between members of a club


And so many others ...
Network : definition

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Detecting groups of vertices with the same behavior ...
Community detection:

Partitioning the vertices of the network into groups, called communities, with many edges within the communities and few links between them.

Utility:

Revealing the structure and the organisation of the network.
Examples

- Network of coauthorship in a university department

• Friendship network at a US high school

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Detecting groups of vertices with the same behavior ...
Suppose that the network has $N$ vertices numbered from 1 to $N$.

The network can be mathematically represented thanks to its adjacency matrix $A \in \mathbb{R}^{N \times N}$:

$$A_{ij} = \begin{cases} 
1 & \text{if vertices } i \text{ and } j \text{ are connected} \\
0 & \text{otherwise}
\end{cases}$$
Suppose that the network has \( N \) vertices numbered from 1 to \( N \).

The network can be mathematically represented thanks to its adjacency matrix \( A \in \mathbb{R}^{N \times N} \):

\[
A_{ij} = \begin{cases} 
1 & \text{if vertices } i \text{ and } j \text{ are connected} \\
0 & \text{otherwise}
\end{cases}
\]

The degree \( k_i \) of vertex \( i \) is the number of edges connected to it, that is:

\[
k_i = \sum_{j=1}^{N} A_{ij}
\]
The structure of the network can also be mathematically represented thanks to the Laplacian, strongly related to the adjacency matrix.

Denote $D$ the diagonal matrix with the degrees of the vertices on its diagonal, that is:

$$D = \begin{pmatrix}
    k_1 & 0 & 0 & \ldots & 0 \\
    0 & k_2 & 0 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & k_N
\end{pmatrix}$$

The Laplacian of the network is then the matrix: $L := D - A$, where $A$ is the adjacency matrix of the network.
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Detecting groups of vertices with the same behavior ...
**Goal**: partitioning the network into two communities \( C_1 \) and \( C_2 \).

The cut function counts the number of edges between communities \( C_1 \) and \( C_2 \), that is:

\[
R = \sum_{i \in C_1, j \in C_2} A_{ij}.
\]
**Goal**: partitioning the network into two communities $C_1$ and $C_2$.

The cut function counts the number of edges between communities $C_1$ and $C_2$, that is:

$$ R = \sum_{i \in C_1, j \in C_2} A_{ij}. $$

$\Rightarrow$ We have to find a partition which minimizes $R$. 


Spectral partitioning method of Fiedler:

Define the following vector $s \in \mathbb{R}^{N \times 1}$:

$$s_i = \begin{cases} 
+1 & \text{if vertex } i \in C_1 \\
-1 & \text{if vertex } i \in C_2 
\end{cases}$$

Then,

$$R = \frac{1}{4} s^T L s,$$

where $L$ is the Laplacian of the network to partition.
Denote $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$ the spectrum of $L$.

Let $v_2$ be the eigenvector related to eigenvalue $\lambda_2$.

The vector $s$ given by Fiedler’s method is:

$$s_i = \begin{cases} +1 & \text{if } [v_2]_i > 0 \\ -1 & \text{if } [v_2]_i < 0 \end{cases}$$

If one entry $i$ of $v_2$ is zero, then $s_i$ can take equivalently value $+1$ or $-1$. 
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Detecting groups of vertices with the same behavior ...
Configuration model :

In this model of random graph:
- the number $N$ of vertices is fixed
- the degree sequence $[k_1, \ldots, k_N]$ is given.

As a consequence, the number $m$ of edges is fixed. Indeed,

$$m = \frac{1}{2} \sum_{i=1}^{N} k_i.$$

Let us place randomly the edges in the graph.

The expected number of edges between vertices $i$ and $j$ is:

$$\frac{k_i k_j}{2m}.$$
The modularity function compares the fraction of edges between two vertices in a same community and the expected fraction of edges (given the degree sequence), that is:

\[ Q = \frac{1}{2m} \sum_{i,j} \left( A_{ij} - \frac{k_i k_j}{2m} \right) \delta(c_i, c_j), \]

where:
- \( m \) is the number of edges in the network
- \( c_i \) is the community index of vertex \( i \)
- \( \delta \) is the Kronecker symbol.

\[ \Rightarrow \text{We have to find a partition which maximizes } Q. \]

There exist several methods. The fastest one is the Louvain method.
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Detecting groups of vertices with the same behavior ...
We would like to detect groups of vertices with the same behavior or of the same type.

Example

A Movie-Actor network.
Actual work:

1. How to define the behavior of a vertex?
2. Developing algorithms to detect such groups of vertices.
Thank you for your attention!