

\mathcal{H}_2 -optimal model reduction of MIMO systems*

P. VAN DOOREN^{†‡}

K. A. GALLIVAN[§]

P.-A. ABSIL[‡]

Abstract

We consider the problem of approximating a $p \times m$ rational transfer function $H(s)$ of high degree by another $p \times m$ rational transfer function $\hat{H}(s)$ of much smaller degree. We derive the gradients of the \mathcal{H}_2 -norm of the approximation error and show how stationary points can be described via tangential interpolation.

Keyword Multivariable systems, model reduction, optimal \mathcal{H}_2 approximation, tangential interpolation.

1 Introduction

In this paper we will consider the problem of approximating a real $p \times m$ rational transfer function $H(s)$ of McMillan degree N by a real $p \times m$ rational transfer function $\hat{H}(s)$ of lower McMillan degree n using the \mathcal{H}_2 -norm as approximation criterion. Since a transfer function has an unbounded \mathcal{H}_2 -norm if it is not strictly proper (a rational transfer function is strictly proper if it is zero at $s = \infty$), we will constrain both $H(s)$ and $\hat{H}(s)$ to be strictly proper. Such transfer functions have state-space realizations $(A, B, C) \in \mathbb{R}^{N^2} \times \mathbb{R}^{Nm} \times \mathbb{R}^{pN}$ and $(\hat{A}, \hat{B}, \hat{C}) \in \mathbb{R}^{n^2} \times \mathbb{R}^{nm} \times \mathbb{R}^{pn}$ satisfying

$$H(s) := C(sI_N - A)^{-1}B \quad \text{and} \quad \hat{H}(s) := \hat{C}(sI_n - \hat{A})^{-1}\hat{B}. \quad (1)$$

The realization $\{\hat{A}, \hat{B}, \hat{C}\}$ is not unique in the sense that the triple $\{\hat{A}_T, \hat{B}_T, \hat{C}_T\} := \{T^{-1}\hat{A}T, T^{-1}\hat{B}, \hat{C}T\}$ for any matrix $T \in GL(n, \mathbb{R})$ defines the same transfer function :

$$\hat{H}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B} = \hat{C}_T(sI_n - \hat{A}_T)^{-1}\hat{B}_T.$$

It is known (see e.g. Theorem 4.7 in Byrnes and Falb [3]) that the geometric quotient of $\mathbb{R}^{n^2} \times \mathbb{R}^{nm} \times \mathbb{R}^{pn}$ under $GL(n, \mathbb{R})$ is a smooth, irreducible variety of dimension $n(m+p)$. This implies that the set $\text{Rat}_{p,m}^n$ of $p \times m$ strictly proper rational transfer functions of degree n can be parameterized with only $n(m+p)$ real parameters in a locally smooth manner.

A possible approach for building a reduced order model $\{\hat{A}, \hat{B}, \hat{C}\}$ from a full order model $\{A, B, C\}$ is tangential interpolation, which can always be achieved (see [4]) by solving two Sylvester equations for the unknowns $W, V \in \mathbb{R}^{N \times n}$

$$AV - V\Sigma_\sigma + BR = 0, \quad (2)$$

$$W^T A - \Sigma_\mu^T W^T + L^T C = 0, \quad (3)$$

and constructing the reduced order model (of degree n) as follows

$$\{\hat{A}, \hat{B}, \hat{C}\} := \{(W^T V)^{-1}W^T AV, (W^T V)^{-1}W^T B, CV\}, \quad (4)$$

[†]Corresponding author. *E-mail address:* paul.vandooren@uclouvain.be

[‡]CESAME, Université catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium

[§]School of Computational Science, Florida State University, Tallahassee FL 32306, USA

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provided the matrix $W^T V$ is invertible (which also implies that V and W must have full rank n). The “interpolation conditions” $\{\Sigma_\sigma, R\}$ and $\{\Sigma_\mu, L\}$ (where $\Sigma_\mu, \Sigma_\sigma \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times n}$ and $L \in \mathbb{R}^{p \times n}$) are known to uniquely determine the projected system $\{\hat{A}, \hat{B}, \hat{C}\}$ [4]. The equations above can be expressed in another coordinate system by applying invertible transformations of the type $\{Q^{-1}\Sigma_\sigma Q, RQ\}$ and $\{P^{-1}\Sigma_\mu P, LP\}$ to the interpolation conditions, which yields transformed matrices VP and WQ but does not affect the transfer function of the reduced order model $\{\hat{A}, \hat{B}, \hat{C}\}$ (see [4]). Therefore, the interpolation conditions essentially impose $n(m+p)$ real conditions, since Σ_σ and Σ_μ can be transformed to their Jordan canonical form. In the case that both matrices are simple (no Jordan blocks of size larger than 1) we can assume Σ_σ and Σ_μ to be block diagonal with a 1×1 diagonal block σ_i or μ_i for each real condition and a 2×2 diagonal block $\begin{bmatrix} \sigma_i & \sigma_{i+1} \\ -\sigma_{i+1} & \sigma_i \end{bmatrix}$ or $\begin{bmatrix} \mu_i & \mu_{i+1} \\ -\mu_{i+1} & \mu_i \end{bmatrix}$ for each pair of complex conjugate conditions. We refer to [1] for a more elaborate discussion on this and for a discrete-time version of the results of this paper.

In this paper we first compute the gradients of the \mathcal{H}_2 error of the approximation problem and then show that its stationary points satisfy special tangential interpolation conditions that generalize earlier results for SISO systems and help understand numerical algorithms to solve this model reduction problem.

2 The \mathcal{H}_2 approximation problem

Let $E(s)$ be an arbitrary strictly proper transfer function, with realization triple $\{A_e, B_e, C_e\}$. If $E(s)$ is unstable, its \mathcal{H}_2 -norm is defined to be ∞ . Otherwise, its squared \mathcal{H}_2 -norm is defined as the trace of a matrix integral [2] :

$$\mathcal{J} := \|E(s)\|_{\mathcal{H}_2}^2 := \text{tr} \int_{-\infty}^{\infty} E(j\omega)^H E(j\omega) \frac{d\omega}{2\pi} = \text{tr} \int_{-\infty}^{\infty} E(j\omega) E(j\omega)^H \frac{d\omega}{2\pi}.$$

By Parseval’s identity, this can also be expressed using the state space realization as (see [2])

$$\mathcal{J} = \text{tr} \int_0^{\infty} [C_e \exp^{A_e t} B_e] [C_e \exp^{A_e t} B_e]^T dt = \text{tr} \int_0^{\infty} [C_e \exp^{A_e t} B_e]^T [C_e \exp^{A_e t} B_e] dt.$$

This can also be related to an expression involving the Gramians P_e and Q_e defined as

$$P_e := \int_0^{\infty} [\exp^{A_e t} B_e] [\exp^{A_e t} B_e]^T dt, \quad Q_e := \int_0^{\infty} [C_e \exp^{A_e t}]^T [C_e \exp^{A_e t}] dt,$$

which are also known to be the solutions of the Lyapunov equations

$$A_e P_e + P_e A_e^T + B_e B_e^T = 0, \quad Q_e A_e + A_e^T Q_e + C_e^T C_e = 0. \quad (5)$$

Using these, it easily follows that the squared \mathcal{H}_2 -norm of $E(s)$ can also be expressed as

$$\mathcal{J} = \text{tr} B_e^T Q_e B_e = \text{tr} C_e P_e C_e^T. \quad (6)$$

We now apply this to the error function $E(s) := H(s) - \hat{H}(s)$. A realization of $E(s)$ in partitioned form is given by

$$\{A_e, B_e, C_e\} := \left\{ \begin{bmatrix} A & \\ & \hat{A} \end{bmatrix}, \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \begin{bmatrix} C & -\hat{C} \end{bmatrix} \right\},$$

and the Lyapunov equations (5) become

$$P_e := \begin{bmatrix} P & X \\ X^T & \hat{P} \end{bmatrix}, \quad \begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} \begin{bmatrix} P & X \\ X^T & \hat{P} \end{bmatrix} + \begin{bmatrix} P & X \\ X^T & \hat{P} \end{bmatrix} \begin{bmatrix} A^T & \\ & \hat{A}^T \end{bmatrix} + \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \begin{bmatrix} B^T & \hat{B}^T \end{bmatrix} = 0, \quad (7)$$

and

$$Q_e := \begin{bmatrix} Q & Y \\ Y^T & \hat{Q} \end{bmatrix}, \quad \begin{bmatrix} A^T & \\ & \hat{A}^T \end{bmatrix} \begin{bmatrix} Q & Y \\ Y^T & \hat{Q} \end{bmatrix} + \begin{bmatrix} Q & Y \\ Y^T & \hat{Q} \end{bmatrix} \begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} + \begin{bmatrix} C^T \\ -\hat{C}^T \end{bmatrix} \begin{bmatrix} C & -\hat{C} \end{bmatrix} = 0. \quad (8)$$

To minimize the \mathcal{H}_2 -norm, \mathcal{J} , of the error function $E(s)$ we must minimize

$$\mathcal{J} = \text{tr} \left(\begin{bmatrix} B^T & \widehat{B}^T \end{bmatrix} \begin{bmatrix} Q & Y \\ Y^T & \widehat{Q} \end{bmatrix} \begin{bmatrix} B \\ \widehat{B} \end{bmatrix} \right) = \text{tr} \left(B^T Q B + 2B^T Y \widehat{B} + \widehat{B}^T \widehat{Q} \widehat{B} \right), \quad (9)$$

where Q , Y and \widehat{Q} depend on A , \widehat{A} , C and \widehat{C} through the Lyapunov equation (8), or equivalently

$$\mathcal{J} = \text{tr} \left(\begin{bmatrix} C & -\widehat{C} \end{bmatrix} \begin{bmatrix} P & X \\ X^T & \widehat{P} \end{bmatrix} \begin{bmatrix} C^T \\ -\widehat{C}^T \end{bmatrix} \right) = \text{tr} \left(C P C^T - 2C X \widehat{C}^T + \widehat{C} \widehat{P} \widehat{C}^T \right), \quad (10)$$

where P , X and \widehat{P} depend on A , \widehat{A} , B and \widehat{B} through the Lyapunov equation (7). Note that the terms $B^T Q B$ and $C P C^T$ in the above expressions are constant, and hence can be discarded in the optimization.

3 Optimality conditions

The expansions above can be used to express first order optimality conditions for the squared \mathcal{H}_2 -norm in terms of the gradients of \mathcal{J} versus \widehat{A} , \widehat{B} and \widehat{C} . We define a gradient as follows.

Definition 3.1 *The gradient of a smooth real scalar function $f(X)$ of a real matrix variable $X \in \mathbb{R}^{n \times p}$ is the real matrix $\nabla_X f(X) \in \mathbb{R}^{n \times p}$ defined by*

$$[\nabla_X f(X)]_{i,j} = \frac{d}{dX_{i,j}} f(X), \quad i = 1, \dots, n, \quad j = 1, \dots, p.$$

It yields the expansion

$$f(X + \Delta) = f(X) + \langle \nabla_X f(X), \Delta \rangle + O(\|\Delta\|^2), \quad \text{where} \quad \langle M, N \rangle := \text{tr}(M^T N).$$

The following lemma (see [7]) is useful in the derivation of our results.

Lemma 3.2 *If $AM + MB + C = 0$ and $NA + BN + D = 0$, then $\text{tr}(CN) = \text{tr}(DM)$.*

Starting from the characterizations (7,10) and (8,9) of the \mathcal{H}_2 norm and using Lemma 3.2 we easily derive succinct forms of the gradients. This theorem is originally due to Wilson [8].

Theorem 3.3 *The gradients $\nabla_{\widehat{A}} \mathcal{J}$, $\nabla_{\widehat{B}} \mathcal{J}$ and $\nabla_{\widehat{C}} \mathcal{J}$ of $\mathcal{J} := \|E(s)\|_{\mathcal{H}_2}^2$ are given by*

$$\nabla_{\widehat{A}} \mathcal{J} = 2(\widehat{Q} \widehat{P} + Y^T X), \quad \nabla_{\widehat{B}} \mathcal{J} = 2(\widehat{Q} \widehat{B} + Y^T B), \quad \nabla_{\widehat{C}} \mathcal{J} = 2(\widehat{C} \widehat{P} - CX), \quad (11)$$

where

$$A^T Y + Y \widehat{A} - C^T \widehat{C} = 0, \quad \widehat{A}^T \widehat{Q} + \widehat{Q} \widehat{A} + \widehat{C}^T \widehat{C} = 0, \quad (12)$$

$$X^T A^T + \widehat{A} X^T + \widehat{B} B^T = 0, \quad \widehat{P} \widehat{A}^T + \widehat{A} \widehat{P} + \widehat{B} \widehat{B}^T = 0. \quad (13)$$

Proof. For finding an expression for $\nabla_{\widehat{A}} \mathcal{J}$ we consider the characterization

$$\mathcal{J} = \text{tr} \left(B^T Q B + 2B^T Y \widehat{B} + \widehat{B}^T \widehat{Q} \widehat{B} \right), \quad A^T Y + Y \widehat{A} - C^T \widehat{C} = 0, \quad \widehat{A}^T \widehat{Q} + \widehat{Q} \widehat{A} + \widehat{C}^T \widehat{C} = 0.$$

Then the first order perturbation $\Delta_{\mathcal{J}}$ corresponding to $\Delta_{\widehat{A}}$ is given by

$$\Delta_{\mathcal{J}} = \text{tr} \left(2\widehat{B} B^T \Delta_Y + \widehat{B} \widehat{B}^T \Delta_{\widehat{Q}} \right)$$

where Δ_Y and $\Delta_{\widehat{Q}}$ depend on $\Delta_{\widehat{A}}$ via the equations

$$A^T \Delta_Y + \Delta_Y \widehat{A} + Y \Delta_{\widehat{A}} = 0, \quad \widehat{A}^T \Delta_{\widehat{Q}} + \Delta_{\widehat{Q}} \widehat{A} + \Delta_{\widehat{A}}^T \widehat{Q} + \widehat{Q} \Delta_{\widehat{A}} = 0. \quad (14)$$

It follows from applying Lemma 3.2 to the Sylvester equations (13,14) that

$$\text{tr}\left(\widehat{B}B^T\Delta_Y\right) = \text{tr}\left(X^TY\Delta_{\widehat{A}}\right) \quad \text{and} \quad \text{tr}\left(\widehat{B}\widehat{B}^T\Delta_{\widehat{Q}}\right) = \text{tr}\left(\widehat{P}(\Delta_{\widehat{A}}^T\widehat{Q} + \widehat{Q}\Delta_{\widehat{A}})\right)$$

and therefore

$$\Delta_{\mathcal{J}} = \text{tr}\left(2X^TY\Delta_{\widehat{A}} + \widehat{P}(\Delta_{\widehat{A}}^T\widehat{Q} + \widehat{Q}\Delta_{\widehat{A}})\right) = \text{tr}\left(2X^TY\Delta_{\widehat{A}} + 2\widehat{P}\widehat{Q}\Delta_{\widehat{A}}\right) = \langle 2(\widehat{Q}\widehat{P} + Y^TX), \Delta_{\widehat{A}} \rangle.$$

Since $\Delta_{\mathcal{J}}$ also equals $\langle \nabla_{\widehat{A}}\mathcal{J}, \Delta_{\widehat{A}} \rangle$, it follows that $\nabla_{\widehat{A}}\mathcal{J} = 2(\widehat{Q}\widehat{P} + Y^TX)$.

To find an expression for $\nabla_{\widehat{B}}\mathcal{J}$ we perturb \widehat{B} in the characterization

$$\mathcal{J} = \text{tr}\left(B^TQB + 2B^TY\widehat{B} + \widehat{B}^T\widehat{Q}\widehat{B}\right).$$

which yields the first order perturbation

$$\Delta_{\mathcal{J}} = \text{tr}\left(2B^TY\Delta_{\widehat{B}} + \Delta_{\widehat{B}}^T\widehat{Q}\widehat{B} + \widehat{B}^T\widehat{Q}\Delta_{\widehat{B}}\right) = \langle 2(Y^TB + \widehat{Q}\widehat{B}), \Delta_{\widehat{B}} \rangle.$$

Since $\Delta_{\mathcal{J}}$ also equals $\langle \nabla_{\widehat{B}}\mathcal{J}, \Delta_{\widehat{B}} \rangle$, it follows that $\nabla_{\widehat{B}}\mathcal{J} = 2(\widehat{Q}\widehat{B} + Y^TB)$.

In a similar fashion we can write the first order perturbation of

$$\mathcal{J} = \text{tr}\left(CPC^T - 2CX\widehat{C}^T + \widehat{C}\widehat{P}\widehat{C}^T\right)$$

to obtain $\nabla_{\widehat{C}}\mathcal{J} = 2(\widehat{C}\widehat{P} - CX)$. □

The gradient forms of Theorem 3.3 allow us to derive our fundamental theoretical result.

Theorem 3.4 *At every stationary point of \mathcal{J} where \widehat{P} and \widehat{Q} are invertible, we have the following identities*

$$\widehat{A} = W^TAV, \quad \widehat{B} = W^TB, \quad \widehat{C} = CV, \quad W^TV = I_n \quad \text{with} \quad W := -Y\widehat{Q}^{-1}, \quad V := X\widehat{P}^{-1} \quad (15)$$

where X, Y, \widehat{P} and \widehat{Q} satisfy the Sylvester equations (12,13).

Proof. Since we are at a stationary point of \mathcal{J} , the gradients versus \widehat{A}, \widehat{B} and \widehat{C} must be zero :

$$\widehat{Q}\widehat{P} + Y^TX = 0, \quad \widehat{Q}\widehat{B} + Y^TB = 0, \quad \widehat{C}\widehat{P} - CX = 0.$$

Since \widehat{P} and \widehat{Q} are invertible, we can define $W := -Y\widehat{Q}^{-1}$ and $V := X\widehat{P}^{-1}$. It then follows that

$$W^TV = I_n, \quad \widehat{B} = W^TB, \quad \widehat{C} = CV.$$

Multiplying the first equation of (13) with W and using $X^T = \widehat{P}V^T$, yields

$$\widehat{P}V^TA^TW + \widehat{A}\widehat{P}V^TW + \widehat{B}B^TW = 0.$$

Using $V^TW = I$, $B^TW = \widehat{B}^T$ and the second equation of (13) it then follows that $\widehat{A} = W^TAV$. □

If we rewrite the above theorem as a projection problem, then we are constructing a projector $\Pi := VW^T$ (implying $W^TV = I_n$) where V and W are given by the following (transposed) Sylvester equations

$$(\widehat{Q}W^T)A + \widehat{A}^T(\widehat{Q}W^T) + \widehat{C}^TC = 0, \quad A(V\widehat{P}) + (V\widehat{P})\widehat{A}^T + B\widehat{B}^T = 0. \quad (16)$$

Notice that \widehat{P} and \widehat{Q} can be interpreted as normalizations to ensure that $W^TV = I_n$.

It was shown in [4] that projecting a system via Sylvester equations always amounts to satisfying tangential interpolation conditions. The Sylvester equations (16) show that the parameters of reduced order models corresponding to stationary points must have specific relationships with the parameters of the tangential interpolation conditions (2,3,4). First note that $-\hat{A} = \Sigma_\sigma = \Sigma_\mu$ requires that the left and right interpolation points are identical and equal to the negatives of the poles of the reduced order model. For SISO systems, choosing identical left and right interpolation point sets implies that $\hat{H}(s)$ and $H(s)$ and, at least, their first derivatives match at the interpolation points. Theorem 3.4 therefore generalizes to MIMO systems the conditions of [6] on the \mathcal{H}_2 -norm stationary points for SISO systems. The simple additive result for the orders of rational interpolation for SISO systems, however, is replaced by more complicated tangential conditions for MIMO systems that require the definition of tangential direction vectors that can be vector polynomials of s . The Sylvester equations (16) show that these direction vectors are also related to parameters of realizations of $\hat{H}(s)$. If the Sylvester equations are expressed in the coordinate system with \hat{A} in Jordan form then the transformed \hat{B} and \hat{C} contain the parameters that define the tangential interpolation directions.

4 Tangential interpolation revisited

Theorem 3.4 provides the fundamental characterization of the stationary points of \mathcal{J} via tangential interpolation conditions and their relationship to the realizations of $\hat{H}(s)$. It is instructive to illustrate those relationships in a particular coordinate system and derive an explicit form of the tangential interpolation conditions. We assume here that all poles of $\hat{H}(s)$ are distinct but possibly complex (the so-called generic case). Hence the transfer functions $H(s)$ and $\hat{H}(s)$ have real realizations $\{A, B, C\}$ and $\{\hat{A}, \hat{B}, \hat{C}\}$ with \hat{A} diagonalizable. The interpretation of these conditions for multiple poles or higher order poles becomes more involved and can be found in an extended version of this paper [1].

Given our assumptions, we have for $\hat{H}(s)$ the partial fraction expansion

$$\hat{H}(s) = \sum_{i=1}^n \frac{\hat{c}_i \hat{b}_i^H}{s - \hat{\lambda}_i}, \quad (17)$$

where $\hat{b}_i \in \mathbb{C}^m$ and $\hat{c}_i \in \mathbb{C}^p$ and where $(\hat{\lambda}_i, \hat{b}_i, \hat{c}_i), i = 1, \dots, n$ is a self-conjugate set.

We must keep in mind that the number of parameters in $\{\hat{A}, \hat{B}, \hat{C}\}$ is not minimal and hence that the gradient conditions of Theorem 3.3 must be redundant. We make this more explicit in the theorem below. For this we will need s_i, t_i^H , the (complex) left and right eigenvectors of the (real) matrix \hat{A} corresponding to the (complex) eigenvalue $\hat{\lambda}_i$. Because of the expansion (17), we then have :

$$\hat{A}s_i = \hat{\lambda}_i s_i, \quad \hat{C}s_i = \hat{c}_i, \quad t_i^H \hat{A} = \hat{\lambda}_i t_i^H, \quad t_i^H \hat{B} = \hat{b}_i^H.$$

Theorem 4.1 *Let $\hat{H}(s) = \sum_{i=1}^n \hat{c}_i \hat{b}_i^H / (s - \hat{\lambda}_i)$ have distinct first order poles where $(\hat{\lambda}_i, \hat{b}_i, \hat{c}_i), i = 1, \dots, n$ is self-conjugate. Then*

$$\frac{1}{2}(\nabla_{\hat{B}} \mathcal{J})^T s_i = [H^T(-\hat{\lambda}_i) - \hat{H}^T(-\hat{\lambda}_i)] \hat{c}_i \quad (18)$$

$$\frac{1}{2} t_i^H (\nabla_{\hat{C}} \mathcal{J})^T = \hat{b}_i^H [H^T(-\hat{\lambda}_i) - \hat{H}^T(-\hat{\lambda}_i)] \quad (19)$$

$$\frac{1}{2} t_i^H (\nabla_{\hat{A}} \mathcal{J})^T s_i = \hat{b}_i^H \frac{d}{ds} [H^T(s) - \hat{H}^T(s)] \Big|_{s=-\hat{\lambda}_i} \hat{c}_i \quad (20)$$

$$\frac{1}{2} t_i^H (\nabla_{\hat{A}} \mathcal{J})^T s_j = \frac{1}{2(\hat{\lambda}_i - \hat{\lambda}_j)} [\hat{b}_i^H (\nabla_{\hat{B}} \mathcal{J})^T s_j - t_i^H (\nabla_{\hat{C}} \mathcal{J})^T \hat{c}_j] \quad (21)$$

Proof. Define $y_i := Y s_i$, $\hat{q}_i := -\hat{Q} s_i$, $x_i := -X t_i$ and $\hat{p}_i := -\hat{P} t_i$. Then from (12,13) we have

$$\begin{aligned} (A^T + \hat{\lambda}_i I) y_i &= C^T \hat{c}_i, & (\hat{A}^T + \hat{\lambda}_i I) \hat{q}_i &= \hat{C}^T \hat{c}_i, \\ x_i^H (A^T + \hat{\lambda}_i I) &= \hat{b}_i^H B^T, & \hat{p}_i^H (\hat{A}^T + \hat{\lambda}_i I) &= \hat{b}_i^H \hat{B}^T. \end{aligned}$$

It follows that

$$y_i = (A^T + \hat{\lambda}_i I)^{-1} C^T \hat{c}_i, \quad \hat{q}_i = (\hat{A}^T + \hat{\lambda}_i I)^{-1} \hat{C}^T \hat{c}_i, \quad (22)$$

$$x_i^H = \hat{b}_i^H B^T (A^T + \hat{\lambda}_i I)^{-1}, \quad \hat{p}_i^H = \hat{b}_i^H \hat{B}^T (\hat{A}^T + \hat{\lambda}_i I)^{-1}, \quad (23)$$

from which we obtain

$$\frac{1}{2} (\nabla_{\hat{B}} \mathcal{J})^T s_i = (\hat{B}^T \hat{Q} + B^T Y) s_i = [H^T(-\hat{\lambda}_i) - \hat{H}^T(-\hat{\lambda}_i)] \hat{c}_i,$$

$$\frac{1}{2} t_i^H (\nabla_{\hat{C}} \mathcal{J})^T = t_i^H (\hat{P} \hat{C}^T - X^T C^T) = \hat{b}_i^H [H^T(-\hat{\lambda}_i) - \hat{H}^T(-\hat{\lambda}_i)].$$

From the (22,23) it also follows that

$$\frac{1}{2} t_i^H (\nabla_{\hat{A}} \mathcal{J})^T s_j = t_i^H (\hat{P} \hat{Q} + X^T Y) s_j = \hat{b}_i^H [\hat{B}^T (\hat{A}^T + \hat{\lambda}_i I)^{-1} (\hat{A}^T + \hat{\lambda}_j I)^{-1} \hat{C}^T - B^T (A^T + \hat{\lambda}_i I)^{-1} (A^T + \hat{\lambda}_j I)^{-1} C^T] \hat{c}_i.$$

If we use $\frac{d}{ds} H(s) = -C(sI - A)^{-2} B$ and $\frac{d}{ds} \hat{H}(s) = -\hat{C}(sI - \hat{A})^{-2} \hat{B}$, then for $i = j$ we obtain

$$\frac{1}{2} t_i^H (\nabla_{\hat{A}} \mathcal{J})^T s_i = \hat{b}_i^H \frac{d}{ds} [H^T(s) - \hat{H}^T(s)] \Big|_{s=-\hat{\lambda}_i} \hat{c}_i.$$

For $i \neq j$ we use the identity

$$(M + \hat{\lambda}_j I)^{-1} (M + \hat{\lambda}_i I)^{-1} = \frac{1}{\hat{\lambda}_i - \hat{\lambda}_j} [(M + \hat{\lambda}_j I)^{-1} - (M + \hat{\lambda}_i I)^{-1}]$$

to obtain

$$\frac{1}{2} t_i^H (\nabla_{\hat{A}} \mathcal{J})^T s_j = \frac{1}{\hat{\lambda}_i - \hat{\lambda}_j} t_i^H \left([H^T(-\hat{\lambda}_i) - \hat{H}^T(-\hat{\lambda}_i)] - [H^T(-\hat{\lambda}_j) - \hat{H}^T(-\hat{\lambda}_j)] \right) s_j$$

and finally

$$\frac{1}{2} t_i^H (\nabla_{\hat{A}} \mathcal{J})^T s_j = \frac{1}{2(\hat{\lambda}_i - \hat{\lambda}_j)} [\hat{b}_i^H (\nabla_{\hat{B}} \mathcal{J})^T s_j - t_i^H (\nabla_{\hat{C}} \mathcal{J})^T \hat{c}_j].$$

□

Let $S := [s_1 \ \dots \ s_n]$, then the above theorem shows that the off-diagonal elements of $S^{-1} (\nabla_{\hat{A}} \mathcal{J})^T S$ vanish when $(\nabla_{\hat{B}} \mathcal{J})^T$ and $(\nabla_{\hat{C}} \mathcal{J})^T$ vanish. Therefore we need to impose only conditions on $\text{diag} S^{-1} (\nabla_{\hat{A}} \mathcal{J})^T S$, on $(\nabla_{\hat{B}} \mathcal{J})^T$ and on $(\nabla_{\hat{C}} \mathcal{J})^T$ to characterize stationary points of \mathcal{J} . These are exactly $n(m+p)$ conditions since the vectors \hat{b}_i^H or \hat{c}_i can be scaled as indicated in Section 1. Moreover one can view them as $n(m+p)$ real conditions since the poles $\hat{\lambda}_i$ come in complex conjugate pairs. The following corollary easily follows.

Corollary 4.2 *If $(\nabla_{\hat{B}} \mathcal{J})^T = 0$, $(\nabla_{\hat{C}} \mathcal{J})^T = 0$ and $\text{diag} S^{-1} (\nabla_{\hat{A}} \mathcal{J})^T S = 0$ then $\nabla_{\hat{A}} \mathcal{J} = 0$ and the following tangential interpolation conditions are satisfied for all $\hat{\lambda}_i, i = 1, \dots, n$:*

$$[H^T(-\hat{\lambda}_i) - \hat{H}^T(-\hat{\lambda}_i)] \hat{c}_i = 0, \quad \hat{b}_i^H [H^T(-\hat{\lambda}_i) - \hat{H}^T(-\hat{\lambda}_i)] = 0, \quad \hat{b}_i^H \frac{d}{ds} [H^T(s) - \hat{H}^T(s)] \Big|_{s=-\hat{\lambda}_i} \hat{c}_i = 0. \quad (24)$$

Notice that we retrieve the conditions of [6] for the SISO case since then \hat{b}_i^H and \hat{c}_i are just nonzero scalars that can be divided out. The conditions above then become the familiar $2n$ interpolation conditions

$$H(-\hat{\lambda}_i) = \hat{H}(-\hat{\lambda}_i), \quad \frac{d}{ds} H(s) \Big|_{s=-\hat{\lambda}_i} = \frac{d}{ds} \hat{H}(s) \Big|_{s=-\hat{\lambda}_i}, \quad i = 1, \dots, n.$$

5 Concluding remarks

The \mathcal{H}_2 norm of a stable strictly proper transfer function $E(s)$ is a smooth function of the parameters $\{A_e, B_e, C_e\}$ of its state-space realization because the squared norm of $E(s)$ is differentiable versus the parameters $\{A_e, B_e, C_e\}$ as long as A_e is stable (the Lyapunov equations are then invertible linear maps and the trace is a smooth function of its parameters). If $\hat{H}(s)$ is an isolated local minimum of the error function $\|H(s) - \hat{H}(s)\|_{\mathcal{H}_2}^2$, then the continuity of the norm implies that a small perturbation of $H(s)$ will induce also only a small perturbation of that local minimum. This explains why we can construct a characterization of the optimality conditions without assuming anything about the structure of the poles of the transfer functions $H(s)$ and $\hat{H}(s)$.

Those ideas also lead to algorithms. One can view (12,13) and (15) as two coupled equations

$$(X, Y, \hat{P}, \hat{Q}) = F(\hat{A}, \hat{B}, \hat{C}) \quad \text{and} \quad (\hat{A}, \hat{B}, \hat{C}) = G(X, Y, \hat{P}, \hat{Q})$$

for which we have a fixed point $(\hat{A}, \hat{B}, \hat{C}) = G(F(\hat{A}, \hat{B}, \hat{C}))$ at every stationary point of $\mathcal{J}(\hat{A}, \hat{B}, \hat{C})$. This automatically suggests an iterative procedure

$$(X, Y, \hat{P}, \hat{Q})_{i+1} = F(\hat{A}, \hat{B}, \hat{C})_{i+1}, \quad (\hat{A}, \hat{B}, \hat{C})_{i+1} = G(X, Y, \hat{P}, \hat{Q})_i,$$

which is expected to converge to a nearby fixed point. This is essentially the idea behind existing algorithms using Sylvester equations in their iterations (see [2]). Another approach would be to use the gradients (or the interpolation conditions of Theorem 4.1) to develop descent methods or even Newton-like methods, as was done for the SISO case in [5].

The two fundamental contributions of this paper are, first, the characterization of the stationary points of \mathcal{J} via tangential interpolation conditions and their relationship to the realizations of $\hat{H}(s)$ given by Theorem 3.4, and, second, the fact that this can be done using Sylvester equations without assuming anything about the structure of either $H(s)$ or $\hat{H}(s)$ thereby providing a framework to relate existing algorithms and to develop and understand new ones. During the submission of this paper we were also informed that similar results for discrete-time systems had been obtained independently in [9].

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