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PLACING ZEROES AND THE KRONECKER CANONICAL FORM*

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Abstract. Given a linear time-invariant control system, it is well known that the transmission zeroes are the generalized eigenvalues of a matrix pencil. Adding outputs to place additional zeroes is equivalent to appending rows to this pencil to place new generalized eigenvalues. Adding inputs is likewise equivalent to appending columns. Since both problems are dual to each other, in this paper we only show how to choose the new rows to place the new zeroes in any desired locations. The process involves the extraction of the individual right Kronecker blocks of the pencil, accomplished entirely with unitary transformations. In particular, when adding one new output, i.e., appending a single row, the maximum number of new zeroes that can be placed is exactly the largest right Kronecker index.

1. Introduction

The placement of transmission of zeroes via synthesis of new outputs and/or inputs has been studied from the point of view of system theory, and certain algorithms have been developed [8], [9], [1], [11]. As for the assignment of zeroes via feedback design, the assignment of zeroes via output synthesis can be analyzed in terms of the theory of matrix pencils, so that a complete characterization of the number of zeroes that can be placed in any given case can be obtained. In [8], [9] an algorithm to synthesize outputs to assign the zeroes was proposed based on the desired form of the transfer function. In [1], a method was proposed to assign zeroes for a SISO system as well as to assign zeroes to the input/output maps from individual inputs to individual outputs. In this paper, we study this problem using the Kronecker theory of pencils [5]. Specifically, we study the problem of zero placement for an arbitrary matrix pencil by the addition of new rows or columns in terms of the structure of the Kronecker Canonical Form (KCF). We show how additional rows or columns can be appended to a pencil to place as many zeroes

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as possible, and discuss the limits on this placement. As in [1], [11], our approach ends up reducing the problem to a problem of pole placement, for which several algorithms are available (see e.g., [7], [10]). However, we are able to synthesize outputs to place the zeroes for an entire MIMO system, as well as determining the limits on the number of new zeroes that can be placed.

We treat this problem by studying the problem of assigning the generalized eigenvalues of a general matrix pencil. The zero placement problem will then be a special case. The methods in this paper are all based on the transformation of the original pencil to one that can be partitioned into the various components of the Kronecker canonical form. The transformations are carried out entirely using unitary transformations, and hence enjoy some numerical stability properties. The computations are based on the so-called staircase algorithm in [12], which separates the left and right Kronecker parts and computes the values of the individual Kronecker indices. We propose a new extension to this algorithm, still based on unitary transformations, that can actually extract the individual Kronecker blocks. Once the individual Kronecker blocks have been extracted, the zeroes may be placed within each Kronecker block in a manner very similar to that of [1].

This paper is organized as follows. First we describe the basic theory that relates the Kronecker theory of matrix pencils to the problem of placing zeroes or more generally placing the generalized eigenvalues for a pencil. Next we describe our computational procedure for extracting the Kronecker blocks and placing the zeroes. We include in an Appendix a step-by-step description of the new process used to extract the individual Kronecker blocks.

2. Basic theory

Consider a linear time-invariant generalized state-space system of dimension n:

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad \mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t), \tag{1}$$

which is *irreducible*, i.e., where

$$\begin{pmatrix} A - \lambda E & B \end{pmatrix} \text{ and } \begin{pmatrix} A - \lambda E \\ C \end{pmatrix}$$

both have full rank n for all finite λ (this also means reachable and observable at finite points), and where

$$\begin{pmatrix} E & B \end{pmatrix} \text{ and } \begin{pmatrix} E \\ C \end{pmatrix}$$

both have full rank n (this also means reachable and observable at infinity). It is well known that the transmission zeroes of the system are also the zeroes of the matrix pencil [4], [13], [6]:

$$G - \lambda F = \begin{pmatrix} A - \lambda E & B \\ C & D \end{pmatrix} \tag{2}$$

We would like to add outputs or inputs to (1) to place new zeroes in desired locations in the complex plane. This corresponds to appending rows or columns, respectively, to (2) to place the zeroes of the embedded system. We discuss the general question of how many zeroes can be placed by appending rows, and outline a procedure to compute the rows to append to place the zeroes at given locations.

To fix ideas, we analyze how many zeroes we can place by a judicious choice of additional rows appended to (2). We write the generalized Schur form for (2) [5]:

$$P(G-\lambda F)Q = \begin{pmatrix} G_r-\lambda F_r & * & * \\ 0 & G_{reg}-\lambda F_{reg} & * \\ 0 & 0 & G_\ell-\lambda F_\ell \end{pmatrix}, \tag{3}$$

where P and Q are unitary (orthogonal in the real case) and $G_r-\lambda F_r$ contains the right (short fat) Kronecker blocks, $G_\ell-\lambda F_\ell$ contains the left (tall thin) Kronecker blocks, and $G_{reg}-\lambda F_{reg}$ is the regular part. The blocks are characterized by the properties that $G_r-\lambda F_r$ has full row rank and $G_\ell-\lambda F_\ell$ has full column rank for all values of λ in the complex plane (including infinity), and $G_{reg}-\lambda F_{reg}$ is square and nonsingular except at a finite number of isolated values of λ, the *eigenvalues* of the pencil. The finite eigenvalues are the *finite zeroes* of the pencil. The infinite eigenvalues correspond to the *infinite zeroes* of the pencil, except that each $(k\times k)$ Jordan block $I_k-\lambda J$ at infinity has only $k-1$ zeroes at infinity (but k infinite eigenvalues) [13]. Notice that this definition implies also that the total number of zeroes equals $\operatorname{rank}(F_{reg})$ [13]. In the Kronecker Canonical Form P, Q are nonsingular matrices, the entries $*$ are zero, and $G_r-\lambda F_r$ has the block "diagonal" form

$$G_r-\lambda F_r = \begin{pmatrix} R_1(\lambda) & & 0 \\ & \ddots & \\ 0 & & R_k(\lambda) \end{pmatrix}, \tag{4}$$

where each $R_i(\lambda)$ is $s_i\times(s_i+1)$, has full row rank for all λ, and represents a single Kronecker block. The $\{s_i\}$'s are the *right Kronecker indices* of the pencil, and we assume without loss of generality that these indices are in nondecreasing order. In the sequel, we will show how to obtain an upper triangular version of the overall form (4), but with nonzero entries above the diagonal blocks, using only unitary transformations. In any case, $G_\ell-\lambda F_\ell$ will have a similar upper triangular form, but with rectangular diagonal blocks with one more row than column and full column rank.

We append some number p of new rows to (2) to obtain

$$\begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}\left[\begin{pmatrix} G \\ Z \end{pmatrix} - \lambda\begin{pmatrix} F \\ 0 \end{pmatrix}\right]Q = \begin{pmatrix} G_r-\lambda F_r & * & * \\ 0 & G_{reg}-\lambda F_{reg} & * \\ 0 & 0 & G_\ell-\lambda F_\ell \\ Z_r & Z_{reg} & Z_\ell \end{pmatrix}. \tag{5}$$

The object is to choose these p new rows so as to place as many zeroes as possible. The rightmost block column $(*, *, G_\ell^T - \lambda F_\ell^T, Z_\ell^T)^T$ has full column rank regardless of the choice of Z_ℓ. The middle block column $(*, G_{reg}^T - \lambda F_{reg}^T, 0, Z_{reg}^T)^T$ has lose rank only at values of λ where $G_{reg} - \lambda F_{reg}$ already loses rank (i.e., only at existing generalized eigenvalues) and only for certain choices of Z_{reg}. The entry Z_{reg} can sometimes be chosen so that the middle block column does not lose rank (or loses less rank than does $G_{reg} - \lambda F_{reg}$ alone) at any particular existing eigenvalue. In the presence of a $G_\ell - \lambda F_\ell$ block, the result may be that the existing eigenvalue disappears from the augmented pencil (5) (or the eigenvalue remains with a smaller multiplicity). But in any case, neither Z_{reg} nor Z_ℓ can be used to place any new zeroes or to increase the multiplicity of any existing zeroes. Some of these effects are explained in more detail in the following subsections.

Hence, only Z_r can be used to place new zeroes. The choice of Z_r is independent of Z_{reg}, Z_ℓ, so we may set the latter to zero. Actually, if we don't set those to zero, there will be coupling between the parts. The effect of this coupling is discussed in the following subsections, but in general it will not affect newly placed zeroes, unless they happen to coincide with zeroes already present in $G_{reg} - \lambda F_{reg}$. We need only consider the subpencil of (5)

$$\begin{pmatrix} G_r - \lambda F_r \\ Z_r \end{pmatrix} = \begin{pmatrix} R_1(\lambda) & & * \\ & \ddots & \\ 0 & & R_k(\lambda) \\ Z_1 & \cdots & Z_k \end{pmatrix}, \tag{6}$$

where each R_i is $s_i \times (s_i + 1)$ and represents a single Kronecker block. It will be seen that the zeroes can be placed by choosing the p rows to be appended, $Z_r = (Z_1, \ldots, Z_k)$, to have the form

$$Z_r = \begin{pmatrix} 0 & \cdots & 0 & \mathbf{z}_{k-p+1}^T & 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & 0 & \mathbf{z}_{k-p+2}^T & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \mathbf{z}_k^T \end{pmatrix}, \tag{7}$$

where each row vector $\mathbf{z}_i^T$ is computed so that the individual $(s_i + 1) \times (s_i + 1)$ pencil $\begin{pmatrix} R_i(\lambda) \\ \mathbf{z}_i^T \end{pmatrix}$ has a subset of the desired zeroes, for $i = k - p + 1, \cdots, k$.

The rest of this section and the next section are devoted to filling in many of the theoretical and computational details, respectively, behind the zero-placement algorithm.

2.1. *How many zeroes can be placed?*

We discuss the specific question: How many zeroes can be placed by appending one row or multiple rows? For this we first need to recall some basic results on pencils and their polynomial null spaces.

Let r be the *normal rank* of the $(m \times n)$-pencil $G - \lambda F$, then it has $n_r \stackrel{\text{def}}{=} n - r$ right null vectors $v_i(\lambda)$ and $m_r \stackrel{\text{def}}{=} m - r$ left null vectors $u_j(\lambda)$, which can be chosen to be polynomial. Collecting these vectors in an $(n \times n_r)$ polynomial matrix $V(\lambda)$ and in an $(m \times m_r)$ polynomial matrix $U(\lambda)$ we thus have:

$$(G - \lambda F)V(\lambda) = 0, \quad U^T(\lambda)(G - \lambda F) = 0. \tag{8}$$

Now the columns of $V(\lambda)$ and $U(\lambda)$ are said to be a *minimal basis* for the respective null spaces if their column degrees are *minimal*. This is the case if and only if [6, p. 458]:

- $V(\lambda)$, respectively $U(\lambda)$, has full column rank or all finite λ
- the highest column degree coefficient matrix of $V(\lambda)$, respectively $U(\lambda)$, has full column rank.

One proves [5] that if $V(\lambda)$, respectively $U(\lambda)$, is minimal then its column degrees are (up to a permutation) equal to the right Kronecker indices $\{s_i\}$, respectively left Kronecker indices $\{t_i\}$, of $G - \lambda F$. Moreover, the *minimality* of the bases refers to the fact that any other polynomial basis for these null spaces must have *higher* column degrees. A consequence of all this is also that the number k of right Kronecker indices is equal to n_r, and the number of left Kronecker indices is equal to m_r. One defines then the *orders* o_r and o_ℓ of the right and left null spaces to be the sum of the column degrees of their minimal bases, i.e., $o_r = \sum_{i=1}^{n_r} s_i$ and $o_\ell = \sum_{i=1}^{m_r} t_i$. A simple consequence of this is (see [12]):

- $G_r - \lambda F_r$ has dimension $o_r \times (o_r + n_r)$
- $G_\ell - \lambda F_\ell$ has dimension $(o_\ell + m_r) \times o_\ell$.

In order to use this for bounding the number of assignable zeroes when appending rows or columns we need the following result, proved in [13]:

Lemma 1. *Let $G - \lambda F$ be a pencil with null space orders o_ℓ and o_r and number of finite and infinite zeros o_f and o_∞ (multiplicities counted), then* rank(F) $=$ $o_\ell + o_r + o_f + o_\infty$.

Notice that in the above result we count *zeroes* at infinity, not *eigenvalues*, to be compatible with their system theoretic interpretation (see text above (4) or [13]).

Since now appending constant rows or columns does not change rank(F) we can only increase the number of zeroes by *minimizing* the null space orders. We now give certain inequalities that will lead to the main result.

Theorem 1. *Let $G - \lambda F$ be an $(m \times n)$-pencil with normal rank r and with Kronecker indices and null space orders $o_r = \sum_{i=1}^{n_r} s_i$ and $o_\ell = \sum_{j=1}^{m_r} t_j$. Then appending p constant rows and denoting this pencil by $G' - \lambda F'$ yields new normal rank and null space orders r', o'_r, and o'_ℓ satisfying:*

$$r \le r' \le r + p,$$

$$o_\ell \le o'_\ell, \quad \textit{with equality only if } r' = r + p,$$

$$\sum_{j=1}^{i} s_j \le \sum_{j=1}^{i} s'_j \quad \textit{for } 1 \le i \le n - r'.$$

Proof. The first result is trivial because the normal rank of a pencil is its rank for almost any value of λ, and appending p rows in a constant matrix then immediately gives the bounds $r \le r' \le r + p$. For the second bound we start from (5) and perform a generalized Schur decomposition on the subpencil consisting of the upper left part to get:

$$\begin{pmatrix} \hat{P} & 0 \\ 0 & I \end{pmatrix} \left(\begin{array}{cc|c} Z_r & Z_{reg} & Z_\ell \\ G_r - \lambda F_r & * & * \\ 0 & G_{reg} - \lambda F_{reg} & * \\ \hline 0 & 0 & G_\ell - \lambda F_\ell \end{array} \right) \begin{pmatrix} \hat{Q} & 0 \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} \hat{G}_r - \lambda \hat{F}_r & * & * & * \\ 0 & \hat{G}_{reg} - \lambda \hat{F}_{reg} & * & * \\ 0 & 0 & \hat{G}_\ell - \lambda \hat{F}_\ell & * \\ 0 & 0 & 0 & G_\ell - \lambda F_\ell \end{pmatrix}$$

Since the subpencil

$$\begin{pmatrix} \hat{G}_\ell - \lambda \hat{F}_\ell & * \\ 0 & G_\ell - \lambda F_\ell \end{pmatrix}$$

has full column rank for all values of λ (including infinity) its number of columns equals the new left null space order o'_ℓ and hence $o'_\ell \ge o_\ell$. Moreover, equality is only met when $\hat{G}_\ell - \lambda \hat{F}_\ell$ is void. But then we also have that the new dimension of the left null space equals the old one, i.e., $m + p - r' = m - r$, which yields the required result.

For the last inequality, let $V'(\lambda)$ be an $(m + p) \times (m + p - r')$ minimal basis for the right null space of $G' - \lambda F'$. Then obviously, we also have

$$(G - \lambda F)V'(\lambda) = 0$$

which implies that the right null space $V'(\lambda)$ of $(G' - \lambda F')$ is a subspace of the right null space $V(\lambda)$ of $(G - \lambda F)$. As a subspace, it then follows from the theory of minimal bases [6, §6.5.4] that there exists a *polynomial* matrix $M(\lambda)$ such that

$$V'(\lambda) = V(\lambda) \cdot M(\lambda).$$

Let V'_h and V_h be the coefficient matrices of the highest column degrees in $V'(\lambda)$ and $V(\lambda)$ (these are the column coefficients of $\lambda^{s'_i}$ and λ^{s_i}, respectively). Since $V'(\lambda)$ and $V(\lambda)$ are minimal bases, we know that V'_h and V_h both have full column rank. From this it follows that element $m_{j,i}(\lambda)$ of the matrix $M(\lambda)$ can not have degree larger than $d_{j,i} \stackrel{\text{def}}{=} s'_i - s_j$. Moreover, the coefficient matrix M_h with the coefficient of $\lambda^{s'_i - s_j}$ as the (j, i)th entry, also has full column rank, since [6]:

$$V'_h = V_h \cdot M_h.$$

For every nonzero element $m^h_{j,i}$ in M_h we know that $s'_i = s_j + d_{j,i} \geq s_j$ since $d_{j,i}$ must be nonnegative and no cancellation can occur between columns in this matrix product due to the linear independence of the columns in V'_h, V_h, and M_h. Since M_h has full column rank, there must exist *distinct* indices $j_1, \ldots, j_{n-r'}$ such that $m^h_{j_i,i} \neq 0$ and hence

$$\sum_{i=1}^{n-r'} s'_i \geq \sum_{i=1}^{n-r'} s_{j_i}.$$

Since the sequence $\{s_j\}$ is increasing, this implies that

$$\sum_{i=1}^{n-r'} s'_i \geq \sum_{i=1}^{n-r'} s_i.$$

Finally, the same reasoning can be applied for the first i columns of the matrices V'_h and M_h, yielding the desired third bound. □

This then automatically leads to the following main result.

Theorem 2. *Let $G - \lambda F$ be a pencil with right Kronecker indices $s_1 \leq \cdots \leq s_{n_r}$. Suppose we append p rows to obtain*

$$\begin{pmatrix} G \\ Z \end{pmatrix} - \lambda \begin{pmatrix} F \\ 0 \end{pmatrix}.$$

The maximum number of new zeroes that can be placed is

$$s_{n_r-p+1} + \cdots + s_{n_r},$$

and a matrix Z can be found to place that many zeroes at any previously chosen locations in the complex plane. This can be achieved by embedding the p largest Kronecker blocks only. The other right Kronecker indices $s_1, \cdots, s_{n_r-p}$ of the augmented pencil will then be unchanged.

Proof. From the inequalities in the previous theorem it is clear that

$$o'_\ell + o'_r \geq o_\ell + \sum_{j=1}^{n-r'} s_j = o_\ell + o_r - \sum_{j=n-r'+1}^{n-r} s_j.$$

Since $\mathrm{rank}(F)$ is not affected by the embedding, it follows from Lemma 1 that the maximum increase in number of zeroes satisfies

$$o'_f + o'_\infty - (o_f + o_\infty) \leq \sum_{j=n-r'+1}^{n-r} s_j.$$

The right hand side of this inequality is maximized by taking as many terms as possible, i.e., by taking $r' = r + p$ and hence:

$$o'_f + o'_\infty - (o_f + o_\infty) \leq \sum_{j=n-r-p+1}^{n-r} s_j.$$

Moreover, by using the embedding suggested in (5), (6), and (7) with $Z_\ell = 0 = Z_r$, this upper bound is actually met. Indeed, each block $P_i(\lambda) \overset{\text{def}}{=} \begin{pmatrix} R_i(\lambda) \\ \mathbf{z}_i^T \end{pmatrix}$ with $\mathbf{z}_i^T \neq 0$ is regular and has s_i zeroes. After a permutation of rows in (6) we have that each $P_i(\lambda)$ appears on the diagonal and becomes part of the new regular part $G'_{reg} - \lambda J'_{reg}$ of $G' - \lambda F'$.

The blocks for which $\mathbf{z}_i^T = 0$ decouple, and the corresponding right Kronecker blocks remain intact in the augmented pencil. This can be seen by noting that the right annihilating vectors corresponding to these blocks in the original pencil (4) (or its upper triangular equivalent) remain so for the augmented pencil (6), with the same degrees in λ. Similarly, the left Kronecker blocks remain unaffected for the same reason.

To place the zeroes for an individual Kronecker block, suppose that $R(\lambda) = (\mathbf{b}, A) - \lambda(0, U)$ (with A, U square) is a single right Kronecker block and hence has full row rank for all λ. Suppose we add the single row $\mathbf{z}^T = (\gamma, \mathbf{y}^T)$. Then observe that the finite zeroes of $P(\lambda) \overset{\text{def}}{=} \begin{pmatrix} R(\lambda) \\ \mathbf{z}^T \end{pmatrix}$ are exactly the eigenvalues of the pencil $A + \mathbf{b}\gamma^{-1}\mathbf{y}^T - \lambda U$. We can choose $\gamma = 1$ and choose $\mathbf{y}^T$ by standard pole placement techniques [7], [10]. The vector $\mathbf{y}^T$ always exists and is generally unique. When we choose $\gamma = 0$, $P(\lambda)$ has at least one *infinite* zero. In fact, as long as $\mathbf{z}^T \neq 0$ the number of trailing zeros in that row indicates the number of infinite zeroes in $P(\lambda)$. □

In order to compute the proper rows Z_r, it is necessary to extract the individual right Kronecker blocks. The procedure to do this is described in detail in the next section, but a brief outline is as follows. We first apply the staircase algorithm [12] to extract the right Kronecker part and compute the corresponding indices. We then permute the rows and columns to extract the smallest right Kronecker block into the upper left corner and decouple this block from the rest of the pencil. On the remaining collection of right Kronecker blocks we repeat this step to extract the next smallest right Kronecker block, until all the right Kronecker blocks have been extracted. At each step, to decouple the upper left from the lower right, we annihilate the entries in the lower left block—in a very particular order that completely fills in the upper right block. All the transformations applied are unitary transformations, and the result will be an upper triangular version of the pencil (4), where $s_1 \leq s_2 \leq \cdots \leq s_\ell$.

2.2. *The effect of coupling*

In this section, we illustrate some of the variations in the Kronecker structure that can occur when a row is appended. The scheme suggested above implies that the matrix Z_r has a decoupled form as in (7), which is not strictly required. Since the method proposed below adds one block $P_i(\lambda)$ at a time to the regular part, let us analyze what happens when we append a single row $\mathbf{z}^T = (\mathbf{z}_1^T, \mathbf{z}_2^T)$ to a right Kronecker block $R(\lambda)$ and a regular block $G_{reg} - \lambda F_{reg}$ as in:

$$P(\lambda) = \begin{pmatrix} R(\lambda) & 0 \\ 0 & G_{reg} - \lambda F_{reg} \\ \mathbf{z}_1^T & \mathbf{z}_2^T \end{pmatrix}. \tag{9}$$

If one of $\mathbf{z}_1^T$ or $\mathbf{z}_2^T$ is zero, then the two parts decouple, so let us assume that $\mathbf{z}_1^T \neq 0$ and $\mathbf{z}_2^T \neq 0$. Let λ_* be a zero of $P(\lambda)$ with corresponding left eigenvector $\mathbf{u}^T = (\mathbf{u}_1^T, \mathbf{u}_2^T, v)$. Then $(\mathbf{u}_1^T, v)$ must be a left eigenvector of $\begin{pmatrix} R(\lambda) \\ \mathbf{z}_1^T \end{pmatrix}$ corresponding to eigenvalue λ_*. Once this condition is satisfied, one can always find a $\mathbf{u}_2^T$ satisfying $\mathbf{u}_2^T(G_{reg} - \lambda_* F_{reg}) + v\mathbf{z}_2^T = 0$, to make $\mathbf{u}^T$ the left eigenvector of $P(\lambda_*)$, assuming λ_* is not an eigenvalue of $(G_{reg} - \lambda F_{reg})$. For then $(G_{reg} - \lambda_* F_{reg})$ has full column rank, and a solution always exists.

If λ_*, $\mathbf{u}_2^T$ is an eigenvalue and left eigenvector of $(G_{reg} - \lambda F_{reg})$, then a left eigenvector of $P(\lambda)$ corresponding to λ_* is $(0, \mathbf{u}_2^T, 0)$ regardless of whether or not the pencil $\begin{pmatrix} R(\lambda) \\ \mathbf{z}_1^T \end{pmatrix}$ also has an eigenvalue λ_*. We illustrate this case with the following (3×3) example:

$$P(\lambda) = \begin{pmatrix} R(\lambda) & 0 \\ 0 & G_{reg} - \lambda F_{reg} \\ \mathbf{z}_1^T & \zeta_2 \end{pmatrix} = \begin{pmatrix} (\lambda \;\; 1) & \\ & \lambda \\ \mathbf{z}_1^T & \zeta_2 \end{pmatrix}.$$

Regardless of the choice of $\mathbf{z}_1^T$, ζ_2, the entire pencil has an eigenvalue 0 with left eigenvector $(0, 1, 0)$. If we set $\mathbf{z}_1^T = (0, 1)$ so that the pencil $\begin{pmatrix} R(\lambda) \\ \mathbf{z}_1^T \end{pmatrix}$ also has an eigenvalue 0, and we set $\zeta_2 = 1$ to couple the two parts together, the resulting pencil is

$$P(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & 0 & \lambda \\ 0 & 1 & 1 \end{pmatrix}.$$

This is a regular pencil with characteristic polynomial $\det P(\lambda) = -\lambda^2$. It has a double, defective eigenvalue at zero. If the two parts are decoupled by setting $\zeta_2 = 0$, the characteristic polynomial remains unchanged, but the double eigenvalue at zero becomes nondefective. If instead we set $\mathbf{z}_t^T = (1, 1)$, the characteristic polynomial becomes $\lambda - \lambda^2$, yielding sample eigenvalues at 0 and 1, independent of the choice of ζ_2. So if both individual pencils $\begin{pmatrix} R(\lambda) \\ \mathbf{z}_1^T \end{pmatrix}$ and $(G_{reg} - \lambda F_{reg})$ have a common eigenvalue, that eigenvalue may remain in the full pencil $P(\lambda)$ with a Jordan chain combined from the Jordan chains from the individual pencils, or else the common eigenvalue may have a new independent Jordan chain. This implies that when placing new zeroes, it is best to avoid any existing zeroes if one wants to keep Jordan chains as short as possible.

We can summarize some of this discussion with the following theorem.

Theorem 3. *Consider the augmented pencil* (5). *As long as the newly placed zeroes do not coincide with any zeroes already existing in* $G_{reg} - \lambda F_{reg}$, *the block* Z_r *may be computed to place those zeroes independent of* Z_{reg}, Z_ℓ. *If new zeroes are placed over existing ones, their Jordan chains might or might not coalesce.*

The same comment applies whenever common zeroes are chosen between added blocks $P_i(\lambda) = \begin{pmatrix} R_i(\lambda) \\ \mathbf{z}_i^T \end{pmatrix}$ since coupling is likely to occur via the nonzero elements above diagonal in (6) as well. Since this paper focuses only on the placement of zeroes and not their Jordan structure, we do not pursue this discussion here.

3. Computational procedure

We sketch an algorithm to place the zeroes. The algorithm consists of three stages. In the first stage, we essentially compute the generalized Schur form (3), separating the left Kronecker part from the regular part and the right Kronecker part. This can be carried out using the staircase algorithm [12] [2]; we will not discuss this process in detail. In the second stage, we extract the individual Kronecker blocks, via a new elimination procedure proposed in this paper. Finally, in the third stage we compute the rows to be appended in order to place the zeroes. The entire extraction process (the first two stages) is carried out using only unitary transformations, hence it enjoys a backward stability property. The zero placing part also enjoys a backward stability property, hence the numerical stability of the whole process is favorable.

The broad steps of the process are as follows.

Algorithm 1. Zero Placement

1. Use the staircase algorithm [12], [2] to extract the left and right Kronecker part from the pencil. Assume the left Kronecker part of the pencil is the $(m \times n)$-pencil $G_\ell - \lambda F_\ell$, with $n = m + k$. Choose $p \leq k$ as the number of rows to appends.

2. Extract the individual Kronecker blocks from the staircase form.

3. To the largest Kronecker blocks compute the row(s) that must be appended in order to place the desired zeroes.

4. Append the computed rows and back-transform through all the accumulated basis transformations back to the original basis for the given pencil $G - \lambda F$.

We fill in some of the details for each step in turn.

3.1. Staircase form

The staircase form is a form that can be obtained from the original pencil that exposes the various Kronecker indices and orders of a pencil, as well as exposing any regular part. In fact, the presence of a regular part can be determined by the staircase algorithm, except that it may not be as numerically reliable as the approach in [3]. (This is still an open area of research.) The algorithm used to extract the left and right Kronecker part is the variant described in [2] to which we refer the reader for all the details, particularly on how to handle the presence of a regular part or left Kronecker blocks.

The result of this algorithm is of the typical form shown in Figure 1.

$$
\begin{aligned}
G_\ell - \lambda F_\ell = & \begin{bmatrix} G_{1,1} & G_{1,2} & G_{1,3} & \cdots & G_{1,k} & G_{1,k+1} \\ 0 & G_{2,2} & G_{2,3} & \cdots & G_{2,k} & G_{2,k+1} \\ 0 & 0 & G_{3,3} & \cdots & G_{3,k} & G_{3,k+1} \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & G_{k,k} & G_{k,k+1} \end{bmatrix} \\
& - \lambda \begin{bmatrix} 0 & F_{1,2} & F_{1,3} & F_{1,4} & \cdot & F_{1,k+1} \\ 0 & 0 & F_{2,3} & F_{2,4} & \cdot\ \cdot & F_{2,k+1} \\ 0 & 0 & 0 & F_{3,4} & \cdot\ \cdot & F_{3,k+1} \\ 0 & 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & F_{k,k+1} \end{bmatrix},
\end{aligned}
\tag{10}
$$

Figure 1. Typical staircase form.

where the matrices $F_{i,i+1}$ are $n_i \times n_i$ nonsingular and the matrices $G_{i,i}$ are $n_i \times n_{i-1}$ and of full row rank n_i. Therefore, the sequence $\{n_i, i = 1, \ldots, k\}$ is *nonincreasing* and its *dual* sequence consists of the right Kronecker indices $\{s_i, i - 1, \ldots, n_r\}$, i.e.,

- there are $n_{i+1} - n_i$ indices equal to i for $i = 0, \ldots, k$

where we have assumed $n_{k+1} = 0$. Notice that this implies that if the smallest Kronecker index is $s_1 = q$, then the first q matrices $G_{i,i}$ for $i = 1, \ldots, q$ are square invertible as well.

Finally, we need to use a variant of the above form where the square matrices $F_{i,i+1}$ are *upper-triangular* and the rectangular matrices $G_{i,i}$ have leading zero columns and a trailing *upper-triangular* matrix. This form can always be obtained as explained in [2] and will be exploited in the subsequent steps.

3.2. Extract individual Kronecker blocks

The next step is to extract the individual Kronecker blocks. We do this by permuting toward the upper left the entries of the matrix corresponding to the Kronecker block of interest, and then annihilating the coupling elements in the appropriate off-diagonal block. Because of the nature of the Kronecker structure, it is necessary to first extract the smallest Kronecker block, then extract the next smallest from what is left, and so on.

It is easier to illustrate the process and then describe it formally. Let q be the number of leading diagonal G blocks that are square. That is, let q be the numbers such that $G_{11}, \ldots, G_{qq}$ in (10) are square, but $G_{q+1,q+1}$ is not. Then the smallest Kronecker index is $s_1 = q$. In the example worked out below we assumed that there are 3 such blocks, so the smallest Kronecker index is $s_1 = 3$. In this particular example, we must thus form a (3×4)-block. We form this block from the 1,1 entries of those first 3 square G blocks together with the 1,1 entries of the corresponding F blocks. That is, we permute the rows and columns of G, F to collect together the upper left entries of all the leading square G blocks. This will form a leading (3×4)-submatrix. Then we must decouple this leading (3×4)-submatrix by eliminating the coupling entries.

To show how this works in more detail, we partition all the blocks of Figure 1 to expose the 1,1 scalar entries, showing the result as Figure 2. We denote scalar entries by g, f, column vectors by $\mathbf{g}$, $\mathbf{f}$, row vectors by $\mathbf{g}'$, $\mathbf{f}'$ (completely unrelated to the transpose of any corresponding column vector), and submatrices by G, F. Note that potentially the row, column, and matrix blocks could be empty. We permute the leading 1,1 entries into the upper left position to obtain Figure 3 in which the (3×4)-block is exposed.

We then decouple the (3×4) Kronecker block from the rest by annihilating one by one the column entries in the lower left part. This is done with (almost) alternating left and right unitary transformations. The entries in the lower left are annihilating in a very particular order, starting with the "outer" diagonal strip.

In this example, the first items eliminated are the entries in the outer diagonal (marked with $\bar{}$ in Figure 3), eliminated in order: $\bar{\mathbf{g}}_{34}$, $\bar{\mathbf{f}}_{24}$, $\bar{\mathbf{g}}_{23}$, $\bar{\mathbf{f}}_{13}$, $\bar{g}_{12}$. Then the next diagonal entries are eliminated in order: $\mathbf{g}_{24}$, $\mathbf{f}_{14}$, $\mathbf{g}_{13}$. Then finally $\mathbf{g}_{14}$ is eliminated. The entries in G are eliminated using unitary transformations from the right, and the entries in F using transformations from the left, as illustrated in the appendix. Each elimination results in a fill in the corresponding position in the upper right block. This example is sufficiently general to show the pattern of fills in the $\mathbf{g}$, $\mathbf{f}$ part for the general case.

The result is shown in Figure 4. In Figure 4, $\hat{}$ denotes entries that were modified from Figure 3, $\hat{0}$ denotes entries that were purposely eliminated, and $\hat{\times}$ denotes entries that were filled in during this process.

We summarize the process as follows.

Algorithm 2. Kronecker Block Extraction

0. Start with an $(m \times n)$-pencil $G - \lambda F$ in staircase form, with $k = n - m$. Let $q = s_1$ $(0 \leq q \leq k)$ be the number of leading diagonal G blocks that are square in the staircase form (Figure 2).

$$
\left[\begin{array}{cc|cc|cc|cc|c|c}
g_{11} & \mathbf{g}'_{11} & g_{12} & \mathbf{g}'_{12} & g_{13} & \mathbf{g}'_{13} & g_{14} & \mathbf{g}'_{14} & \mathbf{g}'_{15} & \cdots \\
0 & G_{11} & \mathbf{g}_{12} & G_{12} & \mathbf{g}_{13} & G_{13} & \mathbf{g}_{14} & G_{14} & G_{15} & \cdots \\
\hline
0 & 0 & g_{22} & \mathbf{g}'_{22} & g_{23} & \mathbf{g}'_{23} & g_{24} & \mathbf{g}'_{24} & \mathbf{g}'_{25} & \cdots \\
0 & 0 & 0 & G_{22} & \mathbf{g}_{23} & G_{23} & \mathbf{g}_{24} & G_{24} & G_{25} & \cdots \\
\hline
0 & 0 & 0 & 0 & g_{33} & \mathbf{g}'_{33} & g_{34} & \mathbf{g}'_{34} & \mathbf{g}'_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & G_{33} & \mathbf{g}_{34} & G_{34} & G_{35} & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
$$

$$
\left[\begin{array}{cc|cc|cc|cc|c|c}
0 & 0 & f_{12} & \mathbf{f}'_{12} & f_{13} & \mathbf{f}'_{13} & f_{14} & \mathbf{f}'_{14} & \mathbf{f}'_{15} & \cdots \\
0 & 0 & 0 & F_{12} & \mathbf{f}_{13} & F_{13} & \mathbf{f}_{14} & F_{14} & F_{15} & \cdots \\
\hline
0 & 0 & 0 & 0 & f_{23} & \mathbf{f}'_{23} & f_{24} & \mathbf{f}'_{24} & \mathbf{f}'_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & F_{23} & \mathbf{f}_{24} & F_{24} & F_{25} & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & f_{34} & \mathbf{f}'_{34} & \mathbf{f}'_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
$$

Figure 2. Partitioned staircase form.

$$
\left[\begin{array}{cccc|cccccc}
g_{11} & g_{12} & g_{13} & g_{14} & \mathbf{g}'_{11} & \mathbf{g}'_{12} & \mathbf{g}'_{13} & \mathbf{g}'_{14} & \mathbf{g}'_{15} & \cdots \\
0 & g_{22} & g_{23} & g_{24} & 0 & \mathbf{g}'_{22} & \mathbf{g}'_{23} & \mathbf{g}'_{24} & \mathbf{g}'_{25} & \cdots \\
0 & 0 & g_{33} & g_{34} & 0 & 0 & \mathbf{g}'_{33} & \mathbf{g}'_{34} & \mathbf{g}'_{35} & \cdots \\
\hline
0 & \bar{\mathbf{g}}_{12} & \mathbf{g}_{13} & \mathbf{g}_{14} & G_{11} & G_{12} & G_{13} & G_{14} & G_{15} & \cdots \\
0 & 0 & \bar{\mathbf{g}}_{23} & \mathbf{g}_{24} & 0 & G_{22} & G_{23} & G_{24} & G_{25} & \cdots \\
0 & 0 & 0 & \bar{\mathbf{g}}_{34} & 0 & 0 & G_{33} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
$$

$$
\left[\begin{array}{cccc|cccccc}
0 & f_{12} & f_{13} & f_{14} & 0 & \mathbf{f}'_{12} & \mathbf{f}'_{13} & \mathbf{f}'_{14} & \mathbf{f}'_{15} & \cdots \\
0 & 0 & f_{23} & f_{24} & 0 & 0 & \mathbf{f}'_{23} & \mathbf{f}'_{24} & \mathbf{f}'_{25} & \cdots \\
0 & 0 & 0 & f_{34} & 0 & 0 & 0 & \mathbf{f}'_{34} & \mathbf{f}'_{35} & \cdots \\
\hline
0 & 0 & \bar{\mathbf{f}}_{13} & \mathbf{f}_{14} & 0 & F_{12} & F_{13} & F_{14} & F_{15} & \cdots \\
0 & 0 & 0 & \bar{\mathbf{f}}_{24} & 0 & 0 & F_{23} & F_{24} & F_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
$$

Figure 3. Permuted form.

1. Permute rows and columns of the pencil so that the "1, 1" entries of the blocks G_{ij}, F_{ij}, $i = 1, \cdots, q$, $j = 1, \cdots, q+1$, are in the upper left, as in Figure 3. Denote the partitioned $(m \times n)$-pencil by

$$
\begin{pmatrix} G^{(1,1)} & G^{(1,2)} \\ G^{(2,1)} & G^{(2,2)} \end{pmatrix} - \lambda \begin{pmatrix} F^{(1,1)} & F^{(1,2)} \\ F^{(2,1)} & F^{(2,2)} \end{pmatrix},
$$

where each block is partitioned as in Figure 3. The leading block $G^{(1,1)} -$

$$\left[\begin{array}{cccc|cccccc}
g_{11} & \hat{g}_{12} & \hat{g}_{13} & \hat{g}_{14} & \hat{\mathbf{g}}'_{11} & \hat{\mathbf{g}}'_{12} & \hat{\mathbf{g}}'_{13} & \mathbf{g}'_{14} & \mathbf{g}'_{15} & \cdots \\
0 & \hat{g}_{22} & \hat{g}_{23} & \hat{g}_{24} & \hat{\times} & \hat{\mathbf{g}}'_{22} & \hat{\mathbf{g}}'_{23} & \hat{\mathbf{g}}'_{24} & \hat{\mathbf{g}}'_{25} & \cdots \\
0 & 0 & \hat{g}_{33} & \hat{g}_{34} & \hat{\times} & \hat{\times} & \hat{\mathbf{g}}'_{33} & \hat{\mathbf{g}}'_{34} & \hat{\mathbf{g}}'_{35} & \cdots \\
\hline
0 & \hat{0} & \hat{0} & \hat{0} & \hat{G}_{11} & \hat{G}_{12} & \hat{G}_{13} & \hat{G}_{14} & \hat{G}_{15} & \cdots \\
0 & 0 & \hat{0} & \hat{0} & 0 & \hat{G}_{22} & \hat{G}_{23} & \hat{G}_{24} & \hat{G}_{25} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{G}_{33} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]$$

$$\left[\begin{array}{cccc|cccccc}
0 & \hat{f}_{12} & \hat{f}_{13} & \hat{f}_{14} & \hat{\times} & \hat{\mathbf{f}}'_{12} & \hat{\mathbf{f}}'_{13} & \mathbf{f}'_{14} & \mathbf{f}'_{15} & \cdots \\
0 & 0 & \hat{f}_{23} & \hat{f}_{24} & \hat{\times} & \hat{\times} & \hat{\mathbf{f}}'_{23} & \hat{\mathbf{f}}'_{24} & \hat{\mathbf{f}}'_{25} & \cdots \\
0 & 0 & 0 & \hat{f}_{34} & \hat{\times} & \hat{\times} & \hat{\times} & \hat{\mathbf{f}}'_{34} & \hat{\mathbf{f}}'_{35} & \cdots \\
\hline
0 & 0 & \hat{0} & \hat{0} & 0 & \hat{F}_{12} & \hat{F}_{13} & \hat{F}_{14} & \hat{F}_{15} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{F}_{23} & \hat{F}_{24} & \hat{F}_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]$$

Figure 4. Result from extraction of smallest Kronecker block.

$\lambda F^{(1,1)}$ is $q \times (q+1)$. In the partitioning of Figure 3 for $1 \le i \le q$, G_{ii} are $k \times k$, so that $G_{ii}^{(1,1)}, G_{ii}^{(1,2)}, G_{ii}^{(2,1)}, G_{ii}^{(2,2)}$ are, respectively, 1×1, $1 \times (k-1)$, $(k-1) \times 1$, $(k-1) \times (k-1)$.

2. Eliminate the entries $G_{ij}^{(2,1)}$, $F_{ij}^{(2,1)}$, in the permuted matrices. This modifies parts of all four blocks (1,1), (1,2), (2,1), (2,2).
   2.1 For $i = q, \cdots, 2, 1$:
   2.2 For $j = 1, \cdots, 2, 1$:
   2.3 Push $G_{j,j+1+k-i}^{(2,1)}$ right into $G_{j,j}^{(2,2)}$.
   2.4 If $j > 1$, Push $F_{j-1,j+1+k-i}^{(2,1)}$ up into $F_{i+k-i,j+1+k-i}^{(1,1)}$.

   The resulting modified "(1,1)" block is not further modified by this algorithm, so we denote it by $G_{11}^{[2]} - \lambda F_{11}^{[2]}$. The modified "(2,2)" block is still in staircase form. The modified "(1,2)" block is full and the modified "(2,1)" block is all zero.

3. If $k \ge 1$, apply this algorithm recursively to the $(m-k) \times (m-k-1)$-pencil $G^{(2,2)} - \lambda F^{(2,2)}$. Apply all the resulting unitary transformations from the right also to the block $G^{(1,2)} - \lambda F^{(1,2)}$.

In the above algorithm description, we use the short hand "push right" and "push up" to mean the following.

- Push $G_{\alpha\beta}^{(2,1)}$ right into $G_{\alpha\gamma}^{(2,2)}$ means: Find a unitary transformation Q_7 such that $(G_{\alpha\beta}^{(2,1)}, G_{\alpha\gamma}^{(2,2)})Q_7 = (0, \tilde{G}_{\alpha\gamma}^{(2,2)})$, where $\tilde{G}_{\alpha\gamma}^{(2,2)}$ is upper triangular. Then apply the transformation to the entire pencil: i.e., for all ι, compute $(\tilde{G}_{\iota\beta}^{(2,1)}$,

$\tilde{G}^{(2,2)}_{\iota\gamma}) = (G^{(2,1)}_{\iota\beta}, G^{(2,2)}_{\iota\gamma})Q_7$, and replace $G^{(2,1)}_{\iota\beta}$, $G^{(2,2)}_{\iota\gamma}$ with $\tilde{G}^{(2,1)}_{\iota\beta}$, $\tilde{G}^{(2,2)}_{\iota\gamma}$. Do likewise with $F^{(2,1)}_{\iota\beta}$, $F^{(2,2)}_{\iota\gamma}$.

- Push $F^{(2,1)}_{\alpha\beta}$ up into $F^{(1,1)}_{\gamma\beta}$ means: Find a unitary transformation Q_8 such that

$$Q_8 \begin{pmatrix} F^{(1,1)}_{\gamma\beta} \\ F^{(2,1)}_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \tilde{F}^{(1,1)}_{\gamma\beta} \\ 0 \end{pmatrix}$$

Then apply the transformation to the entire pencil: i.e., for all ι compute

$$\begin{pmatrix} \tilde{F}^{(1,1)}_{\gamma\iota} \\ \tilde{F}^{(2,1)}_{\alpha\iota} \end{pmatrix} = Q_8 \begin{pmatrix} F^{(1,1)}_{\gamma\iota} \\ F^{(2,1)}_{\alpha\iota} \end{pmatrix}$$

and replace $F^{(1,1)}_{\gamma\iota}$, $F^{(2,1)}_{\alpha\iota}$ with $\tilde{F}^{(1,1)}_{\gamma\iota}$, $\tilde{F}^{(2,1)}_{\alpha\iota}$, respectively. Do likewise with $F^{(1,1)}_{\gamma\iota}$, $F^{(2,1)}_{\alpha\iota}$.

The final form of this extraction process is

$$\begin{aligned} Q^{[2]}_{\mathrm{L}}(G^{[1]} - \lambda F^{[1]})Q^{[2]}_{\mathrm{R}} &= (G^{[2]} - \lambda F^{[2]}) \\ &= \begin{pmatrix} G^{[2]}_{11} & \cdots & G^{[2]}_{1k} \\ & \ddots & \vdots \\ 0 & & G^{[2]}_{kk} \end{pmatrix} - \lambda \begin{pmatrix} F^{[2]}_{11} & \cdots & F^{[2]}_{1k} \\ & \ddots & \vdots \\ 0 & & F^{[2]}_{kk} \end{pmatrix}, \end{aligned} \tag{11}$$

where for each $i = 1, \cdots, k$, $G^{[2]}_{ii}$ is $s_i \times (s_i + 1)$, upper trapezoidal, and $F^{[2]}_{ii} = (0, U_i)$ with U_i $s_i \times s_i$, upper triangular, where $s_i \equiv s_{k+1-i}$ are the right Kronecker indices defined as in Theorem 2, in nondecreasing order. The $s_i \times (s_i + 1)$-pencil $G^{[2]}_{ii} - \lambda F^{[2]}_{ii}$ has full row rank s_i for all values λ. Of course, this is not the Kronecker Canonical Form, but it is an analogous form achievable via unitary transformations. Each diagonal block is equivalent to a single Kronecker block. For almost most any purpose for which the Kronecker form would be required, one can make use of this form just as well.

We remark that in this extraction process, all the entries that are annihilated are never filled in during subsequent steps. Assume we use Givens rotations to annihilate the entries. Applying each rotation costs $O(n)$ operations, including the cost of accumulating the Givens rotations. At most $O(n^2)$ such rotations are generated, since there are no more than $O(n^2)$ entries to annihilate (we can't annihilate more entries than there are in the whole matrix!). Hence the entire extraction process takes $O(n^3)$ operations. Of course, a more precise analysis is possible, but it is difficult because the exact cost will range from free to $O(n^3)$ depending on the exact distribution of the Kronecker indices.

3.3. Place zeroes

To the form (11) we can compute the p rows needed to place $s \stackrel{\text{def}}{=} s_{n_r} + \cdots + s_{n_r - p+1}$ zeroes. Let $\mu_1, \cdots, \mu_s$ be the given set of new zeroes to be placed. Then the rows to append have the following form

$$Z^{[2]} = \begin{pmatrix} 0 & \cdots & 0 & \mathbf{z}^T_{n_r-p+1} & 0 & \cdot & \cdots & 0 \\ \vdots & & \vdots & 0 & \mathbf{z}^T_{n_r-p+2} & \cdot & & \vdots \\ \vdots & & \vdots & \vdots & \ddots & & \ddots & \vdots \\ \vdots & & \vdots & \vdots & & & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \mathbf{z}^T_{n_r} \end{pmatrix}, \tag{12}$$

where we have $n - s = s_1 + \cdots + s_{n_r - p}$ leading columns of zeroes, and the nonzero entries are computed as described below. For each $i = n_r, n_r - 1, \cdots, n_r - p + 1$, each $\mathbf{z}_i^T$ is an s_i-vector chosen so that the square $(s_i + 1) \times (s_i + 1)$-pencil

$$\begin{pmatrix} G^{[2]}_{n_r+1-i, n_r+1-i} \\ \mathbf{z}_i^T \end{pmatrix} - \lambda \begin{pmatrix} F^{[2]}_{n_r+1-i, n_r+1-i} \\ 0 \end{pmatrix} \tag{13}$$

has zeroes $\{\mu_{\hat{s}_i+j}\}_{j=1}^{s_i}$. Recall from (11) that the pencil (13) has the general form $(\mathbf{b}_i, A_i) - \lambda(0, U_i)$ where U_i is square, upper triangular, and nonsingular. Let $\mathbf{z}_i^T = (\gamma, \mathbf{y}_i^T)$. Then observe that the zeroes of $\begin{pmatrix} \mathbf{b}_i & A_i - \lambda U_i \\ \gamma_i & \mathbf{y}_i^T \end{pmatrix}$ are exactly the eigenvalues of the pencil $A_i + \mathbf{b}_i \gamma_i^{-1} \mathbf{y}_i^T - \lambda U_i$. We can choose $\gamma_i = 1$ and choose $\mathbf{y}_i^T$ by standard pole placement techniques [7], [10]. In this case, the new row is generally unique.

To see that the $\{\mathbf{z}_i^T\}$ chosen in this way places the zeroes for the entire pencil, we can permute the rows to put the new regular part of the pencil in the lower right and the remaining right Kronecker structure in the upper left:

$$\begin{pmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ 0 & \tilde{G}_{22} \end{pmatrix} - \lambda \begin{pmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ 0 & \tilde{F}_{22} \end{pmatrix}.$$

where

$$\tilde{G}_{22} - \lambda \tilde{F}_{22} = \begin{pmatrix} \begin{pmatrix} G^{[2]}_{n_r-p+1, n_r-p+1} \\ \mathbf{z}^T_{n_r-p+1} \end{pmatrix} & \cdots & \begin{pmatrix} * \\ 0 \end{pmatrix} \\ & \ddots & \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & & \begin{pmatrix} G^{[2]}_{n_r, n_r} \\ \mathbf{z}^T_{n_r} \end{pmatrix} \end{pmatrix}$$

$$- \lambda \begin{pmatrix} \begin{pmatrix} F^{[2]}_{n_r-p+1, n_r-p+1} \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} * \\ 0 \end{pmatrix} \\ & \ddots & \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} F^{[2]}_{n_r, n_r} \\ 0 \end{pmatrix} \end{pmatrix}$$

is regular with the desired zeroes, and

$$G_{11} - \lambda F_{11} = \begin{pmatrix} G_{11}^{[2]} & \cdots & G_{1,n_r-p}^{[2]} \\ & \ddots & \vdots \\ 0 & & G_{n_r-p,n_r-p}^{[2]} \end{pmatrix} - \lambda \begin{pmatrix} F_{11}^{[2]} & \cdots & F_{1,n_r-p}^{[2]} \\ & \ddots & \vdots \\ 0 & & F_{n_r-p,n_r-p}^{[2]} \end{pmatrix}$$

has the right Kronecker structure left over.

Returning to the original pencil $G - \lambda F$, our original problem was to compute a $(p \times m)$-matrix Z to place s zeroes. We collect together all the transformations to obtain the formula for Z:

$$\begin{pmatrix} Q_{\mathrm{L}}^{[2]} Q_{\mathrm{L}}^{[1]} & \\ & I \end{pmatrix} \left[\begin{pmatrix} G \\ Z \end{pmatrix} - \lambda \begin{pmatrix} F \\ 0 \end{pmatrix} \right] Q_{\mathrm{R}}^{[1]} Q_{\mathrm{R}}^{[2]} = \begin{pmatrix} G^{[2]} \\ Z^{[2]} \end{pmatrix} - \lambda \begin{pmatrix} F^{[2]} \\ 0 \end{pmatrix}$$

so that

$$Z \stackrel{\mathrm{def}}{=} Z^{[2]} (Q_{\mathrm{R}}^{[2]})^H (Q_{\mathrm{R}}^{[1]})^H,$$

where $\Box^H$ denotes the conjugate transpose of $\Box$.

4. Conclusion

We have examined the general problem of placing the generalized eigenvalues to an arbitrary matrix pencil by the addition of new rows of constant coefficients. We found that the number of zeroes that can be placed is limited to the order of the right Kronecker part (the sum of all the right Kronecker indices). In addition, when p rows are added and there are multiple right Kronecker indices, the number of zeroes that can be placed is limited to the sum of the p largest right Kronecker indices. When the number of rows added is just right to make the system square, then the number of zeroes that can be placed is equal to the sum of all the right Kronecker indices.

We have outlined a new method based entirely on unitary transformations to compute the right Kronecker indices and to extract the individual Kronecker blocks. By combining this procedure with pole placement algorithms in the literature, we arrive at a complete method for assigning the generalized eigenvalues for a pencil, that has good numerical stability properties because of the use of unitary transformations. From a control point of view, this method places the transmission zeroes by the synthesis of new outputs. It could also just as easily be used to synthesize inputs.

The new decomposition in which the individual Kronecker blocks are extracted represents a unitary analog to the Kronecker Canonical Form (KCF) in much the same way as the Schur decomposition is a unitary analog to the Jordan canonical form.

Appendix

We show the process to decouple the (3×4) Kronecker block from the rest by annihilating one by one the entries in the lower left, starting with the state of Figure 3. This is done with almost alternating left and right unitary transformations. The entries in the lower left are annihilating in a very particular order, starting with the "outer" diagonal strip (marked with an overbar "–" in Figure 3). We show the step-by-step annihilations for this particular example, where the rows and columns affected by the individual Givens rotation are marked with arrows, the entries used to construct the rotation are enclosed in boxes (e.g., $\boxed{\tilde{0}}$ denotes the entry being annihilated), and all the other entries modified in that step are marked with a wide tilde "~". Fills (zeroes made nonzero) are denoted by "$\times\ \times$" when they occur and "$\hat{\times}$" in subsequent steps. Likewise, entries set to zero are denoted "$\boxed{\tilde{0}}$" the first time and "$\hat{0}$" in subsequent steps. The hat "ˆ" denotes entries changed from Figure 3.

In this particular example, Steps 1 through 5 zero out the outer diagonal strip of G F blocks (marked "–" in Figure 3), Steps 6 through 8 zero out the next strip of each, and Step 9 zeroes out the last, resulting in the situation of Figure 4. Note also that the G_{34}, F_{34} blocks and all the entries below and to their right remain unchanged through this whole process.

Situation of Figure 3:

$$
\left[\begin{array}{cccc|cccccc}
g_{11} & g_{12} & g_{13} & g_{14} & \mathbf{g}'_{11} & \mathbf{g}'_{12} & \mathbf{g}'_{13} & \mathbf{g}'_{14} & \mathbf{g}'_{15} & \cdots \\
0 & g_{22} & g_{23} & g_{24} & 0 & \mathbf{g}'_{22} & \mathbf{g}'_{23} & \mathbf{g}'_{24} & \mathbf{g}'_{25} & \cdots \\
0 & 0 & g_{33} & g_{34} & 0 & 0 & \mathbf{g}'_{33} & \mathbf{g}'_{34} & \mathbf{g}'_{35} & \cdots \\
\hline
0 & \mathbf{g}_{12} & \mathbf{g}_{13} & \mathbf{g}_{14} & G_{11} & G_{12} & G_{13} & G_{14} & G_{15} & \cdots \\
0 & 0 & \mathbf{g}_{23} & \mathbf{g}_{24} & 0 & G_{22} & G_{23} & G_{24} & G_{25} & \cdots \\
0 & 0 & 0 & \mathbf{g}_{34} & 0 & 0 & G_{33} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
\left[\begin{array}{cccc|cccccc}
0 & f_{12} & f_{13} & f_{14} & 0 & \mathbf{f}'_{12} & \mathbf{f}'_{13} & \mathbf{f}'_{14} & \mathbf{f}'_{15} & \cdots \\
0 & 0 & f_{23} & f_{24} & 0 & 0 & \mathbf{f}'_{23} & \mathbf{f}'_{24} & \mathbf{f}'_{25} & \cdots \\
0 & 0 & 0 & f_{34} & 0 & 0 & 0 & \mathbf{f}'_{34} & \mathbf{f}'_{35} & \cdots \\
\hline
0 & 0 & \mathbf{f}_{13} & \mathbf{f}_{14} & 0 & F_{12} & F_{13} & F_{14} & F_{15} & \cdots \\
0 & 0 & 0 & \mathbf{f}_{24} & 0 & 0 & F_{23} & F_{24} & F_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
$$

Step 1: Rotate from right:

$$
\begin{array}{c}
\begin{array}{cccccccccc}
 & & & \Downarrow & & & \Downarrow & & &
\end{array} \\
\left[\begin{array}{cccc|cccccc}
g_{11} & g_{12} & g_{13} & \widetilde{g_{14}} & \mathbf{g}'_{11} & \mathbf{g}'_{12} & \widetilde{\mathbf{g}'_{13}} & \mathbf{g}'_{14} & \mathbf{g}'_{15} & \cdots \\
0 & g_{22} & g_{23} & \widetilde{g_{24}} & 0 & \mathbf{g}'_{22} & \widetilde{\mathbf{g}'_{23}} & \mathbf{g}'_{24} & \mathbf{g}'_{25} & \cdots \\
0 & 0 & g_{33} & \widetilde{g_{34}} & 0 & 0 & \widetilde{\mathbf{g}'_{33}} & \mathbf{g}'_{34} & \mathbf{g}'_{35} & \cdots \\
\hline
0 & \mathbf{g}_{12} & \mathbf{g}_{13} & \widetilde{\mathbf{g}_{14}} & G_{11} & G_{12} & \widetilde{G_{13}} & G_{14} & G_{15} & \cdots \\
0 & 0 & \mathbf{g}_{23} & \widetilde{\mathbf{g}_{24}} & 0 & G_{22} & \widetilde{G_{23}} & G_{24} & G_{25} & \cdots \\
0 & 0 & 0 & \boxed{\tilde{0}} & 0 & 0 & \boxed{\widetilde{G_{33}}} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
 & & & \Downarrow & & & \Downarrow & & &
\end{array} \\
\left[\begin{array}{cccc|cccccc}
0 & f_{12} & f_{13} & \widetilde{f_{14}} & 0 & \mathbf{f}'_{12} & \widetilde{\mathbf{f}'_{13}} & \mathbf{f}'_{14} & \mathbf{f}'_{15} & \cdots \\
0 & 0 & f_{23} & \widetilde{f_{24}} & 0 & 0 & \widetilde{\mathbf{f}'_{23}} & \mathbf{f}'_{24} & \mathbf{f}'_{25} & \cdots \\
0 & 0 & 0 & \widetilde{f_{34}} & 0 & 0 & \times\times & \mathbf{f}'_{34} & \mathbf{f}'_{35} & \cdots \\
\hline
0 & 0 & \mathbf{f}_{13} & \widetilde{\mathbf{f}_{14}} & 0 & F_{12} & \widetilde{F_{13}} & F_{14} & F_{15} & \cdots \\
0 & 0 & 0 & \widetilde{\mathbf{f}_{24}} & 0 & 0 & \widetilde{F_{23}} & F_{24} & F_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
\end{array}
$$

Step 2: Rotate from left:

$$
\left[\begin{array}{cccc|cccccc}
g_{11} & g_{12} & g_{13} & \hat{g}_{14} & \mathbf{g}'_{11} & \mathbf{g}'_{12} & \hat{\mathbf{g}}'_{13} & \mathbf{g}'_{14} & \mathbf{g}'_{15} & \cdots \\
0 & g_{22} & g_{23} & \hat{g}_{24} & 0 & \mathbf{g}'_{22} & \hat{\mathbf{g}}'_{23} & \mathbf{g}'_{24} & \mathbf{g}'_{25} & \cdots \\
0 & 0 & \widetilde{g_{33}} & \widetilde{g_{34}} & 0 & \times\times & \widetilde{\mathbf{g}'_{33}} & \widetilde{\mathbf{g}'_{34}} & \widetilde{\mathbf{g}'_{35}} & \Leftarrow \\
\hline
0 & \mathbf{g}_{12} & \mathbf{g}_{13} & \hat{\mathbf{g}}_{14} & G_{11} & G_{12} & \hat{G}_{13} & G_{14} & G_{15} & \cdots \\
0 & 0 & \widetilde{\mathbf{g}_{23}} & \widetilde{\mathbf{g}_{24}} & 0 & \widetilde{G_{22}} & \widetilde{G_{23}} & \widetilde{G_{24}} & \widetilde{G_{25}} & \Leftarrow \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{G}_{33} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
\left[\begin{array}{cccc|cccccc}
0 & f_{12} & f_{13} & f_{14} & 0 & \mathbf{f}'_{12} & \hat{\mathbf{f}}'_{13} & \mathbf{f}'_{14} & \mathbf{f}'_{15} & \cdots \\
0 & 0 & f_{23} & f_{24} & 0 & 0 & \hat{\mathbf{f}}'_{23} & \mathbf{f}'_{24} & \mathbf{f}'_{25} & \cdots \\
0 & 0 & 0 & \boxed{\widetilde{f_{34}}} & 0 & 0 & \widetilde{\times} & \widetilde{\mathbf{f}'_{34}} & \widetilde{\mathbf{f}'_{35}} & \Leftarrow \\
\hline
0 & 0 & \mathbf{f}_{13} & \hat{\mathbf{f}}_{14} & 0 & F_{12} & \hat{F}_{13} & F_{14} & F_{15} & \cdots \\
0 & 0 & 0 & \boxed{\tilde{0}} & 0 & 0 & \widetilde{F_{23}} & \widetilde{F_{24}} & \widetilde{F_{25}} & \Leftarrow \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
$$

Step 3: Rotate from right:

$$
\begin{array}{c}
\begin{array}{cccccccccc} \phantom{g_{11}} & \phantom{g_{12}} & \Downarrow & \phantom{g_{14}} & \phantom{g_{11}} & \Downarrow & \phantom{g_{13}} & \phantom{g_{14}} & \phantom{g_{15}} & \end{array} \\
\left[\begin{array}{cccc|cccccc}
g_{11} & g_{12} & \widetilde{g_{13}} & \hat{g}_{14} & \mathbf{g}'_{11} & \widetilde{\mathbf{g}'_{12}} & \hat{\mathbf{g}}'_{13} & \mathbf{g}'_{14} & \mathbf{g}'_{15} & \cdots \\
0 & g_{22} & \widetilde{g_{23}} & \hat{g}_{24} & 0 & \mathbf{g}'_{22} & \hat{\mathbf{g}}'_{23} & \hat{\mathbf{g}}'_{24} & \hat{\mathbf{g}}'_{25} & \cdots \\
0 & 0 & \widetilde{g_{33}} & \hat{g}_{34} & 0 & \times & \hat{\mathbf{g}}'_{33} & \hat{\mathbf{g}}'_{34} & \hat{\mathbf{g}}'_{35} & \cdots \\ \hline
0 & \mathbf{g}_{12} & \widetilde{\mathbf{g}_{13}} & \hat{\mathbf{g}}_{14} & G_{11} & \widetilde{G_{12}} & \hat{G}_{13} & G_{14} & G_{15} & \cdots \\
0 & 0 & \boxed{\widetilde{0}} & \hat{\mathbf{g}}_{24} & 0 & \boxed{\widetilde{G_{22}}} & \hat{G}_{23} & \hat{G}_{24} & \hat{G}_{25} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{G}_{33} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
\end{array}
\quad
\begin{array}{c}
\begin{array}{cccccccccc} & \phantom{f_{12}} & \Downarrow & \phantom{f_{14}} & & \Downarrow & \phantom{f_{13}} & \phantom{f_{14}} & \phantom{f_{15}} & \end{array} \\
\left[\begin{array}{cccc|cccccc}
0 & f_{12} & \widetilde{f_{13}} & f_{14} & 0 & \widetilde{\mathbf{f}'_{12}} & \hat{\mathbf{f}}'_{13} & \mathbf{f}'_{14} & \mathbf{f}'_{15} & \cdots \\
0 & 0 & \widetilde{f_{23}} & f_{24} & 0 & \times\times & \hat{\mathbf{f}}'_{23} & \hat{\mathbf{f}}'_{24} & \hat{\mathbf{f}}'_{25} & \cdots \\
0 & 0 & 0 & f_{34} & 0 & 0 & \hat{\times} & \hat{\mathbf{f}}'_{34} & \hat{\mathbf{f}}'_{35} & \cdots \\ \hline
0 & 0 & \mathbf{f}_{13} & \mathbf{f}_{14} & 0 & \widetilde{F_{12}} & \hat{F}_{13} & F_{14} & F_{15} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \hat{F}_{23} & \hat{F}_{24} & \hat{F}_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
\end{array}
$$

Step 4: Rotate from left:

$$
\left[\begin{array}{cccc|cccccc}
g_{11} & g_{12} & \hat{g}_{13} & \hat{g}_{14} & \mathbf{g}'_{11} & \hat{\mathbf{g}}'_{12} & \hat{\mathbf{g}}'_{13} & \mathbf{g}'_{14} & \mathbf{g}'_{15} & \cdots \\
0 & \widetilde{g_{22}} & \widetilde{g_{23}} & \widetilde{g_{24}} & \times\times & \widetilde{\mathbf{g}'_{22}} & \widetilde{\mathbf{g}'_{23}} & \widetilde{\mathbf{g}'_{24}} & \widetilde{\mathbf{g}'_{25}} & \Leftarrow \\
0 & 0 & \hat{g}_{33} & \hat{g}_{34} & 0 & \times & \hat{\mathbf{g}}'_{33} & \hat{\mathbf{g}}'_{34} & \hat{\mathbf{g}}'_{35} & \cdots \\ \hline
0 & \widetilde{\mathbf{g}_{12}} & \widetilde{\mathbf{g}_{13}} & \widetilde{\mathbf{g}_{14}} & \widetilde{G_{11}} & \widetilde{G_{12}} & \widetilde{G_{13}} & \widetilde{G_{14}} & \widetilde{G_{15}} & \Leftarrow \\
0 & 0 & \hat{0} & \hat{\mathbf{g}}_{24} & 0 & \hat{G}_{22} & \hat{G}_{23} & \hat{G}_{24} & \hat{G}_{25} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{G}_{33} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
\quad
\left[\begin{array}{cccc|cccccc}
0 & f_{12} & f_{13} & f_{14} & 0 & \hat{\mathbf{f}}'_{12} & \hat{\mathbf{f}}'_{13} & \mathbf{f}'_{14} & \mathbf{f}'_{15} & \cdots \\
0 & 0 & \boxed{\widetilde{f_{23}}} & \widetilde{f_{24}} & 0 & \widetilde{\times} & \widetilde{\mathbf{f}'_{23}} & \widetilde{\mathbf{f}'_{24}} & \widetilde{\mathbf{f}'_{25}} & \Leftarrow \\
0 & 0 & 0 & f_{34} & 0 & 0 & \hat{\times} & \hat{\mathbf{f}}'_{34} & \hat{\mathbf{f}}'_{35} & \cdots \\ \hline
0 & 0 & \boxed{\widetilde{0}} & \widetilde{\mathbf{f}_{14}} & 0 & \widetilde{F_{12}} & \widetilde{F_{13}} & \widetilde{F_{14}} & \widetilde{F_{15}} & \Leftarrow \\
0 & 0 & 0 & 0 & 0 & 0 & \hat{F}_{23} & \hat{F}_{24} & \hat{F}_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
$$

Step 5: Rotate from right:

$$
\begin{array}{c}
\begin{array}{cccccccccc} \phantom{g_{11}} & \Downarrow & \phantom{g_{13}} & \phantom{g_{14}} & \Downarrow & \phantom{g_{12}} & \phantom{g_{13}} & \phantom{g_{14}} & \phantom{g_{15}} & \end{array} \\
\left[\begin{array}{cccc|cccccc}
g_{11} & \widetilde{g_{12}} & \hat{g}_{13} & \hat{g}_{14} & \widetilde{\mathbf{g}'_{11}} & \hat{\mathbf{g}}'_{12} & \hat{\mathbf{g}}'_{13} & \mathbf{g}'_{14} & \mathbf{g}'_{15} & \cdots \\
0 & \widetilde{g_{22}} & \hat{g}_{23} & \hat{g}_{24} & \times & \hat{\mathbf{g}}'_{22} & \hat{\mathbf{g}}'_{23} & \hat{\mathbf{g}}'_{24} & \hat{\mathbf{g}}'_{25} & \cdots \\
0 & 0 & \hat{g}_{33} & \hat{g}_{34} & 0 & \times & \hat{\mathbf{g}}'_{33} & \hat{\mathbf{g}}'_{34} & \hat{\mathbf{g}}'_{35} & \cdots \\ \hline
0 & \boxed{\widetilde{0}} & \hat{\mathbf{g}}_{13} & \hat{\mathbf{g}}_{14} & \boxed{\widetilde{G_{11}}} & \hat{G}_{12} & \hat{G}_{13} & G_{14} & \hat{G}_{15} & \cdots \\
0 & 0 & \hat{0} & \hat{\mathbf{g}}_{24} & 0 & \hat{G}_{22} & \hat{G}_{23} & \hat{G}_{24} & \hat{G}_{25} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{G}_{33} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
\end{array}
\quad
\begin{array}{c}
\begin{array}{cccccccccc} & \Downarrow & \phantom{f_{13}} & \phantom{f_{14}} & \Downarrow & \phantom{f_{12}} & \phantom{f_{13}} & \phantom{f_{14}} & \phantom{f_{15}} & \end{array} \\
\left[\begin{array}{cccc|cccccc}
0 & \widetilde{f_{12}} & f_{13} & f_{14} & \widetilde{\times\times} & \hat{\mathbf{f}}'_{12} & \hat{\mathbf{f}}'_{13} & \hat{\mathbf{f}}'_{14} & \hat{\mathbf{f}}'_{15} & \cdots \\
0 & 0 & f_{23} & f_{24} & 0 & \hat{\times} & \hat{\mathbf{f}}'_{23} & \hat{\mathbf{f}}'_{24} & \hat{\mathbf{f}}'_{25} & \cdots \\
0 & 0 & 0 & f_{34} & 0 & 0 & \hat{\times} & \hat{\mathbf{f}}'_{34} & \hat{\mathbf{f}}'_{35} & \cdots \\ \hline
0 & 0 & \hat{0} & \mathbf{f}_{14} & 0 & \hat{F}_{12} & \hat{F}_{13} & \hat{F}_{14} & \hat{F}_{15} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{F}_{23} & \hat{F}_{24} & \hat{F}_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
\end{array}
$$

Step 6: Rotate from right:

$$
\begin{array}{c}
\begin{array}{cccccccccc} \phantom{g_{11}} & \phantom{g_{12}} & \phantom{g_{13}} & \Downarrow & \phantom{g_{11}} & \Downarrow & \phantom{g_{13}} & \phantom{g_{14}} & \phantom{g_{15}} & \end{array} \\
\left[\begin{array}{cccc|cccccc}
g_{11} & \hat{g}_{12} & \hat{g}_{13} & \widetilde{g_{14}} & \hat{\mathbf{g}}'_{11} & \widetilde{\mathbf{g}'_{12}} & \hat{\mathbf{g}}'_{13} & \hat{\mathbf{g}}'_{14} & \hat{\mathbf{g}}'_{15} & \cdots \\
0 & \hat{g}_{22} & \hat{g}_{23} & \widetilde{g_{24}} & \hat{\times} & \widetilde{\mathbf{g}'_{22}} & \hat{\mathbf{g}}'_{23} & \hat{\mathbf{g}}'_{24} & \hat{\mathbf{g}}'_{25} & \cdots \\
0 & 0 & \hat{g}_{33} & \widetilde{g_{34}} & 0 & \widetilde{\times} & \hat{\mathbf{g}}'_{33} & \hat{\mathbf{g}}'_{34} & \hat{\mathbf{g}}'_{35} & \cdots \\ \hline
0 & \hat{0} & \hat{\mathbf{g}}_{13} & \widetilde{\mathbf{g}_{14}} & \hat{G}_{11} & \widetilde{G_{12}} & \hat{G}_{13} & G_{14} & \hat{G}_{15} & \cdots \\
0 & 0 & \hat{0} & \boxed{\widetilde{0}} & 0 & \boxed{\widetilde{G_{22}}} & \hat{G}_{23} & \hat{G}_{24} & \hat{G}_{25} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{G}_{33} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
\end{array}
\quad
\begin{array}{c}
\begin{array}{cccccccccc} & \phantom{f_{12}} & \phantom{f_{13}} & \Downarrow & & \Downarrow & \phantom{f_{13}} & \phantom{f_{14}} & \phantom{f_{15}} & \end{array} \\
\left[\begin{array}{cccc|cccccc}
0 & f_{12} & f_{13} & \widetilde{f_{14}} & \hat{\times} & \widetilde{\mathbf{f}'_{12}} & \hat{\mathbf{f}}'_{13} & \hat{\mathbf{f}}'_{14} & \hat{\mathbf{f}}'_{15} & \cdots \\
0 & 0 & f_{23} & \widetilde{f_{24}} & 0 & \widetilde{\times} & \hat{\mathbf{f}}'_{23} & \hat{\mathbf{f}}'_{24} & \hat{\mathbf{f}}'_{25} & \cdots \\
0 & 0 & 0 & \widetilde{f_{34}} & 0 & \widetilde{\times\times} & \hat{\times} & \hat{\mathbf{f}}'_{34} & \hat{\mathbf{f}}'_{35} & \cdots \\ \hline
0 & 0 & \hat{0} & \widetilde{\mathbf{f}_{14}} & 0 & \widetilde{F_{12}} & \hat{F}_{13} & \hat{F}_{14} & \hat{F}_{15} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{F}_{23} & \hat{F}_{24} & \hat{F}_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
\end{array}
$$

Step 7: Rotate from left:

$$
\left[\begin{array}{cccc|cccccc}
g_{11} & \hat{g}_{12} & \hat{g}_{13} & \hat{g}_{14} & \hat{\mathbf{g}}'_{11} & \hat{\mathbf{g}}'_{12} & \hat{\mathbf{g}}'_{13} & \mathbf{g}'_{14} & \hat{\mathbf{g}}'_{15} & \cdots \\
0 & \hat{g}_{22} & \hat{g}_{23} & \hat{g}_{24} & \hat{\times} & \hat{\mathbf{g}}'_{22} & \hat{\mathbf{g}}'_{23} & \hat{\mathbf{g}}'_{24} & \hat{\mathbf{g}}'_{25} & \cdots \\
0 & 0 & \widetilde{g_{33}} & \widetilde{g_{34}} & \widetilde{\times\times} & \widetilde{\times} & \widetilde{\mathbf{g}'_{33}} & \widetilde{\mathbf{g}'_{34}} & \widetilde{\mathbf{g}'_{35}} & \Leftarrow \\ \hline
0 & \hat{0} & \widetilde{\mathbf{g}_{13}} & \widetilde{\mathbf{g}_{14}} & \widetilde{G_{11}} & \widetilde{G_{12}} & \widetilde{G_{13}} & \widetilde{G_{14}} & \widetilde{G_{15}} & \Leftarrow \\
0 & 0 & \hat{0} & \hat{0} & 0 & \hat{G}_{22} & \hat{G}_{23} & \hat{G}_{24} & \hat{G}_{25} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{G}_{33} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
\quad
\left[\begin{array}{cccc|cccccc}
0 & f_{12} & f_{13} & f_{14} & \hat{\times} & \hat{\mathbf{f}}'_{12} & \hat{\mathbf{f}}'_{13} & \hat{\mathbf{f}}'_{14} & \hat{\mathbf{f}}'_{15} & \cdots \\
0 & 0 & f_{23} & f_{24} & 0 & \hat{\times} & \hat{\mathbf{f}}'_{23} & \hat{\mathbf{f}}'_{24} & \hat{\mathbf{f}}'_{25} & \cdots \\
0 & 0 & 0 & \boxed{\widetilde{f_{34}}} & 0 & \widetilde{\times} & \widetilde{\times} & \widetilde{\mathbf{f}'_{34}} & \widetilde{\mathbf{f}'_{35}} & \Leftarrow \\ \hline
0 & 0 & \hat{0} & \boxed{\widetilde{0}} & 0 & \widetilde{F_{12}} & \widetilde{F_{13}} & \widetilde{F_{14}} & \widetilde{F_{15}} & \Leftarrow \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{F}_{23} & \hat{F}_{24} & \hat{F}_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
$$

Step 8: Rotate from right:

$$
\begin{array}{c}
\begin{array}{cccccccccc} \phantom{g_{11}} & \phantom{g_{12}} & \Downarrow & \phantom{g_{14}} & \Downarrow & \phantom{g_{12}} & \phantom{g_{13}} & \phantom{g_{14}} & \phantom{g_{15}} & \end{array} \\
\left[\begin{array}{cccc|cccccc}
g_{11} & \hat{g}_{12} & \widetilde{g_{13}} & \hat{g}_{14} & \widetilde{\mathbf{g}'_{11}} & \hat{\mathbf{g}}'_{12} & \hat{\mathbf{g}}'_{13} & \hat{\mathbf{g}}'_{14} & \hat{\mathbf{g}}'_{15} & \cdots \\
0 & \hat{g}_{22} & \widetilde{g_{23}} & \hat{g}_{24} & \hat{\times} & \hat{\mathbf{g}}'_{22} & \hat{\mathbf{g}}'_{23} & \hat{\mathbf{g}}'_{24} & \hat{\mathbf{g}}'_{25} & \cdots \\
0 & 0 & \widetilde{g_{33}} & \hat{g}_{34} & \widetilde{\times} & \times & \hat{\mathbf{g}}'_{33} & \hat{\mathbf{g}}'_{34} & \hat{\mathbf{g}}'_{35} & \cdots \\ \hline
0 & \hat{0} & \boxed{\widetilde{0}} & \hat{\mathbf{g}}_{14} & \boxed{\widetilde{G_{11}}} & \hat{G}_{12} & \hat{G}_{13} & \hat{G}_{14} & \hat{G}_{15} & \cdots \\
0 & 0 & \hat{0} & \hat{0} & 0 & \hat{G}_{22} & \hat{G}_{23} & \hat{G}_{24} & \hat{G}_{25} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{G}_{33} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
\end{array}
\quad
\begin{array}{c}
\begin{array}{cccccccccc} & \phantom{f_{12}} & \Downarrow & \phantom{f_{14}} & \Downarrow & \phantom{f_{12}} & \phantom{f_{13}} & \phantom{f_{14}} & \phantom{f_{15}} & \end{array} \\
\left[\begin{array}{cccc|cccccc}
0 & f_{12} & \widetilde{f_{13}} & f_{14} & \widetilde{\times} & \hat{\mathbf{f}}'_{12} & \hat{\mathbf{f}}'_{13} & \hat{\mathbf{f}}'_{14} & \hat{\mathbf{f}}'_{15} & \cdots \\
0 & 0 & \widetilde{f_{23}} & f_{24} & \widetilde{\times\times} & \hat{\times} & \hat{\mathbf{f}}'_{23} & \hat{\mathbf{f}}'_{24} & \hat{\mathbf{f}}'_{25} & \cdots \\
0 & 0 & 0 & f_{34} & 0 & \hat{\times} & \hat{\times} & \hat{\mathbf{f}}'_{34} & \hat{\mathbf{f}}'_{35} & \cdots \\ \hline
0 & 0 & \hat{0} & 0 & 0 & \hat{F}_{12} & \hat{F}_{13} & \hat{F}_{14} & \hat{F}_{15} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \hat{F}_{23} & \hat{F}_{24} & \hat{F}_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
\end{array}
$$

Step 9: Rotate from right, yielding situation of Figure 4:

$$
\begin{array}{cc}
\begin{array}{cccccccccc} & & & \Downarrow & \Downarrow & & & & & \end{array} & \begin{array}{cccccccccc} & & & \Downarrow & \Downarrow & & & & & \end{array} \\
\left[\begin{array}{cccc|cccccc}
e_{11} & \hat{e}_{12} & \hat{e}_{13} & \widetilde{e_{14}} & \widetilde{\mathbf{g}'_{11}} & \hat{\mathbf{g}}'_{12} & \hat{\mathbf{g}}'_{13} & \mathbf{g}'_{14} & \mathbf{g}'_{15} & \cdots \\
0 & \hat{g}_{22} & \hat{g}_{23} & \widetilde{g_{24}} & \tilde{\times} & \hat{\mathbf{g}}'_{22} & \hat{\mathbf{g}}'_{23} & \hat{\mathbf{g}}'_{24} & \hat{\mathbf{g}}'_{25} & \cdots \\
0 & 0 & \hat{g}_{33} & \widetilde{g_{34}} & \tilde{\times} & \hat{\times} & \hat{\mathbf{g}}'_{33} & \hat{\mathbf{g}}'_{34} & \hat{\mathbf{g}}'_{35} & \cdots \\
\hline
0 & \hat{0} & \hat{0} & \boxed{\tilde{0}} & \boxed{\widetilde{G_{11}}} & \hat{G}_{12} & \hat{G}_{13} & \hat{G}_{14} & \hat{G}_{15} & \cdots \\
0 & 0 & \hat{0} & \hat{0} & 0 & \hat{G}_{22} & \hat{G}_{23} & \hat{G}_{24} & \hat{G}_{25} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{G}_{33} & G_{34} & G_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & G_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} & \cdots
\end{array}\right]
&
\left[\begin{array}{cccc|cccccc}
0 & \hat{f}_{12} & \hat{f}_{13} & \widetilde{f_{14}} & \tilde{\times} & \hat{\mathbf{f}}'_{12} & \hat{\mathbf{f}}'_{13} & \hat{\mathbf{f}}'_{14} & \hat{\mathbf{f}}'_{15} & \cdots \\
0 & 0 & \hat{f}_{23} & \widetilde{f_{24}} & \tilde{\times} & \hat{\times} & \hat{\mathbf{f}}'_{23} & \hat{\mathbf{f}}'_{24} & \hat{\mathbf{f}}'_{25} & \cdots \\
0 & 0 & 0 & \widetilde{f_{34}} & \widetilde{\times\times} & \hat{\times} & \hat{\times} & \hat{\mathbf{f}}'_{34} & \hat{\mathbf{f}}'_{35} & \cdots \\
\hline
0 & 0 & \hat{0} & \hat{0} & 0 & \hat{F}_{12} & \hat{F}_{13} & \hat{F}_{14} & \hat{F}_{15} & \cdots \\
0 & 0 & 0 & \hat{0} & 0 & 0 & \hat{F}_{23} & \hat{F}_{24} & \hat{F}_{25} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{34} & F_{35} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right]
\end{array}
$$

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