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A PENCIL APPROACH FOR EMBEDDING A POLYNOMIAL MATRIX INTO A UNIMODULAR MATRIX*

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Abstract. In this paper a new method for constructing the unimodular embedding of a polynomial matrix $P(\lambda)$ is derived. As proposed by Eising, the problem can be transformed to one of embedding a pencil, derived from the polynomial matrix $P(\lambda)$. The actual embedding of the pencil is performed here via the staircase form of this pencil, which shortcuts Eising's construction. This then leads to a *new, fast, and numerically reliable* algorithm for embedding a polynomial matrix. The new method uses a fast variant of the staircase algorithm and only requires $O(p^3)$ operations in contrast to the $O(p^4)$ methods proposed up to now (where p is the largest dimension of the pencil). At the same time we also treat the connected problem of finding the (right) null space and (right) inverse of a polynomial matrix $P(\lambda)$.

Key words. polynomial matrix, unimodular matrix, staircase form

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1. Introduction. Let $P(\lambda)$ be an $m \times n$ (with $m < n$) polynomial matrix of degree d:

$$P(\lambda) \doteq P_0 + P_1\lambda + P_2\lambda^2 + \cdots + P_d\lambda^d \tag{1}$$

where each P_i is a real or complex $m \times n$ matrix. In this paper we develop a new algorithm to construct an embedding of this polynomial matrix into a unimodular one, i.e., to find a second polynomial matrix $Q(\lambda)$ of dimension $(n-m) \times n$:

$$Q(\lambda) \doteq Q_0 + Q_1\lambda + \cdots + Q_{d_q}\lambda^{d_q} \tag{2}$$

such that the compound matrix

$$U(\lambda) \doteq \begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix} \tag{3}$$

is unimodular.

Since a unimodular matrix is by definition invertible for all $\lambda \in \mathbf{C}$ (where $\mathbf{C}$ is the *finite* complex plane), the submatrix $P(\lambda)$ must necessarily have full row rank m for all $\lambda \in \mathbf{C}$ in order for a solution of the embedding problem to exist. It turns out that this is also a sufficient condition for a solution to exist and that, moreover, there always exists a solution $Q(\lambda)$ of degree $d_q \leqq d - 1$ [4]. (Here we assumed that $d \geqq 1$ since otherwise the problem degenerates into one involving constant matrices only and becomes trivial.) Although this result was known it is nice to see how easily it is also derived from our algorithmic construction.

The constructive method developed in this paper is then shown to be easily extended to one that also provides the right inverse of $P(\lambda)$, i.e., an $n \times m$ polynomial matrix $M(\lambda)$ satisfying

$$P(\lambda) \cdot M(\lambda) = I_m \tag{4}$$

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and the right null space of $P(\lambda)$, i.e., an $n \times (n-m)$ polynomial matrix $N(\lambda)$ of full column rank and satisfying

$$P(\lambda) \cdot N(\lambda) = 0_{m,n-m}. \tag{5}$$

Our method reformulates the embedding problem of $P(\lambda)$ as an embedding of a *pencil*, an idea which was, e.g., used by Eising [2]. After recalling this in § 2, we show in the next section how the embedding problem for pencils can be trivially solved via the staircase algorithm [10] of pencils. In § 4, we then use these ideas to provide algorithms for solving the related equations for the right inverse and right null space of $P(\lambda)$. Finally we conclude in § 5 with some considerations of complexity and numerical stability of our method and with some numerical examples.

2. Reduction to a pencil problem. The idea developed here is borrowed from Eising [2]. Consider the $dm \times \{(d-1)m+n\}$ pencil $\lambda B - A$ where the matrices B and A are defined as

$$B \doteq \begin{bmatrix} 0 & & & -P_d \\ I_m & \ddots & & -P_{d-1} \\ & \ddots & 0 & \vdots \\ & & & -P_2 \\ & & I_m & -P_1 \end{bmatrix}, \qquad A \doteq \begin{bmatrix} I_m & & & \\ & \ddots & & \\ & & I_m & \\ & & & P_0 \end{bmatrix}. \tag{6}$$

We first show that the pencil $\lambda B - A$ has full row rank for all $\lambda \in \mathbf{C}$ if and only if the polynomial matrix $P(\lambda)$ has full row rank for all $\lambda \in \mathbf{C}$. For this we introduce the $dm \times dm$ unimodular matrices $C(\lambda)$ and $D(\lambda) = C^{-1}(\lambda)$ defined as

$$C(\lambda) \doteq \begin{bmatrix} I & & & & \\ \lambda I & & & & \\ \lambda^2 I & & \ddots & & \\ \vdots & & & & \\ \lambda^{d-1} I & \cdots & \lambda^2 I & \lambda I & I \end{bmatrix}, \qquad D(\lambda) \doteq \begin{bmatrix} I & & & \\ -\lambda I & \ddots & & \\ & \ddots & & \\ & & -\lambda I & I \end{bmatrix} \tag{7}$$

where all identity matrices are of order m. Indeed, by straightforward calculations we find

$$C(\lambda)(A - \lambda B) = C(\lambda) \begin{bmatrix} I & & & \lambda P_d \\ -\lambda I & \ddots & & \vdots \\ & \ddots & I & \lambda P_2 \\ & & -\lambda I & \lambda P_1 + P_0 \end{bmatrix} = \begin{bmatrix} I & & & R_d(\lambda) \\ & \ddots & & \vdots \\ & & I & R_2(\lambda) \\ & & & P(\lambda) \end{bmatrix} \tag{8}$$

where we define $R_{d+1}(\lambda) \doteq 0$, $R_i(\lambda) \doteq \lambda R_{i+1}(\lambda) + \lambda P_i$, $i = d, \cdots, 2$ and $R_1(\lambda) \doteq P(\lambda)$. Using this, and the fact that $C(\lambda)$ is unimodular (and hence invertible for all $\lambda \in \mathbf{C}$) we indeed easily derive that the pencil $\lambda B - A$ has full row rank for all $\lambda \in \mathbf{C}$ if and only if $P(\lambda)$ has full row rank for all $\lambda \in \mathbf{C}$.

Suppose now that we are able to provide an embedding for the pencil $\lambda B - A$, which we denote as

$$\begin{bmatrix} \lambda B - A \\ K(\lambda) \end{bmatrix} \tag{9}$$

and let us partition $K(\lambda)$ as follows:

$$K(\lambda) \doteq [K_1(\lambda), \cdots, K_{d-1}(\lambda), K_d(\lambda)] \tag{10}$$

where $K_i(\lambda)$ has dimensions $(n-m) \times m$, for $i < d$ and $K_d(\lambda)$ dimensions $(n-m) \times n$. Combining (8) and (9), we thus have that

$$G(\lambda) \doteq \left[\begin{array}{cccc} I_m & & & R_d(\lambda) \\ & \ddots & & \vdots \\ & & I_m & R_2(\lambda) \\ 0 & \cdots & 0 & P(\lambda) \\ \hline K_1 & \cdots & K_{d-1} & K_d \end{array}\right] = \left[\begin{array}{c|c} -C(\lambda) & \\ \hline & I_{n-m} \end{array}\right] \cdot \left[\begin{array}{c} \lambda B - A \\ \hline K(\lambda) \end{array}\right] \tag{11}$$

is also unimodular. Introducing the unimodular matrix $H(\lambda)$ of order $(d-1)m + n$ as

$$H(\lambda) \doteq \left[\begin{array}{cccc|c} I_m & & & & \\ & \ddots & & & \\ & & I_m & & \\ & & & I_m & \\ \hline -K_1 & \cdots & -K_{d-1} & 0 & I_{n-m} \end{array}\right] \tag{12}$$

and premultiplying $G(\lambda)$ by $H(\lambda)$ gives

$$S(\lambda) \doteq H(\lambda)G(\lambda) = \left[\begin{array}{ccc|c} I_m & & & R_d(\lambda) \\ & \ddots & & \vdots \\ & & I_m & R_2(\lambda) \\ \hline & & & P(\lambda) \\ & & & Q(\lambda) \end{array}\right] \tag{13}$$

where $Q(\lambda)$ is given by

$$Q(\lambda) \doteq K_d(\lambda) - \sum_{i=1}^{d-1} K_i(\lambda) R_{d-i+1}(\lambda). \tag{14}$$

It is now obvious from (13) that

$$\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix} \tag{15}$$

is unimodular if and only if the embedding (9) is unimodular. This thus shows that the problem of embedding a polynomial matrix (provided this is possible) can always be reformulated as that of embedding a pencil. The reason of reformulating the problem as one for a pencil is that it can be embedded by a *constant* matrix K, as was, e.g., shown by Eising [2]. In the next section we give a simple alternative proof of this result and also show how to construct such a *constant* solution K.

3. Embedding a pencil in a unimodular one. Kronecker (see [3]) has shown that any pencil $\lambda B - A$ can be transformed via *constant* invertible column and row transformations to a canonical block diagonal form $\lambda B_c - A_c$

$$S \cdot (\lambda B - A) \cdot T = \lambda B_c - A_c = \operatorname{diag}\{L_{\varepsilon_1}, \cdots, L_{\varepsilon_p}, L^T_{\eta_1}, \cdots, L^T_{\eta_q}, \lambda N - I, \lambda I - J\} \tag{16}$$

where

(1) L_ε is the $\varepsilon \times (\varepsilon + 1)$ bidiagonal pencil

$$\begin{bmatrix} -1 & \lambda & & \\ & \ddots & \ddots & \\ & & -1 & \lambda \end{bmatrix}. \tag{17}$$

(2) L_η^T is the $(\eta + 1) \times \eta$ bidiagonal pencil

$$\begin{bmatrix} -1 & & \\ \lambda & \ddots & \\ & \ddots & -1 \\ & & \lambda \end{bmatrix}. \tag{18}$$

(3) N is a nilpotent Jordan matrix, and hence $\lambda N - I$ consists of a diagonal block pencil with $\delta_i \times \delta_i$ blocks of the type

$$\begin{bmatrix} -1 & \lambda & & \\ & -1 & \ddots & \\ & & \ddots & \lambda \\ & & & -1 \end{bmatrix}. \tag{19}$$

(4) J is in Jordan canonical form.

The matrix $\lambda I - J$ contains the *finite elementary divisors* and $\lambda N - I$ the *infinite elementary divisors* of $\lambda B - A$. The blocks L_{ε_i} and $L_{\eta_j}^T$ contain the *singularity* of the pencil. The indices ε_i and η_j are called the *Kronecker column* and *row indices*, respectively, and δ_i are called the *degrees* of the infinite elementary divisors.

Using this canonical form we now easily derive the following theorem about the unimodular embedding of a pencil.

THEOREM 1. *A pencil $\lambda B - A$ has a unimodular embedding*

$$\begin{bmatrix} \lambda B - A \\ K(\lambda) \end{bmatrix} \tag{20}$$

if and only if it has no finite elementary divisors and no Kronecker row indices. Moreover, there always exists a constant matrix K such that the new infinite elementary divisors of the embedding are equal to the union of the infinite elementary divisors $\{\delta_i\}$ and of the Kronecker column indices $\{(\varepsilon_j + 1)\}$ of $\lambda B - A$.

Proof. The necessity of the condition is trivial as noted in the Introduction. Indeed the unimodular embedding has full (row) rank for all $\lambda \in \mathbf{C}$, and thus this is also implied for the rows of $\lambda B - A$. Using the block decomposition (16) we easily find then that $\lambda B - A$ can have no finite elementary divisors or Kronecker row indices, since the corresponding blocks do not obey the row rank property for all $\lambda \in \mathbf{C}$.

The sufficiency of the condition is now proved via the construction of a solution K, which at the same time satisfies the second part of the theorem. Indeed, choose K_c to be a matrix whose rows are unit vectors, each with a -1 at the location corresponding to the last column of one of the L_{ε_j} of $\lambda B_c - A_c$. Then obviously the embedding

$$\begin{bmatrix} \lambda B_c - A_c \\ K_c \end{bmatrix} \tag{21}$$

has a Kronecker canonical form with blocks (19) of sizes δ_i and $(\varepsilon_j + 1)$ as requested. This form is indeed obtained by a mere permutation of the rows of (21). Then, defining $K \doteq K_c \cdot T^{-1}$ and using

$$\left[\begin{array}{c|c} S & \\ \hline & I \end{array}\right] \cdot \left[\begin{array}{c} \lambda B - A \\ \hline K \end{array}\right] \cdot T = \left[\begin{array}{c} \lambda B_c - A_c \\ \hline K_c \end{array}\right] \tag{22}$$

we find that (20) and (21) have the same Kronecker canonical form. The fact that a pencil with only infinite elementary divisors is unimodular [4] then completes the proof. □

COROLLARY 1. *A polynomial matrix $P(\lambda)$ of degree d has a unimodular embedding*

$$\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix} \tag{23}$$

if and only if it has full row rank for all $\lambda \in \mathbb{C}$. Moreover, there always exists an embedding with a polynomial matrix $Q(\lambda)$ of degree $(d-1)$.

Proof. As above, the necessity of the condition is trivial. Sufficiency is proved via the construction of K above and the subsequent derivation of $Q(\lambda)$ in (9)–(14). If K is chosen constant, then the construction (14) and the recurrence relation for $R_i(\lambda)$ in (8) yields the following explicit formula for $Q(\lambda)$ in terms of the coefficient matrices P_i of $P(\lambda)$:

$$Q(\lambda) \doteq K_d - \sum_{i=1}^{d-1} K_i \sum_{j=0}^{i-1} \lambda^{i-j} P_{d-j} = K_d - \sum_{k=1}^{d-1} \lambda^k \sum_{i=k}^{d-1} K_i P_{d+k-i}. \tag{24}$$

This clearly shows that $Q(\lambda)$ has degree $(d-1)$ and thus completes the proof. □

While apparently the problem is thus solved via the above construction, it is not a recommended procedure from a numerical point of view. The transformations S and T in the decomposition (16) may indeed be very badly conditioned and thus give rise to a significant loss of accuracy. An alternative decomposition that does not suffer from this drawback is the so-called staircase form of $\lambda B - A$ [10]. For a pencil $\lambda B - A$ with *only* column Kronecker indices $\{\varepsilon_j\}$ and infinite elementary divisors $\{\delta_j\}$, we obtain the following staircase form (which we denote by $\lambda B_{\varepsilon\infty} - A_{\varepsilon\infty}$) via *unitary* transformations U and V [10]:

$$U(\lambda B - A)V = \lambda B_{\varepsilon\infty} - A_{\varepsilon\infty}$$
$$= \begin{bmatrix} -A_{1,1} & \lambda B_{1,2} - A_{1,2} & X & \cdots & X \\ & -A_{2,2} & \lambda B_{2,3} - A_{2,3} & & \vdots \\ & & \ddots & \ddots & X \\ & & & -A_{k,k} & \lambda B_{k,k+1} - A_{k,k+1} \\ & & & & -A_{k+1,k+1} \end{bmatrix}. \tag{25}$$

This form is characterized by the fact that the blocks $A_{i,i}$ $(i = 1, \cdots, k+1)$ have full row rank and the blocks $B_{i,i+1}$ $(i = 1, \cdots, k)$ have full column rank. Notice that the blocks indicated by X in (25) are in fact pencils as well. Let the matrices $A_{i,i}$ $(i = 1, \cdots, k+1)$ and $\lambda B_{i,i+1} - A_{i,i+1}$ $(i = 1, \cdots, k)$ have dimensions $m_i \times n_i$ $(m_i \leqq n_i)$ and $m_i \times n_{i+1}$ $(n_{i+1} \leqq m_i)$, respectively. Then the following theorem, proved in [10], relates these dimensions to the Kronecker canonical form of $\lambda B_{\varepsilon\infty} - A_{\varepsilon\infty}$ (or $\lambda B - A$).

THEOREM 2. *The pencil $\lambda B - A$ with staircase form as in* (25), *has*

$$\begin{array}{ll} n_i - m_i & \text{Kronecker column indices } \varepsilon_j \text{ equal to } i-1, \\ m_i - n_{i+1} & \text{infinite elementary divisors } \delta_j \text{ equal to } i. \end{array} \tag{26}$$

□

At first sight we thus have the requested information to find a constant matrix K for the embedding, using the decomposition (25) as well. That this is in fact very simple is now shown below. Corresponding to each nonsquare $A_{i,i}$ we can easily find and $(n_i - m_i) \times n_i$ matrix C_i such that

$$\begin{bmatrix} C_i \\ A_{i,i} \end{bmatrix} \tag{27}$$

is square invertible and does not depend on λ. Thus, by adding a block row of the type

$$[0 \cdots 0 \quad -C_i \quad X \cdots X] \tag{28}$$

to each corresponding block row

$$[0 \cdots 0 \quad -A_{i,i} \quad \lambda B_{i,i+1} - A_{i,i+1} \quad X \cdots X] \tag{29}$$

in (25) for ($i = 1, \cdots, k$) and adding $[0 \cdots 0 \quad -C_{k+1}]$ to $[0 \cdots 0 \quad -A_{k+1,k+1}]$, the pencil (25) can be embedded into a pencil

$$\begin{bmatrix} \lambda B_{\epsilon\infty} - A_{\epsilon\infty} \\ K_{\epsilon\infty} \end{bmatrix} \tag{30}$$

with

$$K_{\epsilon\infty} \doteq \begin{bmatrix} -C_1 & X & & \cdots & X \\ & -C_2 & & & \vdots \\ & & \ddots & -C_k & X \\ & & & & -C_{k+1} \end{bmatrix}. \tag{31}$$

This matrix $K_{\epsilon\infty}$ has dimensions

$$\sum_{i=1}^{k+1} (n_i - m_i) \times \sum_{i=1}^{k+1} n_i = (n-m) \times \{(d-1)m + n\}.$$

It should be noted that the blocks indicated by X in (31) can be chosen arbitrarily, even as a function of λ, so that the matrix (13) is highly nonunique. For the sequel we assume $K_{\epsilon\infty}$ to be chosen *constant*. It is easily seen that the pencil (30), up to a row permutation Π_r, is again in staircase form:

$$\Pi_r \cdot \begin{bmatrix} \lambda B_{\epsilon\infty} - A_{\epsilon\infty} \\ K_{\epsilon\infty} \end{bmatrix}$$

$$= \begin{bmatrix} -\begin{bmatrix} C_1 \\ A_{1,1} \end{bmatrix} & \begin{bmatrix} X \\ \lambda B_{1,2} - A_{1,2} \end{bmatrix} & \cdots & & X \\ & -\begin{bmatrix} C_2 \\ A_{2,2} \end{bmatrix} & & & \vdots \\ & & \ddots & & \\ & & \ddots & -\begin{bmatrix} C_k \\ A_{k,k} \end{bmatrix} & \begin{bmatrix} X \\ \lambda B_{k,k+1} - A_{k,k+1} \end{bmatrix} \\ & & & & -\begin{bmatrix} C_{k+1} \\ A_{k+1,k+1} \end{bmatrix} \end{bmatrix} \tag{32}$$

since the (new) blocks $\left[\begin{smallmatrix} C_i \\ A_{i,i} \end{smallmatrix}\right]$ have full row rank by construction, and the (new) blocks $\left[\begin{smallmatrix} 0 \\ B_{i,i+1} \end{smallmatrix}\right]$ still have full column rank. The fact that this "embedded" pencil is unimodular easily follows from the full rank property for all $\lambda \in \mathbf{C}$ (guaranteed by the diagonal blocks). The preserved staircase form shows moreover that the pencil (32) has

$$n_i - n_{i+1} \quad \text{infinite elementary divisors } \hat{\delta}_j \text{ equal to } i \tag{33}$$

which, according to Theorem 2, is exactly the same result as in Theorem 1. Just as in (22), we then define $K \doteq K_{\epsilon\infty} V^{-1} = K_{\epsilon\infty} V^*$ (.* denotes the conjugate transpose) and use

$$\left[\begin{array}{c|c} U & \\ \hline & I \end{array}\right] \cdot \left[\begin{array}{c} \lambda B - A \\ \hline K \end{array}\right] \cdot V = \left[\begin{array}{c} \lambda B_{\epsilon\infty} - A_{\epsilon\infty} \\ \hline K_{\epsilon\infty} \end{array}\right] \tag{34}$$

to show that the matrix K obtained via this construction also satisfies the conditions of Theorem 1. This construction thus implicitly provides an embedding satisfying Theorem 1, *without* passing via the numerically sensitive Kronecker canonical form.

Remark 3.1. The staircase algorithm described for general pencils in [10] in fact also tests whether or not a given pencil only possesses infinite elementary divisors and Kronecker column indices. Applied to the pencil (6) it thus tests for the existence of an embedding, and at the same time provides a convenient form for constructing such an embedding in case it exists.

Remark 3.2. While the general staircase algorithm, e.g., described in [10] or [9] has an operation count that is *quartic* in the maximal dimension dm of $\lambda B - A$ (i.e., $O(m^4d^4)$ flops), an improved method has recently been proposed in [1] which has an operation count that is only *cubic* (i.e., $O(m^3d^3)$ flops). Moreover, it is shown there that the "rank carrying stairs" $A_{i,i}$ and $B_{i,i+1}$ can be chosen *triangular* when appropriately updating U and V.

Remark 3.3. It is well known that in general there is no unique solution $Q(\lambda)$ to the embedding problem. The method described above also does not yield a unique solution $Q(\lambda)$. This is clearly reflected by the freedom in choosing the block rows in (28). A possible selection criterion could be to minimize the effort for determining matrix K. When the $m_i \times n_i$ matrices $A_{i,i}$ in (25) are assumed to be upper triangular the $(n_i - m_i) \times n_i$ matrices C_i ($i \leqq k + 1$) can be chosen as

$$C_i = [I, 0] \tag{35}$$

with the remaining X matrices in each row of $K_{\epsilon\infty}$ equal to 0. In that case, the determination of K is of course trivial.

To conclude this section we now summarize the computational procedure.

ALGORITHM EMBED.
(1) Construct the pencil $\lambda B - A$ defined by (6).
(2) Compute the staircase form of $\lambda B - A$ giving (25) with upper triangular matrices $A_{i,i}$.
(3) Construct matrices C_i satisfying (27).
(4) Compute matrix $K_{\epsilon\infty}$ given in (31).
(5) Determine matrix $Q(\lambda)$ via (34) and (24).

4. Inversion of a unimodular matrix. In this section we consider the problem of inversion of a unimodular matrix from a numerical point of view. Throughout this section we denote by $U(\lambda)$ an $n \times n$ polynomial matrix of degree $d \geqq 1$ that is assumed to be *unimodular*, i.e., such that

$$\det U(\lambda) = \text{a nonzero constant}. \tag{36}$$

The determination of $U^{-1}(\lambda)$ is an important step in several problems dealing with polynomial matrices. For example, this inversion problem arises when solving certain polynomial matrix equations which we now first describe.

Computing a right inverse and a right null space of a full row rank polynomial matrix. Let $P(\lambda)$ denote an $m \times n$ polynomial matrix ($m < n$) which has full row rank for all $\lambda \in \mathbf{C}$. Any polynomial matrix $M(\lambda)$ such that $P(\lambda)M(\lambda) = I_m$ is called a right inverse of $P(\lambda)$. Any polynomial matrix $N(\lambda)$ of full column rank (for *some* λ) and such that $P(\lambda)N(\lambda) = 0_{m,n-m}$ is said to span the right null space of $P(\lambda)$. In order to find such matrices $M(\lambda)$ and $N(\lambda)$, we start with *any* unimodular embedding of $P(\lambda)$

$$U(\lambda) \doteq \begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}. \tag{37}$$

This is done using the procedure described in the previous section. Hereafter we determine the inverse of $U(\lambda)$. (To this end, we present a new numerical method in this section.) It is well known (see [4, Lemma 6.3-1]) that this inverse is a polynomial matrix $V(\lambda)$ which we partition as $V(\lambda) = [M(\lambda)|N(\lambda)]$ where $M(\lambda)$ and $N(\lambda)$ have dimensions $n \times m$ and $n \times (n-m)$, respectively. Obviously, we have $U(\lambda)V(\lambda) = I_n$, or equivalently,

$$\left[\begin{array}{c} P(\lambda) \\ \hline Q(\lambda) \end{array}\right] [M(\lambda)|N(\lambda)] = \left[\begin{array}{c|c} I_m & 0 \\ \hline 0 & I_{n-m} \end{array}\right]. \tag{38}$$

Hence,

$$P(\lambda)M(\lambda) = I_m, \qquad P(\lambda)N(\lambda) = 0_{m,n-m}. \tag{39}$$

Clearly, $M(\lambda)$ is a right inverse of $P(\lambda)$ and $N(\lambda)$ spans its right null space since $N(\lambda)$ has full column rank for all $\lambda \in \mathbf{C}$ (being a submatrix of the unimodular matrix $V(\lambda)$).

From an algebraic point of view the computation of $U^{-1}(\lambda)$ is rather simple. Indeed, let $V(\lambda) = U^{-1}(\lambda)$; then we have to solve

$$U(\lambda)V(\lambda) = I_n. \tag{40}$$

Matrix $U(\lambda)$ can, e.g., be reduced by elementary row operations to the so-called triangular Hermite form (see [4, § 6.3] for details). This form can now be used to solve for $V(\lambda)$ by backward substitution. Of course, other methods for determining $V(\lambda)$ can be applied including those for inverting arbitrary polynomial or rational matrices (see, e.g., [8]). However, most of these general inversion methods are not recommended from a numerical point of view. The main reason is that in fact they rely on the Euclidean division algorithm (when reducing the Hermite form) or on formulas that can cause severe loss of significant digits.

Below we present a new (numerically more reliable) algorithm for computing the inverse of a unimodular matrix. Let us denote the $n \times n$ unimodular matrix $U(\lambda)$ of degree d by

$$U(\lambda) \doteq U_0 + U_1\lambda + U_2\lambda^2 + \cdots + U_d\lambda^d \tag{41}$$

where each U_i is a real or complex $n \times n$ matrix. Here again we assume that $d \geqq 1$, since otherwise the problem degenerates into one involving constant matrices only and becomes trivial.

As in the previous section we reformulate the problem as a pencil problem by defining the $dn \times dn$ pencil $\lambda\hat{B} - \hat{A}$ where the matrices $\hat{B}$ and $\hat{A}$ are defined as

$$\hat{B} \doteq \begin{bmatrix} 0 & & & -U_d \\ I_n & \ddots & & -U_{d-1} \\ & \ddots & & \vdots \\ & \ddots & 0 & -U_2 \\ & & I_n & -U_1 \end{bmatrix}, \qquad \hat{A} \doteq \begin{bmatrix} I_n & & & \\ & \ddots & & \\ & & I_n & \\ & & & U_0 \end{bmatrix}. \tag{42}$$

When defining $C(\lambda)$ and $D(\lambda)$ as in (7) but now with identity matrices of order n, we find

$$C(\lambda)(\hat{A}-\lambda\hat{B}) = \begin{bmatrix} I & & & \hat{R}_d(\lambda) \\ & \ddots & & \vdots \\ & & I & \hat{R}_2(\lambda) \\ & & & U(\lambda) \end{bmatrix} \tag{43}$$

where we define $\hat{R}_{d+1}(\lambda) \doteq 0$, $\hat{R}_i(\lambda) \doteq \lambda \hat{R}_{i+1}(\lambda) + \lambda U_i$, $i = d, \cdots, 2$ and $\hat{R}_1(\lambda) \doteq U(\lambda)$.

It is easily seen that (43) is unimodular. Hence, the pencil $\lambda\hat{B} - \hat{A}$ is also unimodular. It follows from (43) that

$$\begin{aligned} U^{-1}(\lambda) &= -[0, \cdots, 0, I_n](\lambda\hat{B}-\hat{A})^{-1}C^{-1}(\lambda)[0, \cdots, 0, I_n]^T \\ &= -[0, \cdots, 0, I_n](\lambda\hat{B}-\hat{A})^{-1}D(\lambda)[0, \cdots, 0, I_n]^T \\ &= -[0, \cdots, 0, I_n](\lambda\hat{B}-\hat{A})^{-1}[0, \cdots, 0, I_n]^T. \end{aligned} \tag{44}$$

This thus shows that inverting a unimodular polynomial matrix is easily reformulated as inverting a unimodular pencil.

In order now to solve the inversion problem of the unimodular pencil $\lambda\hat{B} - \hat{A}$, we first note that the Kronecker canonical form of $\lambda\hat{B} - \hat{A}$ merely consists of $I - \lambda\hat{N}$ where $\hat{N}$ is nilpotent:

$$\hat{S}\cdot(\lambda\hat{B}-\hat{A})\cdot\hat{T} = I - \lambda\hat{N}. \tag{45}$$

From this the inverse is trivially obtained as

$$(\lambda\hat{B}-\hat{A})^{-1} = \hat{T}^{-1}\cdot(I+\lambda\hat{N}+\lambda^2\hat{N}^2+\cdots+\lambda^l\hat{N}^l)\cdot\hat{S}^{-1} \tag{46}$$

where $l + 1$ is the size of the largest infinite elementary divisor in (16) (i.e., the largest $\delta_i \times \delta_i$ block of the type (19) in $I - \lambda\hat{N}$). If we define the polynomial matrix $V(\lambda) \doteq U(\lambda)^{-1}$ as

$$V(\lambda) \doteq V_0 + V_1\lambda + \cdots + V_l\lambda^l, \tag{47}$$

then the combination of (44) and (46) gives us

$$V_i = -[0, \cdots, 0, I_n]\cdot\hat{T}^{-1}\cdot\hat{N}^i\cdot\hat{S}^{-1}\cdot[0, \cdots, 0, I_n]^T \qquad (i = 0, \cdots, l) \tag{48}$$

which thus solves the problem. But since the Kronecker decomposition is a sensitive tool from a numerical point of view, we again turn to the staircase form of $\lambda\hat{B} - \hat{A}$. This can be obtained under *unitary transformations* Q and Z:

$$Q\cdot(\lambda\hat{B}-\hat{A})\cdot Z \doteq \lambda\hat{B}_\infty - \hat{A}_\infty$$

$$= \begin{array}{c} \left[\begin{array}{c|c|c|c} -\hat{A}_{1,1} & \lambda\hat{B}_{1,2}-\hat{A}_{1,2} & \cdots & X \\ \hline & -\hat{A}_{2,2} & & \vdots \\ \hline & & \ddots & \lambda\hat{B}_{l,l+1}-\hat{A}_{l,l+1} \\ \hline & & & -\hat{A}_{l+1,l+1} \end{array}\right] \begin{array}{l} \} n_1 \\ \\ \} n_l \\ \} n_{l+1} \end{array} \\ \begin{array}{cccc} \underbrace{\qquad}_{n_1} & \underbrace{\qquad\qquad}_{n_2} & & \underbrace{\qquad\qquad}_{n_{l+1}} \end{array} \end{array} \tag{49}$$

where the matrices $\hat{A}_{i,i}$ are upper triangular matrices of full row rank, and the matrices $\hat{B}_{i,i+1}$ have full column rank. Since $\lambda\hat{B} - \hat{A}$ has only infinite elementary divisors, the $\hat{A}_{i,i}$ are square invertible and so is $\hat{A}_\infty$. Let us now introduce

(50) $$\hat{N}_\infty \doteq \hat{B}_\infty \hat{A}_\infty^{-1};$$

then $\hat{N}_\infty$ has exactly the same block structure as $\hat{B}_\infty$ since $\hat{A}_\infty^{-1}$ is upper triangular. Thus, $\hat{N}_\infty$ is nilpotent and we then have that

(51) $$(\lambda\hat{B}-\hat{A})^{-1} = Z^*\cdot(\lambda\hat{B}_\infty - \hat{A}_\infty)^{-1}\cdot Q^* = Z^*\hat{A}_\infty^{-1}\cdot(I+\hat{N}_\infty\lambda+\hat{N}_\infty^2\lambda^2+\cdots+\hat{N}_\infty^l\lambda^l)\cdot Q^*.$$

The computation of $\hat{A}_\infty^{-1}$ is rather simple since it is triangular and so is the construction of $\hat{N}_\infty$. By Theorem 2 we find that the index $l+1$ obtained from the Kronecker canonical decomposition (i.e., the index of nilpotency of $\hat{N}$) equals the number of "stairs" $\hat{A}_{i,i}$ in (49), and hence also the index of nilpotency of $\hat{N}_\infty$.

The combination of (44), (47), and (51) now gives us

(52) $$V_i = Z_{\text{left}}\hat{N}_\infty^i Q_{\text{right}} \qquad (i=0,\cdots,l)$$

where

(53) $$Z_{\text{left}} \doteq -[0,\cdots,0,I_n]Z^*\hat{A}_\infty^{-1}$$

and

(54) $$Q_{\text{right}} \doteq Q^*[0,\cdots,0,I_n]^T.$$

Here Z_{left} and Q_{right} have dimensions $n \times dn$ and $dn \times n$, respectively.

Remark 4.1. If the unimodular matrix $U(\lambda)$ results from an embedding problem, then the construction of the previous section immediately yields a staircase form of the type (49). The possibility of choosing the diagonal blocks triangular in this embedding (see Remark 3.3) is thus appropriate here.

Remark 4.2. Since the index of nilpotency of $\hat{N}_\infty$ determines the number $l+1$ of coefficients V_i to be computed, trying to minimize l when dealing with the embedding problem is recommended. This is in fact done in the construction of Theorem 1: the lengths of the Jordan chains of the infinite elementary divisors—i.e., the number of stairs in the resulting staircase form (32)—is kept minimal, namely equal to the number of stairs in the staircase form (25) we are starting from. It is important to note here that not all V_i are necessarily nonzero, although the $\hat{N}^i$ and $\hat{N}_\infty^i$ matrices in (46) and (51) are nonzero for $i = 0, \cdots, l$. This thus means that l is in fact only an *upper bound* for the actual degree of $V(\lambda)$. This is, e.g., seen in the examples below.

We conclude this section with a summary of this procedure.

ALGORITHM INVERT.

(1) Construct the pencil $\lambda\hat{B} - \hat{A}$ defined by (42).
(2) Compute the staircase form of $\lambda\hat{B} - \hat{A}$ giving (49) with upper triangular $\hat{A}_{i,i}$ and compute $\hat{N}_\infty$ via (50).
(3) Compute Z_{left} and Q_{right} via (53) and (54).
(4) Compute the coefficients V_i of $V(\lambda)$ using (52).

5. Computational aspects. In the design of any numerical algorithm we are mainly concerned with two aspects: numerical reliability and computational speed.

As far as numerical precision is concerned, we can certainly say that the methods are based on the use of the staircase forms (25) or (49), which can be obtained by numerically stable algorithms [10], [11], [9], [12]. For the embedding problem this guarantees a rather good numerical behavior since the determination of $K_{e\infty}$, and subsequently of $Q(\lambda)$ via (35) and (24), does not seem to introduce any numerical difficulty. The method is, we believe, certainly to be preferred over the method using the Kronecker canonical form described in (21)–(22) or Eising's method [2], since these both require inverses of matrices that can be badly conditioned.

For the inversion problem the situation is somewhat different. There the use of the staircase form again avoids the use of the numerically sensitive Kronecker canonical form, but there is still an inversion problem involved. That this cannot be avoided is easily seen from the following recursions for the coefficients V_i of the inverse of a unimodular matrix:

$$V_0 = U_0^{-1}, \quad V_1 = -U_0^{-1} \cdot (U_1 V_0), \quad V_2 = -U_0^{-1} \cdot (U_2 V_0 + U_1 V_1), \cdots,$$

(55)
$$V_k = -U_0^{-1} \cdot \left(\sum_{i=0}^{k-1} U_{k-i} V_i \right).$$

If we *know* that the matrix $U(\lambda)$ is unimodular and that the degree of its inverse will be k, then this is probably the most direct (and also most reliable) method to compute the coefficients of $V(\lambda)$. But Algorithm Invert also provides a test for the unimodularity of $U(\lambda)$ and computes a (usually close) upper bound l for the degree k of its inverse. The algorithm is probably not much more sensitive than the mere application of (55), and it is certainly recommended for problems that are coming from an embedding since there $U(\lambda)$ is not directly available, whereas $\lambda \hat{B} - \hat{A}$ is.

Remark 5.1. It should be noted here that Eising also proposes a number of variants of his method which normally improve the numerical sensitivity of the problem, while allowing the embedding $U(\lambda)$ to have larger infinite elementary divisors than the minimum required. This is particularly important for the subsequent inversion problem where a trade-off between degree and sensitivity of the solution $V(\lambda)$ is pointed out by Eising [2].

As far as the computational complexity is concerned, we have already remarked that a cubic algorithm is available [1] for computing the staircase form of an arbitrary pencil, in contrast to the quartic methods that are available up to now [13], [10], [9], [6], [5]. For the embedding problem this decomposition constitutes the bulk of the work (namely $O(m^3 d^3)$ flops) since the construction of $K \doteq K_{e\infty} \cdot V^*$ and $Q(\lambda)$ using (24) only require $O(m^2 d^2 n)$ flops and $O(md^2 n(n-m))$ flops, respectively.

For the inversion problem we suppose first that it is connected to an embedding and, hence, that (49) is available. The computation of $\hat{N}_\infty$ and Z_{left} takes $O(n^3 d^3)$ and $O(m^2 d^2 n)$ flops, respectively (Q_{right} is obtained at no cost). Starting with these data, the V_i are then computed recursively using

(56) $\quad X_0 = Q_{\text{right}}, \quad V_0 = Z_{\text{left}} \cdot X_0, \quad \text{for } i = 1, \cdots, l\text{:} \quad X_i = \hat{N}_\infty \cdot X_{i-1}, \quad V_i = Z_{\text{left}} \cdot X_i$

which takes $O(lm^2 d^2 n)$ flops for the total recursion. Here it is clear that it is very important to keep l as small as possible, since otherwise the complexity of this step may well become the larger part of the work (l may be as large as md!). If now the inversion problem is independent of an embedding, then the staircase form (49) has to be computed also which requires an additional $O(n^3 d^3)$ flops. Moreover, one then has $m = n$.

We conclude this section with some numerical examples. The embedding problem largely relies on the staircase form, which has already been treated by various authors [13], [10], [1]. Therefore we restrict ourselves here to the inversion part.

Numerical examples. We give here some numerical examples of the Invert Algorithm. They were run on a VAX-750 computer with relative machine precision $EPS = 2^{-56} \approx 0.14 * 10^{-16}$. The notation is consistent with formulas (41)–(54). For brevity, we only list the nontrivial matrices. The computations were performed with the interactive matrix manipulation package MATLAB [7].

Example 1.

$$U(\lambda) \doteq \begin{bmatrix} 1 & \lambda & \lambda^2 \\ & 1 & \lambda \\ & & 1 \end{bmatrix}.$$

For $\lambda \hat{B}_\infty - \hat{A}_\infty = Q(\lambda \hat{B} - \hat{A})Z$ and $\hat{N}_\infty = \hat{B}_\infty \hat{A}_\infty^{-1}$ the following results were found up to 16 correct digits:

$$\hat{B}_\infty = \begin{bmatrix} 0 & 0 & -2\alpha & 0 & 0 & 0 \\ & & 0 & -2\alpha & -\alpha & 0 \\ & & & & \alpha & 0 \\ & & & & 0 & 1 \\ & & & & & 0 \\ & & & & & 0 \end{bmatrix}, \quad \hat{A}_\infty = \begin{bmatrix} -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & +1 & & \\ & & & & -1 & \\ & & & & & -1 \end{bmatrix},$$

$$\hat{N}_\infty = \begin{bmatrix} 0 & 0 & 2\alpha & 0 & 0 & 0 \\ & & 0 & -2\alpha & \alpha & 0 \\ & & & & -\alpha & 0 \\ & & & & 0 & -1 \\ & & & & & 0 \\ & & & & & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 & \alpha & 0 \\ \alpha & 0 & 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & -\alpha & -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

where $\alpha = \sqrt{2}/2$ and $l = 2$. Straightforward computation of $U^{-1}(\lambda)$ using (52) gives

$$V(\lambda) = \begin{bmatrix} 1 & -\lambda & 0 \\ & 1 & -\lambda \\ & & 1 \end{bmatrix}.$$

Example 2.

$$U(\lambda) = \begin{bmatrix} 0 & \lambda^2 & 1 \\ 0 & 1 & 0 \\ 1 & \lambda+7 & \lambda^2+7\lambda+3 \end{bmatrix}.$$

In this case we obtained (up to 16 correct digits)

$$\hat{B}_\infty = \begin{bmatrix} 0 & -1 & 7 & 0 & 1 & 0 \\ & & -1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & -1 \\ & & & & & 0 \end{bmatrix}, \quad \hat{A}_\infty = \begin{bmatrix} -1 & 0 & -3 & 0 & -7 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & -1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & -1 \end{bmatrix},$$

$$\hat{N}_\infty = \begin{bmatrix} 0 & -1 & 7 & 0 & 1 & 0 \\ & & -1 & 0 & 0 & 0 \\ & & & -1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \\ & & & & & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Moreover, $l = 5$ and

$$V_0 = \begin{bmatrix} -3 & -7 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad V_1 = \begin{bmatrix} -7 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$

$$V_3 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_5 = 10^{-17} * \begin{bmatrix} 0 & -0.3469 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, when neglecting the term $\lambda^5 V_5$ (recall $EPS \approx 0.14 * 10^{-16}$) we indeed find the exact formula for the inverse of $U(\lambda)$, i.e.,

$$V(\lambda) = \begin{bmatrix} (-\lambda^2 - 7\lambda - 3) & (\lambda^4 + 7\lambda^3 + 3\lambda^2 - \lambda - 7) & 1 \\ 0 & 1 & 0 \\ 1 & -\lambda^2 & 0 \end{bmatrix}.$$

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