# Reducing subspaces : definitions, properties and algorithms.

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Abstract

In this paper we introduce the new concept of reducing subspaces of a singular pencil, which extends the notion of deflating subspaces to the singular case. We briefly discuss uniqueness of such subspaces and we give an algorithm for computing them. The algorithm also gives the Kronecker canonical form of the singular pencil.

## 1. Introduction

The last few years, the numerical literature has started to show some interest in the computation of the Kronecker canonical form of singular pencils because of its relevance in several applications [1][6][10][12][13] and stable algorithms have been developed recently [2][6][9][10][15][16]. Since this problem is (often) ill conditioned an appropriate reformulation of conditioning has to be made in order to give a meaning to the computed results (see e.g. [10]).

Other concepts that were not extended to the singular case are those of eigenvectors and invariant subspaces, or more generally, of deflating subspaces. In Section 2, we introduce the concept of reducing subspaces and show it generalizes the notion of deflating subspaces to the singular case. Such subspaces reduce the eigenstructure problem of a singular pencil $\lambda B-A$ to the eigenstructure problem of two smaller pencils. Under eigenstructure we understand here all the invariants of the $m\times n$ pencil $\lambda B-A$ under equivalence transformations (i.e. invertible column and row transformations). The eigenstructure of $\lambda B-A$ is retrieved in its Kronecker canonical form [4]:

$$M(\lambda B-A)N = \mathrm{diag}\{L_{\ell_1},\ldots,L_{\ell_s}, L^T_{r_1},\ldots,L^T_{r_t}, I-\lambda N, \lambda I-J\} \quad (1.1)$$

where i) $L_k$ is the $(k+1)\times k$ bidiagonal pencil

$$L_k = \underbrace{\begin{bmatrix} \lambda & & \\ -1 & \ddots & \\ & \ddots & \lambda \\ & & -1 \end{bmatrix}}_{k} \Big\} k+1 \quad (1.2)$$

and ii) $N$ is nilpotent and both $N$ and $J$ are in Jordan canonical form.

The 'eigenstructure' of the pencil $\lambda B-A$ is then given by :

i) the finite elementary divisors $(\lambda-\alpha_i)^{d_j}$ of $\lambda B-A$, reflected by the Jordan blocks of size $d_j$ at $\alpha_i$ in $J$.

ii) the infinite elementary divisors $(\mu)^{d_j}$ of $\lambda B-A$, reflected by the Jordan blocks of size $d_j$ at $0$ in $N$.

iii) the left and right minimal indices $\{\ell_1,\ldots,\ell_s\}$ and $\{r_1,\ldots,r_t\}$ of $\lambda B-A$, reflected by the blocks $L_{\ell_i}$ and $L^T_{r_j}$, respectively.

For the computation of the eigenstructure of a singular pencil, algorithms have been developed previously [9][12][15][16]. They compute a quasi triangular form :

$$M(\lambda B-A)N = \begin{bmatrix} \lambda B_r-A_r & * & * & * \\ 0 & \lambda B_i-A_i & * & * \\ 0 & 0 & \lambda B_f-A_f & * \\ 0 & 0 & 0 & \lambda B_\ell-A_\ell \end{bmatrix} \quad (1.3)$$

where $\lambda B_r-A_r$ and $\lambda B_\ell-A_\ell$ are singular pencils containing the right and left minimal indices of $\lambda B-A$, respectively, and $\lambda B_f-A_f$ and $\lambda B_i-A_i$ are regular pencils containing the finite and infinite elementary divisors of $\lambda B-A$, respectively. It is shown in [9] that this form can be obtained under unitary transformations $M$ and $N$, thus guaranteeing the numerical stability of the method. In Section 3 we show how to combine this algorithm with a recent algorithm derived for the computation of deflating subspaces of a regular pencil, in order to obtain an algorithm for computing reducing subspaces of a singular pencil.

The following notation and conventions will be used throughout the paper. We denote a block diagonal matrix by $\mathrm{diag}\{A_{11},\ldots,A_{kk}\}$. We use $A^*$ for the conjugate transpose of a matrix $A$ and $A^T$ for the transpose of a matrix $A$. A complex (real) square matrix $A$ is called unitary (orthogonal) when $A^*A=I$ ($A^TA=I$). When no explicit distinction is made between the complex and the real case, we use the term unitary and the notation $A^*$ for the real case as well. Script is used for vectorspaces. $\mathcal{H}_n$ will denote the spaces $\mathbb{C}^n$ or $\mathbb{R}^n$, depending on the context of the problem. $A\mathcal{X}$ is the image of $\mathcal{X}$ under $A$ ; $\mathcal{Im}\, A$ and $\mathcal{Ker}\, A$ are the image and kernel of $A$, respectively. $\mathcal{X}+\mathcal{Y}$ and $\mathcal{X}\oplus\mathcal{Y}$ are the sum and the direct sum, respectively of the spaces $\mathcal{X}$ and $\mathcal{Y}$. Two vectorspaces of special interest are $\mathcal{N}_r(\lambda B-A)$ and $\mathcal{N}_\ell(\lambda B-A)$, the right and left null spaces, respectively, of a pencil $\lambda B-A$. These are vectorspaces over the field of rational functions in $\lambda$ and are of dimension

$$\dim.\mathcal{N}_r(\lambda B-A)=n-r;\quad \dim.\mathcal{N}_\ell(\lambda B-A)=m-r \quad (1.4\ a;b)$$

respectively, when $\lambda B-A$ is a $m\times n$ pencil of <u>normal rank</u> $r$ [3]. These dimensions are also called the right and left nullity of $\lambda B-A$, respectively, and the pencil $\lambda B-A$ is said to be right and left invertible, respectively, when the corresponding nullity is zero. When the columns of a matrix $X$ form a basis for the space $\mathcal{X}$, this is denoted by $\mathcal{X} = \langle X \rangle$. The space spanned by the null vector only is denoted by $\{0\}$. By $\Lambda(B,A)$ we denote the <u>spectrum</u> of the pencil $\lambda B-A$, i.e. the collection of generalized eigenvalues, multiplicities counted. By $E(B,A)$ we denote the complete <u>eigenstructure</u> of the pencil $\lambda B-A$, i.e. all the structural elements as described in (1.1).

## 2. Deflating and reducing subspaces.

Let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of $\mathcal{H}_n$ and $\mathcal{H}_m$, respectively, such that

(2.1) $$\mathcal{Y} = B\mathcal{X} + A\mathcal{X}$$

Let $\ell$ and $k$ be their respective dimensions and construct the unitary matrices $Q$ and $Z$, partitioned as :

(2.2) $$Z = [\underbrace{Z_1}_{\ell} \mid Z_2] \quad ; \quad Q = [\underbrace{Q_1}_{k} \mid Q_2]$$

such that

(2.3) $$\mathcal{X} = \langle Z_1 \rangle \;;\; \mathcal{Y} = \langle Q_1 \rangle$$

Then it follows from (2.1) that $Q_2^* A Z_1 = Q_2^* B Z_1 = 0$ and thus

(2.4) $$Q^*(\lambda B-A)Z \triangleq \lambda\hat{B}-\hat{A} \triangleq \lambda\left[\begin{array}{c|c} \hat{B}_{11} & \hat{B}_{12} \\ \hline 0 & \hat{B}_{22} \end{array}\right] - \left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hline 0 & \hat{A}_{22} \end{array}\right]$$

In this new coordinate system $\mathcal{X}$ and $\mathcal{Y}$ are now represented by

(2.5 a;b) $$\mathcal{X} = \left\langle \begin{bmatrix} I_\ell \\ 0 \end{bmatrix} \right\rangle ; \quad \mathcal{Y} = \left\langle \begin{bmatrix} I_k \\ 0 \end{bmatrix} \right\rangle$$

The map $\lambda B-A$ restricted to the spaces $\mathcal{X}$ and $\mathcal{Y}$ and its spectrum are also denoted by

(2.6 a;b) $$\lambda\hat{B}_{11}-\hat{A}_{11} = (\lambda B-A)\big|_{\mathcal{X},\mathcal{Y}} \quad ; \quad \Lambda(\hat{B}_{11},\hat{A}_{11}) = \Lambda(B,A)\big|_{\mathcal{X},\mathcal{Y}}$$

In the regular case, i.e. when $m$ and $n$ are equal to the normal rank, the dimensions of $\mathcal{X}$ and $\mathcal{Y}$ satisfy the inequality [8] :

(2.7) $$\dim \mathcal{Y} \geq \dim \mathcal{X}$$

and it is only in the case of equality that such spaces become of interest. They are called <u>deflating subspaces</u> and possess the following property (see [8] for a proof)

<u>Theorem 2.1.</u>

Let $\mathcal{X},\mathcal{Y}$ be a pair of deflating subspaces and perform the corresponding transformation (2.4), then the diagonal pencils $\lambda\hat{B}_{ii}-\hat{A}_{ii}$, $i=1,2$ are regular and $\Lambda(\hat{B}_{11},\hat{A}_{11}) \cup \Lambda(\hat{B}_{22},\hat{A}_{22}) = \Lambda(B,A)$. □

This theorem justifies the terminology "deflating subspaces", since the problem of computing $\Lambda(B,A)$ is now deflated to two eigenvalue problems of smaller dimension. The following results are important for the characterization of some specific pairs of deflating subspaces,

<u>Lemma 2.1.</u> [8]

The equation in $M$ and $L$ :

(2.8) $$M(\lambda B-A) + (\lambda\hat{B}-\hat{A})L = \lambda\tilde{B}-\tilde{A}$$

where $\lambda B-A$ and $\lambda\hat{B}-\hat{A}$ are <u>regular</u> pencils and $\lambda\tilde{B}-\tilde{A}$ is arbitrary, has a <u>unique</u> solution when $\Lambda(B,A) \cap \Lambda(\hat{B},\hat{A}) = \emptyset$. □

<u>Lemma 2.2.</u>

Let the pencils $\lambda B-A$ and $\lambda\hat{B}-\hat{A}$ be conformably partitioned and upper block triangular:

(2.9 a;b) $$\lambda B-A = \left[\begin{array}{c|c} \lambda B_{11}-A_{11} & \lambda B_{12}-A_{12} \\ \hline 0 & \lambda B_{22}-A_{22} \end{array}\right]; \quad \lambda\hat{B}-\hat{A} = \left[\begin{array}{c|c} \lambda\hat{B}_{11}-\hat{A}_{11} & \lambda\hat{B}_{12}-\hat{A}_{12} \\ \hline 0 & \lambda\hat{B}_{22}-\hat{A}_{22} \end{array}\right]$$

and let $\lambda B-A$ and $\lambda\hat{B}-\hat{A}$ be equivalent, i.e. there exist invertible matrices $M$ and $N$ such that :

(2.10) $$M(\lambda B-A)N = \lambda\hat{B}-\hat{A}$$

Then $M$ and $N$ are also upper block triangular if $\Lambda(B_{11},A_{11})$ and $\Lambda(\hat{B}_{22},\hat{A}_{22})$ have no common points.

Proof :

Using $L=M^{-1}$, we rewrite (2.10) with a conformable partitioning of M and L :

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \cdot \begin{bmatrix} \lambda B_{11}-A_{11} & \lambda B_{12}-A_{12} \\ 0 & \lambda B_{22}-A_{22} \end{bmatrix} = \begin{bmatrix} \lambda\hat{B}_{11}-\hat{A}_{11} & \lambda\hat{B}_{12}-\hat{A}_{12} \\ 0 & \lambda\hat{B}_{22}-\hat{A}_{22} \end{bmatrix} \cdot \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \qquad (2.11)$$

This yields the equation

$$M_{21}(\lambda B_{11}-A_{11}) - (\lambda\hat{B}_{22}-\hat{A}_{22})L_{21} = 0 . \qquad (2.12)$$

Because of Lemma 2.1 and $\Lambda(B_{11},A_{11}) \cap \Lambda(\hat{B}_{22},\hat{A}_{22})=\emptyset$, (2.12) has a unique solution for $M_{21}$ and $L_{21}$ which is clearly $M_{21}=L_{21}=0$. This completes the proof.

□

This lemma leads directly to the following theorem.

Theorem 2.2.

Let $\Lambda_1$ be a subset of $\Lambda(B,A)$ disjoint from the remaining eigenvalues $\Lambda_2=\Lambda(B,A)\setminus\Lambda_1$. Then there exists a unique pair of deflating subspaces such that $\Lambda(B,A)|_{X,Y} = \Lambda_1$.

Proof :

Let $\ell_1$ be the number of generalised eigenvalues in $\Lambda_1$. It is known by construction (see [11]) that there always exist one pair $X_1,Y_1$ of dimension $\ell_1$ satisfying $\Lambda(B,A)|_{X_1,Y_1} = \Lambda_1$. Its uniqueness follows from Lemma 2.2. Indeed, without loss of generality we may assume that (see (2.3)-(2.5) for the appropriate coordinate system)

$$X_1 = Y_1 = \left\langle \begin{bmatrix} I_{\ell_1} \\ 0 \end{bmatrix} \right\rangle \qquad (2.13)$$

and thus that $\lambda B-A$ has the block triangular form (2.9a). If there is a second pair $X_2$, $Y_2$ satisfying $\Lambda(B,A)|_{X_2,Y_2} = \Lambda_1$, then there exist updating transformations Q and Z as in (2.3) (2.4) and such that

$$X_2 = \langle Z_1 \rangle \quad ; \quad Y_2 = \langle Q_1 \rangle \qquad (2.14\ a;b)$$

$$Q \cdot \begin{bmatrix} \lambda B_{11}-A_{11} & \lambda B_{12}-A_{12} \\ 0 & \lambda B_{22}-A_{22} \end{bmatrix} \cdot Z = \begin{bmatrix} \lambda\hat{B}_{11}-\hat{A}_{11} & \lambda\hat{B}_{12}-\hat{A}_{12} \\ 0 & \lambda\hat{B}_{22}-\hat{A}_{22} \end{bmatrix} \qquad (2.14c)$$

Since $\Lambda_1 \cap \Lambda_2 = \emptyset$ we are in the situation of Lemma 2.2 and thus Q and Z are both upper block triangular. Therefore

$$\langle Z_1 \rangle = \langle Q_1 \rangle = \left\langle \begin{bmatrix} I_{\ell_1} \\ 0 \end{bmatrix} \right\rangle \qquad (2.15)$$

which establishes the unicity of the deflating subspaces.

□

Theorem 2.2. can also be retrieved from the work of Stewart [8] but Lemma's 2.1 and 2.2 are also useful for the extension of the above results to the singular case. Let $\lambda B-A$ be a m×n singular pencil with normal rank r. We will show that for any pair $X,Y$ as in (2.1), the following inequality is always satisfied :

$$\dim.Y \geq \dim.X-\dim.N_r \qquad (2.16)$$

In the case of equality the pair $X,Y$ plays a role comparable to that of deflating subspaces in the regular case. Such spaces are given the name of reducing subspaces of the pencil $\lambda B-A$. Notice that this concept reduces to that of deflating subspaces in the regular case since then $N_r = \{0\}$ . We first prove the following extension of Lemma 2.2 :

Lemma 2.3.

The equation in M and L :

$$M(\lambda B-A) + (\lambda D-C)L = \lambda F-E \qquad (2.17)$$

where $\lambda B-A$ and $\lambda D-C$ are left and right invertible respectively, has a solution when $\Lambda(B,A) \cap \Lambda(D,C) = \emptyset$ .

Proof :

First transform $\lambda B-A$ and $\lambda D-C$ to their Kronecker canonical form via the equivalence transformations :

$$M_1(\lambda B-A)N_1 = \lambda B_c-A_c \quad ; \quad M_2(\lambda D-C)N_2 = \lambda D_c-C_c \qquad (2.18\ a;b)$$

which reduces the equation (2.17) to the equivalent equation

$$(2.19) \qquad M_c(\lambda B_c - A_c) + (\lambda D_c - C_c)L_c = \lambda F_c - E_c$$

with $M_c = M_2 M M_1^{-1}$, $L_c = N_2^{-1} L N_1$ and $\lambda F_c - E_c = M_2(\lambda F - E)N_1$.
When partitioning $M_c, L_c$ and $\lambda F_c - E_c$ conformably with the blocks on the diagonal forms $\lambda B_c - A_c$ and $\lambda D_c - E_c$, equation (2.19) reduces to the set of (independent) equations :

$$(2.20) \qquad [M_c]_{ij}[\lambda B_c - A_c]_j + [\lambda D_c - C_c]_i [L_c]_{ij} = [\lambda F_c - E_c]_{ij}$$

When the canonical blocks $[\lambda B_c - A_c]_j$ and $[\lambda D_c - C_c]_i$ are regular, we are in the situation of Lemma 2.1 and (2.20) has a unique solution since by assumption these blocks have disjoint spectrum. When one or both blocks are singular we now show how to reduce the problem to a regular one. Because of the assumptions of left and right invertibility, the only singular blocks that may occur are :

$$(2.21\ a;b) \qquad [\lambda B_c - A_c]_j = L_k \quad ; \quad [\lambda D_c - C_c]_i = L_\ell^T$$

By deleting the first or last row in $L_k$ the truncated block $[\overline{\lambda B_c - A_c}]_j$ is regular with k eigenvalues at $\infty$ or zero, respectively. This corresponds to taking the first or last column in $[M_c]_{ij}$ equal to zero and solving for the truncated matrix $[\overline{M_c}]_{ij}$. A dual technique can be used for the second term in (2.20) such that this equation is replaced by :

$$(2.22) \qquad [\overline{M_c}]_{ij}[\overline{\lambda B_c - A_c}]_j + [\overline{\lambda D_c - C_c}]_i [\overline{L_c}]_{ij} = [\lambda F_c - E_c]_{ij}$$

where the upperbar indicates that the matrix has been truncated if needed. As indicated above, the truncation(s) may always be performed such that the blocks $[\overline{\lambda B_c - A_c}]_j$ and $[\overline{\lambda D_c - C_c}]_i$ have disjoint spectrum. We thus satisfy Lemma 2.1 and the solution $[\overline{M_c}]_{ij}$, $[\overline{L_c}]_{ij}$ of (2.22) yields also a solution to (2.20) by merely adding a zero column and row to reconstruct $[M_c]_{ij}$ and $[L_c]_{ij}$, respectively. Putting all these solutions together, we thus constructed (nonunique) matrices $M_c$ and $L_c$ satisfying (2.19).

□

This now leads to the following generalization of Theorem 2.1.

<u>Theorem 2.3.</u>

Let $X, Y$ be a pair of reducing subspaces and perform the coordinate transformation (2.3)(2.4), then the diagonal pencils have zero left and right nullity, respectively and $\Lambda(\hat{B}_{11}, \hat{A}_{11}) \cup \Lambda(\hat{B}_{22}, \hat{A}_{22}) = \Lambda(B, A)$ .

<u>Proof:</u>

Let $r_i$ be the normal rank of the pencils $\lambda \hat{B}_{ii} - \hat{A}_{ii}$, i=1,2. First we prove the inequality (2.16) and show that equality also implies $r_1 = \ell$ and $r_2 = n-k$. Clearly

$$(2.23\ a;b) \qquad r_1 \leq \ell \quad ; \quad r_2 \leq n-k$$

and, because of the structure of (2.4), the following holds :

$$(2.24\ a) \qquad k - r_1 = \dim. N_r(\lambda \hat{B}_{11} - \hat{A}_{11}) \leq \dim. N_r(\lambda B - A) = n - r$$

$$(2.24\ b) \qquad m - \ell - r_2 = \dim. N_\ell(\lambda \hat{B}_{22} - \hat{A}_{22}) \leq \dim. N_\ell(\lambda B - A) = m - r$$

Combining these inequalities we find

$$(2.25\ a) \qquad k - \ell \leq k - r_1 \leq n - r$$

$$(2.25\ b) \qquad m - \ell - n + k \leq m - \ell - r_2 \leq m - r$$

From this it easily follows that (2.16) holds since $k - \ell = \dim.X - \dim.Y$ and $n - r = \dim.N_r$. Moreover equality implies the middle terms in (2.25) to be equal to their upper and lower bounds, which then gives $r_1 = \ell$ and $r_2 = n-k$. In order to prove the second part of the theorem, we show the existence of conformable transformations M and N such that :

$$(2.26) \qquad \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline 0 & M_{22} \end{array}\right] \cdot \left[\begin{array}{c|c} \lambda \hat{B}_{11} - \hat{A}_{11} & \lambda \hat{B}_{12} - \hat{A}_{12} \\ \hline 0 & \lambda \hat{B}_{22} - \hat{A}_{22} \end{array}\right] \cdot \left[\begin{array}{c|c} N_{11} & N_{12} \\ \hline 0 & N_{22} \end{array}\right] =$$

$$\left[\begin{array}{cc|cc} \lambda \tilde{B}_r - \tilde{A}_r & 0 & 0 & 0 \\ 0 & \lambda \tilde{B}_{11} - \tilde{A}_{11} & \lambda \tilde{B}_{12} - \tilde{A}_{12} & 0 \\ \hline 0 & 0 & \lambda \tilde{B}_{22} - \tilde{A}_{22} & 0 \\ 0 & 0 & 0 & \lambda \tilde{B}_\ell - \tilde{A}_\ell \end{array}\right]$$

where $\lambda \tilde{B}_{ii} - \tilde{A}_{ii}$, i=1,2 are regular and $\lambda \tilde{B}_r - \tilde{A}_r$ and $\lambda \tilde{B}_\ell - \tilde{A}_\ell$ contain the right and left minimal indices of $\lambda B - A$. For this we first choose $M_{ii}, N_{ii}$, i=1,2 such that the pencils $\lambda \hat{B}_{ii} - \hat{A}_{ii}$ are transformed to their Kronecker canonical form

$$(2.27\ a) \qquad M_{11}(\lambda \hat{B}_{11} - \hat{A}_{11})N_{11} = \mathrm{diag}(\lambda \tilde{B}_r - \tilde{A}_r,\ \lambda \tilde{B}_{11} - \tilde{A}_{11})$$

(2.27 b) $$M_{22}(\lambda\hat{B}_{22}-\hat{A}_{22})N_{22} = \mathrm{diag}(\lambda\tilde{B}_{22}-\tilde{A}_{22}, \lambda\tilde{B}_{\ell}-\tilde{A}_{\ell})$$

The remaining three zero blocks in (2.26) are then obtained by an appropriate choice of $M_{12}$ and $N_{12}$. This is possible by virtue of the previous lemma since the ... of the pairs $\lambda\tilde{B}_r-\tilde{A}_r$ and $\lambda\tilde{B}_{22}-\tilde{A}_{22}$, $\lambda\tilde{B}_{11}-\tilde{A}_{11}$ and $\lambda\tilde{B}_{\ell}-\tilde{A}_{\ell}$, $\lambda\tilde{B}_r-\tilde{A}_r$ and $\lambda\tilde{B}_{\ell}-\tilde{A}_{\ell}$, are mutually disjoint. From the form (2.26) we now easily see that the central pencil

(2.28) $$\left[\begin{array}{c|c} \lambda\tilde{B}_{11}-\tilde{A}_{11} & \lambda\tilde{B}_{12}-\tilde{A}_{12} \\ \hline 0 & \lambda\tilde{B}_{22}-\tilde{A}_{22} \end{array}\right]$$

is the regular part of $\lambda B-A$, and the second part of this theorem then immediately follows from Theorem 2.1.

□

Notice also that there is a one to one correspondence between pairs of reducing subspaces of the pencil $\lambda B-A$ and pairs of deflating subspaces of its regular part, as shown in the proof of the above theorem. This remark leads directly to a generalization of Theorem 2.2. to the singular case.

<u>Theorem 2.4.</u>

Let $\Lambda_1$ be a subset of $\Lambda(B,A)$ disjoint from the remaining spectrum $\Lambda_2 = \Lambda(B,A) \setminus \Lambda_1$. Then there exists a <u>unique</u> pair of reducing subspaces such that $\Lambda(B,A)\big|_{X,Y} = \Lambda_1$.

<u>Proof</u> :

This follows immediately from Theorem 2.2. and the observation that to every pair of reducing subspaces there corresponds a pair of deflating subspaces of the regular part of $\lambda B-A$.

□

When a set of reducing subspaces of a pencil $\lambda B-A$ performs a separation in the spectrum $\Lambda(B,A) = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, then $M_{21}$ and $N_{21}$ in (2.26) may be chosen such that $\lambda\tilde{B}_{12}-\tilde{A}_{12}$ is eliminated as well since $\Lambda(\tilde{B}_{11},\tilde{A}_{11})$ and $\Lambda(\tilde{B}_{22},\tilde{A}_{22})$ are disjoint. This proves thus that the reduction (2.4) obtained by this pair of subspaces has the property

(2.29) $$E(B_{11},A_{11}) \cup E(B_{22},A_{22}) = E(B,A)$$

We thus proved the following result.

<u>Corollary 2.1.</u>
When a set of reducing subspaces performs a separation in the spectrum of the pencil $\lambda B-A$, then $E(B,A)\big|_{X,Y}$ is a subset of $E(B,A)$

□

As shown in Theorem 2.3, the right and left null space structures are always separated by a pair of reducing subspaces. The minimal and maximal pairs of reducing subspaces are easily seen to be those separating $\lambda\tilde{B}_r-\tilde{A}_r$ and $\lambda\tilde{B}_{\ell}-\tilde{A}_{\ell}$, respectively, from the rest of the pencil. We also have that any pair of reducing subspaces $X,Y$ satisfies

(2.30 a) $$\{0\} \subset X_{min} \subset X \subset X_{max} \subset \mathcal{H}_n$$

(2.30 b) $$\{0\} \subset Y_{min} \subset Y \subset Y_{max} \subset \mathcal{H}_m$$

as easily follows from the proof of Theorem 2.3.
The computation of deflating subspaces with specified spectrum $\Lambda_1$ has been described in [11] and a stable algorithm, based on an updating of the QZ decomposition, was given there. This can be used to compute reducing subspaces with specified spectrum, as soon as one has an algorithm to compute the pairs $X_{min}$, $Y_{min}$ and $X_{max}$, $Y_{max}$, or in other words, as soon as one has an algorithm to extract the regular part $\lambda B_{reg}-A_{reg}$ of the pencil $\lambda B-A$. Since to each pair of deflating subspaces $X_{reg}$, $Y_{reg}$ of this regular part there corresponds a pair of reducing subspaces $X,Y$ with the property

(2.31) $$\Lambda(B,A)\big|_{X,Y} = \Lambda(B_{reg},A_{reg})\big|_{X_{reg},Y_{reg}}$$

this indeed solves the problem of computing reducing subspaces with specified spectrum.

## 3. <u>Algorithms</u>.

In this section we show how ideas of previous algorithms [9][11] can be combined to yield an algorithm for computing pairs of reducing subspaces. We first show that the constructions of the pairs $X_{min}$, $Y_{min}$ and $X_{max}$, $Y_{max}$ can be solved recursively by building a chain of decompositions of the type (2.4) but where only the last decomposition of this chain corresponds to a pair of reducing subspaces. At each stage of the recursion, information about the structure of the pencils $\lambda\hat{B}_{11}-\hat{A}_{11}$ and $\lambda\hat{B}_{22}-\hat{A}_{22}$ is recovered. The results relie on the following theorem, implicitly proved in [9].

<u>Theorem 3.1.</u>

Let $X = Ker\ B$, $Y = AX$, then the corresponding decomposition (2.4) has the property that $E(\hat{B}_{22},\hat{A}_{22})$ and $E(B,A)$ are <u>equal</u> except for the infinite elementary divisors and right minimal indices of $E(\hat{B}_{22},\hat{A}_{22})$ which are those of $E(B,A)$ reduced by 1.

□

Such a decomposition is easily obtained by the following construction. Choose $Z$ such that $\langle Z_1 \rangle = \mathrm{Ker}\, B$. We then have (with $s_1 = \dim \mathrm{Ker}\, B$) :

$$(3.1) \qquad (\lambda B - A)Z = \left[ \underbrace{-A_1}_{s_1} \;\middle|\; \lambda B_2 - A_2 \right]$$

where $B_2$ has full column rank. Choose then $Q$ such that $\langle Q_1 \rangle = \mathrm{Im}\, A_1 = AX$. We then have (with $r_1 = \dim \mathrm{Im}\, A_1$) :

$$(3.2) \qquad Q^*(\lambda B - A)Z = \left[ \begin{array}{c|c} -\hat{A}_{11} & \lambda B_{12} - A_{12} \\ \hline \underbrace{0}_{s_1} & \lambda B_{22} - A_{22} \end{array} \right] \begin{array}{l} \}r_1 \\ \\ \end{array}$$

where $\hat{A}_{11}$ has full row rank and $\begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix}$ has full column rank.

This reduction step (3.1)(3.2) can be performed on the bottom pencil $\lambda B_{22} - A_{22}$ and can be repeated recursively like this, until one obtains, say at step $\ell$, a pencil $\lambda B_{\ell+1,\ell+1} - A_{\ell+1,\ell+1}$ where $\mathrm{Ker}\ B_{\ell+1,\ell+1} = \{0\}$. No further reduction can thus be obtained and the complete decomposition then looks like (see [9] for more details) :

$$(3.3\ a) \qquad \hat{Q}^*(\lambda B - A)\hat{Z} = \left[ \begin{array}{c|c} \lambda B_{ri} - A_{ri} & * \\ \hline 0 & \lambda B_{\ell\ell} - A_{\ell\ell} \end{array} \right] =$$

$$(3.3\ b) \qquad \left[ \begin{array}{cccc|c} -\hat{A}_{11} & * & \cdots & * & \\ & -\hat{A}_{22} & \cdots & * & \\ & & \ddots & \vdots & * \\ & 0 & & -\hat{A}_{\ell\ell} & \\ \hline \underbrace{\phantom{0}}_{s_1} & \underbrace{\phantom{0}}_{s_2} & \underbrace{0}_{\cdots} & \underbrace{\phantom{0}}_{s_\ell} & \lambda B_{\ell+1,\ell+1} - A_{\ell+1,\ell+1} \end{array} \right] \begin{array}{l} \}r_1 \\ \}r_2 \\ \vdots \\ \}r_\ell \\ \\ \end{array}$$

Because of the construction, all $\hat{A}_{ii}$ have full row rank and $B_{\ell+1,\ell+1}$ has full column rank. This ensures the following inequalities to hold (when defining $s_{\ell+1} = 0$) :

$$(3.4) \qquad s_1 \geq r_1 \geq s_2 \geq r_2 \geq \ldots \geq s_\ell \geq r_\ell \geq s_{\ell+1}$$

The dimensions $\{s_i\}$ and $\{r_i\}$ produced by the algorithm can be shown to yield the following information about the eigenstructure of the top pencil : $\lambda B_{ri} - A_{ri}$ has

$$\begin{cases} e_i = (s_i - r_i) \text{ right minimal indices equal to } i-1 \\ d_i = (r_i - s_{i+1}) \text{ infinite elementary divisors of degree } i \end{cases} \qquad (i = 1, \ldots, \ell)$$

and has no other structural elements. Furthermore, $\lambda B_{ri} - A_{ri}$ and $\lambda B_{\ell\ell} - A_{\ell\ell}$ are easily seen to be right invertible and left invertible, respectively, because of the rank properties of the $\hat{A}_{ii}$ and of $B_{\ell+1,\ell+1}$, respectively. We are then in the situation of Theorem 2.3. and have thus constructed a pair of reducing subspaces. Finally it follows from the full column rank of $B_{\ell+1,\ell+1}$ that the bottom pencil $\lambda B_{\ell\ell} - A_{\ell\ell}$ has <u>no</u> infinite elementary divisors. The constructed pair of reducing subspaces is thus the <u>unique</u> pair $X_i$, $Y_i$ whose spectrum contains <u>all</u> the infinite eigenvalues of $\lambda B - A$ :

$$(3.5) \qquad (\lambda B - A)\Big|_{X_i, Y_i} = \{\infty, \ldots, \infty\}$$

The spaces are spanned by the first $\hat{s}$ columns of $\hat{Z}$ and $\hat{r}$ columns of $\hat{Q}$, respectively, where

$$(3.6\ a;b) \qquad \hat{s} = \sum_{i=1}^{\ell} e_i \quad ; \quad \hat{r} = \sum_{i=1}^{\ell} r_i$$

It is well known that the minimal indices of the pencils

$$(3.7\ a;b;c;d) \qquad \lambda B - A \ ; \ \mu A - B \ ; \ \mu'(A - \alpha B) - B ; \ \lambda'(B - 1/\alpha A) - A$$

are all the same and that their elementary divisors are related via the transformations $\mu = 1/\lambda$ ; $\mu' = 1/(\lambda - \alpha)$ ; $1/\lambda' = 1/\lambda - 1/\alpha$ .

Therefore the infinite elementary divisors of $\mu'(A - \alpha B) - B$ and $\lambda'(B - 1/\alpha A) - A$ are those of $\lambda B - A$ at $\lambda = \alpha$ . Notice that the cases (3.7b) and (3.7a) are special cases of the latter two for $\alpha = 0$ and $\alpha = \infty$ , respectively. We can thus obtain the unique pair of reducing subspaces $X_\alpha, Y_\alpha$ whose spectrum contains <u>all</u> the eigenvalues $\alpha$ of $\lambda B - A$ :

$$(3.8) \qquad (\lambda B - A)\Big|_{X_\alpha, Y_\alpha} = \{\alpha, \ldots, \alpha\}$$

by using the above method on the pencils $\mu'(A - \alpha B) - B$ or $\lambda'(B - 1/\alpha A) - A$. The pencil (3.7c) is chosen when $|\alpha| < \|B\| / \|A\|$ and the pencil (3.7d) is chosen otherwise (here $\|.\|$ stands for any norm invariant under unitary transformations). This choice ensures the backward stability of the method. Indeed, the algorithm implementing the decomposition (3.3) on a pencil $\lambda D - C$ can be shown to yield <u>exactly</u>

$$\bar{Q}^*(\lambda\bar{D}-\bar{C})\bar{Z} = \left[\begin{array}{c|c} \lambda D_{11}-C_{11} & \lambda D_{12}-C_{12} \\ \hline 0 & \lambda D_{22}-C_{22} \end{array}\right] \tag{3.9}$$

for a slightly perturbed pencil $\lambda\bar{D}-\bar{C}$ satisfying :

$$\|\bar{C}-C\| < \Pi.\varepsilon.\|C\| \qquad \|\bar{D}-D\| < \Pi.\varepsilon.\|D\| \tag{3.10}$$

Here $\varepsilon$ is the machine accuracy of the computer and $\Pi$ is some polynomial expression in the dimensions m and n of the pencil $\lambda D-C$. Moreover, the matrices $\bar{Q}$ and $\bar{Z}$ are $\Pi.\varepsilon$-close to being unitary. With the choice of pencil (3.7c) or (3.7d) proposed above, this yields that the computed spaces $\bar{X}_\alpha$, $\bar{Y}_\alpha$, spanned by the nearly orthonormal columns of $\bar{Q}_1$ and $\bar{Z}_1$, are the exact spaces with spectrum at $\alpha$ of a slightly perturbed pencil $\lambda\bar{B}-\bar{A}$, satisfying

$$\|\bar{A}-A\| < 3\Pi.\varepsilon\,\|A\| \qquad \|\bar{B}-B\| < 3\Pi.\varepsilon\,\|B\| \tag{3.11}$$

In going from (3.10) to (3.11) it is important that $\alpha$ or $1/\alpha$, respectively, can be appropriately bounded, which explains the appropriate choice of pencil (3.7c) or (3.7d) (see [9]). This thus yields a stable algorithm for the computation of a pair of reducing subspaces whose spectrum consists of <u>one</u> point $\alpha$ only.

When one wants to compute pairs of reducing subspaces corresponding to more of less points or, more generally, to compute <u>all</u> reducing subspaces with specified spectrum, one may proceed as follows. First one extracts the regular part of the pencil $\lambda B-A$ via the above algorithm and its dual form. The 'dual' algorithm consists of inverting the role of columns and rows in the above method. This is obtained by using the above method on the 'pertransposed' (i.e. transposed over the second diagonal) and then 'pertransposing' the obtained result (see [9]). This then yields a decomposition of the type :

$$\tilde{Q}^*(\lambda B-A)\tilde{Z} = \left[\begin{array}{c|c} \lambda B_{rf}-A_{rf} & * \\ \hline 0 & \lambda B_{i\ell}-A_{i\ell} \end{array}\right] \tag{3.12}$$

where now $\lambda B_{rf}-A_{rf}$ contains all the <u>finite</u> elementary divisors and the <u>right</u> minimal indices of $\lambda B-A$, and where $\lambda B_{i\ell}-A_{i\ell}$ contains all the <u>infinite</u> elementary divisors and the <u>left</u> minimal indices of $\lambda B-A$. The constructed <u>pair</u> of reducing subspaces $X_f, Y_f$ is thus the unique pair whose spectrum contains <u>all</u> the finite eigenvalues $\lambda B-A$ :

$$(\lambda B-A)\Big|_{X_f,Y_f} = \{\alpha_1,\dots,\alpha_k\} \tag{3.13}$$

Using this dual decomposition on the diagonal blocks $\lambda B_{ri}-A_{ri}$ and $\lambda B_{f\ell}-A_{f\ell}$ of (3.3a) one then separates the right minimal indices and infinite elementary divisors of $\lambda B_{ri}-A_{ri}$ in two diagonal blocks $\lambda B_r-A_r$ and $\lambda B_i-A_i$, and the finite elementary divisors and left minimal indices of $\lambda B_{f\ell}-A_{f\ell}$ in two diagonal blocks $\lambda B_f-A_f$ and $\lambda B_\ell-A_\ell$ :

$$Q^*(\lambda B-A)Z = \begin{bmatrix} \lambda B_r-A_r & * & * & * \\ 0 & \lambda B_i-A_i & * & * \\ 0 & 0 & \lambda B_f-A_f & * \\ 0 & 0 & 0 & \lambda B_\ell-A_\ell \end{bmatrix} \begin{matrix} \}m_1 \\ \}m_2 \\ \}m_3 \\ \}m_4 \end{matrix} \qquad \text{columns: } n_1,\ n_2,\ n_3,\ n_4 \tag{3.14}$$

This decomposition yields the regular part of $\lambda B-A$ (see [9]) :

$$\lambda B_{reg}-A_{reg} = \left[\begin{array}{c|c} \lambda B_i-A_i & * \\ \hline 0 & \lambda B_f-A_f \end{array}\right] \Big\}\, d=m_2+m_3, \qquad d=n_2+n_3 \tag{3.15}$$

and the normal rank r of $\lambda B-A$ is given by $r=m_1+d+n_4$. The reducing subspaces $X_{min}$, $Y_{min}$, $X_{max}$ and $Y_{max}$ are spanned by the first $n_1$ columns of Z, $m_1$ columns of Q, $(n_1+n_2+n_3)$ columns of Z and $(m_1+m_2+m_3)$ columns of Q, respectively.

As discussed in the previous section, there corresponds a pair of reducing subspaces for $\lambda B-A$ to each pair of deflating subspaces of its regular part $\lambda B_{reg}-A_{reg}$, and conversely.The problem is thus reduced now to the computation of <u>deflating</u> subspaces of a <u>regular</u> pencil, which is essentially solved in [11] . These subspaces are obtained by an efficient update of the QZ decomposition [5] in order to obtain any requested ordering of eigenvalues along the diagonal of the decomposition. The method is also adapted to cope with the specific problem of real pencils (see [11] for more details). The numerical stability of this QZ update is proved in [11], which together with the above mentioned method thus yield a stable method for computing pairs of reducing subspaces of an arbitrary singular pencil.

## 4. Concluding remarks.

In the previous section we have presented a method to compute pairs of reducing subspaces with prescribed spectrum, as introduced in Section 2. The method consists of two steps : first, the extraction of the regular part of the pencil $\lambda B-A$, and, second, the computation of a pair of deflating subspaces of this regular part. The latter part can be performed in $O(d^3)$ operations (where d is the dimension of the regular part) using the QZ algorithm [5] and the update in [11] for obtaining the correct spectrum. The method described here for the extraction of the regular part, though, may require a number of operations which is not cubic in the dimensions m and/or n of $\lambda B-A$ but quartic (see [9]) since up to $O(\min(m,n))$ rank determinations of full matrices may be required. When efficiently exploiting the computations of previous rank determinations at each step, to overall amount of operations may be reduced to $O(mn^2)$. This is e.g. done in [2] for a specific class of pencils often occurring in linear system theory, but the idea can be extrapolated to the general case. Similar ideas may be found in the work of Kublanovskaya on dense and sparse pencils [15][16] .

Another link with linear system theory is of a more theoretic nature. All the geometric concepts introduced by Wonham [14] can be shown to be special cases of the concept of reducing subspaces introduced here. Reducing subspaces also enter the picture naturally when trying to extend some results of factorization to the singular case (see e.g. [10]). These remarks thus tend to indicate that the concept of reducing subspaces, as defined here, is an appropriate extension of the concept of deflating subspaces, since it occurs in several practical problems.

A last remark ought to be made about the possible ill-posedness of the spaces we are trying to compute. It is indeed shown via some simple examples in [9] that singular pencils may have an ill-posed eigenstructure and that one must be careful when interpreting the computed results. Yet when one fixes the normal rank of a pencil $\lambda B-A$ to the minimal possible one within $\epsilon$ perturbations of A and B, then the problem of computing reducing subspaces becomes well-posed (in a 'restricted' sense, of course, [10]). This is comparable to the problem of computing a generalized inverse $A^e$ of a $m\times n$ matrix A which becomes well-posed when fixing its $\epsilon$-rank. Moreover, there is hope to derive perturbation bounds for reducing subspaces in the style of Stewart's work on deflating subspaces [7][8] since there is a strong parallelism between both concepts.

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