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DEADBEAT CONTROL: A SPECIAL INVERSE EIGENVALUE PROBLEM

P. VAN DOOREN

Philips Research Laboratory Brussels, Av. Van Becelaere, 2, B-1170 Brussels, Belgium

Dedicated to Professor Germund Dahlquist: on the occasion of his 60th birthday

Abstract.

In this paper we give a numerical method to construct a rank m correction BF (where the $n \times m$ matrix B is known and the $m \times n$ matrix F is to be found) to a $n \times n$ matrix A, in order to put all the eigenvalues of $A+BF$ at zero. This problem is known in the control literature as "deadbeat control". Our method constructs, in a recursive manner, a unitary transformation yielding a coordinate system in which the matrix F is computed by merely solving a set of linear equations. Moreover, in this coordinate system one easily constructs the minimum norm solution to the problem. The coordinate system is related to the Krylov sequence $A^{-1}B$, $A^{-2}B$, $A^{-3}B, \ldots$. Partial results of numerical stability are also obtained.

1. Introduction.

The problem of "deadbeat control" arises in the area of discrete-time control systems, where one considers the following system of difference equations:

$$x_{i+1} = Ax_i + Bu_i. \tag{1}$$

Here, n is the dimension of the "state-vector" x_i and m the dimension of the "input-vector" u_i. The problem of deadbeat control is to find a "state feedback" $u_i = Fx_i + v_i$ such that the resulting system:

$$x_{i+1} = (A+BF)x_i + v_i \tag{2}$$

has a *nilpotent* matrix $(A+BF)$, i.e. $(A+BF)^k = 0$, for some *minimal* power k. The solution of the homogeneous part of the system (2) then "dies out" after k steps [9], whence the name "deadbeat control". Because of the relation with control theory, we will frequently use its jargon to denote concepts that are familiar to this area (such as "feedback", "controllable subspace", etc.).

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This problem has been considered by several authors and several efforts have been undertaken recently to come up with numerically reliable methods to solve the problem [4, 5, 7, 11, 12, 21]. The method presented in this paper is very similar to the one developed in [11, 12]. In our special case, though, a simplified algorithm can be obtained which allows analysis of the numerical behavior of the method and permits the construction of the minimum norm solution to the problem.

2. Problem (re)formulation.

The method described in this paper is based on the use of unitary transformations only. These transformations are chosen because of their invariance property with respect to certain norms:

$$\|UAV\| = \|A\| \text{ for } U, V \text{ unitary, i.e. } U'U = UU' = I, \qquad V'V = VV' = I$$

where $\|\cdot\|$ stands for both the spectral and Frobenius norms [16], and a prime denotes the conjugate transpose of a matrix. As shown in the next sections, this guarantees that the errors produced by the algorithm do not blow up – therefore resulting in a numerically reliable algorithm – and also that the feedback matrix in the transformed coordinate system has still the same norm.

To start with, the (A, B) pair is transformed via a unitary state-space transformation V to the "staircase form" (see e.g. [2, 4, 8, 15, 17, 19, 20]):

$$\text{(3a)} \qquad \left[\begin{array}{c|c} V'B & V'AV \end{array}\right] = \left[\begin{array}{c|cc} B_c & A_c & X \\ 0 & 0 & A_{\bar{c}} \end{array}\right] =$$

$$\text{(3b)} \qquad \begin{array}{c} \left[\begin{array}{c|cccccc} B_1 & A_{1,1} & A_{1,2} & \cdots & \cdots & A_{1,k} & A_{1,k+1} \\ & A_{2,1} & A_{2,2} & \cdots & \cdots & A_{2,k} & A_{2,k+1} \\ & & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & & \ddots & \ddots & \vdots & \\ & & & & A_{k,k-1} & A_{k,k} & A_{k,k+1} \\ 0 & & & & 0 & & A_{k+1,k+1} \end{array}\right] \begin{array}{l} \}r_1 \\ \}r_2 \\ \\ \\ \}r_k \\ \}n_{k+1} \end{array} \\ \underbrace{}_{m} \quad \underbrace{}_{r_1} \quad \underbrace{}_{r_2} \qquad \underbrace{}_{r_{k-1}} \quad \underbrace{}_{r_k} \quad \underbrace{}_{n_{k+1}} \end{array}$$

Here B_1 and the $A_{i,i-1}$ off-diagonal blocks have full row rank r_i by construction. Note that this transformation does not affect the problem formulation: the feedback FV applied to the pair $(V'AV, V'B)$ yields the same eigenvalues as the feedback F applied to (A, B) since the "closed loop" matrices $V'AV+V'BFV$ and $A+BF$ are similar. Moreover, it is easy to see that in the new coordinate system (3) the eigenvalues of $A_{\bar{c}}$ can not be modified by feedback, since $V'AV+V'BFV$ has the same block structure as $V'AV$. These eigenvalues are also called the "uncontrollable modes" of the (A, B) pair.

Let us denote the ith Krylov subspace generated by A and B as:

$$R_i(A, B) = \langle B\rangle + A\langle B\rangle + \cdots + A^{i-1}\langle B\rangle \tag{4}$$

where $\langle\cdot\rangle$ denotes the range of a matrix. This subspace is also called the "ith reachable subspace" of the system (1) [23]. Because of the rank properties of the B_1 and $A_{i,i-1}$ blocks, one easily checks that (with $r_0 = m$, $r_i = 0$ for $i > k$, and $d_i = r_1 + r_2 + \ldots + r_i$) [18, 19]:

$$\dim R_i(A, B) = \dim R_i(A_c, B_c) = d_i \tag{5}$$

and

$$R_i(V'AV, V'B) = \left\langle \begin{pmatrix} I_{d_i} \\ 0 \end{pmatrix} \right\rangle. \tag{6}$$

The latter shows in fact that the columns of V yield orthogonal bases for the growing subspaces $R_i(A, B)$, whence the form (3b) can be obtained with a block Lanczos process (see [3] for more details). The subsystem (A_c, B_c) of dimension $d_k \times d_k$ eventually yields a Krylov subspace of full dimension d_k. In the control literature [23] it is therefore called "controllable", and its "controllability indices" are given by the rule (see e.g. [19]): there are $(r_i - r_{i+1})$ controllability indices c_j equal to i for $i = 1, \ldots, k$. Notice that the number k of non zero ranks r_i equals c_m, the largest controllability index.

It is known [23] that for a "controllable" (sub)system one can always assign the spectrum of the closed loop matrix arbitrarily by feedback (this is also implicitly proved by the constructive algorithm described below). The above "staircase form" therefore answers the question of solvability by separating the "controllable" (i.e. assignable) part of system (1) from its "uncontrollable" (i.e. unassignable) part: the problem is indeed solvable if all the uncontrollable modes are already at zero (i.e. if $A_{\bar{c}}$ is nilpotent). In the sequel we assume the problem to be solvable. $A_{\bar{c}}$, being irrelevant, can then be omitted and we therefore identify (A, B) with (A_c, B_c).

Consider now the spaces (called "ith controllable subspace" in [10]):

$$S_i(A, B) = \{x \mid A^i x \in A^{i-1}\langle B\rangle + \ldots + \langle B\rangle\}, \qquad i = 1, \ldots, k. \tag{7}$$

This linear subspace is the set of all initial conditions x_0 to (1), that can be "driven" to zero in time i by an appropriate choice of inputs u_j, $j = 0,\ldots,i-1$. This follows immediately from the following formula for x_i derived recursively from (1):

$$x_i = A^i x_0 + A^{i-1}Bu_0 + \ldots + ABu_{i-2} + Bu_{i-1}. \tag{8}$$

Let A^{-1} denote the functional inverse of a map. Thus, applied to a subspace S this means:

$$A^{-1}S = \{x | Ax \in S\}. \tag{9}$$

It is shown in [1, 14] that the spaces S_i satisfy the recursion:

$$\begin{aligned} S_{i+1} &= A^{-1}(S_i + \langle B \rangle) \\ \{0\} &= S_0 \subset S_1 \subset \ldots \subset S_l = S_{l+1} = \ldots \end{aligned} \tag{10}$$

where

$$l = \min \{i | S_i = S_{i+1}\}. \tag{11}$$

The spaces R_i and S_i are known to be invariant under feedback [1, 14]:

$$\begin{aligned} R_i(A, B) &= R_i(A+BF, B) \\ S_i(A, B) &= S_i(A+BF, B). \end{aligned} \tag{12}$$

Then for a controllable system (A, B) this leads to:

$$l = k \tag{13a}$$

$$\dim S_i = \dim R_i = d_i = \sum_{j=1}^{i} r_j \tag{13b}$$

$$S_i(A, B) = A^{-1}\langle B \rangle + A^{-2}\langle B \rangle + \ldots + A^{-i}\langle B \rangle. \tag{13c}$$

This immediately follows [1, 14] from the fact that $\bar{A} = A + BF$ can be chosen to be invertible [23] and that then:

$$R_i = \bar{A}^i S_i. \tag{14}$$

The properties above will be used in the sequel. A feedback matrix F is now a solution to the deadbeat control problem if ([14, 1]):

$$(15) \qquad (A+BF)S_i \subset S_{i-1}, \qquad i = 1,\ldots k.$$

This is easily seen by recursively applying (15) to obtain that $(A+BF)^k x$ must lie in S_0, and therefore be zero, for any x. Although (15) is not a necessary condition for deadbeat (see [6] for a counter example), one usually looks for a feedback satisfying this condition since it yields the additional property of driving to zero any state x in a minimum number of steps: all states x that can be driven to zero in, say, i steps (where $i < k$) belong to S_i by definition (7) and are indeed driven to zero in that many steps, because of (15). As is often done, we therefore also assume (15) to hold when talking about deadbeat.

Let now U be a unitary transformation partitioned in k blocks of r_k columns:

$$(16) \qquad U = [\underbrace{\bar{U}_1}_{r_1}|\cdots|\underbrace{\bar{U}_k}_{r_k}]$$

such that:

$$(17) \qquad S_i = \langle[\bar{U}_1|\cdots|\bar{U}_i]\rangle.$$

Let F be any solution of (15), then:

$$(18) \qquad U'(A+BF)U = \begin{array}{c} \left[\begin{array}{ccccc} 0 & A_{1,2} & \cdot\ \cdot & \cdot\ \cdot & A_{1,k} \\ & 0 & \cdot & & \cdot \\ & & \ddots & & \vdots \\ & & & \ddots & \\ & 0 & & & A_{k-1,k} \\ & & & & 0 \end{array}\right] \begin{array}{l} \}r_1 \\ \}r_2 \\ \vdots \\ \\ \}r_{k-1} \\ \}r_k \end{array} \\ \underbrace{}_{r_1}\underbrace{}_{r_2} \qquad\qquad \underbrace{\qquad}_{r_k} \end{array}$$

This follows from the fact that in the new coordinate system (i.e. after the similarity transformation $U'.U$) the spaces S_i are spanned by:

$$(19) \qquad S_i = \left\langle \begin{bmatrix} I_{d_i} \\ 0 \end{bmatrix} \right\rangle, \qquad i = 1,\ldots,k.$$

The transformation U thus transforms the original coordinate system to that spanned by the Krylov sequence (13c). If one starts with the coordinate system (3b) of the staircase form, U therefore transforms the Krylov sequence (4) into the

Krylov sequence (13c) (this is possible since their dimensions are equal). In the next section we derive an algorithm to do this efficiently and to solve meanwhile the deadbeat control problem.

3. A recursive method.

We now describe how to construct U and F such that $U'(A+BF)U$ has the form (18), when starting with (A, B) as in (3b). The algorithm is recursive and consists of k steps, where k is as defined above in (3b). We will show that at the end of each step i the following form is obtained:

$$U_i'(A+BF_i)U_i = \begin{array}{c} \left[\begin{array}{cc} A_d^i & X \\ 0 & A_s^i \end{array}\right] \end{array} \begin{array}{l} \}\, d_i = \sum_{j=1}^{i} r_j \\ \}\, n-d_i \end{array}$$

$$(21) \qquad U_i'B = \left[\begin{array}{c} B_d^i \\ B_s^i \end{array}\right] \begin{array}{l} \}\, d_i \\ \}\, n'-d_i. \end{array}$$

Here the subsystem (A_d^i, B_d^i) is already "beaten to death":

$$(22) \qquad [B_d^i | A_d^i] = \left[\begin{array}{c|cccc} B_1^i & 0 & A_{1,2}^i & \cdots & A_{1,i} \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & A_{i-1,i}^i \\ B_i^i & & & & 0 \end{array}\right] \begin{array}{l} \}\, r_i \\ \\ \}\, r_{i-1} \\ \}\, r_i \end{array}$$

and the subsystem (A_s^i, B_s^i) is still in staircase form:

$$[B_s^i | A_s^i] = \left[\begin{array}{c|cccc} B_{i+1}^i & A_{i+1,i+1}^i & \cdots & & A_{i+1,k}^i \\ & A_{i+2,i+1}^i & & & A_{i+2,k}^i \\ 0 & & \ddots & & \vdots \\ & & & A_{k,k-1}^i & A_{k,k}^i \end{array}\right] \begin{array}{l} \}\, r_{i+1} \\ \}\, r_{i+2} \\ \\ \}\, r_k \end{array}$$

where the leading blocks B^i_{i+1} and $A^i_{j,j-1}$, $j = i+2,\ldots,k$ have full row rank. At the beginning of step 1 (i.e. at the end of step 0) this is indeed satisfied since this is merely the staircase from (3b). We now derive step i of the recursive algorithm. At the beginning of this step we thus have the configuration (20)–(23) with i decremented by 1. We then construct transformation and feedback matrices

$$\hat{U}_i = \begin{bmatrix} I_{d_{i-1}} & 0 \\ 0 & U^i_s \end{bmatrix}, \qquad \hat{F}_i = [0 \quad F^i_s] \tag{24}$$

that only affect the subsystem (A^{i+1}_s, B^{i+1}_s). We are thus trying to find matrices F^i_s and U^i_s such that:

$$U^{\prime}_i(A^{i-1}_s + B^{i-1}_s F^i_s)U^i_s = \left[\begin{array}{c|c} 0 & X \\ \hline 0 & X \end{array}\right] \begin{array}{l} \\ \end{array} \tag{25}$$
$$\qquad\qquad\qquad\qquad\qquad \underbrace{}_{r_i} \quad \underbrace{}_{n-d_i}$$

or equivalently:

$$(A^{i-1}_s + B^{i-1}_s F^i_s)U^i_s = [\underbrace{0}_{r_i} \mid \underbrace{X}_{n-d_i}]. \tag{26}$$

This is obtained as follows: let U^i_s be a unitary transformation triangularizing A^{i-1}_s (with r_j, $j = i,\ldots,k$ as in (3b)):

$$A^{i-1}_s U^i_s = R^i = \begin{array}{c} \left[\begin{array}{ccc} R^i_{i,i} & \cdots & R^i_{i,k} \\ & \ddots & \vdots \\ & & R^i_{k,k} \end{array}\right] \begin{array}{l} \}r_i \\ \vdots \\ \}r_k \end{array} \\ \underbrace{}_{r_i} \cdots\cdots \underbrace{}_{r_k} \end{array} \tag{27}$$

Here again the result was partitioned conformably with A. Then solve the equation

$$R^i_{i,i} = -B^{i-1}_i . X \tag{28}$$

This equation has a solution since B^{i-1}_i has full row rank and thus has a left inverse [16]. A minimum norm solution G_i is given by

$$G_i = -(B^{i-1}_i)^+ . R^i_{i,i} \tag{29}$$

where the plus sign denotes the Moore-Penrose inverse of a matrix. Notice that this is a minimum norm solution in the spectral norm and moreover the *unique* minimum norm solution in the Frobenius norm.

Using the feedback:

$$F_s^i = [G_i \ 0 \cdots \cdots 0] U_s^{i\prime} \tag{30}$$

it is easily seen that $(A_s^{i-1} + B_s^{i-1} F_s^i) U_s^i$ is equal to

$$(A_s^{i-1} + B_s^{i-1} F_s^i) U_s^i = \begin{array}{c} \begin{bmatrix} 0 & R_{i,i+1}^i & \cdots & R_{i,k}^i \\ & R_{i+1,i+1}^i & \cdots & R_{i+1,k} \\ & & \ddots & \vdots \\ & & & R_{k,k}^i \end{bmatrix} \begin{matrix} \}r_i \\ \}r_{i+1} \\ \vdots \\ \}r_k \end{matrix} \\ \begin{matrix} \underbrace{}_{r_i} & \underbrace{\phantom{R_{i,i+1}}}_{r_{i+1}} & \cdots & \underbrace{\phantom{R_{k,k}}}_{r_k} \end{matrix} \end{array} \tag{31}$$

and thus satisfies (26).

We now prove that $(U_s^i)'$ has the same block structure as A_s^{i-1}, i.e.:

$$(U_s^i)' = \begin{bmatrix} U_{i,i} & U_{i,i+1} & \cdots & U_{i,k} \\ U_{i+1,i} & U_{i+1,i+1} & \cdots & U_{i+1,k} \\ & \ddots & \ddots & \vdots \\ 0 & & U_{k,k-1} & U_{k,k} \end{bmatrix} \begin{matrix} \}r_i \\ \}r_{i+1} \\ \\ \}r_k \end{matrix} \tag{32}$$

Indeed, when writing (27) as:

$$A_s^{i-1} = R^i \cdot (U_s^i)' \tag{33}$$

it follows that the bottom $n-d_i$ rows of R^i are linearly independent since those of A_s^{i-1} are, and since the invertible column transformation $(U_s^i)'$ does not affect the independence of the rows of the matrix R^i. Thus the blocks $R_{j,j}^i$, $j = i+1, \ldots, k$, in (27) are invertible. Let S be the upper triangular inverse of the $(n-d_i) \times (n-d_i)$ bottom part of R^i, then multiplying the bottom $n-d_i$ rows of (33) with S from the left, we obtain:

$$\text{(34a)}\qquad S\cdot\begin{array}{c}\begin{bmatrix} A^i_{i+1,i} & A^i_{i+1,i+1} & \cdots & A^i_{i+1,k} \\ \vdots & \ddots & & \vdots \\ & & \ddots & \vdots \\ 0 & & A^i_{k,k-1} & A^i_{k,k} \end{bmatrix}\end{array}\begin{array}{l}\}r_{i+1}\\ \\ \\ \}r_k\end{array} =$$

$$\text{(34b)}\qquad = \begin{bmatrix} U_{i+1,i} & U_{i+1,i+1} & \cdots & U_{i+1,k} \\ \vdots & \ddots & & \vdots \\ & & \ddots & \vdots \\ U_{k,i} & \cdots & U_{k,k-1} & U_{k,k} \end{bmatrix}\begin{array}{l}\}r_{i+1}\\ \\ \\ \}r_k\end{array}.$$

Since S is an upper triangular invertible matrix we indeed find that the blocks $U_{j,l}$ with $j > l+1$ are zero and that the blocks $U_{j,j-1}$ have linearly independent rows, which completes the proof. Because of the above structure of the transformations $(U^i_s)'$, after this step i, the following form is obtained:

$$[(U^i_s)' B^{i-1}_s | (U^i_s)'(A^{i-1}_s + B^{i-1}_s F^i_s)U^i_s] =$$

$$\text{(35)}\qquad \left[\begin{array}{c|c|cccc} B^i_i & 0 & A^i_{i,i+1} & \cdots & & A^i_{i,k} \\ \hline B^i_{i+1} & & A^i_{i+1,i+1} & \cdots & & A^i_{i+1,k} \\ & & A^i_{i+2,i+1} & \ddots & & A^i_{i+2,k} \\ & 0 & & \ddots & & \vdots \\ & & & & \ddots & \vdots \\ & & & & A^i_{k,k-1} & A^i_{k,k} \end{array}\right].$$

$$F^i_s U^i_s = [\ G_i \mid \qquad 0 \qquad]$$

Moreover the identities:

$$\text{(36)}\qquad B^i_{i+1} = U_{i+1,i}\cdot B^{i-1}_i;\ A^i_{j+1,j} = U_{j+1,j}\cdot R^i_{j,j}, \qquad j = i+1,\ldots,k-1$$

follow from (31), (32) and these imply that the blocks in (36) have full column rank.

Embedding this in (20)–(23) for i decremented by 1, we clearly retrieve the form (20)–(23) at the end of step i. The updating of the transformation and feedback matrices are easily checked to be:

$$U_i = U_{i-1}\hat{U}_i \tag{37a}$$

$$F_iU_i = F_{i-1}U_i + \hat{F}_i\hat{U}_i = F_{i-1}U_{i-1} + \hat{F}_i\hat{U}_i. \tag{37b}$$

The last equality follows from the fact that the feedback matrices $F_{i-1}U_{i-1}$ have their last $(n-d_i)$ columns equal to zero and are therefore unaffected by the subsequent transformations $\hat{U}_j$, for $j > i-1$.

After k steps of this recursion, one finally obtains:

$$U_k'(A+BF_k)U_k = \left[\begin{array}{c|ccccc} B_1^k & 0 & A_{1,2}^k & \cdots & \cdots & A_{1,k}^k \\ B_2^k & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & A_{k-1,k}^k \\ B_k^k & & & & & 0 \end{array}\right] \tag{38}$$

where

$$U_k = \hat{U}_1\hat{U}_2\cdots\hat{U}_k \tag{39a}$$

$$\hat{U}_i = \begin{bmatrix} I_{d_{i-1}} & 0 \\ 0 & U_s^i \end{bmatrix} \tag{39b}$$

$$F_k = \hat{F}_1 + \hat{F}_2 + \cdots + \hat{F}_k \tag{39c}$$

$$\hat{F}_iU_k = [0\cdots 0\ G_i\ 0\cdots 0] \tag{39d}$$

This is now clearly in the form (18) as requested in section 2.

4. Numerical considerations.

In this section we discuss the numerical stability of the method above and show that it yields a minimum norm solution to the problem when solutions are not unique.

To analyze the stability of the algorithm, we first remark that the transformation matrix U obtained by the algorithm is independent of the feedback matrix F and that it satisfies (16), (17), i.e. the first d_i columns of U span the ith controllable subspace S_i in the coordinate system of (3). This follows from (15), (16), (17) and the fact that the spaces S_i are defined independently of the feedback matrix F (see e.g. in (7) or (13c)). It can also be derived by induction from the algorithm: in (35), the term $(U_s^i)'B_s^{i-1}F_s^iU_s^i$ indeed only affects the (zero) blocks $A_{i,i}^i$ and $A_{i+1,i}^i$ which are not used in the subsequent steps.

Let us now look at the problem in the coordinate system of (A_u, B_u), partitioned conformably with the original (A, B) pair:

$$\text{(40)} \qquad A_u = U'AU = \begin{bmatrix} A_{11}^u & \cdots & A_{1k}^u \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_{k1}^u & \cdots & A_{kk}^u \end{bmatrix}; B_u = U'B = \begin{bmatrix} B_1^u \\ \cdot \\ \cdot \\ \cdot \\ B_k^u \end{bmatrix} = \begin{bmatrix} B_1^k \\ \cdot \\ \cdot \\ \cdot \\ B_k^k \end{bmatrix}$$

Since in this coordinate system there exists a feedback matrix F_u such that $A_u + B_uF_u$ has the form (18), we have:

$$\text{(41)} \qquad < \begin{bmatrix} A_{ii}^u \\ \cdot \\ \cdot \\ \cdot \\ A_{ki}^u \end{bmatrix} > \subset < \begin{bmatrix} B_i^u \\ \cdot \\ \cdot \\ \cdot \\ B_k^u \end{bmatrix} >$$

Because of the special structure of the transformation matrices $\hat{U}_i$ (39b) it follows that:

$$\text{(42)} \qquad \left[\begin{array}{c|c} B_i^u & A_{ii}^u \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ B_k^u & A_{ki}^u \end{array}\right] = \hat{U}_k' \cdots \hat{U}_i' \cdot \left[\begin{array}{c|c} B_i^{i-1} & R_{ii}^i \\ 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 \end{array}\right]$$

with B_i^{i-1} of full row rank. The minimum norm solution G_i in (29) of the system (28) is thus also the minimum norm solution of

$$\text{(43)} \qquad \begin{bmatrix} A_{ii}^u \\ \cdot \\ \cdot \\ \cdot \\ A_{ki}^u \end{bmatrix} + \begin{bmatrix} B_i^u \\ \cdot \\ \cdot \\ \cdot \\ B_k^u \end{bmatrix} G_i = 0$$

and this is thus the corresponding ith submatrix of F_u. Let us now write the analogous perturbed equations, where computed and therefore perturbed quantities are denoted with an upper bar. Unitary transformations can be performed in a *backward* stable manner [16, 22] and thus equation (40) yields:

$$\bar{A}_u = \bar{U}'A\bar{U},\ \bar{B}_u = \bar{U}'B \tag{44a}$$

with

$$\bar{U} = U + \Delta U, \qquad U'U = I_n, \qquad \|\Delta U\| = \varepsilon_u \tag{44b}$$

$$< \begin{bmatrix} \bar{A}^u_{ii} \\ \cdot \\ \cdot \\ \cdot \\ \bar{A}^u_{ki} \end{bmatrix} > \subset < \begin{bmatrix} \bar{B}^u_i \\ \cdot \\ \cdot \\ \cdot \\ \bar{B}^u_k \end{bmatrix} > \tag{44c}$$

and with ε_u of the order of the relative precision ε of the computer.

That the inclusion (44c) still holds for the perturbed $(\bar{A}_u, \bar{B}_u)$ pair is due to the fact that the left hand side of (42) is in fact not computed. Appropriate perturbations can thus be assumed to make (44c) hold.

This then means that the Krylov subspaces or "controllable subspaces" $S_i(A, B)$, $i = 1, \ldots, k$, can be computed in a *backward* stable manner by the algorithm above: the computed spaces $\bar{S}_i(A, B)$ are indeed the *exact* Krylov subspaces corresponding to the slightly perturbed system:

$$\bar{A} = \bar{U} \cdot \bar{A}_u \cdot \bar{U}^{-1} = A + \Delta_u A, \qquad \|\Delta_u A\| = \varepsilon_a \|A\| \tag{45a}$$

$$\bar{B} = \bar{U} \cdot \bar{B}_u = B + \Delta_u B, \qquad \|\Delta_u B\| = \varepsilon_b \|B\|. \tag{45b}$$

A similar result of backward stability can be obtained when starting with an (A, B) pair that is not in the staircase form (3). This is because (3) can be obtained with a unitary transformation V in a backward stable manner [19]: the errors (45) can then be transformed back to the original (A, B) pair and added up to those due to the staircase reduction (3). Nothing, of course, is claimed about forward errors since both V and U can be very sensitive to rounding errors [19].

For the computation of the feedback matrix F_u itself only a weaker result can be derived, which does not amount to backward nor to forward stability. Notice that G_i is computed by solving the equations:

$$\bar{R}^i_{i,i} + \bar{B}^{i-1}_i \cdot G_i = 0, \qquad i = 1, \ldots, k \tag{46}$$

instead of the (equivalent) systems:

$$\begin{bmatrix} \bar{A}^u_{ii} \\ \cdot \\ \cdot \\ \cdot \\ \bar{A}^u_{ki} \end{bmatrix} + \begin{bmatrix} \bar{B}^u_i \\ \cdot \\ \cdot \\ \cdot \\ \bar{B}^u_k \end{bmatrix} \cdot G_i = 0, \qquad i = 1,\ldots,k. \tag{47}$$

Each separate column of the computed solutions $\bar{G}_i$, $i = 1,\ldots,k$, is then obtained in a backward stable manner, but one can *not* guarantee [16] that there exists a unique $(\bar{A}_f, \bar{B}_f)$ pair obtained by adding an ε-perturbation to the original system, such that the "closed loop" matrix has the zero structure described by (18):

$$(\bar{A}_f + \bar{B}_f \bar{F}_u) = \begin{bmatrix} 0 & x & \cdots & x \\ & & \cdot & \cdot \\ \cdot & & \cdot\ \cdot & \cdot \\ \cdot & & \cdot\ \cdot & \cdot \\ \cdot & & \cdot & x \\ 0 & \cdot & \cdot\ \cdot & 0 \end{bmatrix} \tag{48}$$

Yet we can prove that there exists an $(\bar{A}_f, \bar{B}_f)$ pair as in (48), exactly similar to the system $(A + \Delta_f A, B + \Delta_f B)$, but satisfying the weaker bounds:

$$\bar{A} = \bar{U} \cdot \bar{A}_f \cdot \bar{U}^{-1} = A + \Delta_f A, \qquad \|\Delta_f A\| = \eta_a \|A\| + \eta_{bf} \|B\| \cdot \|F\| \tag{49a}$$

$$\bar{B} = \bar{U} \cdot \bar{B}_f = B + \Delta_f B, \qquad \|\Delta_f B\| = \eta_b \|B\| \tag{49b}$$

$$\bar{F} = \bar{F}_u \bar{U}^{-1} \tag{49c}$$

where η_a, η_b and η_{bf} are of the order of ε, and where $\bar{U}$ and $\bar{F}_u$ are the exact matrices stored in the computer (notice that we do not have $\bar{F}$ in the computer, and that $\bar{U}$ is no longer unitary, but close to it). This is shown as follows: the jth column of $\bar{F}_u$ is a column of a $\bar{G}_i$, say, $\bar{g}_l$ which is the backward stable solution of the corresponding equation:

$$\begin{bmatrix} \bar{A}^u_{ii} \\ \cdot \\ \cdot \\ \cdot \\ \bar{A}^u_{ki} \end{bmatrix} + \begin{bmatrix} \bar{B}^u_i \\ \cdot \\ \cdot \\ \cdot \\ \bar{B}^u_k \end{bmatrix} \cdot g_l = 0. \tag{50}$$

The residue r_l of the computed solution $\bar{g}_l$ is therefore bounded by [16, 22] (with η_r of the order of ε):

(51) $$\|r_i\| \leqslant \eta_r \cdot \|\bar{B}\| \cdot \|\bar{g}_i\| \leqslant \eta_r \cdot \|\bar{B}\| \cdot \|\bar{F}\|.$$

Taking $\bar{B}_f = \bar{B}_u$ and adding the residuals r_i to the corresponding columns of $\bar{A}_u$ to yield $\bar{A}_f$, we then obtain (48), (49) after transforming back with the state-space transformation $\bar{U} . \bar{U}^{-1}$. Therefore the following kth power vanishes exactly:

(52) $$[(A+\Delta_f A)+(B+\Delta_f B)\bar{F}]^k = 0.$$

Although we cannot prove backward stability for this part of the algorithm, one obtains bounds for the norm of $[A+B\cdot \bar{F}]^k$ that are of the same order as those that would be obtained with a backward stable solution $\bar{F}_{bs}$. Indeed, because of (49), (52) one easily shows that for $i > k$:

(53a) $$\|(A+B\bar{F})^i\| \sim k\cdot \{\eta_a\|A\| + (\eta_{bf}+\eta_b)\|B\|\cdot\|\bar{F}\|\}\cdot\|A+B\bar{F}\|^{i-1}$$

(53b) $$\|(A+B\bar{F}_{bs})^i\| \sim k\cdot \{\varepsilon_a\|A\| + \varepsilon_b\|B\|\cdot\|\bar{F}_{bs}\|\}\cdot\|A+B\bar{F}_{bs}\|^{i-1}$$

for any norm satisfying the product inequality $\|S\cdot T\| \leqslant \|S\|\cdot\|T\|$. This easily follows from:

$$(A+B\bar{F})^k = (A+B\bar{F})^k - (\bar{A}+\overline{B\bar{F}})^k = -k(\Delta_f A + \Delta_f B\cdot \bar{F})(A+B\bar{F})^{k-1} + O(\varepsilon^2)$$

$$(A+B\bar{F}_{bs})^k = (A+BF_{bs})^k - (\bar{A}+\overline{BF}_{bs})^k = -k(\Delta A + \Delta B\cdot \bar{F}_{bs})(A+B\bar{F}_{bs})^{k-1} + O(\varepsilon^2).$$

Here $\bar{A}$, $\bar{B}$ denote twice the appropriate perturbed A, B pair for which the considered feedback is an *exact* solution. This comparison therefore shows that our method does not behave worse than a stable method in this sense.

In order to prove that the obtained feedback matrix F is the unique minimum (Frobenius) norm solution to the problem we observe that this is the case by construction for each of the submatrices G_i of F_u in (39). Since for the Frobenius norm we have

(54) $$\|F\|^2 = \|F_u\|^2 = \sum_i \|G_i\|^2$$

this then also holds for F_u and F in their respective coordinate systems. A priori bounds for the norm of F are not easy to obtain since F is highly problem dependent ($\|F\|$ depends heavily on "how controllable" (A, B) is).

Finally, a few words can be said about the complexity of this algorithm. When using the variant of the staircase algorithm described in [12] (here, a unitary input transformation is used as well), all the blocks $A_{i+1,i}$ and B_1 are triangular, which results in a somewhat faster implementation of the subsequent deadbeat recursion. The complexity of the staircase reduction and deadbeat algorithm are then of the order of $(\frac{8}{3})n^3+(\frac{11}{2})mn^2+2m^2n$ and $2n^3+2mn^2+m^2n$, respectively. Both are thus cubic in the dimensions of the (A, B) pair and the deadbeat algorithm turns out to be even cheaper than the preliminary staircase reduction.

We terminate this section with some examples.

Example 1.

Let

$$A=\begin{bmatrix}1&0&1\\0&1&1\\0&1&0\end{bmatrix},\quad B=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix},\quad r_1=2,\ r_2=1$$

then we find

$$F_u=\begin{bmatrix}-1&-1&0\\0&-1&1\end{bmatrix},\quad U=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix},\quad F=\begin{bmatrix}-1&0&-1\\0&-1&-1\end{bmatrix}.$$

The general solution F_u^g in fact equals:

$$F_u^g=\begin{bmatrix}-1&-1&g\\0&-1&1\end{bmatrix},\quad A_u+B_uF_u^g=\begin{bmatrix}0&0&g\\0&0&-1\\0&0&0\end{bmatrix}$$

and it is easily seen that the F_u above is the minimum Frobenius norm choice for F_u^g. However, F_u is not the minimum norm solution for the 2-norm. Indeed, while $\|F_u\|_2=3^{1/2}=1.732$, it can be proved that:

$$\min_g \|F_u^g\|_2=(5^{1/2}+1)/2=1.6180, \qquad \text{for } g=(5^{1/2}-1)/2=0.6180.$$

Example 2.

Let

$$A=\begin{bmatrix}0&0&1\\0&1&0\\1&0&1\end{bmatrix},\quad B=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix},\quad r_1=2,\ r_2=1$$

then we find (with $a=2^{-1/2}$):

$$F_u=\begin{bmatrix}a&0&3a\\0&-1&3a\end{bmatrix},\quad U=\begin{bmatrix}a&0&-a\\0&-1&0\\-a&0&-a\end{bmatrix},\quad F=\begin{bmatrix}-1&0&-2\\0&-1&0\end{bmatrix}.$$

The general solution F_u^g in fact equals:

$$F_u^g = \begin{bmatrix} a & 0 & 3a \\ 0 & -1 & g \end{bmatrix}, \quad A_u + B_u F_u^g = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & -g \\ 0 & 0 & 0 \end{bmatrix}$$

and the minimum norm solution is now obtained for $g = 0$ in both the 2-norm and the Frobenius norm ($\|F\| = 5^{1/2}$). Numerically one obtains (on a VAX11/780 with $\varepsilon = 2^{-56} = 1.387778780781446D-17$):

$$U = \begin{bmatrix} 0.707106781186548 & 0.000000000000000 & -0.707106781186458 \\ 0.000000000000000 & -1.000000000000000 & 0.000000000000000 \\ -0.707106781186548 & 0.000000000000000 & -0.707106781186548 \end{bmatrix}$$

$$F_u = \begin{bmatrix} 0.707106781186548 & 0.000000000000000 & 2.121320343559643 \\ 0.000000000000000 & 1.000000000000000 & 0.000000000000000 \end{bmatrix}$$

$\|(A + BF_uU')^2\| = 1.710375635613043D-16$

Example 3.

This is a randomly generated A, B pair with $n = 5$, $m = 2$, $r_1 = 2$, $r_2 = 2$ and $r_3 = 1$. The computations were performed with MATLAB [13] running on a VAX11/780 (with ε as above.) Only 4 significant digits are given.

$$A = \begin{bmatrix} 0.2113 & 0.6284 & 0.5608 & 0.2321 & 0.3076 \\ 0.7560 & 0.8497 & 0.6624 & 0.2312 & 0.9330 \\ 0.0002 & 0.6857 & 0.7264 & 0.2165 & 0.2146 \\ 0.3303 & 0.8782 & 0.1985 & 0.8834 & 0.3126 \\ 0.6654 & 0.0684 & 0.5443 & 0.6525 & 0.3616 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.2922 & 0.5015 \\ 0.5664 & 0.4369 \\ 0.4826 & 0.2693 \\ 0.3322 & 0.6326 \\ 0.5935 & 0.4052 \end{bmatrix}$$

Reduction to staircase form $(A_s, B_s) = (V_a' \cdot A \cdot V_a, V_a' \cdot B \cdot V_b)$, involving an input transformation V_b also as in the variant described in [12].

$$A_s = \begin{bmatrix} 0.2179 & 0.1833 & 0.0715 & -0.4907 & -0.0216 \\ -0.0163 & 2.4486 & -0.3437 & -0.0281 & 0.1353 \\ 0.7046 & -0.2001 & 0.2098 & 0.2102 & 0.0847 \\ 0.0000 & -0.3836 & -0.1116 & -0.0484 & 0.4536 \\ 0.0000 & 0.0000 & 0.0000 & -0.4958 & 0.2046 \end{bmatrix}$$

$$B_s = \begin{bmatrix} -0.3411 & -0.0921 \\ 0.0000 & -1.4339 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}$$

$$V_a = \begin{bmatrix} 0.4698 & 0.3589 & -0.4508 & 0.4590 & -0.4863 \\ -0.2230 & 0.5103 & -0.3057 & -0.7319 & -0.2463 \\ -0.4080 & 0.3992 & -0.4025 & 0.3633 & 0.6166 \\ 0.6664 & 0.4296 & 0.3136 & -0.1973 & 0.4838 \\ -0.3450 & 0.5165 & 0.6655 & 0.2875 & -0.2977 \end{bmatrix}$$

$$V_b = \begin{bmatrix} 0.6915 & -0.7224 \\ -0.7224 & -0.6915 \end{bmatrix}$$

Deadbeat correction:

$$A_u = \begin{bmatrix} 0.0834 & 0.2060 & 0.6562 & -0.2194 & 0.4892 \\ 0.4124 & 1.3353 & 0.2279 & -0.4733 & 0.9615 \\ -0.0106 & 0.1477 & 0.4741 & 0.3668 & -0.0764 \\ 0.0024 & 0.0079 & 0.0005 & 0.0025 & -0.5161 \\ 0.3841 & 1.2531 & 0.0719 & 0.3965 & 1.1372 \end{bmatrix}$$

$$B_u = \begin{bmatrix} -0.1159 & -0.1909 \\ -0.0177 & -1.0405 \\ 0.3203 & -0.0284 \\ 0.0000 & -0.0061 \\ 0.0000 & -0.9720 \end{bmatrix}$$

$$U = \begin{bmatrix} 0.3399 & 0.0518 & -0.9390 & 0.0000 & 0.0000 \\ 0.1113 & 0.7223 & 0.0801 & 0.0043 & 0.6778 \\ -0.9217 & -0.0059 & -0.3339 & 0.0012 & 0.1972 \\ -0.0572 & 0.2631 & -0.0062 & 0.9227 & -0.2760 \\ -0.1387 & 0.6375 & -0.0151 & -0.3856 & -0.6523 \end{bmatrix}$$

$$F_u = \begin{bmatrix} 0.0683 & -0.3467 & -1.4736 & -1.1091 & 0.0000 \\ 0.3952 & 1.2892 & 0.0740 & 0.4079 & 1.1700 \end{bmatrix}$$

$$A_u + B_u \cdot F_u = \begin{bmatrix} 0.0000 & 0.0000 & 0.8129 & -0.1687 & 0.2659 \\ 0.0000 & 0.0000 & 0.1769 & -0.8781 & -0.2559 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.1097 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.5232 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$F_s = F_u \, . \, U' = \begin{bmatrix} 1.3890 & -0.3656 & 0.4299 & -1.1092 & 0.2193 \\ 0.1316 & 1.7759 & -0.1654 & 0.3695 & -0.1545 \end{bmatrix}$$

$$\|(A_s + B_s F_s)^3\|_2 = 3.4552D-17$$

$$F = V_b \, . \, F_s \, . \, V_a' = \begin{bmatrix} -0.9352 & -0.4122 & -1.3473 & 0.3791 & -1.1900 \\ -0.4960 & -0.5746 & 0.3072 & -1.3376 & -0.0786 \end{bmatrix}$$

$$\|(A+BF)^3\|_2 = 3.7697D-17$$

These errors clearly illustrate the stability properties discussed above.

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