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FACTORIZATION OF A RATIONAL MATRIX : THE SINGULAR CASE.

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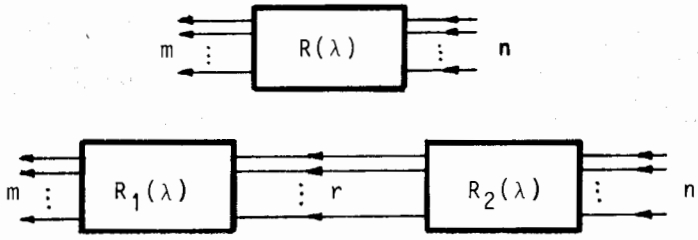
We study the problem of minimal factorization of an arbitrary rational matrix $R(\lambda)$, i.e. where $R(\lambda)$ is not necessarily square or invertible. Following the definition of minimality used here, we show that the problem can be solved via a generalized eigenvalue problem which will be singular when $R(\lambda)$ is singular. The concept of invariant subspace, which has been used in the solution of the minimal factorization problem for regular matrices, is now replaced by a reducing subspace, a recently introduced concept which is a logical extension of invariant and deflating subspaces to the singular pencil case.

INTRODUCTION

The minimal factorization problem of a general $m\times n$ rational matrix $R(\lambda)$ has circuit theory as one of its origin [5] [7][38], but can also be encountered in areas such as control theory [36], filtering [2], etc... The factorization problem consists of finding two rational matrices $R_1(\lambda)$ and $R_2(\lambda)$, respectively of dimensions $m\times r$ and $r\times n$, such that :

$$R(\lambda) = R_1(\lambda) \cdot R_2(\lambda) \tag{0.1}$$

In circuit theory, this corresponds to breaking up a circuit with n inputs and m outputs into a cascade of two circuits with respectively r inputs and m outputs, and n inputs and r outputs :



(0.2)

Recently [4], a complete description has been obtained of the solution of the minimal factorization problem for a regular rational matrix (i.e. $m=n=r$ and $\det[R(\lambda)] \not\equiv 0$). The condition of minimality of the factorization (0.1) is, in the regular case, simply that the McMillan degrees δ_1 and δ_2 of the two factors add up to δ, the McMillan degree of $R(\lambda)$:

$$\delta_1 + \delta_2 = \delta \tag{0.3}$$

(In general, we always have that $\delta_1 + \delta_2 \geq \delta$, which justifies the term "minimal".) This condition guarantees that all the structural elements (i.e. poles, zeros and minimal indices) of $R(\lambda)$ are retrieved in the two factors $R_1(\lambda)$ and $R_2(\lambda)$. The approach used in [4] is geometric and based on a minimal state-space realization of $R(\lambda)$:

$$R(\lambda) = C(\lambda I_\delta - A)^{-1} B + D \tag{0.4}$$

The poles and zeros of $R(\lambda)$ are then described by the eigenvalues of A and $A^\times \triangleq A-BD^{-1}C$, respectively, and the conditions for factorizability are phrased in terms of the invariant subspaces of the two operators A and $A^\times$. Different types of extensions have also been obtained in [3][6][31] and were based on the same principles.

In order to extend these results to the singular case (i.e. $R(\lambda)$ not necessarily square or invertible), we first look at the condition of minimality for a factorization of the type (0.1) where $R(\lambda)$ is $m\times n$, $R_1(\lambda)$ is $m\times r$ and $R_2(\lambda)$ is $r\times n$. We show that the structural elements of $R(\lambda)$- which are now its poles, zeros and left and right minimal indices since $R(\lambda)$ is singular - are distributed over the two factors if r equals the normal rank r_n of $R(\lambda)$ or, in other words, if r is also minimal. Indeed, in general one has for factorization of the type (0.1) that $r \geq r_n$. If the condition

$$r = r_n \tag{0.5}$$

is not added to the degree condition (0.3), then some of the structural properties of $R(\lambda)$ may be lost in the factorization and the factorization is not considered anymore to be "minimal". This is illustrated in the next section.
Secondly, the operator $A^{\times}$ is replaced by a pencil $\lambda\hat{E}-\hat{A}$ in the singular case. Indeed, $R(\lambda)$ may now have a defectuous set of finite zeros, which are then replaced by infinite zeros and/or left and right minimal indices [33] . The standard eigenvalue problem (SEP) $A^{\times}$ has thus to be replaced by a generalized eigenvalue problem (GEP) $\lambda\hat{E}-\hat{A}$. The recently introduced concept of reducing subspaces is recalled in Section 3.
The factorization problem is then solved in terms of the SEP $\lambda I-A$ - describing the poles of $R(\lambda)$ - and the GEP $\lambda\hat{E}-\hat{A}$ - describing the zeros of $R(\lambda)$ - using the geometric ideas of [4]. We also extend these results to the cascade factorization problem and briefly state some related problems in the area of linear system theory.

The following notation will be used throughout the paper. We use A^{*} for the conjugate transpose and A^{T} for the transpose of a matrix A. A complex (real) square matrix A is called unitary (orthogonal) when $A^{*}A=I$ ($A^{T}A=I$). When no explicit distinction is made between the complex and the real case, we use the term unitary and the notation A^{*} for the real case as well. Script is used for vectorspaces. $\mathbb{H}_n$ will denote the spaces $\mathbb{C}^n$ or $\mathbb{R}^n$, depending on the context of the problem. $A\mathcal{X}$ is the image of $\mathcal{X}$ under A ; $\mathcal{I}m\ A$ and $\mathcal{K}er\ A$ are the image and kernel of A, repectively, $\mathcal{X}+\mathcal{Y}$ and $\mathcal{X}\oplus\mathcal{Y}$ are the sum and the direct sum, respectively, of the spaces $\mathcal{X}$ and $\mathcal{Y}$. When the columns of a matrix X form a basis for the space $\mathcal{X}$, this is denoted $\mathcal{X} = \langle X\rangle$. The space spanned by the null vector only is denoted by $\{0\}$.

1. Eigenstructure of a rational matrix

Here we remind in a few words some results about the structural elements of rational matrices and their state-space realizations and we discuss the conditions of minimality of a factorization in terms of these elements.

DEFINITION 1.1

A rational matrix $R(\lambda)$ is called *regular at* α if the constant matrix $R(\lambda)$ is square, bounded and invertible at α. □

From the work of McMillan [16] , it follows that for an arbitrary $m \times n$ rational matrix $R(\lambda)$, there exist decompositions of the type :

$$M_\alpha(\lambda).R(\lambda).N_\alpha(\lambda) = \left[\begin{array}{ccc|c} (\lambda-\alpha)^{\sigma_1} & & & \\ & \ddots & & 0_{r,n-r} \\ & & (\lambda-\alpha)^{\sigma_r} & \\ \hline & 0_{m-r,r} & & 0_{m-r,n-r} \end{array}\right] \qquad (1.1)$$

where $M_\alpha(\lambda)$ and $N_\alpha(\lambda)$ are regular at α and the $\{\sigma_i(\alpha) \mid i=1,\ldots,r\}$ form a non-decreasing sequence, and

$$M_\infty(\lambda).R(\lambda).N_\infty(\lambda) = \left[\begin{array}{ccc|c} (1/\lambda)^{\sigma_1} & & & \\ & \ddots & & 0_{r,n-r} \\ & & (1/\lambda)^{\sigma_r} & \\ \hline & 0_{m-r,r} & & 0_{m-r,n-r} \end{array}\right] \qquad (1.2)$$

where $M_\infty(\lambda)$ and $N_\infty(\lambda)$ are regular at infinity and the $\{\sigma_i(\infty) \mid i=1,\ldots,r\}$ form a non-decreasing sequence. The right hand sides of these decomposition are unique and yield the following definition.

DEFINITION 1.2 [32]

The indices $\sigma_i(\alpha)$ at a (finite or infinite) point α, defined as in (1.1), (1.2), are called the structural indices of $R(\lambda)$ at this point α. □

At almost all points α these indices are trivially zero [16]. This shows that $R(\alpha)$ has rank r almost everywhere, which is therefore called the *normal rank* of $R(\lambda)$. The points α with nontrivial indices are the *poles* (when $\sigma_i(\alpha) < 0$) and *zeros* (when $\sigma_i(\alpha) > 0$) of $R(\lambda)$. We then have

DEFINITION 1.3 [16][20]

The polar degree $\delta_p(\alpha)$ and zero degree $\delta_z(\alpha)$ at a point α is defined as :

$$\delta_p(\alpha) = -\sum_{\sigma_i<0} \sigma_i(\alpha) \; ; \; \delta_z(\alpha) = \sum_{\sigma_i>0} \sigma_i(\alpha) \qquad (1.3)$$

□

When the normal rank r is smaller than m and/or n, one defines the vectorspaces

$$\begin{aligned} N_r(R) &\triangleq \{v(\lambda) \mid R(\lambda)v(\lambda) = 0\} \\ N_\ell(R) &\triangleq \{u(\lambda) \mid u^T(\lambda)R(\lambda) = 0\} \end{aligned} \qquad (1.4)$$

called the right and left nullspaces, respectively, of the rational matrix $R(\lambda)$. These are vectorspaces over the field of rational functions in λ of dimension

$$\text{null}_r \triangleq \dim.N_r(R(\lambda))=n-r, \quad \text{null}_\ell \triangleq \dim.N_\ell(R(\lambda))=m-r \qquad (1.5)$$

respectively, when $R(\lambda)$ is a $m\times n$ rational matrix of normal rank r [9]. These dimensions are also called the right and left nullity of $R(\lambda)$, respectively, and the matrix $R(\cdot)$ is said to be right and left invertible, respectively, when the corresponding nullity if zero. Moreover, it is always possible for a vectorspace S over the field of rational functions in λ, to choose a polynomial basis

$$\{p_1(\lambda) \; , \; \ldots \; , \; p_k(\lambda)\} \qquad (1.6)$$

Let us define the *index* d_i of a polynomial vector $p_i(\lambda)$ as the maximum polynomial degree in its components, then (1.6) is called a *minimal polynomial basis* for S if the sum of the indices d_i is minimal over all polynomial bases for S. These indices are invariant for a given space S, except for their ordering [9]. When corresponding to the spaces $N_r(R)$ and $N_\ell(R)$, they are called the *right and left minimal indices* of $R(\cdot)$. This now leads to the

following

DEFINITION 1.4

The right and left null orders of $R(\cdot)$ are defined as the sums of the right and left minimal indices, respectively :

$$\delta_r(R) = \sum_i r_i \; ; \quad \delta_\ell(R) = \sum_i \ell_i \qquad \square \tag{1.7}$$

Under *eigenstructure* of a rational matrix we now understand all the structural elements described above, i.e. the *polar* and *zero structure* and the *right* and *left null space structure*.

Let now the rational matrix $R(\cdot)$ be bounded at infinity, then it can be <u>realized</u> by a quadruple $\{A,B,C,D\}$:

$$R(\lambda) = C(\lambda I_\delta - A)^{-1} B + D \tag{1.8}$$

Note that $R(\lambda)$ in (1.8) has no poles at infinity since $R(\infty)$ is bounded at infinity. This can always be obtained by a bilinear transformation on λ, which does not affect factorizability. The quadruple $\{A,B,C,D\}$ may be chosen of the same field H as the coefficients of $R(\lambda)$. In the sequel we implicitly assume H to be the field of complex numbers $\mathbb{C}$ and we mention the case $\mathbb{R}$ only when it needs a special treatment. Using the above definitions of poles, zeros and minimal indices of a rational matrix, we have the following theorem [12][20][33] relating the structure of $R(\lambda)$ to that of the pencils

$$S_p(\lambda) \triangleq [\lambda I_\delta - A]; \; S_z(\lambda) \triangleq \left[\begin{array}{c|c} \lambda I_\delta - A & B \\ \hline -C & D \end{array}\right] \tag{1.9}$$

THEOREM 1.1 [33]

Let the quadruple $\{A,B,C,D\}$ be a minimal realization of the rational matrix $R(\lambda)$ and let $S_p(\lambda)$ and $S_z(\lambda)$ be as defined above in (1.9). Then

i) *the poles $\{p_i\}$ of $R(\lambda)$ are the zeros of $S_p(\lambda)$ and so are their nontrivial indices.*

ii) *the zeros* $\{z_i\}$ *of* $R(\lambda)$ *are the zeros of* $S_z(\lambda)$ *and so are their nontrivial indices.*

iii) *the left minimal indices* $\{\ell_i \mid i=1,\ldots,\text{null}_\ell\}$ *and right minimal indices* $\{r_i \mid i=1,\ldots,\text{null}_r\}$ *of* $R(\lambda)$ *and* $S_z(\lambda)$ *are the same* □

This relation between the structural elements of $R(\lambda)$ and of the pencils $S_p(\lambda)$ and $S_z(\lambda)$ reduces the eigenstructure problem of a rational matrix to the problem of retrieving the Kronecker structure of two pencils, i.e. to two eigenvalue problems. In Section 4 it is shown how to exploit this for providing a geometrical solution to the factorization problem. From the above relation one also derives the following result

THEOREM 1.2.

Let $\delta_p(.)$, $\delta_z(.)$ *denote the total polar and zero degrees, respectively, of a rational matrix* $R(\lambda)$, *then the McMilman degree* $\delta(.)$ *of a rational matrix equals :*

$$\delta(R) = \delta_p(R) = \delta_z(R) + \delta_\ell(R) + \delta_r(R) \qquad (1.10)$$

PROOF. See [33]. □

The following conditions are then easily shown to be sufficient for the preservation of the eigenstructure in a factorization.

THEOREM 1.3.

The eigenstructure of the rational matrix $R(\lambda)$ *is retrieved in the factors* $R_1(\lambda)$ *and* $R_2(\lambda)$ *of the factorization (0.1) if*

$$\delta(R) = \delta(R_1) + \delta(R_2) \qquad (1.11)$$

and

$$r = r_n(R) \ , \qquad (1.12)$$

the normal rank of $R(\lambda)$.

PROOF. From (1.10)(1.11) it follows that

$$\delta_p(R) = \delta_p(R_1) + \delta_p(R_2) \tag{1.13}$$

This implies that none of the poles of R_1 and R_2 are cancelled with zeros in the product (0.1). Therefore the poles of $R(\lambda)$ are invariably retrieved in the two factors R_1 and R_2. From (1.12) it follows that r is also the normal rank of R_1 and R_2 and thus that R_1 is left invertible and R_2 right invertible. It then immediately follows that

$$\begin{aligned} N_\ell(R_1) &= N_\ell(R) \quad ; \quad N_r(R_1) = \{0\} \\ N_r(R_2) &= N_r(R) \quad ; \quad N_\ell(R_2) = \{0\} \end{aligned} \tag{1.14}$$

The left minimal indices of R_1 are thus those of R and the right minimal indices of R_2 are those of R, since these indices are completely characterized by the corresponding null spaces [9]. Finally, from (1.10)(1.14) it now follows that

$$\delta_z(R) = \delta_z(R_1) + \delta_z(R_2) \tag{1.15}$$

and, as in the case of the poles, this also guarantees that the zeros of $R(\lambda)$ are invariable retrieved in the two factors. □

Notice that the above conditions are not *necessary* as illustrated by the following counter example (condition (1.12) is *not* satisfied) :

$$\begin{bmatrix} 1/\lambda^2 \\ 1/\lambda^2 \end{bmatrix} = \begin{bmatrix} 1 & 1/\lambda \\ 0 & 1/\lambda \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1/\lambda \end{bmatrix} \tag{1.16}$$

Here all poles are at $\lambda=0$ and zeros at $\lambda = \infty$. The polar and zero degrees are $\delta_p(R) = \delta_z(R) = 2$, $\delta_p(R_1) = \delta_z(R_1) = 1$, $\delta_p(R_2) = \delta_z(R_2) = 1$. $R_1(\lambda)$ is a regular matrix, but both $R(\lambda)$ and $R_2(\lambda)$ have a *constant* left null space of dimension 1, indicating a single left minimal index equal to zero :

$\ell_1(R) = \ell_1(R_2) = 0$. The eigenstructure of $R(\cdot)$ is thus indeed retrieved in its two factors, although only condition (1.11) is *fulfilled. Both conditions* (1.11) and (1.12) are satisfied in the factorization

$$\begin{bmatrix} 1/\lambda^2 & \\ & 1/\lambda^2 \end{bmatrix} = \begin{bmatrix} 1/\lambda & \\ & 1/\lambda \end{bmatrix} \cdot \begin{bmatrix} 1/\lambda & \end{bmatrix} \tag{1.17}$$

where again poles and zeros are distributed in the same manner over the two factors, but now $R_2(\lambda)$ is regular and $\ell_1(R) = \ell_1(R_1) = 0$. In the sequel such factorizations where both conditions (1.11) and (1.12) are satisfied, will be termed *minimal*.

REMARK 1.1

The term "preserved" ought to be clarified here : the left and right minimal indices of $R(\lambda)$ are indeed invariably retrieved in $R_1(\lambda)$ and $R_2(\lambda)$, respectively, while the poles and zeros of $R(\lambda)$ are distributed over the two factors with preservation of their total multiplicity, but not necessarily of their structural indices as described in definition 1.2 (see the above example). □

2. Kronecker structure and zero pencil

In this section we relate Theorem 1.1 to the Kronecker structure of the pencil $S_z(\lambda)$ and define the "zero pencil" of a realization, which will play a fundamental role in the sequel.

Under Kronecker structure of a m n pencil $\lambda S-T$ with entries in H we understand here all the invariants of $\lambda S-T$ under equivalence transformations (i.e. invertible column and row transformations). The Kronecker structure of $\cdot S-T$ is retrieved in its Kronecker canonical form [10] :

$$M(\lambda S-T)P=\lambda S_c-T_c = \mathrm{diag}\{L_{\ell_1},\ldots,L_{\ell_s}, L^T_{r_1},\ldots,L^T_{r_t}, I-\lambda N,\lambda I-J\} \tag{2.1}$$

where i) L_k is the $(k+1)\times k$ bidiagonal pencil

$$L_k = \underbrace{\begin{bmatrix} \lambda & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & \lambda \\ & & & -1 \end{bmatrix}}_{k} \Big\} k+1 \qquad (2.2)$$

and ii) N is nilpotent and both N and J are in Jordan canonical form. The Kronecker structure of the pencil $\lambda S-T$ is then given by:

i) the *finite elementary divisors* $(\lambda-\alpha_i)^{d_j}$ of $\lambda S-T$, reflected by the Jordan blocks of size d_j at α_i in J.

ii) the *infinite elementary divisors* $(\mu)^{d_j}$ of $\lambda S-T$, reflected by the Jordan blocks of size d_j at 0 in N.

iii) the *left and right Kronecker indices* $\{\ell_1,\ldots,\ell_s\}$ and $\{r_1,\ldots,r_t\}$ of $\lambda S-T$, reflected by the blocks L_{ℓ_i} and $L^T_{r_j}$, respectively.

The Kronecker structure of a pencil $\lambda S-T$ and its structural elements as defined in section 1 are closely related [32][33] as shown below.

THEOREM 2.1

i) *Each finite elementary divisor* $(\lambda-\alpha_i)^{d_j}$ *of* $\lambda S-T$ *corresponds exactly to a (positive) index* d_j *at* α_i.

ii) *Each infinite elementary divisor* $(\mu)^{d_j}$ *of* $\lambda S-T$ *corresponds exactly to a (non-negative) index* (d_j-1) *at* ∞.

iii) *The left and right Kronecker indices of* $\lambda S-T$ *are exactly its left and right minimal indices.*

PROOF. See [32][33]. □

The difference between *elementary divisors* (i.e. the multiplicity of a *generalized eigenvalue* as defined by the Kronecker structure) and *structural indices* (i.e. the multiplicity of a *zero* as defined in Section 1) lies thus only in the point at infinity and they only differ by 1. Let now U be <u>any</u> invertible transformation compressing the rows of the last block column

of the system matrix $S_z(\lambda)$ of a rational matrix $R(\lambda)$:

$$U.S_z(\lambda) = U.\left[\begin{array}{c|c} \lambda I-A & B \\ \hline -C & D \end{array}\right] = \left[\begin{array}{c|c} \lambda\hat{E}-\hat{A} & 0 \\ \hline * & \hat{D} \end{array}\right] \tag{2.3}$$

where $\hat{U}$ has linearly independent rows. Then the following holds.

THEOREM 2.2.

The Kronecker structure of the pencil $\lambda\hat{E}-\hat{A}$ and $S_z(\lambda)$ are the same, except for the infinite elementary divisors and right Kronecker indices, which are reduced by 1 in $\lambda\hat{E}-\hat{A}$.

PROOF See [24][29] □

THEOREM 2.3.

The generalized eigenvalues of the pencil $\lambda\hat{E}-\hat{A}$ are the zeros of $R(\lambda)$, multiplicities counted. More specifically, one has that each Jordan chain of length d at a certain generalized eigenvalue of $\lambda\hat{E}-\hat{A}$ corresponds to a structural index d at that zero of $R(\lambda)$.

PROOF. Follows readily from Theorems 1.1, 2.1 and 2.2. Indeed, for the finite zeros, the (positive) structural indices of $R(\lambda)$ and $S_z(\lambda)$ are the same (Theorem 1.1), and are also equal to the Jordan chain lengths of $S_z(\lambda)$ (Theorem 2.1) and of $\lambda\hat{E}-\hat{A}$ (Theorem 2.2). For the infinite zeros, the positive structural indices of $R(\lambda)$ and $S_z(\lambda)$ are still the same (Theorem 1.1), but the corresponding Jordan chains of $S_z(\lambda)$ are 1 larger (Theorem 2.1) while those of $\lambda\hat{E}-\hat{A}$ are then again equal (Theorem 2.2) □

Notice that $\lambda\hat{E}-\hat{A}$ is not uniquely defined since neither are the coordinate system for the realization $\{A,B,C,D\}$ or the transformation U used in (2.3). For a given realization $\{A,B,C,D\}$, $\lambda\hat{E}-\hat{A}$ only depends on the choice of transformation U since

$$\lambda\hat{E}-\hat{A} = U_1 \,.\, S_z(\lambda) \,.\, \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{2.4}$$

where U_1 is the top block row of U and spans the left null space of the compound matrix $\begin{bmatrix} B \\ D \end{bmatrix}$. Because of the latter U_1 is uniquely defined up to an invertible left transformation.

REMARK 2.1

For the case D invertible, a specific choice of U_1, namely $[I_\delta, -BD^{-1}]$ yields the pencil :

$$\lambda I_\delta - A^\times = \lambda I_\delta - (A - BD^{-1}C) \tag{2.5}$$

for $\lambda\hat{E}-\hat{A}$. This is a familiar result which was also used to treat the factorization problem in the regular case [3]. In practice, though, one would prefer to use a <u>unitary</u> transformation U in (2.3) because of the advantageous numerical properties inherited from such a choice (see also Remark 3.1). The rows of U_1 would then be an orthonormal basis for the left nullspace of $\begin{bmatrix} B \\ D \end{bmatrix}$ □

When, on the other hand, a state-space transformation is performed, one obtains the following result.

THEOREM 2.4

When a state-space transformation T is performed on the realization $\{A,B,C,D\}$ *of* $R(\lambda)$, *i.e.*

$$\{A,B,C,D\} \xrightarrow{T} \{T^{-1}AT,\ T^{-1}B,\ C^T,\ D\} \tag{2.6}$$

then the pencils $\lambda I-A$ *and* $\lambda\hat{E}-\hat{A}$ *are transformed as :*

$$\lambda I - A \xrightarrow{T} \lambda I - A_t = \lambda I - T^{-1}AT \tag{2.7a}$$

$$\lambda\hat{E}-\hat{A} \xrightarrow{T} \lambda\hat{E}_t - \hat{A}_t = \lambda R\hat{E}T - R\hat{A}T \tag{2.7b}$$

where R *is an invertible transformation depending on the choice of transformation in* (2.3).

PROOF. The result for $\lambda I-A$ is trivial. For $\lambda\hat{E}-\hat{A}$, let

$$\left[\begin{array}{c|c} \lambda\hat{E}-\hat{A} & 0 \\ \hline * & \hat{D} \end{array}\right] = U \cdot S_z(\lambda) \tag{2.8}$$

$$\left[\begin{array}{c|c} \lambda\hat{E}_t-\hat{A}_t & 0 \\ \hline * & \hat{D}_t \end{array}\right] = U_t \cdot \left[\begin{array}{c|c} T^{-1} & \\ \hline & I_n \end{array}\right] \cdot S_z(\lambda) \cdot \left[\begin{array}{c|c} T & \\ \hline & I_n \end{array}\right]$$

$$= V \cdot S_z(\lambda) \cdot \left[\begin{array}{c|c} T & \\ \hline & I_n \end{array}\right] \tag{2.9}$$

be the corresponding reductions (2.3) in the two coordinate systems. When partitioning U and V conformably with the left hand sides of (2.8) and (2.9), we find that their top part U_1 and V_1, respectively, both span the left null space of $\begin{bmatrix} B \\ D \end{bmatrix}$. Therefore there exists an invertible transformation R, depending on the choice of U and U_t in (2.8)(2.9), such that :

$$V_1 = R \cdot U_1 \tag{2.10}$$

Comparing (2.8)(2.9)(2.10) one than finds (2.7b). □

As follows from Remark 2.1 and Theorems 2.3 and 2.4, several properties of the *pole operator* A and *zero operator* $A^{\times}$ [3], carry over in the singular case to the pencils (2.7a) and (2.7b). It is then also to be expected that, in the singular case, they play an important role in the analysis of the poles and zeros of the factors of $R(\lambda)$, as is shown in the next section. The pencils $\lambda I-A$ and $\lambda\hat{E}-\hat{A}$ are therefore termed the *pole pencil* and *zero pencil*, respectively, of the rational matrix $R(\cdot)$.

3. Reducing subspaces

In this section we develop the geometrical background of the generalized eigenvalue problem, needed for computing minimal factorizations.

The concept of *reducing subspace* [28][29] reviewed in this section, separates the Kronecker structure of an arbitrary pencil into that of two subpencils. These 'substructures' will be exactly those of the two factors $R_1(\lambda)$ and $R_2(\cdot)$ to be computed.

By $\Lambda(S,T)$, we denote the *spectrum* of the pencil $\lambda S-T$, i.e. the collection of generalized eigenvalues, multiplicities counted. By $E(S,T)$ we denote the complete *Kronecker structure* of the pencil $\lambda S-T$, i.e. all the structural elements as described by the Kronecker canonical form (see Section 2).

REMARK 3.1

All the decompositions used for explicitating the geometric concepts defined in this section are chosen to be unitary, although there is no need for it. The reason for this choice is that stable numerical methods for computing these spaces are based on these "unitary" decompositions [17][23][24][25][26][29] (see also [13][14][35] for alternative methods). □

Let X and Y be subspaces of H_n and H_m, respectively, such that

$$Y = SX + TX \tag{3.1}$$

Let ℓ and k be their respective dimensions and construct the unitary matrices Q and Z, partitioned as :

$$Z = [Z_1 \mid \underbrace{Z_2}_{\ell}] \; ; \quad Q = [Q_1 \quad \underbrace{Q_2}_{k}] \tag{3.2}$$

such that, in the given coordinate system

$$X = \langle Z_2 \rangle \; ; \; Y = \langle Q_2 \rangle \tag{3.3}$$

Then it follows from (3.1) that $Q_1^* S Z_2 = Q_1^* T Z_2 = 0$ and thus

$$Q^*(\lambda S-T)Z \triangleq \lambda\hat{S}-\hat{T} \triangleq \lambda \begin{bmatrix} \hat{S}_{11} & 0 \\ \hat{S}_{21} & \hat{S}_{22} \end{bmatrix} - \begin{bmatrix} \hat{T}_{11} & 0 \\ \hat{T}_{21} & \hat{T}_{22} \end{bmatrix} \tag{3.4}$$

In this new coordinate system, $\mathcal{X}$ and $\mathcal{Y}$ are now represented by

$$\mathcal{X} = \left\langle \begin{bmatrix} 0 \\ \\ I_\ell \end{bmatrix} \right\rangle \quad ; \quad \mathcal{Y} = \left\langle \begin{bmatrix} 0 \\ \\ I_k \end{bmatrix} \right\rangle \tag{3.5}$$

The map $\lambda S-T$ restricted to the spaces $\mathcal{X}$ and $\mathcal{Y}$, its spectrum and its eigenstructure are also denoted by

$$\lambda\hat{S}_{22}-\hat{T}_{22}=(\lambda S-T)|_{\mathcal{X},\mathcal{Y}};\Lambda(\hat{S}_{22},\hat{T}_{22})=\Lambda(S,T)|_{\mathcal{X},\mathcal{Y}};E(\hat{S}_{22},\hat{T}_{22})=E(S,T)|_{\mathcal{X},\mathcal{Y}} \tag{3.6}$$

Notice that the orthogonal complements $\mathcal{X}^\perp$ of $\mathcal{X}$ and $\mathcal{Y}^\perp$ of $\mathcal{Y}$ also yield a triangular decomposition of the pencil λS^*-T^*. Indeed by taking the conjugate transpose of (3.4) and rearranging the blocks, one obtains :

$$\left[Z_2|Z_1\right]^* (\lambda S^*-T^*) \left[Q_2|Q_1\right]= \lambda \begin{bmatrix} \hat{S}_{22}^* & 0 \\ \\ \hat{S}_{21}^* & \hat{S}_{11}^* \end{bmatrix} - \begin{bmatrix} \hat{T}_{22}^* & 0 \\ \\ \hat{T}_{21}^* & \hat{T}_{11}^* \end{bmatrix} \tag{3.7}$$

In analogy to (3.6), the map λS^*-T^* restricted to the spaces $\mathcal{Y}^\perp$ and $\mathcal{X}^\perp$, its spectrum and its eigenstructure are denoted in terms of the subpencil $\lambda\hat{S}_{11} - \hat{T}_{11}$, or also, using a "dual" notation :

$$\lambda\hat{S}_{11}-\hat{T}_{11}=_{\mathcal{Y}^\perp,\mathcal{X}^\perp}|(\lambda S-T);\Lambda(\hat{S}_{11},\hat{T}_{11})=_{\mathcal{Y}^\perp,\mathcal{X}^\perp}\Lambda(S,T) ; \quad E(\hat{S}_{11},\hat{T}_{11}) =_{\mathcal{Y}^\perp,\mathcal{X}^\perp}| E(S,T) \tag{3.8}$$

In [29] , it is shown that a pair of subspaces $\mathcal{X},\mathcal{Y}$ as in (3.1) always satisfies the inequality

$$\dim.\mathcal{Y} \geqslant \dim.\mathcal{X} - \dim. N_r \tag{3.9}$$

When equality is met, i.e. when

$$\mathcal{Y} = S\mathcal{X} + T\mathcal{X} \ ; \ \dim.\mathcal{Y} = \dim.\mathcal{X} - \dim.\mathcal{N}_r \tag{3.10}$$

the spaces are called a pair of *(right) reducing subspaces* of the pencil $\lambda S-T$. Notice that, because of (1.5), we then automatically have

$$\mathcal{X}^{\perp} = S^{*}\mathcal{Y}^{\perp} + T^{*}\mathcal{Y}^{\perp}; \ \dim.\mathcal{X}^{\perp} = \dim.\mathcal{Y}^{\perp} - \dim.\mathcal{N}_{\ell} \tag{3.11}$$

For this reason the spaces $\mathcal{Y}^{\perp},\mathcal{X}^{\perp}$ are called a pair of *left* reducing subspaces of $\lambda S-T$. Notice also that all the subspaces in (3.10) and (3.11) can be retrieved from $\mathcal{X}$ only, which is therefore called <u>a</u> reducing subspace of $\lambda S-T$.

REMARK 3.2

The development of reducing subspaces followed in [29] was based on upper block triangular forms in contrast with the lower block triangular form (3.4) used here. This is inessential since the spaces can be defined in a coordinate free manner, but it will prove useful in the sequel. □

In [29] we proved that the above generalization is a logical one since it carries over properties of deflating and invariant subspaces to the singular pencil case. These properties are recalled in the following theorem (for a proof see [29]) :

THEOREM 3.1

a) *Let $\mathcal{X},\mathcal{Y}$ be a pair of subspaces satisfying* (3.1) . *Then the reduced pencils* $(\lambda S-T)|_{\mathcal{X},\mathcal{Y}}$ *and* $\mathcal{Y}^{\perp},\mathcal{X}^{\perp}|(\lambda S-T)$ *are right and left invertible, respectively, iff it is a pair of reducing subspaces. For such a pair the following spectral reduction is then obtained :*

$$\Lambda(S,T)=\Lambda(S,T)\Big|_{\mathcal{X},\mathcal{Y}} \ \dot{\cup} \ \Lambda(S,T)\Big|_{\mathcal{Y}^{\perp},\mathcal{X}^{\perp}} \tag{3.12}$$

where $\dot{\cup}$ *denotes union with any common elements repeated.*

b) *If, moreover, the reduced spectra are disjoint :*

$$\Lambda(S,T)\big|_{X,Y} \cap \Lambda\big|_{Y^\perp,X^\perp}(S,T) = \emptyset \ , \qquad (3.13)$$

then we also have the following eigenstructure reduction :

$$E(S,T) = E(S,T)\big|_{X,Y} \cup E\big|_{Y^\perp,X^\perp}(S,T) \ . \qquad (3.14)$$

and the corresponding pairs of right and left reducing subspaces X,Y *and* $Y^\perp,X^\perp$ *are then the* <u>*unique*</u> *pairs with spectrum*

$$\Lambda_1 = \Lambda(S,T)\big|_{X,Y} \ ; \ \Lambda_2 = \Lambda\big|_{Y^\perp,X^\perp}(S,T) \ , \qquad (3.15)$$

respectively. □

REMARK 3.3

In [28][29] it is shown that the set of all possible reducing subspaces X form a lattice with "smallest" element X_{min} and "largest" element X_{max} (the same holds for the corresponding Y spaces). From Theorem 2.2 in [28], it also follows that for a pencil $\lambda S_c - T_c$ in Kronecker canonical form (2.1), a reducing subspace X_c must be a direct sum of the columnspace of all $L_{r_i}^T$ blocks and of an (arbitrary) invariant subspace of $N \oplus J$. For an equivalent pencil $\lambda S - T = M^{-1}(\lambda S_c - T_c)P^{-1}$ a reducing subspace X then of course is given by $P\,X_c$. This also explains the lattice structure of these spaces. □

In the next section reducing subspaces of the zero pencil $\lambda\hat{E}-\hat{A}$ are used to solve the factorization problem. In this context we can make here already the following remarks.

REMARK 3.4

a) Notice that a representation $\mathcal{X}$ of a reducing subspace X of $\lambda\hat{E}-\hat{A}$ is <u>not</u> affected by the degree of freedom in the construction (2.3) of $\lambda\hat{E}-\hat{A}$. This follows from Theorem 2.4 since for a <u>given</u> coordinate system, any two zero pencils are related by a left transformation R only.

b) Instead, when a state-space transformation T is performed on the realization $\{A,B,C,D\}$ of $R(\cdot)$, i.e.

$$\{A,B,C,D\} \xrightarrow{T} \{T^{-1}AT,\ T^{-1}B,\ CT,\ D\} \tag{3.16}$$

then representations X_p of an invariant subspace $\mathcal{X}_p$ of $\lambda I-A$, and X_z of a reducing subspace $\mathcal{X}_z$ of (any) $\lambda\hat{E}-\hat{A}$, are transformed *simultaneously* as :

$$X_p \xrightarrow{T} T^{-1}X_p \ ;\ X_z \xrightarrow{T} T^{-1}X_z \tag{3.17}$$

according to Theorem 2.4 again.

c) Using the notation (3.8) on $\lambda S-T = S_z(\cdot)$ as in (2.3), we have that with $\mathcal{X} = Ker\ S$ and $\mathcal{Y} = T\mathcal{X}$ one can write :

$$\lambda\hat{E}-\hat{A} = {}_{\mathcal{Y}^\perp,\mathcal{X}^\perp}\big|S_z(\cdot) \tag{3.18}$$

since $Ker\ S = Im \begin{bmatrix} 0 \\ I_n \end{bmatrix}$ and $T\mathcal{X} = Im \begin{bmatrix} B \\ D \end{bmatrix}$. The degrees of freedom mentioned in Theorem 2.4 are reflected here in the possible choice of representation for $\mathcal{Y}^\perp$ and $\mathcal{X}^\perp$. □

4. Factorization.

Here we give necessary and sufficient conditions for the existence of a minimal factorization (0.1) (0.3) (0.5), based on the spectral subspaces defined in the previous section. Algorithms are thereby obtained to generate any possible factorization of a given matrix.

It follows from conditions (0.3) (0.5) that no pole zero cancellations occur between $R_1(\lambda)$ and $R_2(\cdot)$ if the factorization is minimal. The poles (respectively zeros) of $R(\lambda)$ are then those of $R_1(\lambda)$ and $R_2(\lambda)$. Therefore the factors $R_i(\lambda)$ may have no infinite poles and can thus be realized as :

$$R_i(\lambda) = D_i+C_i(\lambda I_{\delta_i}-A_{ii})^{-1}B_i \ ;\ i=1,2 \tag{4.1}$$

We then have the following two results whose proofs (see [4]) also hold for nonsquare matrices.

LEMMA 4.1

Let the factors $R_i(\lambda)$ $i=1,2$ *be realized as in* (4.1) *then a realization for* $R(\lambda) = R_1(\lambda) \,.\, R_2(\lambda)$ *is given by*

$$A \triangleq \left[\begin{array}{c|c} A_{11} & B_1C_2 \\ \hline 0 & A_{22} \end{array}\right] \quad B \triangleq \left[\begin{array}{c} B_1D_2 \\ \hline B_2 \end{array}\right] \begin{array}{l} \}\delta_1 \\ \\ \}\delta_2 \end{array} \tag{4.2}$$

$$C \triangleq \left[\underbrace{C_1}_{\delta_1} \;\middle|\; \underbrace{D_1C_2}_{\delta_2}\right] \quad D \triangleq \left[\underbrace{D_1D_2}_{n}\right] \quad \}m$$

PROOF. See [4] □

THEOREM 4.1

A transfer function $R(\lambda)$ *has a minimal factorization* (0.1)(0.3)(0.5) *iff it has a realization*

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] \quad B = \left[\begin{array}{c} B_{11} \\ \hline B_{22} \end{array}\right] \begin{array}{l} \}\delta_1 \\ \\ \}\delta_2 \end{array} \tag{4.3}$$

$$C = \left[\underbrace{C_{11}}_{\delta_1} \;\middle|\; \underbrace{C_{22}}_{\delta_2}\right] \quad D = \left[\underbrace{D}_{n}\right] \quad \}m$$

with

$$\text{rank} \,.\, \begin{bmatrix} A_{12} & B_{11} \\ C_{22} & D \end{bmatrix} = r_n \tag{4.4}$$

PROOF

<u>If</u> : If rank condition (4.4) is satisfied, then there exists a factorization of the type

$$\begin{bmatrix} A_{12} & B_{11} \\ C_{22} & D \end{bmatrix} = \underbrace{\begin{bmatrix} B_1 \\ D_1 \end{bmatrix}}_{r_n} \cdot \begin{bmatrix} C_2 & D_2 \end{bmatrix} \quad \} r_n \qquad (4.5)$$

Now put $C_1=C_{11}$, $B_2=B_{22}$, then $\{A_{11},B_1,C_1,D_1\}$ and $\{A_{22},B_2,C_2,D_2\}$ are realizations of factors $R_1(\lambda)$, $R_2(\lambda)$ according to Lemma 4.1. Conditions (0.3) and (0.5) are clearly satisfied.

<u>Only if</u> : If there exists a factorization $R_1(\lambda).R_2(\lambda)$ satisfying (0.3) and (0.5), then let (4.1) be a realization of the factors. The realization (4.2) of Lemma 4.1 then satisfies (4.3)(4.4). □

In [4] D was assumed to be square and invertible. The existence of such a *factorable realization* (4.3)(4.4) was analyzed with the invariant subspaces of A and $A-BD^{-1}C$. The following theorem extends this result to the case where D is not necessarily square or invertible, by now considering the pencils $\lambda I-A$ and $\lambda\hat{E}-\hat{A}$ defined above (note that the reducing subspaces of $\lambda\hat{E}-\hat{A}$ become the invariant subspaces of $A-BD^{-1}C$ when D is square and invertible). The following lemma links the rank condition of the previous theorem to the concept of reducing subspaces.

LEMMA 4.2

Let the realization $\{A,B,C,D\}$ *be partitioned as :*

$$A = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_{11} \\ 0 & A_{22} & B_{22} \\ \hline \underbrace{C_{11}}_{\delta_1} & \underbrace{C_{22}}_{\delta_2} & \underbrace{D}_{n} \end{array}\right] \begin{array}{l} \}\delta_1 \\ \}\delta_2 \\ \}m \end{array} \qquad (4.6)$$

and let $\lambda\hat{E}-\hat{A}$ *be the zero pencil of this realization, then* $X_z = Im\begin{bmatrix} 0 \\ I_{\delta_2} \end{bmatrix}$ *is a deflating subspace of* $\lambda\hat{E}-\hat{A}$ *iff*

$$\text{rank} \cdot \begin{bmatrix} A_{12} & B_{11} \\ C_{22} & D \end{bmatrix} = r \tag{4.7}$$

where r is the normal rank of $R(\lambda)$

PROOF

<u>If</u> : Let rank condition (4.7) be satisfied, then there exist a unitary transformation V such that

$$V^{\star} \cdot \begin{bmatrix} -A_{12} & B_{11} \\ -C_{22} & D \end{bmatrix} = \begin{bmatrix} V_{11}^{\star} & V_{31}^{\star} \\ V_{13}^{\star} & V_{33}^{\star} \end{bmatrix} \cdot \begin{bmatrix} -A_{12} & B_{11} \\ -C_{22} & D \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 & 0 \\ -C_2 & D_2 \end{bmatrix} \begin{array}{l} \}\ \delta_1+m-r \\ \}\ r \end{array} \end{array} \tag{4.8}$$

Notice that $m-r = \text{null}_{\ell}$ because of Theorem 1.1. This can now be embedded into :

$$\left[\begin{array}{cc|c} V_{11}^{\star} & 0 & V_{31}^{\star} \\ 0 & I_{\delta_2} & 0 \\ \hline V_{13}^{\star} & 0 & V_{33}^{\star} \end{array}\right] \cdot \left[\begin{array}{cc|c} \lambda I_{\delta_1}-A_{11} & -A_{12} & B_{11} \\ 0 & \lambda I_{\delta_2}-A_{22} & B_{22} \\ \hline -C_{11} & -C_{22} & D \end{array}\right] =$$

$$\left[\begin{array}{cc|c} \lambda\hat{E}_{11}-\hat{A}_{11} & 0 & 0 \\ 0 & \lambda I_{\delta_2}-A_{22} & B_{22} \\ \hline * & -C_2 & D_2 \end{array}\right] \begin{array}{l} \}\ \delta_1+\text{null}_{\ell} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \delta_2+m-\text{null}_{\ell} \end{array} \tag{4.9}$$

and further reduced by an additional transformation on the two lower block rows to the form (where $\hat{D}$ has linearly independent rows) :

$$\left[\begin{array}{cc|c} \lambda\hat{E}_{11}-\hat{A}_{11} & 0 & 0 \\ * & \lambda\hat{E}_{22}-\hat{A}_{22} & 0 \\ \hline * & * & \hat{D} \end{array}\right] \begin{array}{l} \}\delta_1+\text{null}_\ell \\ \left.\begin{array}{l} \\ \\ \end{array}\right\}\delta_2+m-\text{null}_\ell \\ \\ \end{array} \tag{4.10}$$

According to (3.7)(3.11), the space $\mathcal{Y}_z^{\perp}=Im\begin{bmatrix} I_{\delta_1+\text{null}_\ell} \\ 0 \end{bmatrix}$ is thus a left reducing subspace, and $\mathcal{X}_z = Im\begin{bmatrix} 0 \\ I_{\delta_2} \end{bmatrix}$ a (right) reducing subspace of the zero pencil $\lambda\hat{E}-\hat{A}$ of the realization (4.6)(4.7).

<u>Only if</u> :

Let $\mathcal{X}_z$ be spanned by $\begin{bmatrix} 0 \\ I_{\delta_2} \end{bmatrix}$. Then there exists a unitary transformation U, conformably partitioned with the realization $\{A,B,C,D\}$ such that

$$\begin{array}{l} \delta_1+\text{null}_\ell \\ \delta_2+m-\text{null}_\ell \end{array} \left\{\begin{array}{c} \left[\begin{array}{cc|c} U_{11}^* & U_{21}^* & U_{31}^* \\ U_{12}^* & U_{22}^* & U_{32}^* \\ \hline U_{13}^* & U_{23}^* & U_{33}^* \end{array}\right] \\ \begin{array}{ccc} \delta_1 & \delta_2 & m \end{array} \end{array}\right. \cdot \left[\begin{array}{cc|c} \lambda I_{\delta_1}-A_{11} & -A_{12} & B_{11} \\ 0 & \lambda I_{\delta_2}-A_{22} & B_{22} \\ \hline -C_{11} & -C_{22} & D \end{array}\right] = \tag{4.11}$$

$$\left[\begin{array}{cc|c} \lambda\hat{E}_{11}-\hat{A}_{11} & 0 & 0 \\ * & \lambda\hat{E}_{22}-\hat{A}_{22} & 0 \\ \hline * & * & \hat{D} \\ \delta_1 & \delta_2 & n \end{array}\right] \begin{array}{l} \}\delta_1+\text{null} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\}\delta_2+m-\text{null} \\ \\ \end{array}$$

This form is obtained by first compressing the rows of the last block column $[B_{11}^T, B_{22}^T, D^T]^T$ as in (2.3), and then updating the left transformation in order to obtain the lower block triangular form of $\lambda\hat{E}-\hat{A}$ corresponding to $\mathcal{X}_z$, as in (3.4). From this is follows that $U_{21}=0$ since

$$[\, U_{11}^* \quad U_{21}^* \quad U_{31}^* \,] \cdot \begin{bmatrix} 0 \\ \lambda I_{\delta_2} \\ 0 \end{bmatrix} = 0 \tag{4.12}$$

Therefore we have that the $\delta_1+\text{null}_\ell$ rows of $[U_{11}^\star \quad U_{31}^\star]$ are linearly independent and that :

$$\delta_1+\text{null}_\ell \left\{ \begin{bmatrix} U_{11}^\star & U_{31}^\star \end{bmatrix} \right. \underbrace{\begin{bmatrix} A_{12} & B_{11} \\ C_{22} & D \end{bmatrix}}_{\delta_2 \quad n} \begin{matrix} \delta_1 \\ m \end{matrix} = 0 \tag{4.13}$$

This implies that

$$\text{rank} \begin{bmatrix} A_{12} & B_{11} \\ C_{22} & D \end{bmatrix} \leq m-\text{null}_\ell = r \tag{4.14}$$

The inequality can not hold, though, since this would imply, through Lemma 4.1, the existence of a factorization with smaller normal rank than r. Condition (4.7) is thus satisfied. □

This now allows to give a spectral description of the existence of a factorization of a rational matrix.

THEOREM 4.2.

Let $\lambda I-A$ *and* $\lambda\hat{E}-\hat{A}$ *be the pole and zero pencils of a realization* $\{A,B,C,D\}$ *of* $R(\lambda)$. *Then* $R(\lambda)$ *has a factorization realization* $\{A_t, B_t, C_t, D\} = \{T^{-1}AT,\ T^{-1}B, CT, D\}$ *iff there exist independent subspaces* X_p *and* X_z *such that*

$$\text{(i)}\quad AX_p \subset X_p \quad \text{(ii)}\ \dim.(\hat{E}X_z+\hat{A}X_z)=\dim.X_z-\text{null}_\ell \quad \text{(iii)}\ X_p + X_z = H_\delta \tag{4.15}$$

PROOF.

<u>If</u> : Because of (iii), the transformation $T= [X_p \mid X_z]$, where the columns of X_p and X_z span the spaces X_p and X_z, is invertible. Since (i) holds, the transformed system $\{A_t,B_t,C_t,D_t\}$ has the form

$$A_t = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] ;\ B_t = \begin{bmatrix} B_{11} \\ \hline B_{22} \end{bmatrix} \tag{4.16}$$

$$C_t = \left[\begin{array}{c|c} C_1 & C_{22} \end{array}\right] ;\ D_t = \left[\ D\ \right]$$

In this coordinate system the pole pencil $\lambda I-A_t$ has the required form (4.3) and the rank condition (4.4) is satisfied because of (ii) and Lemma 4.2.

Only if :

Let (4.3)(4.4) be a factorable realization of $R(\lambda)$. Clearly $X_p = Im \begin{bmatrix} I_{\delta_1} \\ 0 \end{bmatrix}$ satisfies (i) in that coordinate system.

Since rank condition (4.4) holds, $X_z = Im \begin{bmatrix} 0 \\ I_{\delta_2} \end{bmatrix}$ satisfies (ii) because of Lemma 4.2. The third condition (iii) is also clearly satisfied. □

5. Cascade factorization

Extending the results of the previous section to several factors, one is led to the problem of minimal cascade factorization :

$$R(\lambda) = R_1(\lambda) \,.\, R_2(\lambda) \,.\, \ldots \,.\, R_k(\lambda) \tag{5.1}$$

where all $R_i(\lambda)$ have small degree. This factorization is said to be *minimal* when the degrees δ_i of the factors $R_i(\lambda)$ add up to the degree δ of $R(\lambda)$:

$$\delta = \sum_{i=1}^{k} \delta_i \tag{5.2}$$

and when all the intermediate factors $R_i(\lambda)$, $i=2,\ldots,k-1$, have dimensions $r \times r$, with r the normal rank of $R(\lambda)$. Such factorizations are obtained by repeating the factorization described in the previous section (k-1) times. It easily follows that the minimality conditions still imply the absence of pole/zero cancellations between the factors $R_i(\lambda)$ in (5.1). The poles (respectively zeros) of $R(\lambda)$ are then those of the factors $R_i(\lambda)$; the left null space of $R(\lambda)$ is that of $R_1(\lambda)$ and the right null space of $R(\lambda)$ is that of $R_k(\lambda)$. Since here also the factors $R_i(\lambda)$ have no infinite poles, they can be realized as follows :

$$R_i(\lambda) = D_i + C_i(\lambda I_{\delta_i} - A_{ii})^{-1} B_i \tag{5.3}$$

The following lemma [30] holds also for the nonsquare case :

LEMMA 5.1.

Let the factors $R_i(\lambda)$ *be realized by* (5.3) *for* $i=1,\ldots,k$ *and define*

$$A_{ij} \triangleq B_i D_{i+1} \cdots D_{j-1} C_j \qquad (\text{for } i < j) \tag{5.4a}$$

$$B_{jj} \triangleq B_j D_{j+1} \cdots D_k \tag{5.4b}$$

$$C_{jj} \triangleq D_1 \cdots D_{j-1} C_j \tag{5.4c}$$

$$D_{kk} \triangleq D_1 \cdots D_k \tag{5.4d}$$

Then the product (5.1) *is realized by* $\{A,B,C,D\}$ *with*

$$A \triangleq \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ 0 & \ddots & \vdots \\ & & A_{kk} \end{bmatrix} ; \; B \triangleq \begin{bmatrix} B_{11} \\ \vdots \\ B_{kk} \end{bmatrix} \begin{matrix} \} \, 1 \\ \\ \} \, k \end{matrix} \tag{5.5a;b}$$

$$C \triangleq \begin{bmatrix} \underbrace{C_{11}}_{\delta_k} & \cdots & \underbrace{C_{kk}}_{\delta_k} \end{bmatrix} ; \; D \triangleq \begin{bmatrix} \underbrace{D_{kk}}_{n} \end{bmatrix} \} \, m \tag{5.5c;d}$$

PROOF. See [30] . □

Let us now define for the realization (5.5) the compound matrices

$$\left[\begin{array}{c|c} A^{(j)} & B^{(j)} \\ \hline C^{(j)} & D^{(j)} \end{array}\right] \triangleq \left[\begin{array}{ccc|c} A_{1,j+1} & \cdots & A_{1,k} & B_{1,1} \\ \vdots & & \vdots & \vdots \\ A_{j,j+1} & \cdots & A_{j,k} & B_{j,j} \\ \hline C_{j+1,j+1} & \cdots & C_{k,k} & D_{k,k} \end{array}\right] \qquad j=1,\ldots,k-1 \tag{5.6}$$

The following theorem discusses the existence of a cascade factorization (5.1)(5.2) for a realization in the form (5.5).

THEOREM 5.1

Let $R(\lambda)$ *be a* $m \times n$ *proper regular transfer matrix. Then* $R(\lambda)$ *has a factorization* (5.1) *with factors* $R_i(\lambda)$ *of degrees* δ_i, *iff it has a minimal realization* $\{A,B,C,D\}$ *of the form* (5.5) *with*

$$\operatorname{rank}\left[\begin{array}{c|c} A^{(j)} & B^{(j)} \\ \hline C^{(j)} & D^{(j)} \end{array}\right] = r \quad \text{for } j=1,\dots,k-1 \tag{5.7}$$

PROOF.

Only if : Let the factors $R_j(\lambda)$ of the factorization (5.1) be realized as in (5.3) and take the realization (5.4)(5.5) for $R(\lambda)$. Using (5.4) one easily checks that :

$$\left[\begin{array}{c|c} A^{(j)} & B^{(j)} \\ \hline C^{(j)} & D^{(j)} \end{array}\right] = \left[\begin{array}{ccc} B_1D_2 & \dots & D_j \\ B_2 & \dots & D_j \\ & \vdots & \\ & B_j & \\ \hline D_1D_2 & \dots & D_j \end{array}\right] \cdot \left[\begin{array}{c|c} C_{j+1}, D_{j+1}C_{j+2}, \dots, D_{j+1} & \dots D_{k-1}C_k \\ D_{j+1} \dots D_k & \end{array}\right] \tag{5.8}$$

This proves the rank condition (5.7) since the factors $R_j(\lambda)$ have dimensions $r\times r$ for $j=2,\dots,k-1$.

If : For $k=2$, this is Theorem 4.1. We now prove by induction that it holds for larger k if it also holds for $k'=k-1$. Partition (5.5) as :

$$A \triangleq \left[\begin{array}{c|c} A_{1,1} & A^{(1)} \\ \hline 0 & A'_{22} \end{array}\right] \quad B \triangleq \left[\begin{array}{c} B^{(1)} \\ \hline B'_2 \end{array}\right] \quad \begin{array}{l} 1 \\ k \\ i \end{array} \tag{5.9}$$

$$C \triangleq \left[\underbrace{C_{1,1}}_{\delta_1} \;\middle|\; \underbrace{C^{(1)}}_{\delta'}\right] \quad D \triangleq \left[\underbrace{D^{(1)}}_{n}\right] \quad m$$

Then condition (5.6) for $j=1$ says that this matrix can be factorized as :

$$\begin{array}{l} \delta_1\{ \\ m\{ \end{array} \left[\begin{array}{c|c} A^{(1)} & B^{(1)} \\ \hline \underbrace{C^{(1)}}_{\delta'} & \underbrace{D^{(1)}}_{n} \end{array}\right] \triangleq \underbrace{\left[\begin{array}{c} B_1 \\ \hline D_1 \end{array}\right]}_{r} \cdot \left[C'_2 \quad D'_2\right] \quad r \tag{5.10}$$

Putting $C_1=C_{1,1}$, this yields :

$$A = \left[\begin{array}{c|c} A_{1,1} & B_1C_2' \\ \hline 0 & A_{22}' \end{array}\right] ; \; B = \left[\begin{array}{c} B_1D_2' \\ \hline B_2' \end{array}\right]$$

$$C = \left[\begin{array}{c|c} C_1 & D_1C_2' \end{array}\right] ; \; D = \left[\begin{array}{c} D_1D_2' \end{array}\right] \tag{5.11}$$

From Theorem 4.1 it follows that this corresponds to a factorization

$$R(\lambda) = R_1(\lambda) \; . \; R'(\lambda) \tag{5.12}$$

with

$$\begin{aligned} R_1(\lambda) &= C_1(\lambda I_{\delta_1} - A_{11})^{-1} \; B_1 + D_1 \quad ; \\ R'(\lambda) &= C_2'(\lambda I_{\delta'} - A_{22}') \; B_2' + D_2' \end{aligned} \tag{5.13}$$

The system $\{A_{22}', B_2', C_2', D_2'\}$ now has a block structure as in (5.5) but with $k'=k-1$ blocks on the diagonal of A_{22}'. Because of (5.10), the rank condition (5.7) for $j=2,\ldots,k-1$ imposed on $\{A,B,C,D\}$ are equivalent to the corresponding rank conditions on the subsystem $\{A_{22}', B_2', C_2', D_2'$ for $j=1,\ldots,k'-1$. Since the theorem is assumed to hold for k', we thus have that the blocks of $\{A_{22}', B_2', C_2', D_2'\}$ satisfy (5.4) for k'. Using (5.11), it is easily seen that (5.4) now also holds for k, which completes the induction step. □

Notice that (5.9)-(5.13) also suggests an algorithm to construct factors $R_i(\lambda) = D_i + C_i(\lambda I_{\delta_i} - A_{i,i})^{-1} \; B_i$ from a realization $\{A,B,C,D\}$ satisfying Theorem 5.1. Indeed, let the unitary matrix

$$U_1 = \begin{array}{c} \left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] \begin{array}{l} \}\delta_1 \\ \}m \end{array} \\ \begin{array}{cc} \underbrace{\phantom{U_{21}}}_{\delta_1+m-r} & \underbrace{\phantom{U_{22}}}_{r} \end{array} \end{array} \tag{5.14}$$

satisfy

$$\left[\begin{array}{c|c} A^{(1)} & B^{(1)} \\ \hline C^{(1)} & D^{(1)} \end{array}\right] \triangleq \left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] \cdot \left[\begin{array}{c|c} 0 & 0 \\ \hline C_2' & D_2' \end{array}\right] = \begin{array}{c} \left[\begin{array}{c} U_{12} \\ \hline U_{22} \end{array}\right] \\ \underbrace{\phantom{U_{22}}}_{r} \end{array} \cdot \left[\begin{array}{c|c} C_2' & D_2' \end{array}\right] \}r \tag{5.15}$$

Such a unitary matrix is easily constructed with Householder transformations. Then with $B_1 \triangleq U_{12}$, $C_1 \triangleq C_{1,1}$, $D_1 \triangleq U_{22}$ we have constructed the factor $R_1(\lambda) = D_1 + C_1(I_{\delta_1} - A_{1,1})^{-1} B_1$ according to (5.9)-(5.13).

If we perform this unitary transformation in an "embedded form" $\hat{U}_1$ on the system matrix of (5.9), we have :

$$\left[\begin{array}{c|c|c} U_{11}^* & 0 & U_{21}^* \\ \hline 0 & I_{\delta'} & 0 \\ \hline U_{12}^* & 0 & U_{22}^* \end{array}\right] \cdot \left[\begin{array}{c|c|c} \lambda I_{\delta_1} - A_{1,1} & -A^{(1)} & B^{(1)} \\ \hline 0 & \lambda I_{\delta'} - A_{22}' & B_2' \\ \hline -C_{1,1} & -C^{(1)} & D^{(1)} \end{array}\right] = \tag{5.16}$$

$$\left[\begin{array}{c|c|c} \lambda \hat{E}_{1,1} - \hat{A}_{1,1} & 0 & 0 \\ \hline * & \lambda I_{\delta'} - A_{22}' & B_2' \\ \hline * & -C_2' & D_2' \end{array}\right]$$

Notice also that $\lambda \hat{E}_{1,1} - \hat{A}_{1,1}$ is in fact the zero pencil of the realization $\{A_{1,1}, B_1, C_1, D_1\}$ since

$$\left[\begin{array}{c|c} U_{11}^* & U_{21}^* \\ \hline U_{12}^* & U_{22}^* \end{array}\right] \cdot \left[\begin{array}{c|c} \lambda I - A_{1,1} & B_1 \\ \hline -C_1 & D_1 \end{array}\right] = \left[\begin{array}{c|c} \lambda \hat{E}_{1,1} - \hat{A}_{1,1} & 0 \\ \hline * & I_r \end{array}\right] \tag{5.17}$$

Continuing the above ideas recursively we finally obtain that a sequence of "embedded" unitary transformations $\hat{U}_i$ on the system of (5.9) yields :

$$\hat{U}_{k-1}^* \cdot \ldots \cdot \hat{U}_1^* \cdot \left[\begin{array}{c|c} \lambda I_\delta - A & B \\ \hline -C & D \end{array}\right] = \tag{5.18}$$

$$= \left[\begin{array}{ccc|cc} \lambda\hat{E}_{1,1}-\hat{A}_{1,1} & & & & \\ & \ddots & 0 & & \\ * & & \ddots & & \\ & & \lambda\hat{E}_{k-1,k-1}-\hat{A}_{k-1,k-1} & 0 & 0 \\ \hline & * & & \lambda I_{\delta_k}-A_{kk} & B_k \\ \hline & * & & -C_k' & D_k' \end{array}\right]$$

The factors $R_i(\lambda)=D_i-C_i(\lambda I_{\delta_i}-A_{ii})^{-1}B_i$ are extracted at each stage for $i=1,\ldots,k-1$ and the last factor $R_k(\lambda)$ is realized by the leftover quadruple $\{A_{kk}, B_k, C_k', D_k'\}$. Its zero pencil can be obtained by compressing the rows of the compound matrix $\begin{bmatrix} B_k \\ D_k' \end{bmatrix}$:

$$U_k \cdot \left[\begin{array}{c|c} \lambda I_{\delta_k}-A_{k,k} & B_k \\ \hline -C_k' & D_k' \end{array}\right] = \left[\begin{array}{c|c} \lambda\hat{E}_{k,k}-\hat{A}_{k,k} & 0 \\ \hline * & \hat{D} \end{array}\right] \tag{5.19}$$

where $\hat{D}$ has linearly independent rows. This, of course, could also be embedded in the form (5.18). It would yield the zero pencil of $R(\lambda)$, but in block triangular form, where each $\lambda\hat{E}_{ii}-\hat{A}_{ii}$ is the zero pencil of the corresponding factor $R_i(\lambda)$. The poles and zeros of these factors are thus given by the (generalized) eigenvalues of the pencils $\lambda I-A_{i,i}$ and $\lambda\hat{E}_{i,i}-\hat{A}_{i,i}$. The construction of these factors only requires unitary transformations on the system matrix of $R(\lambda)$. This algorithm can therefore be shown to be backward stable [34] .

Since the factors $R_i(\lambda)$ can easily be obtained from a realization satisfying Theorem 5.1, we call this a *cascaded* realization. We now analyze the existence of such a realization for a given $R(\lambda)$, which is defined by an *arbitrary* minimal realiza-

tion $\{A,B,C,D\}$. The results are inspired from [30][31].

THEOREM 5.2.

Let $\lambda I-A$ *and* $\lambda\hat{E}-\hat{A}$ *be the pole and zero pencils of a minimal realization* $\{A,B,C,D\}$ *of* $R(\lambda)$. *Then* $R(\cdot)$ *has a minimal cascade factorization* (5.1)(5.2) *iff there exist* (k-1) *pairs of independent spaces nested as follows :*

$$\{0\} \subset X_1 \subset X_2 \subset \ldots \subset X_{k-1} \subset H_\delta \tag{5.20}$$

$$H^\delta \supset Y_1 \supset Y_2 \supset \ldots \supset Y_{k-1} \supset 0 \tag{5.21}$$

and such that the following conditions are satisfied for each pair

$$\text{(i)}\quad AX_i \subset X_i \quad \text{(ii)}\ \dim(\hat{E}Y_i+\hat{A}Y_i)=\dim Y_i-\text{null}_\ell \quad \text{(iii)}\ X_i \oplus Y_i = H_\delta \tag{5.22}$$

PROOF.

<u>If</u> : Let us define $X_k \triangleq Y_0 = H_\delta$, $X_0 \triangleq Y_k = \{0\}$ and $S_i \triangleq X_i \cap Y_{i-1}$ for i=1,...,k. It follows then from (5.20)(5.21) and (5.22 iii) that

$$\left.\begin{aligned} X_i &= X_{i-1} \oplus S_i \\ Y_i &= S_{i+1} \oplus Y_{i+1} \end{aligned}\right. \quad \text{for } i=1,\ldots,k-1 \tag{5.23}$$

and hence, by induction, that

$$\left.\begin{aligned} X_i &= S_1 \oplus \ldots \oplus S_i \\ Y_i &= S_{i+1} \oplus \ldots \oplus S_k \end{aligned}\right. \quad \text{for } i=1,\ldots,k-1 \tag{5.24}$$

Let ξ_i, η_i and δ_i be the dimension of X_i, Y_i and S_i, respectively. It follows from (5.22 iii) that

$$S_1 \oplus S_2 \oplus \ldots \oplus S_k = H_\delta \tag{5.25}$$

We thus can choose a coordinate system in which $\mathcal{S}_i$ is spanned by the columns of

$$\begin{bmatrix} 0 \\ I_{\delta_i} \\ 0 \end{bmatrix} \begin{matrix} \}\xi_{i-1} \\ \\ \}\eta_i \end{matrix} \quad \text{for } i=1,\ldots,k \tag{5.26}$$

whence X_i and Y_i are spanned by the columns of

$$\begin{bmatrix} I_{\xi_i} \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ I_{\eta_i} \end{bmatrix} \quad , \text{ respectively}$$

In this coordinate system the pole pencil $\{I-\}$ has the required form (5.5) and the rank conditions (5.6) are satisfied because of (5.22 ii) and Lemma 4.2.

<u>Only if</u> : In the coordinate system of (5.5) the spaces X_i and Y_i defined as in (5.24)(5.26) clearly satisfy all conditions because of (5.7) and Lemma 4.2 □

COROLLARY 5.1

If $R(\lambda)$ *has a realization* $\{A,B,C,D\}$ *and if in this coordinate system the columns of* S_i *are bases for the subspaces* $\mathcal{S}_i$ *of Theorem* 5.2, *then* $\{T^{-1}AT,\ T^{-1}B,\ CT,\ D\}$ *is a cascaded realization when* $T = [S_1|\ldots|S_k]$, *and only when* T *is of that type.*

PROOF.

Under this transformation new bases for $\mathcal{S}_i$ are given by (5.26) as required in Theorem 5.2 (if and only if). □

From (5.5) it follows that the chain (5.22 i) of invariant subspaces of A determines the choice of poles in the factors $R_i(\lambda)$, through the constituting subspaces $\mathcal{S}_i$ of (5.24). From (5.18)(5.19) the same holds for the chain (5.22 ii) of reducing subspaces of $\lambda\hat{E}-\hat{A}$ and the choice of zeros in the factors

$R_i(\lambda)$. Condition (5.22 iii) finally says whether or not a minimal cascade factorization with this distribution of poles and zeros over the factors $R_i(\lambda)$, exists. As discussed in [30] the mathematical condition (5.22 iii) should be replaced by a quantitative condition which is a more realistic criterion for numerical practice. The choice suggested in [30] is cond. T_0, where

$$T_0 \triangleq [S_1^0 \mid \ldots \mid S_k^0] \tag{5.27}$$

and where the S_i are an orthonormal basis for the S_i of Theorem 5.2, thus satisfying Corollary 5.1. It is shown in [30] that if cond. T_0 is above some threshold, one should rather not perform the cascade factorization with that choice of poles and zeros in the factors, because of its inherent numerical difficulty. Condition (5.22 iii) is then "essentially" not satisfied.

We conclude with a final remark on the numerical implications of the approach proposed here.
Numerical algorithms for computing reducing subspaces with specified spectrum have been derived in [24][26][29] and are known to the backward stable, just as the methods for computing invariant subspaces with specified spectrum [23][24]. Sensitivity results can also be found in [22][28][34]. In order to compute a cascade factorization with a given distribution of poles and zeros one proceeds as follows. Starting from a given realization, construct $\lambda\hat{E}-\hat{A}$ and decompose A and $\lambda\hat{E}-\hat{A}$ in their upper and lower Schur forms [34][24] :

$$U^*A\,U \triangleq A_s = \begin{bmatrix} a_{11} & \cdots & & * \\ & \ddots & & \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{bmatrix} \tag{5.28}$$

$$Q^*(\lambda\hat{E}-\hat{A})Z \triangleq \lambda\hat{E}_s-\hat{A}_s \triangleq \lambda\begin{bmatrix} \hat{E}_\ell & & & 0 \\ \vdots & \hat{e}_{11} & & \\ \vdots & & \ddots & \\ \vdots & & & \hat{e}_{kk} \\ * & \cdots & \cdots & \hat{E}_r \end{bmatrix} - \begin{bmatrix} \hat{A}_\ell & & & 0 \\ \vdots & \hat{a}_{11} & & \\ \vdots & & \ddots & \\ \vdots & & & \hat{a}_{kk} \\ * & \cdots & \cdots & \hat{A}_r \end{bmatrix} \tag{5.29}$$

where $\lambda\hat{E}_r-\hat{A}_r$ and $\lambda\hat{E}_\ell-\hat{A}_\ell$ are singular pencils containing the right and left null space structure of $\lambda\hat{E}_s-\hat{A}_s$ [24] . The central part of (5.29) is the regular part of $\lambda\hat{E}_s-\hat{A}_s$ and the ratio's $\hat{a}_{ii}/\hat{e}_{ii}$, $i=1,\ldots,k$ are the generalized eigenvalues of $\lambda\hat{E}-\hat{A}$ and thus the zeros of $R(\lambda)$. We then construct the product of the unitary transformations Z and U :

$$V \triangleq Z^* \cdot U \tag{5.30}$$

Then perform a block LU factorization (without pivoting outside the block partitioning) :

$$V \triangleq Q_\ell \cdot Q_u \tag{5.31}$$

where the block sizes are conformable with the degrees δ_i of the factors $R_i(\lambda)$ to be constructed. Then, performing the state-space transformation :

$$T = Z \cdot Q_\ell = U \cdot Q_u^{-1} \tag{5.32}$$

it can be checked that in the new coordinate system of $\{A_t,B_t,C_t,D\}$, the spaces $S_i = Im \begin{bmatrix} 0 \\ I_{\delta_i} \\ 0 \end{bmatrix}$ satisfy the construction of Theorem 5.2, and thus yield a factorable realization of the form (5.5)(5.7). Notice that all the steps to obtain $\lambda\hat{E}-\hat{A}$, A_s and $\lambda\hat{E}-\hat{A}_s$ can be performed in a backward stable manner [34][24] . The only possible unstability comes from the construction of the state-space transformation T in (5.31)(5.32) and its application in order to obtain $\{A_t,B_t,C_t,D\}$. For this reason, a threshold is imposed on the pivots encountered in the LU decomposition, such that the factorization is rejected when instability occurs : the eigenspaces are then not suited for constructing a minimal factorization (5.1) because their angles are too skew (see [4] [30] for a more elaborate discussion on this). In that case another distribution of poles and zeros over the factors $R_i(\lambda)$ has to be tried out. This is done by updating the decompositions (5.28) and (5.29) in order to reorder the eigenvalues [23][26],

thereby yielding a different V in (5.31) and a different state-space transformation T. After the state-space transformation T (i.e. a numerically acceptable one) has been performed, one again needs only (stable) unitary transformations for the actual construction of the factors $R_i(\lambda)$ as was remarked in (5.18) (5.19).

REMARK 5.1.

Notice that the reducing subspaces X_z of $\hat{E}-\hat{A}$ have a minimal and maximal element X_{min}, X_{max} as discussed in Remark 3.3. Their dimensions are equal to δ_ℓ and $\delta_\ell + \delta_z$, respectively, where δ_ℓ and δ_z are as defined in Section 1. It then readily follows that $\delta_\ell(R)$ and $\delta_r(R) = \delta(R)-\delta_\ell(R)-\delta_z(R)$ are lower bounds for the degrees $\delta(R_1)$ and $\delta(R_k)$ of the left and right factors, according to Theorem 1.2. □

6. Related problems in linear system theory

Several topics in linear systems theory are related to that of minimal factorization, either because they are a special case of it, or because the problem can be reduced to a factorization problem and hence the techniques used to solve it are similar. A direct application is the problem of spectral factorization, occurring in several areas [1][2][5][18][38]. The problem there is to find a factorization :

$$Z(s) + Z^T(-s) = R(s) \,.\, R^T(-s) \qquad (6.1)$$

where Z(s) is a given positive real rational matrix and where R(s) is requested to have its poles and zeros in the left half plane (the formulation here is in continuous time : the discrete time version of the problem is similar). The techniques used for treating the singular case [1][5][38] are rather complicated with respect to the simple geometric approach obtained here (see also [26][27][28]).

A dual problem to the above one is that of optimal control [11][15][18][21][36] : given the stabilizable system

$$\begin{cases} x_{i+1} = A\, x_i + B\, u_i \\ y_i \quad = C\, x_i + D\, u_i \end{cases} \tag{6.2}$$

find the control $u_i = -K\, x_i$ minimizing the functional

$$J = \sum_{i=0}^{\infty} y_i^* y_i \tag{6.3}$$

(here again a continuous time version of the problem can be formulated in similar terms). The link of this problem with spectral factorization if discussed in e.g. [18], where also other related problems are mentioned. Methods to solve this problem with techniques similar to those discussed in this paper are given in [26][27][28].

Another area where factorizations of a certain type occur is control systems design [8][19][36] . There, minimal factorizations of the type

$$R(\lambda) = R_1(\lambda)\, R_2(\lambda) \tag{6.4}$$

are required where the condition $\delta = \delta_1 + \delta_2$ is relaxed in the sense that the degree condition has to be satisfied for the finite points only. This problem could be treated in a similar fashion to the one discussed here in this paper. It is interesting to note that a possible way to solve this problem is via the use of (A,B)-invariant subspaces [8], introduced in [37]. In [28] it is shown that in fact several of the geometrical concepts defined in [37] are special cases of reducing subspaces defined on an appropriate pencil. This makes the bridge between seemingly different approaches for tackling factorization problems. This connection also clarifies the similarity of the numerical algorithms derived for tackling these different problems [25][26][27][28].

REMARK 6.1. Notice that all the matrices occurring in these applications are real and that one also wants the separate factors to be real. This can be ensured when complex conju-

gate poles or zeros are grouped together in the factors [4][30]. The only difference in the analysis here, is that the Schur forms (5.28)(5.29) now have to be replaced by so-called real Schur forms [34][17][24] . Some factors will then necessarily have at least degree two. The software to reorder these real Schur forms is also available [23][26]. □

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