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STABILITY RADIUS OF DISCRETE TIME–VARYING SYSTEMS OF DESCRIPTOR FORM

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Abstract

We derive a simple formula for the unstructured stability radii of a linear discrete time–dependent system of descriptor form $E_k x_{k+1} = A_k x_k + u_k$ subject to additive time–varying perturbations on both E and A.

Keywords: Time-varying systems, Stability radius, Robust stabilization

1 Introduction

In the last three decades the topic of stability radii of systems of differential and difference equations has been studied intensively (see the overview paper [3] and for computational issues [5], [6]). In this paper we study a descriptor system with time–dependent coefficients of the form $E_k x_{k+1} = A_k x_k + u_k$ subject to additive time-varying perturbations $(E_k + \Delta E_k)x_{k+1} = (A_k + \Delta A_k)x_k + u_k$. For such a system we provide a simple analytic expression for the stability radius in terms of the induced norm of the operator $E\sigma - A$, where E and A are seen as multiplication operators acting on l^2 spaces. In the sequel the following notation is used. If $u = (u_k)_{k\in\mathbf{Z}}$, $v = (v_k)_{k\in\mathbf{Z}}$, $u_k, v_k \in \mathbf{C}^m$, and $A = (A_k)_{k\in\mathbf{Z}}$, $B = (B_k)_{k\in\mathbf{Z}}$, $C = (C_k)_{k\in\mathbf{Z}}$, $A_k \in \mathbf{C}^{n\times n}$, $B_k \in \mathbf{C}^{n\times m}$ and $C_k \in \mathbf{C}^{p\times n}$, are two $\mathbf{C}^m$-valued sequences and three matrix sequences, respectively, then $u+v := (u_k+v_k)_{k\in\mathbf{Z}}$, $Bu := (B_k u_k)_{k\in\mathbf{Z}}$ and $CB := (C_k B_k)_{k\in\mathbf{Z}}$. Let σ and σ^{-1} be the *forward* and *backward unit shift* operators. A matrix sequence, say B, is bounded if there exists a bounded constant M such that $\|B_k\| \leq M$, $\forall k \in \mathbf{Z}$. By $l^{2,m}$ we denote the Hilbert space of norm-square (double infinite) $\mathbf{C}^m$-valued sequences. The spectral radius of a linear bounded operator on a Hilbert space will be denoted by $\rho(\cdot)$.

2 Exponential stability and dichotomy

Let $M = (M_k)_{k\in\mathbf{Z}}$, $M_k \in \mathbf{C}^{r\times r}$. Introduce

$$S_{ij}^M := \begin{cases} M_{i-1} \cdots M_j & i > j \\ I & i = j \\ M_i \cdots M_{j-1} & i < j \end{cases} \tag{2.1}$$

$\forall i, j \in \mathbf{Z}$ and call S_{ij}^M the *evolution operator* of M. We say that M defines an *exponentially stable* (ES) (*anticausal exponentially stable* (AES)) evolution if there exist $\rho_0 \geq 1$ and $0 < q < 1$ such that $\|S_{ij}^M\| \leq \rho_0 q^{i-j}$, $\forall i \geq j$ ($\|S_{ij}^M\| \leq \rho_0 q^{j-i}$, $\forall j \geq i$). Let now $M = (M_k)_{k\in\mathbf{Z}}$, $N = (N_k)_{k\in\mathbf{Z}}$, with M_k and $N_k \in \mathbf{C}^{r\times r}$, be given. Assume M and N *bounded* and consider the system of difference equations

$$(M\sigma - N)w = z \tag{2.2}$$

where $z = (z_k)_{k\in\mathbf{Z}}$, $w = (w_k)_{k\in\mathbf{Z}}$ with $z_k, w_k \in \mathbf{C}^r$. Since M and N are bounded $M\sigma - N$ is a linear bounded operator on $l^{2,r}$. We say that (2.2) in *uniquely solvable* in $l^{2,r}$ if for arbitrary $z \in l^{2,r}$ there exists a unique $w \in l^{2,r}$ for which (2.2) is fulfilled. Clearly this is equivalent to the existence of a bounded inverse of $M\sigma - N$, i.e. $w = (M\sigma - N)^{-1}z$, $\forall z \in l^{2,r}$. We say that $M\sigma - N : l^{2,r} \mapsto l^{2,r}$ defines an *exponentially dichotomic* (ED) evolution of order p ($0 \leq p \leq r$) if there exist four matrix sequences $U = (U_k)_{k\in\mathbf{Z}}, Z = (Z_k)_{k\in\mathbf{Z}}, S = (S_k)_{k\in\mathbf{Z}}$, and $T = (T_k)_{k\in\mathbf{Z}}$ with U_k and $Z_k \in \mathbf{R}^{r\times r}$, $S_k \in \mathbf{R}^{p\times p}$, $T_k \in \mathbf{R}^{(r-p)\times(r-p)}$ such that U and Z are bounded, U^{-1} and Z^{-1} are well defined and bounded, S defines an ES evolution, T defines an AES evolution and

$$U(M\sigma - N)Z = \begin{bmatrix} I_{l^{2,p}}\sigma - S & O \\ O & T\sigma - I_{l^{2,r-p}} \end{bmatrix}. \tag{2.3}$$

We say that $M\sigma - N : l^{2,r} \mapsto l^{2,r}$ defines an ES evolution if it defines an ED evolution of order $p = r$. For more details about dichotomy of a system of difference equations see [1]. We have the following key result (see [2], Theorem 2.1).

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Theorem 2.1 *Let $M = (M_k)_{k\in\mathbf{Z}}, N = (N_k)_{k\in\mathbf{Z}}$ be bounded ($M_k, N_k \in \mathbf{C}^{r\times r}$). Then $M\sigma - N$ has a bounded inverse on $l^{2,r}$ if and only if $M\sigma - N$ defines an ED evolution (of order p).*

3 Main result

Consider the system of difference equations

$$E_k x_{k+1} = A_k x_k + u_k, \quad k \in \mathbf{Z} \tag{3.1}$$

where $E_k, A_k \in \mathbf{C}^{n\times n}$, $u_k, x_k \in \mathbf{C}^n$. The system (3.1) can be rewritten compactly as $(E\sigma - A)x = u$. Here $E = (E_k)_{k\in\mathbf{Z}}$, $A = (A_k)_{k\in\mathbf{Z}}$, $u = (u_k)_{k\in\mathbf{Z}}$, $x = (x_k)_{k\in\mathbf{Z}}$. Assume that E and A are bounded and that $E\sigma - A$ defines an ES evolution and therefore, according to Theorem 2.1, is boundedly invertible. Consider now the perturbed system

$$(E_k + \Delta E_k)x_{k+1} = (A_k + \Delta A_k)x_k + u_k, \; k \in \mathbf{Z} \tag{3.2}$$

where $\Delta E_k, \Delta A_k \in \mathbf{C}^{n\times n}$ or, equivalently, $((E + \Delta E)\sigma - (A + \Delta A))x = u$, where $\Delta E = (\Delta E_k)_{k\in\mathbf{Z}}$, $\Delta A = (\Delta A_k)_{k\in\mathbf{Z}}$. We define the stability radius of the discrete time-varying system of descriptor form given in (3.1) as $\mathbf{r}(E, A) := \sup\{\epsilon > 0 \mid \| [\; \Delta A \quad \Delta E \;] \| < \epsilon \Rightarrow (E+\Delta E)\sigma-(A+\Delta A) \textit{ defines an ES evolution}\}$. We give now the main result of the paper.

Theorem 3.1 *Assume that $E\sigma - A$ defines an ES evolution. Then*

$$\mathbf{r}(E, A) = \frac{1}{\sqrt{2}\|(E\sigma - A)^{-1}\|}.$$

The proof strongly relies on the following result concerning robust stabilization of a discrete time–varying system by output dynamic feedback. Consider the discrete time–varying systems

$$\mathcal{S}_\Delta : \left\{ \begin{array}{rcl} \sigma x^\Delta & = & A^\Delta x^\Delta + B^\Delta u^\Delta \\ y^\Delta & = & C^\Delta x^\Delta + D^\Delta u^\Delta \end{array} \right. , \qquad y^\Delta = G^\Delta u^\Delta$$

$$\mathcal{S}_N : \left\{ \begin{array}{rcl} \sigma x^N & = & A^N x^N + \left[\begin{array}{cc} B^{N1} & B^{N2} \end{array} \right] \left[\begin{array}{c} u^{N1} \\ u^{N2} \end{array} \right] \\ \left[\begin{array}{c} y^{N1} \\ y^{N2} \end{array} \right] & = & \left[\begin{array}{c} C^{N1} \\ C^{N2} \end{array} \right] x^N \\ & & + \left[\begin{array}{cc} D^{N11} & D^{N12} \\ D^{N21} & D^{N22} \end{array} \right] \left[\begin{array}{c} u^{N1} \\ u^{N2} \end{array} \right] \\ & & y^N = G^N u^N \end{array} \right.$$

$$\mathcal{S}_K : \left\{ \begin{array}{rcl} \sigma x^K & = & A^K x^K + B^K u^K \\ y^K & = & C^K x^K + D^K u^K \end{array} \right. \qquad y^K = G^K u^K$$

Assume A^Δ defines an ES evolution, the input output operators G^N and G^K of the systems $\mathcal{S}_N$ and $\mathcal{S}_K$ are well defined and bounded (this happens, for example, if the corresponding system defines a rational node), and assume that the feedback systems $\mathcal{S}_{N\Delta}$: $y^{N2} = G^{N\Delta}u^{N2}$ and $\mathcal{S}_{NK}$: $y^{N1} = G^{NK}u^{N1}$ obtained by coupling $u^\Delta = y^{N1}$, $u^{N1} = y^\Delta$ and $u^K = y^{N2}$, $u^{N2} = y^K$, respectively, are well defined, i.e. the above couplings have compatible dimensions and the well–posedness conditions: $(I - D^\Delta D^{N11})^{-1}$ is well defined and bounded and $(I - D^{N22}D^K)^{-1}$ is well defined and bounded, are fulfilled. Then we have the following key result.

Theorem 3.2 *The system $\mathcal{S}_K$ stabilizes the system $\mathcal{S}_{N\Delta}$ for all stable systems $\mathcal{S}_\Delta$ with $\|G^\Delta\| < \epsilon$ if and only if $\mathcal{S}$ stabilizes $\mathcal{S}_{N\Delta}$ for $\Delta = 0$ and $\epsilon \leq \epsilon_{\max} := \|G^{NK}\|^{-1}$. Moreover, there exists a stationary system $\mathcal{S}_\Delta$ of norm $\epsilon_{\max}$ which destabilizes the overall feedback system.*

For a continuous time–invariant system we get

$$\mathbf{r}(E, A) = \left\| \left[\begin{array}{c} (sE - A)^{-1} \\ s(sE - A)^{-1} \end{array} \right] \right\|_\infty^{-1}$$

We may extend our result similarly to any class of systems for which a small–gain type theorem (see [4]) could be proved. However, one should be careful about the class of allowable perturbations to ensure that the perturbed system is still in the admissible class (i.e. is well defined and uniquely solvable).

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