Clustering patterns of urban built-up areas with curves of fractal scaling behavior

Isabelle THOMAS
FRS-FNRS, CORE and Department of Geography,
Université catholique de Louvain, Louvain-la-Neuve, Belgium.
Isabelle.thomas@uclouvain.be

Pierre FRANKHAUSER
THEMA (CNRS UMR 6049), Université de Franche-Comté, Besançon, France.
pierre.frankhauser@univ-fcomte.fr

Benoit FRENAY
Machine Learning Group, Université catholique de Louvain, Louvain-la-Neuve, Belgium.
michel.verleysen@uclouvain.be
Benoit.Frenay@uclouvain.be

Michel VERLEYSEN
Machine Learning Group, Université catholique de Louvain, Louvain-la-Neuve, Belgium.
michel.verleysen@uclouvain.be

Contact address:
Isabelle Thomas
CORE
Voie du Roman Pays, 34
B-1348-Louvain-la-Neuve,
Belgium
Isabelle.thomas@uclouvain.be

Clustering patterns of urban built-up areas with curves of fractal scaling behavior

Abstract. Fractal dimension is an index which can be used for characterizing urban areas. The use of the curve of scaling behavior is less common. However its shape gives local information about the morphology of the built-up area. This paper suggests a method based on a *k*-medoid for clustering these curves. It is applied to 49 ward of European cities, and shows that the curves add interesting intra-ward information to our knowledge of the spatial variation of the urban texture. Moreover, morphological similarities are observed between cities: living, architectural and planning trends are not specific to individual cities.

1 Introduction

The analysis of urban built-up areas is a fascinating and complex topic (Levy, 1999; Conzen, 2001; Batty, 2005). Fractals have considerable potential for describing, measuring, analyzing and modeling complex realities, whatever the field of application (ecology, physics, remote sensing, etc.) (see e.g. Halley et al., 2004). An interesting aspect of fractals for urban geographers is their ability to summarize the complexity, compactness and heterogeneity of a spatial distribution in a single value (the fractal dimension, denoted *D*), that is independent of scale (see e.g. Batty, 2005; Lam and Cola, 2002; Frankhauser 1998a ; 2008; Lorenz, 2003; Salingaros 2003).

This paper discusses the use of a less common fractal output: the curve of scaling behavior, which may provide interesting spatial information about the organization of urban patterns at different scales. The curve of scaling behavior is complementary to the fractal dimension, which is a more global index. When computing curves for several built-up areas, it is interesting to compare their shapes not only visually but also quantitatively. In this paper we suggest the use of a *k*-medoid algorithm for this purpose (Section 2.3) and apply the method to 49 European wards (Section 3). The curve of scaling behavior is seen as an interesting complementary morphometrical measurement, a local index of morphology. Both indicators (fractal dimension and curve of scaling behavior) should be part of the geographer's toolbox for exploratory spatial data analysis (ESDA) of urban morphologies.

Our results differ from those presented by ****** (reference will be filled in after acceptance) where the fractal dimension was simply measured on a larger set of urban wards; it made no reference to curves of scaling behavior. The two papers are complementary.

2 Methodological aspects

Binary images are used here: black pixels correspond to built-up areas and white pixels to open spaces. We tested whether the spatial distribution of the black pixels followed a fractal law. A practical example of our output is presented in Appendix 1. First, the fractal dimension (Section 2.1), and then the curve of scaling behavior (Section 2.2) are considered. These methods are clustered in Section 2.3.

2.1 Fractal dimension(s)

Fractal dimension is a quantity indicating how completely a fractal fills the space being studied as one zooms down to finer and finer scales. There is no unique way of defining and estimating the fractal dimension (D); here it is computed by means of a correlation analysis (see e.g. Grassberger and Procaccia, 1983). A small square window of size ε surrounds each built-up pixel. The number of built-up pixels in each window is then counted, which allows the mean number of pair correlations $N(\varepsilon)$ per window to be computed. This operation is repeated for windows of different sizes. This results in a series of measurements that can be represented on a Cartesian graph, where the X coordinate is the size of the window ε, and the Y coordinate is the mean number of points per window (see Appendix 1).

The next step consists of fitting this empirical curve to a theoretical curve that corresponds to a fractal law, i.e. a power law which links the number of correlations to the size of the window:

$$N(\varepsilon) = a\ \varepsilon\ \varepsilon^{D} \qquad [1]$$

where a is the pre-factor of shape, and summarizes the non-fractal morphological properties of the geometric object being analyzed (see e.g. Gouyet, 1996, Frankhauser, 1998b, or ****** et al., (submitted) for a discussion). It can be interpreted as a synthetic indicator of the local particularities of the pattern across scales, due mainly to the fact that the elements of the built-up structure do not have the same shape. For instance, the size of the buildings in a residential area (such as detached houses) differs from that of an industrial zone. Hence, even if the scaling behavior and the fractal dimension are the same for both patterns, the mass $N(\varepsilon)$ differs because the base lengths of the buildings are different. The meaning of a becomes clearer when the identity $a \equiv b^D$ is introduced into Equation [1]:

$$N = a\,\varepsilon\,\varepsilon^D = b^{\,D}\,\varepsilon\,\varepsilon^D = (b\varepsilon\,)^D = \varepsilon'\varepsilon^D. \qquad [2]$$

In a sense, b corresponds to the average base length of the buildings, which are the constitutive elements of the urban patterns. Differences in b values also appear when two patterns that have been digitized in different ways are compared, i.e. when the size of the pixels differs. Hence, in this study the size of the pixel was controlled and fixed at 4 meters.

In real-world patterns, fractal behavior can change across scales. Frankhauser (1998a; 2008) has shown that such changes often occur within rather small ranges of ε, especially for small distances corresponding to the size of small blocks of houses or courtyards. He suggested introducing an additional parameter c that allows the estimation of D and a by acting on the overall position of the power law curve. The enlarged fractal law then becomes:

$$N(\varepsilon) = a\varepsilon\,\varepsilon^D + c, \qquad [2]$$

A non-linear regression is used to estimate the a, D and c that best fit the empirical curve (Appendix 2). The quality of the estimation is measured by an R^2 coefficient. The fit between the two curves (empirical and estimated) is considered as "poor" when $R^2 <$ 0.9999; in these cases, we can either conclude that the pattern being studied is not fractal, or that it is multi-fractal (see e.g. Tannier and Pumain, 2005).

The fractal dimension of a built-up area can take any value between 0 and 2. When $D = 2$, the built-up pattern is uniformly distributed. $D = 0$ corresponds to a limiting case in which the pattern is made up of one single point (e. g. a single farm building surrounded by fields). $D < 1$ corresponds to a pattern of disconnected elements (a number of built-up clusters separated one from another). $D > 1$ indicates connected elements[1] forming large

1 In this paper, connectivity is always considered from a fractal point of view.

and small clusters, in which isolated elements may also occur. The closer D is to 2, the more the elements are connected to each other and belong to one single large cluster. From experience, we know that D provides quite a good indicator of the morphology of a built-up area (see e.g. De Keersmaecker, Frankhauser and Thomas, 2003; Batty, 2005). The absolute value of D is slightly influenced by the estimation technique, the size of the window, and the centering of the window, but these factors do not affect the relative variations and operational conclusions (see e.g. Thomas et al., 2007).

2.2 Curve of scaling behavior

In this section we introduce an alternative representation of the empirical results of a fractal analysis: the "curve of scaling behavior" (Frankhauser 1998a). Palmer (1988) called this a "fractogram", but this terminology is limited to ecology (Leduc, Prairie & Bergeron, 1994). In urban analyses, the curve of scaling behavior has only been used up to now for defining critical scales where fractal behavior changes; it has often been used to redefine the size of the window (see Frankhauser, 1998a; 1998b; 2004; Tannier and Pumain, 2005). Batty (2001) used this type of representation, which he called "signature" to analyze simulated urban patterns.

In this paper, we consider the potential use of the curve of scaling behavior for further characterizing the morphology of urban areas. Let us first recall the underlying logic of this type of representation. For this purpose, we start with the original fractal law in Equation [1]. Taking the logarithm of this relation yields

$$\log N(\varepsilon) = \log a + D \log \varepsilon$$

The auxiliary variables $y \equiv \log N(\varepsilon)$ and $x \equiv \log \varepsilon$ are now introduced, giving the linear relation

$$y = \log a + D\,x.$$

Thus, the variation of y with respect to x is

$$\frac{dy}{dx} \equiv \frac{d \log N(\varepsilon)}{d \log \varepsilon} = D$$

where D corresponds to the constant slope value. However, as already pointed out, the fractal dimension may depend on the scales of the real-world patterns. Then D becomes a function of the scale ε (i.e. $D = D(\varepsilon)$). It is also possible that the typical shape of the objects depends on the scale. In this case this would mean that the shape of house blocks or town sections is not the same as that of buildings, which implies that the pre-factor *a* is a function of *ε*.

If we assume that the prefactor *a*, and the fractal dimension *D*, both depend on the distance parameter, we obtain:

$$\log N(\varepsilon) = \log a(\varepsilon) + D(\varepsilon)\ \log \varepsilon$$

or

$$y = \log a(\varepsilon) + D(\varepsilon)\, x$$

and thus the variation of *y* with respect to *x* becomes (Frankhauser 1998a):

$$\frac{d \log N(\varepsilon)}{d \log \varepsilon} \equiv \alpha(\varepsilon) = \frac{d \log a(\varepsilon)}{d \log \varepsilon} + \frac{d D(\varepsilon)}{d \log \varepsilon} \log \varepsilon + D(\varepsilon)$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{d \log a(\varepsilon)}{dx} + \frac{d D(\varepsilon)}{dx} x + D(\varepsilon)$$

Given that

$$d \log N(\varepsilon) = \frac{dN(\varepsilon)}{N(\varepsilon)}; \qquad d \log \varepsilon = \frac{d\varepsilon}{\varepsilon}$$

we obtain

$$\alpha(\varepsilon) = \frac{\left(\dfrac{dN(\varepsilon)}{N(\varepsilon)}\right)}{\left(\dfrac{d\varepsilon}{\varepsilon}\right)}$$

Thus the parameter *α(ε)* describes the relative change in the built-up mass *N(ε)* with respect to the relative change in the distance. If the parameters *a* and *D* were constant, *α(ε)* would be equal to *D* and we would find the usual allometric relationship, typical of fractals. However two additional terms now contribute to *α(ε)*: the first refers to variations in the shape of the elements which do not affect the fractal behavior, i.e. the hierarchical organization of the pattern. The second describes the changes of fractal behavior across scales[2]. Due to these terms, the *α* values may exceed the upper limit value of *D* = 2. Hence we may expect the empirical curve of scaling behavior *α(ε) to* provide detailed information about how the spatial organization of an urban pattern changes across scales.

---

[2] Remember that we assume here that *c* is a global parameter which does not vary with scale. It may be interpreted as a general error term which summarizes other random errors. Then we may rewrite the relationship *N(e) = a eD + c* simply as *N(ε) – c ≡ N'(ε)= a eD + c*, which allows us to proceed to the subsequent steps.

2.3 Clustering curves

The shape of the *α(ε)* curve of scaling behavior reveals intra-ward spatial structures. By comparing the curves visually, we can distinguish different types of shapes; but it is not obvious either how to find objective criteria for defining classes, or how to fix the number of clusters. The clustering methods traditionally used by geographers are not applicable as linear correlation does not assess the resemblance between two shapes. The relationship between two shapes can be non-linear (horizontal or vertical shifts, rotations, etc.) whereas correlation only measures linear relationships. To solve this problem we decided to use a *k*-medoid algorithm. This is a simplified *k*-mean algorithm where each cluster is represented by a medoid instead of a centroid. Figure 1 shows 6 data points in a 2-dimensional space. The centroid is a point representing the space which does not belong to the initial set of data points, while the medoid is one of them. This figure illustrates the difference between a medoid and a centroid: the centroid is the point in the space which is on average closest to each data point, while the medoid is the data point which is on average closest to the other data po
ints.

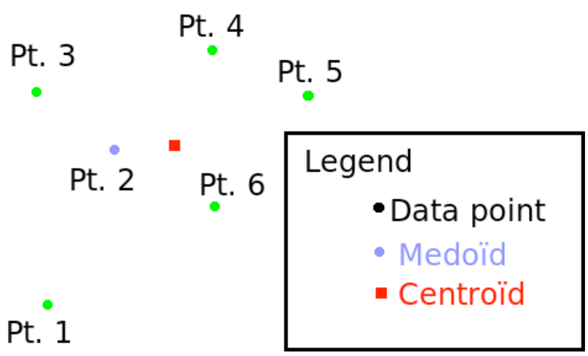


Figure 1: The difference between a centroid and a medoid

Given a set of curves $\alpha_1 \ldots \alpha_n$, the *k*-medoid algorithm (Bishop, 2006) produces a clustering $C = \{C_1 \ldots C_m\}$ where the m clusters are characterized by the medoids $g_1 \ldots g_m$. These medoids are chosen from the *n* curves: C_k contains the curves that are closer to g_k, in a Euclidian sense, than to any other medoid. More precisely, this algorithm finds the (local) minimum of the function

$$J(C) = \sum_{k=1}^{m} J(C_k) = \sum_{k=1}^{m} \sum_{\alpha_i \in C_k} d(\alpha_i, g_k)^2$$

where $d(\alpha_i, g_k)$ is the dissimilarity between the curve α_i and the medoid g_k. Notice that d is not necessarily a distance; hence we cannot use the *k*-mean algorithm to find the centroid.

To compute the clustering corresponding to this description, the algorithm proceeds in two steps: it first computes the dissimilarity $d(\alpha_i, \alpha_j)$ between the pairs of curves α_i and α_j, and then it minimizes $J(C)$. Let us first consider the computation of the dissimilarity $d(\alpha, \alpha')$ between the two curves $\alpha = (\alpha^1 \ldots \alpha^T)$ and $\alpha' = (\alpha'^1 \ldots \alpha'^T)$ where α^i is the i^th point of α. A simple solution (see Figure 2) is to define

$$d(\alpha, \alpha') = \sum_{i=1}^{T} (\alpha^i - \alpha'^i)^2 .$$

However, both curves must have the same number of points, what is not necessarily so. Moreover, this dissimilarity cannot adapt to horizontal shifts between α and α'.

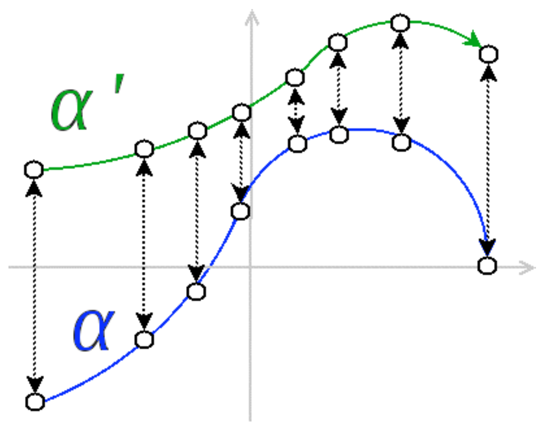


Figure 2: Naive matching between two α curves

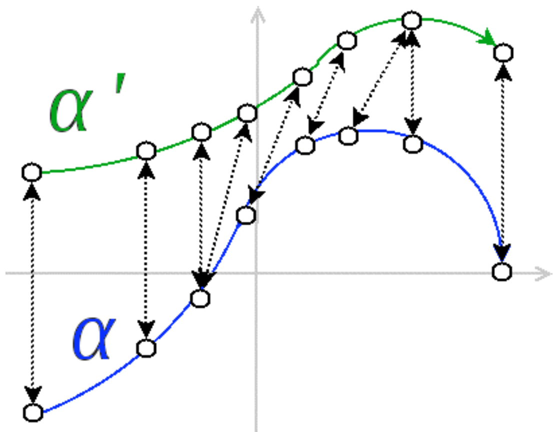


Figure 3: Optimal matching between two α curves

A better solution (see Figure 3) is to (i) find a match between the points of α and α'; and then (ii) compute the distance between the matched points. In others words, we do not necessarily compare the ith feature of α with the ith feature of α': the matching gives the jth feature of α' which seems to correspond to the ith feature of α and then we compare them. In practice, this match is computed using dynamic programming.

Once the distances between the curves have been computed, the k-medoid algorithm starts with a random clustering and proceeds in two steps. Firstly, it finds the medoid g_i of each cluster C_i, i.e. the curve in C_i which is the closest to the other curves in C_i. Secondly, each curve α_i is assigned to the cluster Cj whose medoid g_j is closest to α_i. These two steps are repeated again and again while $J(C)$ is still decreasing (it can be shown that $J(C)$ never increases). Since the result most often depends on the initial clustering (because $J(C)$ may have local minima), the whole process has to be repeated with different initial clusterings.

3 Clustering European urban wards: empirical results

In this section we will cluster curves of scaling behavior using the *k*-medoid method.

3.1 Data

The morphology of the built-up elements of 49 town sections form the data set for this analysis. These wards come from nine European cities: Besançon, Cergy, Lille, Lyon, and Montbéliard in France, Brussels and Charleroi in Belgium, and Stuttgart and the Ruhr area in Germany. A resolution of 4 meters per pixel was adopted for all wards. There were two main guidelines for selecting the wards: similar functional areas and very specific looking like homogeneous wards.

We limited ourselves to considering built-up areas, without knowing the exact function of the buildings (residential, service, industrial, etc.). The open spaces (white pixels) are considered as lacunae, or “green areas” (see Section 1). We know that there is a bias here that we could not avoid: these open areas or empty cells include roads! In order to minimize this bias, we avoided choosing wards containing large transportation infrastructures, such as railway stations or major roads.

A window was specified around each urban section in such a way that it included the built-up are that had been visually selected; the fractal dimension was computed on this window. Very small windows were avoided, following the principle that the error increases as the number of observations falls; the ratio between the size of the object and that of the window is always less than 1, in order to avoid measuring artifacts**.**

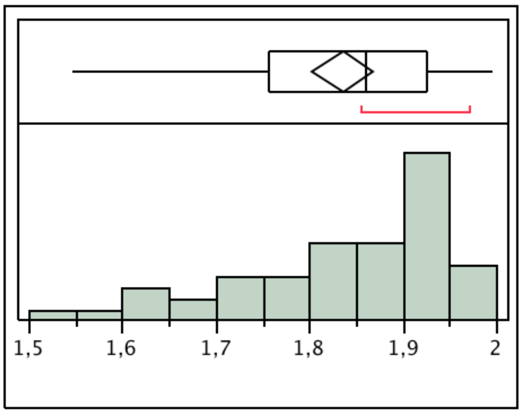
A measuring protocol was defined and applied. This ensured rigorous control of the quality of the estimate and avoided measurement artifacts (De Keersmaecker et al, 2003; Thomas et al., submitted). The same method, with the same control parameters and the same threshold values, was used for all the windows.

3.2 Fractal dimension

The fractal dimension in this data set had an average value of 1.84 and most of the observed values were higher than 1.75 (Figure 4). High D values indicates that the built-up area is homogenous at different scales, while small D values reveal heterogeneity i.e. variety in the built-up areas across scales. In our sample of wards, there is some variation between urban wards. Overall, the wards are globally quite homogeneous: the value of the first quartile is very high (D=1.92), and the value of the median (1.86) is above the mean value. Large values of D are associated with small variations of a (Figure 5) and R^2 (not illustrated here). The a values are here always less smaller than 2, confirming the fractal nature of the phenomenon. As expected, R^2 is always higher than 0.9999 and increases with D: the higher the value of D, the denser and more homogenous is the built-up area (i.e. the larger its mass), and, hence, the more urbanized it is (see also Frankhauser et al., 2008 or Thomas et al, 2007). These first results correspond to our expectations: the history and geography of the city matter, and no clear-cut country effect is visible (see Thomas et al., submitted, for further analyses).

As Figure 5 suggests, even if D reveals the homogeneity/heterogeneity of a built-up space, by itself it is not sufficient to discriminate univocally between two spatial organizations. Different patterns can lead to the same value of D, and a given value of D

may correspond to different spatial patterns (see Thomas et al. 2007 for a demonstration). Hence *D* can only be considered as kind of general index.



Mean: 1.835
Standard deviation: 0.113

Figure 4: Statistical distribution of *D* (histogram and boxplot)

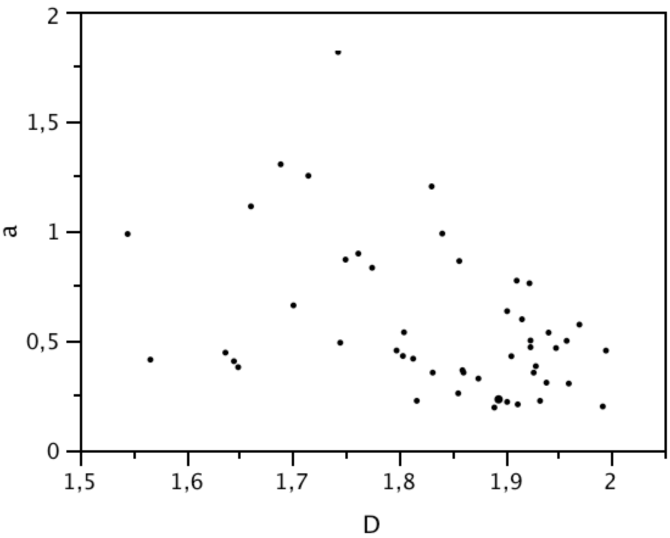


Figure 5: The relationship between the fractal dimension *(D)* and the pre-factor (*a*)

3.3 Clustering curves

In spatial analysis, it is interesting not only to characterize the morphology of each pattern by one or several indices (Section 3.2), but also to see whether some places look alike and why (historical or geographical circumstances). For instance, we expect settlements which grew up during early periods of industrialization to have different patterns from medieval centers or XXth century new towns. Recent observations seem to confirm such hypotheses (see e.g. Frankhauser, 2004; 2008; Salingaros, 2003; Thomas et al., 2007, 2008 a and b). The aim here is to cluster wards on the basis of the shape of their curve of scaling behavior. In other words, a curve such as that illustrated in Appendix 1(c) should be clustered with all other curves having the “same” shape, that is to say curves with a minimum value at low distances, even if this minimum is placed slightly further farther left or right. This last condition makes the problem trickier than simply the computation of a correlation distance. It was tackled using the *k*-medoid method described in Section 2.3.

Figure 6 shows the value of $J(C)$ for different numbers of clusters (k). $J(C)$ decreases as k increases, so that there is no formal criterion for choosing k. For this paper we chose k = 5, as a compromise between model error and model complexity. The left column in Figure 7 gives the composition of the clusters in terms of curves, while the right column gives one example of each cluster (either the medoid or the area that was visually the most typical).

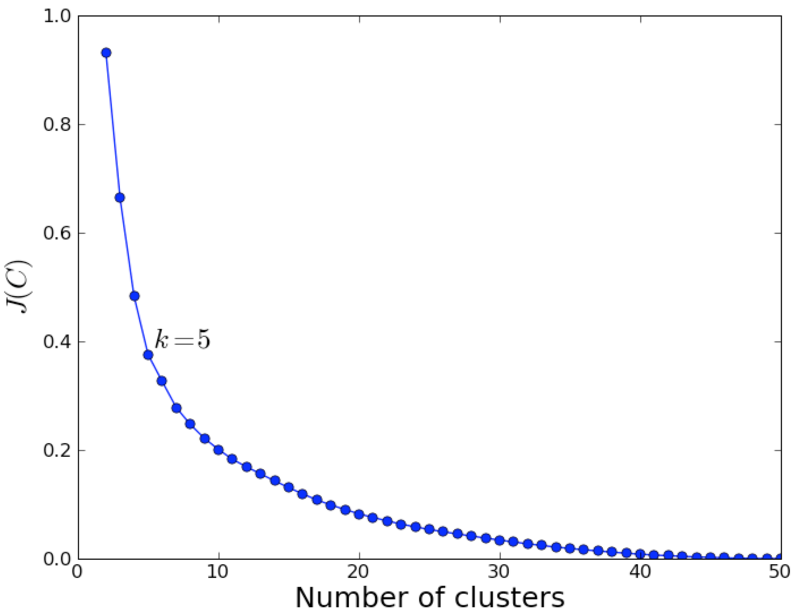
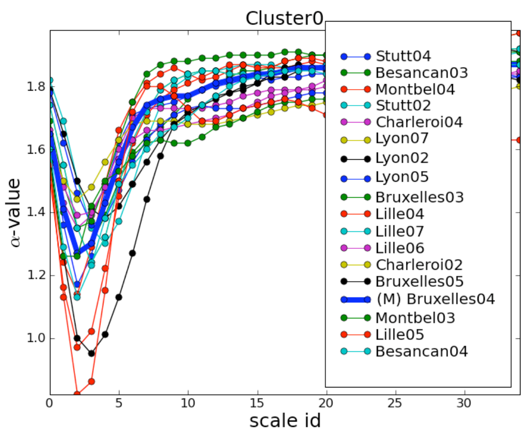
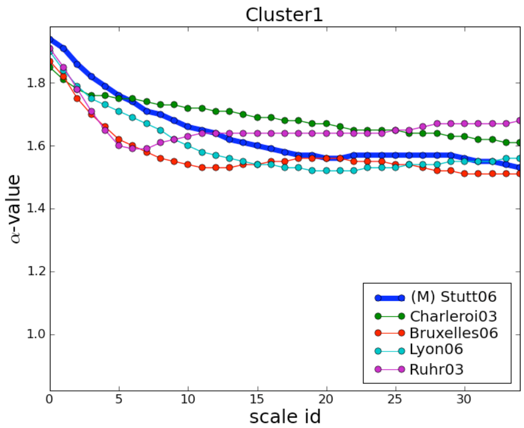
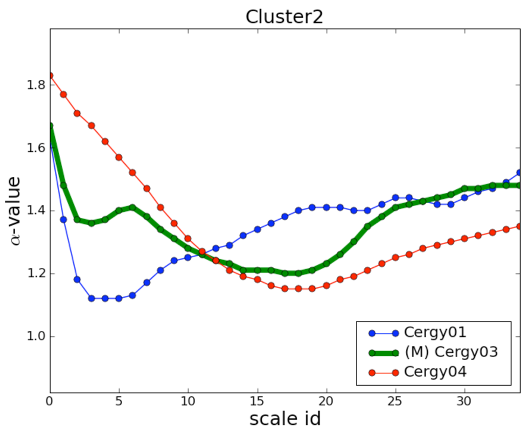
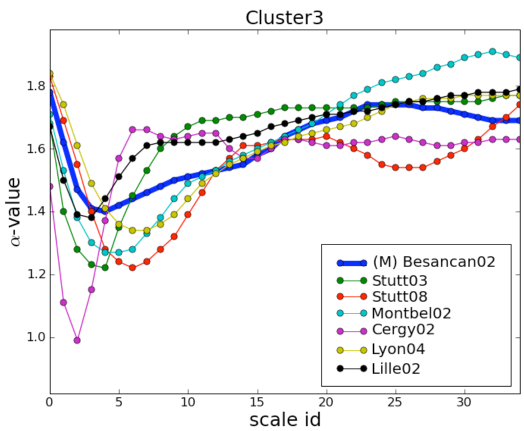


Figure 6: The relationship between $J(C)$ and the number of clusters, k

Cluster	Curves of scaling behavior	An example
0		 *Brussels 04*

1		 Brussels 06
2		 Cergy-Pontoise 01
3		 *Besançon 02*

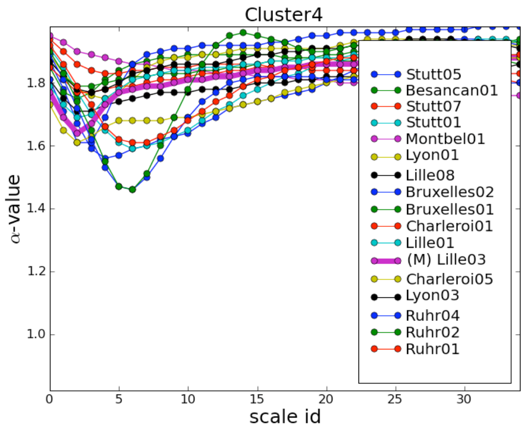
4		 *Lille 03*

Figure 7: Cluster composition when k = 5: curves and one example of each type of urban structure.

Let us now compare the clusters in terms of $J(C\,)$. For better comparison, we will use the square root of the value divided by the number of curves in the cluster, and label this Adj $J(C)$. Adj $J(C)$ indicates the average dissimilarity between a α curve and the medoid of the cluster to which it has been assigned (Table 1).

Cluster 0	0.0878
Cluster 1	0.0668
Cluster 2	0.1136
Cluster 3	0.1026
Cluster 4	0.0773

Table 1: Adj $J(C\,)$ for the five clusters

The highest values of Adj $J(C)$ are observed for Clusters 2 and 3. This means that the curves in these clusters are the most diverse (see also Figure 7), while Clusters 0, 1 and 4 consist of curves that look more alike. Surprisingly, these last 3 groups also have high D values (Figure 8).

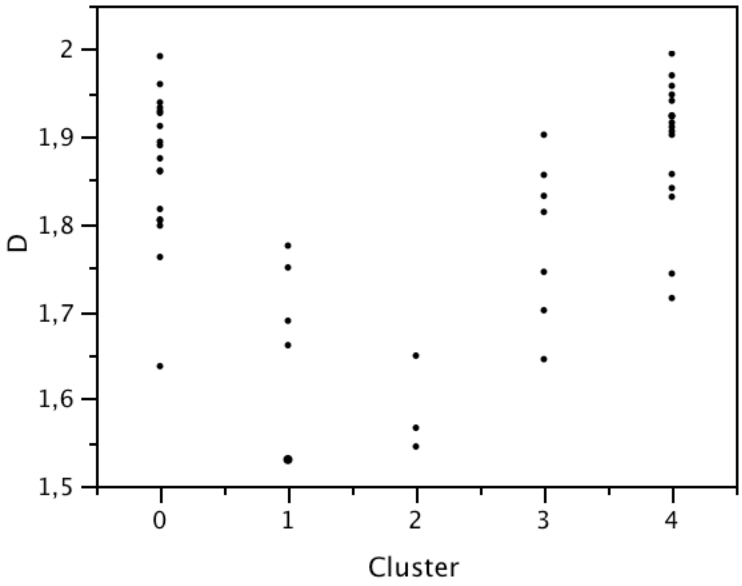


Figure 8: The distribution of the fractal dimensions, *D,* in each cluster.

Clusters 0 and 4 have the highest average D values (D > 1.7) (Figure 8), and these two groups correspond to more or less ”classic” densely urbanized areas. However the shape of the scaling curves is different in Clusters 0 and 4 (see the left part of Figure 7), due to differences in the structure of the built-up areas(right of Figure 7). The analysis of the curves of scaling behavior has thus provided detailed information about the spatial organization of the urban fabric, since the variation in the fractal behavior across scales is taken into account.

By looking in more detail at the features of the wards, we can see that Cluster 0 corresponds to detached houses aligned along roads (regular organization). The distances between the buildings are small, but “white pixels” (open spaces) between the buildings are quite numerous. This explains the substantial drop in the scaling behavior at short distances (Figure 7, left). As pointed out above, parameter α links the relative variation in the built-up areas to that of distance, and in Cluster 0 the relative variation is low at short distances. This type of fabric often characterizes the suburbs of cities. Cluster 4 has similar D values to Cluster 0 (see Figure 8), but a visually different built-up morphology: the curves are much flatter (less variation) and the buildings are more densely packed. They are often terraced. Cluster 4 mostly consists of dwellings in old city centers, mixed with some larger buildings used as offices, schools, shops, etc..

Cluster 2 is heterogeneous in terms of the scaling curves (Table 1). There are only three urban wards in this cluster, and they have atypical scaling curves (Figure 7). All three come from the new town of Cergy-Pontoise in France, which was created in 1969 to manage the development of the Paris Region in terms of habitat, activities, transport, etc. Cergy Pontoise has avoided the both the role of industrial center and that of dormitory town, and has succeeded in maintaining the balance between places of work, culture, and habitation. This has led to a certain diversity in its built-up areas, as illustrated in Figure 7, with a mixture of large apartment blocks (*barres*) and small detached houses (*pavillonnaire*s).

Cluster 3 corresponds to "pure" Corbusian built-up areas. It consists of social housing (apartment blocks), in quite uniform and regular formations. In France these are called *Les Grands Ensembles*.

Last but not least, Cluster 1 consists of areas with buildings covering large irregular areas. These are mainly free-standing industrial or office buildings, where intra-building distances are considerable. As illustrated by the curves in Figure 7, the scaling behavior is large at small distances due to the size of the buildings.

4 Conclusion

This paper has considered scaling behavior curves, an output of fractal analysis. These curves illustrate how the two fractal parameters vary across scales. Distance ranges can be identified at which substantial changes in spatial organization occur, or alternatively, for which the parameters are stable. Hence the information contained in these curves turns out to be complementary to that of the fractal dimension D which remains a useful, but rather general indicator.

An important contribution of this paper is the use of the k-medoid method to cluster the curves of similar shapes. This method is quite similar to that of k-means, but can use dissimilarity measures which are not distances. Here, it allows a dissimilarity to be computed from a matching, which is well-suited to curves with horizontal or vertical shifts. It is useful not only for curves of fractal scaling behavior but also for clustering any other curves in geography (remote sensing, etc.). The application to a set of

European urban wards showed that the clustering results fit well with planning history (areas with similar histories cluster together). Clustering the curves instead of using only the fractal dimension undoubtedly adds accuracy to the final result in terms of morphometry.

Acknowledgements.- (… to be completed after acceptance …) The authors would like to thank the anonymous referees for their helpful comments and suggestions.

References

Batty, M., 2001, "Generating urban forms from diffusive growth" *Environment and Planning A*,23, 511–544.

Batty M., 2005, *Cities and Complexity: Understanding Cities with Cellular Automata, Agent-Based Models, and Fractals*, MIT Press, Cambridge, Massachusetts.

Bishop C., 2006, *Pattern Recognition and Machine Learning, S*pringer.

Conzen M., 2001, "The study of urban form in the United States" *Urban Morphology*, 5:1, 3–14.

De Keersmaecker M.L., Frankhauser P., Thomas I., 2003, "Using fractal dimensions to characterize intra-urban diversity: the example of Brussels" *Geographical Analysis*, 35:4, 310–328.

Frankhauser P., 1998a, "The fractal approach: a new tool for the spatial analysis of urban agglomerations" *Population* 4, 205–240.

Frankhauser P., 1998b, "Fractal geometry of urban patterns and their morphogenesis" *Discrete Dynamics in Nature and Society*, 2, 127–145.

Frankhauser P, 2004, "Comparing the morphology of urban patterns in Europe – a fractal approach" in A. Borsdorf and P. Zembri (eds) *European Cities – Insights on Outskirts*, COST Report, Action 10 Urban Civil Engineering, Vol. 2, Structures, , Brussels, 2004, 79–105.

Frankhauser P., 2008, *Fractal Geometry for Measuring and Modelling Urban Patterns*, in Albeverio S., Andrey D., Giordano P., Vancheri A. (eds), *The Dynamics of Complex Urban Systems: An Interdisciplinary Approach*, Physica-Verlag 213–243.

Gouyet J.F., 1996, *Physique et structures fractals*, Masson, second edition, Paris.

Grassberger P, Procaccia I, 1983, "Measuring the strangeness of a strange attractor" *Physica D*, 9, 189–208.

Halley J., Hartley S., Kallimanis A., Kunin W., Lennon J., Sgardelis S., 2004, "Uses and abuses of fractal methodology in ecology" *Ecology Letters*, 7:3, 254–271.

Lam N., de Cola, L., 2002. *Fractals in Geography*, The Backburn Press, Caldwell, New Jersey.

Leduc A, Prairie Y, Bergeron Y, 1994, "Fractal dimension estimates of a fragmented landscape: sources of variability" *Landscape Ecology*, 9 :4, 279–286.

Levy A., 1999, "Urban morphology and the problem of the modern urban fabric: some questions of research" *Urban Morphology*, 32:2, 79–85.

Lorenz W., 2003, "Fractals and fractal architecture". Available at: http://www.iemar.tuwien.ac.at/fractal_architecture/subpages/01Introduction.html

Palmer M., 1988, "Fractal geometry: a tool for describing spatial patterns of plant communities" *Vegetatio*, 75, 91–102

Salingaros N., 2003, "Connecting the fractal city", Key note speech, Biennial of Towns and Town planners in Europe (Barcelona). Available at http://zeta.math.utsa.edu/~yxk833/connecting.html

Tannier C., Pumain D., 2005, "Fractales et géographie urbaine : aperçu théorique et application pratique" *Cybergeo,* 307, 20pp.

Thomas I., Frankhauser P., De Keersmaecker M.-L. (2007), Fractal dimension versus density of the built-up surfaces in the periphery of Brussels. *Papers in Regional Science*, 86 :2, 287-307.

Thomas I., Frankhauser P., Biernacki C., 2008a, "The morphology of built-up landscapes in Wallonia, Belgium: a classification using fractal indices" *Landscape and Urban Planning* 84, 99–115.

Thomas I., Tannier C., Frankhauser P., 2008b, "Is there a link between fractal dimension and residential choice at a regional level?" *Cybergeo,* 413*.*

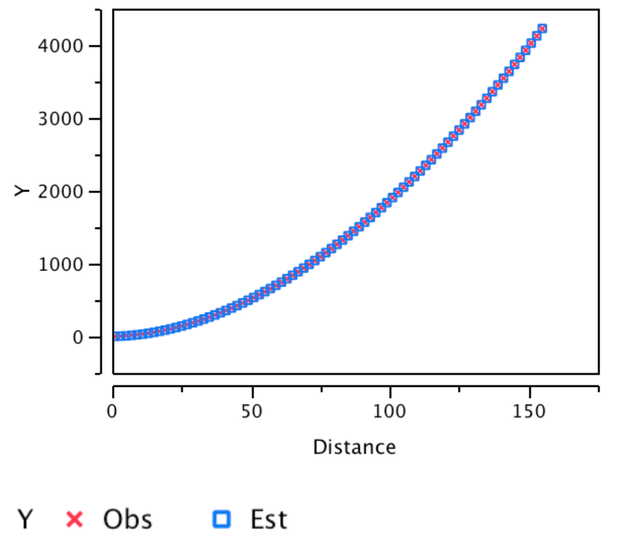
To be added later (names hidden for confidentiality reasons (evaluation) :

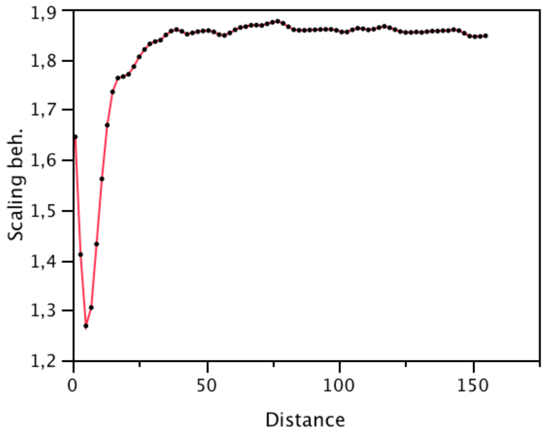
******., submitted, "Comparing the fractality of urban districts: do national processes matter in Europe?" (Paper presently submitted for publication to *Geographical Analysis*)

Appendix 1

Example of a fractal analysis using *Fractalyse* (http://www.fractalyse.org/) on a real world urban area in Brussels (named Bruxelles04) (Belgium)

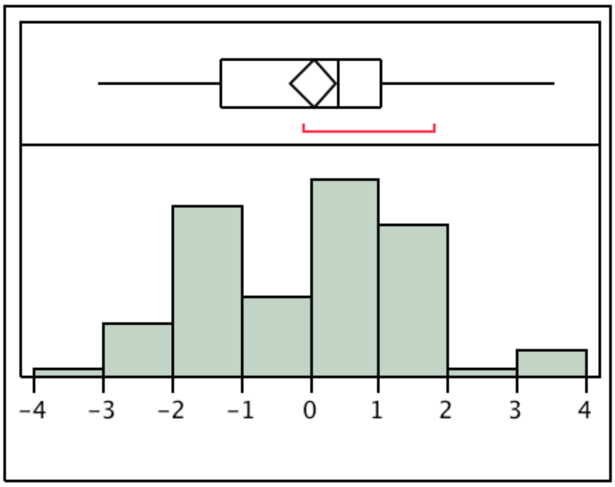
(a) The area





(b) The estimation curve: variation of the observed and estimated values of *D* with distance

(c) The curve of scaling behavior



(d)
Distribution of the difference between the observed and the estimated value (histogram and boxplot)