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SPEECH MODELLING AND THE TRIGONOMETRIC MOMENT PROBLEM

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Abstract

It is shown that the partial trigonometric moment problem provides an appropriate unifying framework for some speech modelling techniques like the line spectral pairs and composite sinusoidal wave model recently proposed by Itakura et al., the eigenmodel of Pisarenko used for formant extraction and the classical autoregressive model on which LPC is based. This moment problem is equivalent to an extension problem in the class of impedance functions and hence has a simple circuit theoretical interpretation. The connection with the classical power moment problem is also established.

1. Introduction

Among the many techniques existing today for the modelling of discrete time signals by rational functions, the linear prediction or autoregressive (AR) model is undoubtedly one of the most powerful, especially in the field of speech processing [1]). By its very nature, this model provides a simple speech synthesis technique as well as an analysis tool for the estimation of the formants or resonant frequencies of the vocal tract. The popularity of the AR method can be justified by many arguments but among these certainly emerges the fact that efficient computational procedures for the derivation of the model are available as well as robust structures for its implementation. In particular, the reflection or partial correlation (PARCOR) coefficients have become a classical characterization of the AR model [2]). A few years ago, Itakura and Sugamura [3]) showed however that an AR model could equivalently be described in terms of the so-called line spectral pairs leading to a characterization which in some respects is more efficient than the classical PARCOR method. At about the same time, Sagayama and Itakura [4]) also proposed a new technique for speech synthesis, called the composite sinusoidal wave model, in which the time signal is represented as a sum of several sinusoidal waves. The amplitudes and frequencies of these sinusoids are adjusted so as to match a limited number of the signal autocorrelation lags. Gueguen [5]) for his part introduced still another speech model based on the eigenvector associated with the smallest eigenvalue of the autocorrelation matrix. This eigenmodel, as it is called, is mainly used for formant extraction in speech analysis and for the investigation of the singular case in linear prediction which may occur for example when insufficient pre-emphasis is applied [6]. The location of the zeros of the corresponding eigenfilter was further examined by Makhoul [7]. It was soon recognized [6,8] that the eigenmodel was in fact closely related to an earlier method developed by Pisarenko [9] in the field of geophysics for the retrieval of harmonics from a discrete time signal corrupted by additive background noise. A modification of Pisarenko's method was proposed by Kung [8] to cope with some of the limitations inherent in its original formulation.

Facing such a variety of methods for the modelling of discrete time signals one is naturally led to ask whether there exists a unifying approach in which these different methods would naturally fit as particular cases of a general solution to some basic modelling problem. The purpose of this paper is precisely to answer this question and to show that a unifying framework is provided by the classical trigonometric moment problem in which the work of Caratheodory, Schur and Szegö play an important role.

In sec. 2, one reviews the most important properties of the AR model and emphasizes the fact that the first autocorrelation lags of the model output signal coincide with those of the original signal. This reconstruction of the power density function by the model is in fact a trigonometric moment problem and is reformulated in sec. 3 as Caratheodory's extension problem in the class of impedance functions. The solution to this problem is generally not unique but can be parametrized in terms of an arbitrary reflectance or Schur function. In sec. 4, one shows that the AR model, Sagayama's model and Pisarenko's model can be derived from this general solution as particular cases corresponding to some special choice for the arbitrary reflectance. The connection with Itakura's line spectral pairs for the estimation of the formants and as a representation of the AR model is also mentioned. In sec. 5 it is shown that the general solution to Caratheodory's extension problem can be represented as a cascade of lossless two-ports closely related to the Richards section and terminated on an arbitrary passive impedance. Finally, one shows in sec. 6 that Sagayama's composite sinusoidal wave model can be reformulated as a Gauss quadrature formula and the exact connection with the classical power moment problem is established.

2. Spectral estimation

The AR model in fig. 1 consists of an all-pole filter $A_p(z) = 1 + \sum_{1}^{p} A_{p,l} z^{-l}$ excited by a periodic pulse train in the case of voiced speech and by white noise for unvoiced speech[1,10]). The importance of this approach lies in the well-known fact that the output signal of the AR model provides in both situations a power density function which reproduces that of the original signal in the following sense. Given a sequence $\{s_n\}$ of windowed speech samples, its power density function $P(\Theta)$ has a Fourier expansion $P(\Theta) = \sum_{-\infty}^{+\infty} c_l e^{il\Theta}$ where

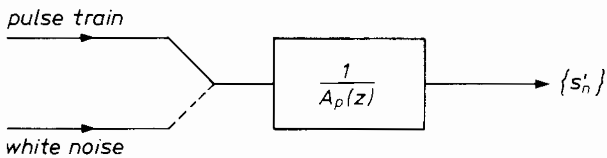

Fig. 1.

the autocorrelation lags c_l are given by

$$c_l = c_{-l} = \sum_{n=-\infty}^{+\infty} s_n s_{n-l}. \tag{1}$$

Note that both summations have in fact finite range, due to the windowing effect. The output of the AR model is a sequence $\{s_n'\}$ and the envelope of its power density function is

$$P'(\Theta) = \sigma_p^2/|A_p(e^{i\Theta})|^2, \tag{2}$$

where σ_p^2 is the energy of the input signal. It was shown by Whittle[11]) that the Fourier expansion $P'(\Theta) = \sum_{-\infty}^{+\infty} c_l' e^{il\Theta}$ satisfies

$$c_l' = c_l \quad \text{for} \quad l \in [-p, p]. \tag{3}$$

Thus, the number of identical autocorrelation lags for the model output and for the actual signal is determined by the degree of the all-pole filter. Before discussing this point in more detail, let us briefly recall some of the important properties of the AR model. The coefficient vector $A_p = [1, A_{p,1}, \ldots, A_{p,p}]^T$ is obtained by solving the linear system

$$C_p A_p = [\sigma_p^2, 0, \ldots, 0]^T, \tag{4}$$

where C_p is the symmetric Toeplitz matrix built on the first $p+1$ autocorrelation lags

$$C = \begin{bmatrix} c_0 & c_1 & \ldots & c_p \\ c_1 & c_0 & \ldots & c_{p-1} \\ \vdots & \vdots & & \vdots \\ c_p & c_{p-1} & \ldots & c_0 \end{bmatrix}. \tag{5}$$

The stability of the model, i.e. the fact that $A_p(z)$ has all its zeros in the open unit disk $|z| < 1$, is a direct consequence of the fact that C_p is positive definite [12]). (The case of singular non-negative definite autocorrelation matrices which occurs in particular for purely harmonic signals will be touched upon in sec. 4.) The zeros of $A_p(z)$ close to the unit circle provide an estimation for the formants. The linear system (4) can efficiently be solved via the Levinson recursion

$$A_0(z) = 1, \quad A_j(z) = A_{j-1}(z) + k_j z^{-1} \hat{A}_{j-1}(z), \quad j \in [1, p], \tag{6}$$

where $\hat{A}_j(z)$ stands for the reciprocal polynomial $z^{-j} A_j(z^{-1})$ and where k_j, known as the j-th reflection coefficient or partial correlation coefficient, is given by

$$k_j = -[c_j + c_{j-1} A_{j-1,1} + \ldots + c_1 A_{j-1,j-1}]/\sigma_{j-1}^2. \tag{7}$$

The stability of $A_p(z)$ is equivalent to the property

$$|k_j| < 1 \qquad j \in [1, p]. \tag{8}$$

Polynomial $A_p(z)$ of the autoregressive model can also be interpreted as the p-step linear predictor which estimates a sample s_n of the speech signal as a linear combination of the p preceding samples,

$$s_n' \simeq -\sum_{l=1}^{p} A_{p,l}\, s_{n-l}. \tag{9}$$

Minimisation of the prediction error $\sum_{-\infty}^{+\infty} (s_n - s_n')^2$ with respect to the coefficients $A_{p,1}, A_{p,2}, \ldots, A_{p,p}$ leads to the same linear system (4) where σ_p^2 must now be interpreted as the energy of the residual prediction error, for which the following recursion formula holds

$$\sigma_0^2 = c_0, \quad \sigma_j^2 = (1 - k_j^2)\, \sigma_{j-1}^2, \quad j \in [1, p]. \tag{10}$$

Let us now go back to the power density spectra of the original signal and of the AR model. In view of (2) and (3) the all-pole filter $A_p(z)$ can be considered as a particular solution to a more general problem, known in the mathematical literature as the *partial trigonometric moment problem* [12]), which consists in finding all non-negative functions $P'(\Theta)$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P'(\Theta)\, \mathrm{e}^{\mathrm{i}l\Theta}\, \mathrm{d}\Theta = c_l, \qquad l \in [-p, p]. \tag{11}$$

Since functions of the distribution type may also be included in the solution, the problem is more accurately stated as that of finding all non-decreasing measures $\mu(\Theta)$ for which the first $p+1$ moments have the prescribed values $c_0, c_1, \ldots, c_p$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{il\Theta}\, d\mu(\Theta) = c_l, \qquad l \in [-p, p]. \tag{12}$$

3. Caratheodory's extension problem

In loose terms, a non-negative function like $P'(\Theta)$ can be thought of as being the real part of a passive impedance $f(z)$ evaluated on the unit circle $z = e^{i\Theta}$. Consequently, it is not surprising that an equivalent formulation for the trigonometric moment problem (12) consists in finding all impedance functions $f(z)$ (i.e. analytic and with non-negative real part in the domain $1 < |z| \leqslant \infty$) having a Maclaurin expansion of the form

$$f(z) = c_0 + 2\sum_{l=1}^{p} c_l z^{-l} + 2\sum_{l=p+1}^{\infty} c_l' z^{-l}. \tag{13}$$

Indeed, with any non-decreasing measure $\mu(\Theta)$ one can associate an impedance function $f(z)$ via the Riesz–Herglotz representation [12])

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\Theta} + z^{-1}}{e^{i\Theta} - z^{-1}}\, d\mu(\Theta), \tag{14}$$

which will satisfy (13) in view of (12). Conversely, given an impedance function $f(z)$ one can derive a unique non-decreasing measure satisfying (14) by a limiting process on the real part, namely

$$\mu(\Theta) = \lim_{r \to 1^+} \int_0^{\Theta} \operatorname{Re} f(r\, e^{i\phi})\, d\phi \tag{15}$$

and one can verify then that (13) is equivalent to (12) in view of (14) and (15). Due to this equivalence, the partial trigonometric moment problem can be converted into *Caratheodory's extension problem in the class of positive functions* namely to find the set of all analytic functions with non-negative real part in $|z| > 1$ and whose first $p+1$ Maclaurin coefficients coincide with $c_0, 2c_1, \ldots, 2c_p$. On the other hand, a positive function is characterized by the fact that the Toeplitz matrices of increasing order built on the Maclaurin coefficients are all non-negative definite [12]). Consequently, Caratheodory's extension problem can be equivalently formulated as finding, for the given positive definite matrix C_p, all non-negative definite Toeplitz extensions

$$C_{p+s} = \begin{bmatrix} c_0 & c_1 & \dots & c_p & c'_{p+1} & \dots & c'_{p+s} \\ c_1 & c_0 & \dots & c_{p-1} & c_p & & c'_{p+s-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_p & c_{p-1} & \dots & c_0 & c_1 & \dots & c'_s \\ c'_{p+1} & c_p & & c_1 & c_0 & \dots & c'_{s-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c'_{p+s} & c'_{p+s-1} & & c'_s & c'_{s-1} & \dots & c_0 \end{bmatrix}, \quad s \in [1, \infty). \quad (16)$$

The solution to this problem can be expressed in terms of polynomial $A_p(z)$ of the AR model and a second polynomial $R_p(z)$ generated by the same recursion formula (6) but with a change of signs of the reflection coefficients,

$$R_0(z) = c_0, \quad R_j(z) = R_{j-1}(z) - k_j z^{-1} \hat{R}_{j-1}(z), \qquad j \in [1, p]. \qquad (17)$$

These polynomials are in fact closely related to Szegö's orthogonal polynomials of the first and second kind [13]). In view of the similarity between (17) and (6), $A_p(z)$ and $R_p(z)$ are obviously not independent. It turns out that they are linked by the relation

$$R_p(z) = A_p(z)\,(c_0 + 2c_1 z^{-1} + \dots + 2c_p z^{-p}) + 0(z^{-p-1}). \qquad (18)$$

Another useful identity, which can be derived from (6) and (17), is the following

$$A_p(z)\,\hat{R}_p(z) + \hat{A}_p(z)\,R_p(z) = 2\sigma_p^2 z^{-p}, \qquad (19)$$

with σ_p^2 defined as in (4). Finally, let us recall that a function $\varphi(z)$ analytic and satisfying $|\varphi(z)| \leqslant 1$ in $|z| > 1$ is called a *Schur function* and corresponds thus to the well-known reflectance function of a passive one-port used in circuit theory. The general solution to Caratheodory's extension problem can be parametrized as a well defined transformation of an arbitrary Schur function $\varphi_{p+1}(z)$ [12]). This has a remarkable expression in terms of the polynomials $A_p(z)$ and $R_p(z)$, namely

$$f(z) = \frac{R_p(z) - z^{-1}\,\varphi_{p+1}(z)\,\hat{R}_p(z)}{A_p(z) + z^{-1}\,\varphi_{p+1}(z)\,\hat{A}_p(z)}\,. \qquad (20)$$

Note, incidentally, that transformations of this type are called homographic transformations. The background of the important result (20) and its circuit theoretical interpretation will be given in sec. 5. For each particular choice of φ_{p+1}, formula (20) yields an impedance function f and hence by (15) a measure satisfying (12). Note that for most applications, φ_{p+1} is restricted to the class of rational Schur functions. In case $f(z)$ is continuous on the unit circle, one has simply $P'(\Theta) = \mathrm{Re} f(\mathrm{e}^{\mathrm{i}\Theta})$ and the spectral factorization of $P'(\Theta)$ yields then the transfer function of the model[14]). As for the formants, they can be estimated by the poles of (20) close to the unit circle. Formula (20) is of central importance for our purpose because Whittle's AR model, Itakura's line spectral pairs and Sagayama's composite sinusoidal wave synthesis as well as Pisarenko's model will be shown to be reducible to special cases of this formula.

4. The models of Whittle, Itakura, Sagayama and Pisarenko

An obviously simple choice in (20) is $\varphi_{p+1}(z) = 0$ giving $f(z) = R_p(z)/A_p(z)$, a result consistent with (18). Hence, one has by (19)

$$P'(\Theta) = \operatorname{Re} f(e^{i\Theta}) = \sigma_p^2/|A_p(e^{i\Theta})|^2, \tag{21}$$

which is exactly the power density spectrum (2) of the AR model shown in fig. 1. A classical structure for this model is the feedback lattice filter[10]) where the multipliers are precisely the reflection coefficients $k_l (l = 1, 2, \ldots, p)$ which can conveniently be computed via formula (7) of Levinson's algorithm.

The next simplest choice is $\varphi_{p+1}(z) = 1$ or -1 giving

$$f(z) = \frac{R_p(z) \mp z^{-1}\hat{R}_p(z)}{A_p(z) \pm z^{-1}\hat{A}_p(z)}. \tag{22}$$

Since $f(z)$ is now a reactance of degree $p + 1$ (see sec. 5) its poles are simple and located on the unit circle. Consequently, $f(z)$ admits a partial fraction expansion which, by combining the contributions of complex conjugate poles, can be put under the form

$$f(z) = \sum_{l=1}^{p+1} \varrho_l \frac{e^{i\Theta_l} + z^{-1}}{e^{i\Theta_l} - z^{-1}}, \tag{23}$$

where $\Theta_1, \Theta_2, \ldots, \Theta_{p+1}$ are distinct points on the interval $[0, 2\pi)$ and $\varrho_1, \varrho_2, \ldots, \varrho_{p+1}$ are positive real numbers. The corresponding power density function obtained via (15) is $P'(\Theta) = 2\pi \sum_{l=1}^{p+1} \varrho_l \delta(\Theta - \Theta_l)$ and consists thus of $p + 1$ spectral lines located at $\Theta_l (l = 1, 2, \ldots, p + 1)$. In view of (11), the first $p + 1$ lags of the autocorrelation function are modelled by

$$c_l = \sum_{k=1}^{p+1} \varrho_k e^{-il\Theta_k}, \qquad l \in [-p, p] \tag{24}$$

and the signal itself can be represented within arbitrary phases by

$$s_n' = \sum_{k=1}^{p+1} \sqrt{\varrho_k}\, e^{-in\Theta_k}. \tag{25}$$

For p odd and $\varphi_{p+1}(z) = 1$, all zeros of the denominator in (22) occur in complex conjugate pairs and (24), (25) then become

$$c_l = 2 \sum_{k=1}^{(p+1)/2} \varrho_k \cos l\,\Theta_k, \qquad l \in [-p, p] \tag{26}$$

$$s_n' = 2 \sum_{k=1}^{(p+1)/2} \sqrt{\varrho_k} \cos n\,\Theta_k\,, \tag{27}$$

which is precisely Sagayama's composite sinusoidal wave model[4]). In practice, the derivation of this model consists of Levinson's recursion followed by a factorization and partial fraction expansion to find successively $A_p(z), R_p(z)$, the zeros $e^{-i\Theta_k}$ of the denominator in (22) and the coefficients ϱ_k. Observe that this algorithm for computing Sagayama's model is simpler than the one proposed in the original presentation[4]) and which involves the solution of Hankel and Vandermonde matrix equations. According to the approach presented here, the formants are closely related to the zeros of either $P_p(z) = A_p(z) - z^{-1}\hat{A}_p(z)$ or $Q_p(z) = A_p(z) + z^{-1}\hat{A}_p(z)$, depending on the choice made for $\varphi_{p+1}(z)$. It is interesting to note that the zeros of these polynomials are precisely Itakura's line spectral pairs for the estimation of the formants[3]). On the other hand, it is clear that the all-pole filter of the classical AR model can completely be characterized by $P_p(z)$ and $Q_p(z)$ since

$$A_p(z) = \tfrac{1}{2}[P_p(z) + Q_p(z)] \tag{28}$$

and this relation is the basis for Itakura's line spectral pairs synthesis filter.

As a preparation to Pisarenko's method, recall that the particular choice $\varphi_{p+1}(z) = \pm 1$ in (20) generates a reactance function f. It is known in that case that the Toeplitz matrices C_{p+s} defined in (16) are all singular for $s = 1, 2, \ldots$[12]). From this point of view, Sagayama's composite sinusoidal wave model can be considered as the *singular extension* of the original positive definite matrix C_p. In addition, this minimal degree singular extension has only two solutions corresponding to the arbitrary choice $\varphi_{p+1}(z) = \pm 1$ (see sec. 5).

Let ϱ_0 be the smallest eigenvalue of C_p and let ν be its multiplicity. In Pisarenko's approach one considers the set of Toeplitz matrices $C_l^* = C_l - \varrho_0 I_{l+1}$. Let us denote by $A_l^*(z)$, $R_l^*(z)$, k_l^* and σ_l^{*2} the polynomials, reflection coefficients and residual energy associated with this set of matrices. Since $C_{p-\nu}^*$ is now positive definite and $C_{p-\nu+1}^*, \ldots, C_p^*$ are non-negative definite but singular by construction, the latter matrices can thus be considered as singular extensions of $C_{p-\nu}^*$. In other words, Pisarenko's problem for C_p has been reduced to Sagayama's problem for $C_{p-\nu}^*$ whose solution is uniquely given by the reactance

$$f^*(z) = \frac{R^*_{p-\nu}(z) - z^{-1}\, k^*_{p-\nu+1}\, \hat{R}^*_{p-\nu}(z)}{A^*_{p-\nu}(z) + z^{-1}\, k^*_{p-\nu+1}\, \hat{A}^*_{p-\nu}(z)}, \tag{29}$$

with $k^*_{p-\nu+1}$ equal to $+1$ or -1. Going back to the original problem and denoting by $\mathrm{e}^{-\mathrm{i}\Theta^*_k}$ the zeros of the denominator in (29), one has by applying (23) to the data $(c_0 - \varrho_0, c_1, \ldots, c_{p-\nu})$ instead of $(c_0, c_1, \ldots, c_p)$

$$f(z) = \varrho_0 + \sum_{l=1}^{p-\nu+1} \varrho^*_l \, \frac{\mathrm{e}^{\mathrm{i}\Theta^*_l} + z^{-1}}{\mathrm{e}^{\mathrm{i}\Theta^*_l} - z^{-1}}. \tag{30}$$

In view of (15), the power spectral density is given by

$$P'(\Theta) = \varrho_0 + \sum_{l=1}^{p-\nu+1} \varrho^*_l \, \delta(\Theta - \Theta^*_l) \tag{31}$$

and formula (11) gives then Pisarenko's model for the autocorrelation lags

$$c_l = \varrho_0\, \delta_{0,l} + \sum_{k=1}^{p-\nu+1} \varrho^*_k \, \mathrm{e}^{-\mathrm{i}l\Theta^*_k}, \quad \text{for} \quad l \in [-p, p]. \tag{32}$$

Applying the recurrence formulas (7) and (10) to the matrices C^*_k one finds $\sigma^{*2}_{p-\nu+1} = 0$ which shows, in view of (6), that the polynomial denominator in (29) is built on the unique eigenvector of the matrix $C_{p-\nu+1}$ associated with the smallest eigenvalue ϱ_0. This remark leads us to Pisarenko's own presentation of his method [9]) as an application of one of Caratheodory's theorems where the representation of a set of complex numbers by a sum of imaginary exponentials is considered. Finally, since (31) is a particular solution to the trigonometric moment problem, we know there exists some $\varphi_{p+1}(z)$ such that formula (20) generates the impedance function shown in (30). However there seems to exist no easy way in which to select this particular Schur function from the outset.

Let us close this section with a general remark on system modelling. Any measure $\mu(\Theta)$ can be decomposed as

$$\mu(\Theta) = \mu_0(\Theta) + \mu_1(\Theta), \tag{33}$$

where $\mu_0(\Theta)$ is the *absolutely continuous part* and $\mu_1(\Theta)$ the *singular* part. For the applications considered here, the singular component is a finite discrete measure $\mu_1(\Theta) = \sum_{k=1}^{K} \varrho_k u(\Theta - \Theta_k)$, where $u(x)$ is the unit step-function and for the absolutely continuous part one has $\mathrm{d}\mu_0(\Theta) = |h(\mathrm{e}^{\mathrm{i}\Theta})|^2 \mathrm{d}\Theta$ where $h(z)$ is a stable rational function with a stable inverse. A general model based on the decomposition (33) is shown in fig. 2, where $s_1'(t)$ and $s_0'(t)$ are called the

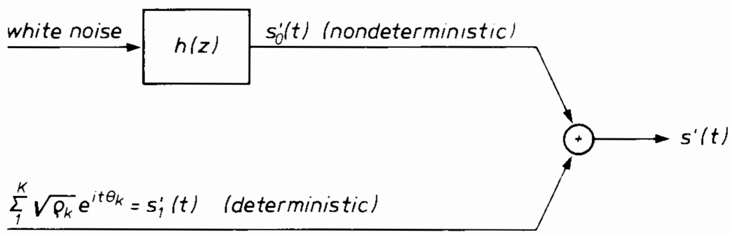

Fig. 2.

deterministic and non-deterministic components of the signal. Whittle's AR model has only a non-deterministic part with $h(z) = A_p^{-1}(z)$, while Sagayama's composite sinusoidal wave model exhibits only a deterministic part with $K = p + 1$. Pisarenko's approach gives a mixed model where $K = p - \nu + 1$ in the deterministic path and where the non-deterministic part reduces to white noise of energy ϱ_0, hence with $h(z) = 1$.

5. Circuit theoretical interpretation

Let us go back to Caratheodory's extension and its general solution (20). With any impedance $f(z)$ one can associate a reflectance or Schur function (for the definition see sec. 3),

$$\varphi(z) = \frac{1 - f(z)}{1 + f(z)} \, . \tag{34}$$

Conversely, if $\varphi(z)$ is a Schur function then $f(z) = [1 - \varphi(z)]/[1 + \varphi(z)]$ is an impedance function. Since the first l coefficients in the Maclaurin expansion of $\varphi(z)$ are uniquely determined by those of $f(z)$, the Caratheodory extension problem can be reformulated as a Schur extension problem, namely to find all Schur functions whose first $p + 1$ Maclaurin coefficients have prescribed values. On the other hand, a Schur function is best characterized in terms of its Schur parameters which are generated through the Schur algorithm as follows. The recurrence formula

$$\varphi_{l+1}(z) = z \, \frac{\varphi_l(z) - \varphi_l(\infty)}{1 - \varphi_l(\infty)\, \varphi_l(z)} \, , \qquad \varphi_0(z) = \varphi(z) \tag{35}$$

defines for $l = 0, 1, 2, \ldots$ a sequence of Schur functions $\varphi_0(z) = \varphi(z)$, $\varphi_1(z)$, $\varphi_2(z)$, ... and a sequence of Schur parameters $\varphi_0(\infty) = (1 - c_0)/(1 + c_0)$, $\varphi_1(\infty)$, $\varphi_2(\infty)$, ... whose moduli are all bounded by unity. A remarkable property of Schur's algorithm is that the Schur parameters $\varphi_1(\infty)$, $\varphi_2(\infty)$, ...

of (34) are precisely the reflection coefficients $k_1, k_2, \ldots$ occurring in Levinson's algorithm [12]). The connection between Levinson's algorithm and a particular type of Darlington synthesis was put forth by Dewilde, Vieira and Kailath [15]). One easily verifies that the first l Schur parameters of $\varphi(z)$ are uniquely determined by the first l Maclaurin coefficients of $\varphi(z)$ and hence by the first l Maclaurin coefficients of $f(z)$. Consequently, the extension problem is that of finding the set of all Schur functions whose first $p+1$ Schur parameters have some prescribed values, given by the reflection coefficients k_l.

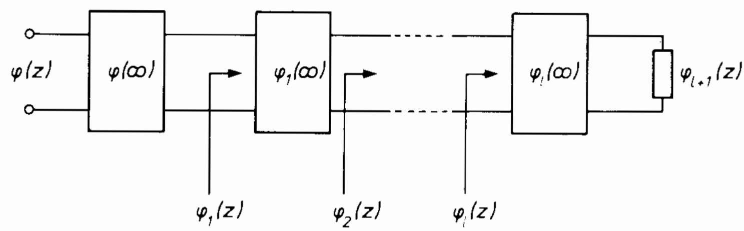

Fig. 3.

From a circuit theory point of view, the recurrence formula (35) corresponds to the cascade synthesis (fig. 3) of the reflectance $\varphi(z)$ by repeated extraction of lossless sections with chain scattering matrices

$$[1 - \varphi_l^2(\infty)]^{-\frac{1}{2}} \begin{bmatrix} z^{-1} & \varphi_l(\infty) \\ z^{-1}\,\varphi_l(\infty) & 1 \end{bmatrix} \tag{36}$$

The reflection coefficient of the remaining impedance after l such steps is $\varphi_{l+1}(z)$. The general solution to Schur's extension problem is thus given by the cascade of length $p+1$ where $\varphi(\infty), \varphi_1(\infty), \varphi_2(\infty), \ldots \varphi_p(\infty)$ have the prescribed values and terminated in an arbitrary reflectance $\varphi_{p+1}(z)$. In view of the remarkable correspondance

$$\varphi_l(\infty) = k_l \qquad (l = 1, 2, \ldots), \tag{37}$$

between Schur's and Levinson's algorithm it is not surprising that the input impedance of this cascade can be expressed as given by (20) in terms of the polynomials $A_p(z)$ and $R_p(z)$ generated by the Levinson recurrences (6), (17). For the AR model, one has $\varphi_{p+1}(z) = 0$ which means that the cascade is terminated in a unit resistor. On the other hand, a reactance of degree $p+1$ is generated when the cascade is either short-circuited ($\varphi_{p+1}(z) = 1$) or open-circuited ($\varphi_{p+1}(z) = -1$) at the far end and this situation corresponds to Sagayama's composite sinusoidal wave model.

6. Connection with Gauss's quadrature formula

It is apparent from the preceding sections that the orthogonal polynomials on the unit circle are basic for the solution of the trigonometric moment problem. A similar role is played by the orthogonal polynomials on the real axis in the power moment problem and the Gauss quadrature formula. This section shows how the trigonometric moment problem can be converted into a power moment problem, an approach which, interestingly, is close to Sagayama's own presentation [4]) of the composite sinusoidal wave model (24). It turns out that there exists a solution with an arbitrary fixed value of Θ_{p+1}, which is not surprising in view of the available number of parameters. For simplicity we shall take $\Theta_{p+1} = 0$ which corresponds exactly to Sagayama's formulation of the problem.

The purpose of this section is to show that Sagayama's solution (24) to the partial trigonometric moment problem can be reduced to a classical solution of the partial power moment problem [12]). Normalizing the angles Θ_k in (24) by putting $\Theta_{p+1} = 0$, we get by subtraction the representation

$$c_0 - c_l = \sum_{k=1}^{p} \varrho_k (1 - \mathrm{e}^{-\mathrm{i}l\Theta_k}) \tag{38}$$

for $l \in [-p, p]$. Let $\mu(\Theta)$ denote some non-decreasing measure satisfying the trigonometric moment problem (12), for instance the solution provided by the AR model. Then (38) is equivalent to the condition

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} x(\mathrm{e}^{\mathrm{i}\Theta}) \, \mathrm{d}\mu(\Theta) = \sum_{k=1}^{p} \varrho_k \, x(\mathrm{e}^{\mathrm{i}\Theta_k}) \tag{39}$$

for all trigonometric polynomials $x(z)$ of support $[\ \ 1p, p]$ and satisfying $x(1) = 0$. Condition (39) is now easily reducible to a power moment problem via the classical change of variable

$$\mathrm{e}^{\mathrm{i}\Theta} = \frac{\mathrm{i}t - 1}{\mathrm{i}t + 1}, \tag{40}$$

which transforms the unit circle into the imaginary axis. Indeed, with the measure $\mu(\Theta)$, the numbers ϱ_k, Θ_k and the trigonometric polynomial $x(z)$, let us associate a set of corresponding quantities defined by

$$\mathrm{d}\mu^*(t) = \frac{\mathrm{d}\mu(2 \operatorname{arc\,cotg} t)}{2\pi(1 + t^2)^p}, \quad t_k = \operatorname{cotg} \frac{\Theta_k}{2}, \tag{41}$$

$$\varrho_k^* = \frac{\varrho_k}{(1 + t_k^2)^p}, \quad x^*(t) = (1 + t^2)^p \cdot \left(\frac{\mathrm{i}t - 1}{\mathrm{i}t + 1} \right), \tag{42}$$

where the constraint $x(1) = 0$ implies $\deg x^*(t) \leq 2p - 1$. With these definitions condition (39) requires that numbers ϱ_k^*, t_k ($k = 1, 2, \ldots, p$) should be found which satisfy the Gauss–Jacobi quadrature formula [12])

$$\int_{-\infty}^{+\infty} x^*(t)\,\mathrm{d}\mu^*(t) = \sum_{k=1}^{p} \varrho_k^*\, x^*(t_k) \tag{43}$$

for all polynomials $x^*(t)$ of degree smaller than or equal to $2p - 1$. The solution to this problem is unique and can be expressed in terms of orthogonal polynomials $q_l^*(t)$ of degree $l = 0, 1, 2, \ldots, p$ relative to the measure $\mu^*(t)$. The t_k are the zeros of the orthogonal polynomial $q_p^*(t)$ (which are known to be real and distinct) and the ϱ_k^* are the Christoffel numbers given by

$$\varrho_k^* = \left[\sum_{l=0}^{p-1} (q_l^*(t_k))^2\right]^{-1}. \tag{44}$$

The solution to Sagayama's representation problem follows then from (41), (42) as

$$\Theta_k = 2 \operatorname{arc\,cotg} t_k, \quad \varrho_k = \varrho_k^*(1 + t_k^2)^p \tag{45}$$

for $k = 1, 2, \ldots, p$ and ϱ_{p+1} results from $c_0 = \sum_{k=1}^{p} \varrho_k + \varrho_{p+1}$. The fact that ϱ_{p+1} is positive can easily be proved.

It is known [12]) that the orthogonal polynomial $q_p^*(t) = \sum_{l=0}^{p} q_{p,l}^* t^l$ is obtained as the solution of a linear system constructed on the Hankel matrix

$$\mathrm{H}_p = \begin{bmatrix} c_0^* & c_1^* & \ldots & c_p^* \\ c_1^* & c_2^* & \ldots & c_{p+1}^* \\ \vdots & & & \\ c_p^* & c_{p+1}^* & \ldots & c_{2p}^* \end{bmatrix}, \tag{46}$$

where c_k^* is the k-th power moment of the measure $\mu^*(t)$

$$c_k^* = \int_{-\infty}^{+\infty} t^k\,\mathrm{d}\mu^*(t). \tag{47}$$

More precisely, the vector $q_p^* = [q_{p,0}^*, q_{p,1}^*, \ldots, q_{p,p}^*]'$ is the last column of H_p^{-1} and hence satisfies the equation

$$\mathrm{H}_p\, q_p^* = [0, 0, \ldots, 0, 1]'. \tag{48}$$

Let us now check how the solution to Sagayama's problem obtained via (45) and (48) is consistent with the result derived in sec. 4 in the framework of Toeplitz systems. The elements of C_p are the trigonometric moments defined in (12) which, in view of (40) and (41), can be rewritten as

$$c_l = (-1)^l \int_{-\infty}^{+\infty} (1 + \mathrm{i}t)^{p+l} (1 - \mathrm{i}t)^{p-l} \, \mathrm{d}\mu^*(t) \tag{49}$$

for $l \in [-p, p]$ thereby showing that the c_l are some linear combinations of the c_l^*. The precise form of this relation is based on the expansion

$$(1 - z)^l (1 + z)^{p-l} = \sum_{k=0}^{p} v_{l,k} \, z^k, \tag{50}$$

where

$$v_{l,k} = \sum_{j=0}^{k} (-1)^j \binom{l}{j} \binom{p-l}{k-j}. \tag{51}$$

Let us define now the diagonal matrix $\Delta = \mathrm{diag}(1, \mathrm{i}, -1, \ldots, \mathrm{i}^p)$ and the matrix $V = [v_{l,k} : 0 \leqslant l,k \leqslant p]$ which often occurs in problems involving the transformation (40); see e.g. Bauer [16]). Using (47), (50) and (51), it appears that (49) defines C_p as a conjunctive transform of H_p

$$C_p = (\Delta^2 \, V \bar{\Delta}) \, \mathrm{H}_p (\Delta \, V^T \, \Delta^2). \tag{52}$$

Let us associate with q_p^* a vector q_p defined as

$$q_p = (2\mathrm{i})^{-p} \, \Delta^2 \, V^T \, \bar{\Delta} \, q_p^*, \tag{53}$$

which in view of (48) is the solution of the Toeplitz system

$$C_p \, q_p = [1, 1, \ldots, 1]^T. \tag{54}$$

Next, if we build on the elements of $q_p = [q_{p,0}, q_{p,1}, \ldots, q_{p,p}]^T$ a polynomial $q_p(z) = \sum_{l=0}^{p} q_{p,l} z^{-l}$ one can show by (53) and (50) that, under the transformation $t = \mathrm{cotg}\, \Theta/2$, the following identity holds

$$q_p(\mathrm{e}^{-\mathrm{i}\Theta}) = (\mathrm{i} - t)^{-p} \, q_p^*(t). \tag{55}$$

Consequently, in view of (45), the numbers $\mathrm{e}^{-\mathrm{i}\Theta_1}, \mathrm{e}^{-\mathrm{i}\Theta_2}, \ldots, \mathrm{e}^{-\mathrm{i}\Theta_p}$ in Sagayama's solution are the zeros of the polynomial $q_p(z)$ built on the solution of (54). Finally one can check that these numbers, augmented with $\mathrm{e}^{\mathrm{i}\Theta_{p+1}} = 1$, are the zeros of $A_p(z) - z^{-1} \hat{A}_p(z)$. For this purpose, consider the following Toeplitz system

$$C_p \begin{bmatrix} q_{p,0} & 0 & A_{p,0} \\ q_{p,1} & q_{p-1,0} & A_{p,1} \\ \vdots & \vdots & \vdots \\ q_{p,p} & q_{p-1,p-1} & A_{p,p} \end{bmatrix} = \begin{bmatrix} 1 & \lambda & \sigma_p^2 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 0 \end{bmatrix}, \tag{56}$$

where $\lambda = c_1 q_{p-1,0} + \ldots + c_p q_{p-1,p-1}$. From (56) one immediately derives the identity

$$q_p(z) - z^{-1} q_{p-1}(z) + \sigma_p^{-2}(\lambda - 1) A_p(z) = 0, \tag{57}$$

where $q_{p-1}(z) = \sum_{l=0}^{p-1} q_{p-1,l} z^{-l}$. Due to the centrosymmetry of C_p, polynomials $q_p(z)$ and $q_{p-1}(z)$ are self-reciprocal and hence the reciprocal version of (57) becomes

$$q_p(z) - q_{p-1}(z) + \sigma_p^{-2}(\lambda - 1) \hat{A}_p(z) = 0. \tag{58}$$

Elimination of $q_{p-1}(z)$ between (57) and (58) yields

$$(1 - z^{-1}) q_p(z) = \sigma_p^{-2}(1 - \lambda) [A_p(z) - z^{-1} \hat{A}_p(z)], \tag{59}$$

which proves the assertion.

7. Conclusion

Due to the variety of rational models presently available for spectral estimation and speech production, it is not always clear how seemingly disconnected methods can lead to equivalent models in terms of power spectrum matching. The paper has shown that many of these methods can be considered as particular solutions to a more general mathematical problem. In this framework, the equivalence becomes obvious and it appears in addition that there exists in fact an infinite number of models, each corresponding to a different choice for the Schur function $\varphi_{p+1}(z)$ in (20). It remains an open question whether some of the "new" models offer interesting properties for practical implementations.

The paper also emphasizes the connection existing between different areas interest like circuit theory, geophysics, speech processing etc. and shows that techniques available in one area can be used in an other. In this line, a more compact and probably more robust computational scheme was derived for the construction of Sagayama's composite sinusoidal wave model.

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