Polynomial matrix factorizations via pole-zero cancelation

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Abstract

In this paper we give an algorithm for obtaining factorizations $P(\lambda) = P_1(\lambda) \cdot P_2(\lambda)$, where $P_1(\lambda)$ and/or $P_2(\lambda)$ are polynomial matrices and $P_1(\lambda)$ is regular. Moreover the factors $P_1(\lambda)$ and $P_2(\lambda)$ are such that either the *poles* of $[P_1(\lambda)]^{-1}$ and $P_2(\lambda)$ are in a prescribed set Γ of the extended complex plane, or their *zeros*. Such factorizations cover e.g. the specific cases of coprime factorization, polynomial factor extraction and GCD extraction. The algorithm works on the state space (or generalized state space) realization of $P(\lambda)$ and derives the corresponding realizations of the factors.

1 Introduction

Several problems occurring in the literature of linear systems theory can be rephrased as a factorization problem of some rational matrix $R(\lambda)$. We consider a certain class of such factorizations, namely where the $p \times m$ rational matrix $R(\lambda)$ is factored into a product of two rational matrices :

$$R(\lambda) = R_1(\lambda) \cdot R_2(\lambda), \qquad (1)$$

where $R_1(\lambda)$ is $p \times p$ regular and of minimal degree and where, given a set Γ of the extended complex plane, one of the following two conditions is satisfied :

a) the *poles* of $R_1^{-1}(\lambda)$ and $R_2(\lambda)$ lie in Γ (2.a)

b) the *zeros* of $R_1^{-1}(\lambda)$ and $R_2(\lambda)$ lie in Γ (2.b)

We give a general algorithm for obtaining such factorizations using the state space (or generalized state space) realization of $R(\lambda)$ and yielding the corresponding realizations of the factors. Applying this to problems involving polynomial matrices, one solves the following standard polynomial factorization problems :

1) Coprime factorization

Given a $p \times m$ rational matrix $R(\lambda)$, one wants to find polynomial matrices $D(\lambda)$ and $N(\lambda)$ with $D(\lambda)$ regular, such that $R(\lambda) = D^{-1}(\lambda) \cdot N(\lambda)$. This fits into the above formulation with condition (2.a), where $\Gamma = \{\infty\}$.

2) GCD extraction

Let $P_i(\lambda)$, $i, 1, \cdots, k$ be a set of polynomial matrices of dimensions $p \times m_i$, then their greatest common (left) divisor (GCD) is defined as the *regular* polynomial matrix $D(\lambda)$ such that : $P(\lambda) = D(\lambda) \cdot Q(\lambda)$ where $P(\lambda) = [P_1(\lambda), \cdots, P_k(\lambda)]$ and $Q(\lambda) = [Q_1(\lambda), \cdots, Q_k(\lambda)]$ and where the *quotients* $Q_i(\lambda)$ form together a polynomial matrix $Q(\lambda)$ with *no* Smith zeros anymore. Hence, both $D^{-1}(\lambda)$ and $Q(\lambda)$ have all their McMillan zeros at ∞, which is thus condition (2.b) with $\Gamma = \{\infty\}$

3) Polynomial factor extraction

Given a $p \times m$ polynomial matrix $P(\lambda)$, one wants to find a $p \times p$ *regular divisor* $P_1(\lambda)$ which contains all the zeros of $P(\lambda)$ inside a set Γ_f *not* containing the point at infinity : $P(\lambda) = P_1(\lambda) \cdot P_2(\lambda)$. The factors $P_1^{-1}(\lambda)$ and $P_2(\lambda)$ have thus their zeros in the complement $\Gamma = \Gamma_f^c$ and this factorization then satisfies (2.b).

We show in the sequel that there always exist such factorizations for any set Γ and that they are in fact far from unique unless some additional conditions are imposed. We typically require the (McMillan) degree δ of $R_1(\lambda)$ to be minimal, which implies that

a) $\delta(R_1) = \#poles$ of $R(\lambda)$ outside Γ (3.a)

b) $\delta(R_1) = \#zeros$ of $R(\lambda)$ outside Γ (3.b)

depending on the choice (2.a) or (2.b) for the factorization (1). Notice also that to each factorization of the above type there corresponds a "dual" factorization where the role of $R_1(\lambda)$ and $R_2(\lambda)$ is interchanged. These are easily obtained by working as above on the transpose of $R(\lambda)$ and transposing back the obtained factors. These dual factorizations are often called a "left" and "right" factorization, respectively.

Although the above theory holds for proper rational matrices (since we suppose $R(\lambda)$ has a state space realization), we can apply it here to polynomial matrices. One can indeed always perform a 1st degree conformal mapping $\lambda = (a\mu + b)/(c\mu + d)$ which does not affect the degree of rational matrices and which turns $P(\lambda)$ into a *proper* rational matrix $R(\mu) \doteq P(\frac{a\mu+b}{c\mu+d})$. The set $\Gamma(\lambda)$ of course has to be transformed accordingly to $\Gamma(\mu)$. Another way of by-passing this difficulty is to use *generalized state space realizations* [14], but since the theory is slightly more involved then, we prefer to stick to state space models. Below, we therefore talk about rational matrices and their state space representations and we only comment in the last section on problems involving polynomial matrices.

2 A recursive approach

For the development of our algorithm we were strongly inspired by [1], [4], [9]. There it is shown that it is always possible to find a regular $p \times p$ rational transfer function $C(\lambda)$ of degree 1, i.e. with one pole γ and one zero δ, such that in the product $R_2(\lambda) = C(\lambda) \cdot R(\lambda)$, *either* δ cancels with a pole (say α_1) of $R(\lambda)$, *or* γ cancels with a zero (say β_1) of $R(\lambda)$. For such a cancelation to occur, one of course needs $\delta = \alpha_1$, respectively, $\gamma = \beta_1$, but in the matrix case some additional vector conditions are required and can always be satisfied as shown in [4]. Let us represent the rational matrix $R(\lambda)$ via its poles $\{\alpha_i \mid i = 1, \cdots, \ell\}$ and zeros $\{\beta_j \mid j = 1, \cdots, k\}$, then the product $R_2(\lambda) = C(\lambda) \cdot R(\lambda)$ can be represented as either of the following two :

$$\left[\frac{\delta}{\gamma}\right]\left(\frac{\beta_1, \cdots, \beta_k}{\alpha_1, \cdots, \alpha_\ell}\right) = \left(\frac{\beta_1, \beta_2, \cdots, \beta_k}{\gamma, \alpha_2, \cdots, \alpha_\ell}\right), \quad \left[\frac{\delta}{\gamma}\right]\left(\frac{\beta_1, \cdots, \beta_k}{\alpha_1, \cdots, \alpha_\ell}\right) = \left(\frac{\delta, \beta_2, \cdots, \beta_k}{\alpha_1, \alpha_2, \cdots, \alpha_\ell}\right). \quad (4,5)$$

Here we use square brackets for *regular* rational matrices and round ones for (possibly) *singular* rational matrices. The number of poles is by definition the McMillan degree of the (regular or singular) rational matrix. Notice that for regular rational matrices, this also equals the number of zeros, while for the singular case, the number of zeros can be less than the McMillan degree [14]. Moreover, the inverse $C^{-1}(\lambda)$ of a regular rational matrix has its poles and zeros interchanged. Therefore, from (4-5) one obtains for $R(\lambda) = C^{-1}(\lambda).R_2(\lambda) = R_1(\lambda) \cdot R_2(\lambda)$:

$$\left(\frac{\beta_1, \cdots, \beta_k}{\alpha_1, \cdots, \alpha_\ell}\right) = \left[\frac{\gamma}{\alpha_1}\right] \cdot \left(\frac{\beta_1, \cdots, \beta_k}{\gamma, \alpha_2, \cdots, \alpha_\ell}\right) \text{ or } \left(\frac{\beta_1, \cdots, \beta_k}{\alpha_1, \cdots, \alpha_\ell}\right) = \left[\frac{\beta_1}{\delta}\right] \cdot \left(\frac{\delta, \beta_2, \cdots, \beta_k}{\alpha_1, \cdots, \alpha_\ell}\right). \quad (6,7)$$

It is shown in [4] that in the construction of $C(\lambda)$ in (4) the pole γ can be chosen *arbitrarily* while δ is fixed since it must cancel with α_1. If α_1 would have been the only *pole* of $R(\lambda)$ outside a given set Γ, then the factorization (6) with $R_1(\lambda) = C^{-1}(\lambda)$ would correspond to a factorization as described in section 1. All conditions would indeed be satisfied iff γ was chosen inside Γ as well, which is always possible as indicated above. The same holds for (5) where $\gamma = \beta_1$ cancels a zero β_1 of $R(\lambda)$. If β_1 was the only *zero* outside Γ, then (7) would satisfy all conditions of the factorization described in section 1, provided δ was chosen inside Γ. The factor $C(\lambda)$ in (4-5) has thus "dislocated" one pole $\alpha_1 = \delta$ (resp. zero

$\beta_1 = \gamma$) outside Γ to a pole γ (resp. zero δ) inside Γ, this using a factor $R_1^{-1}(\lambda) = C(\lambda)$ which itself has a pole γ (resp. zero δ) inside Γ. When several poles $\{\alpha_i \mid i = 1, \cdots, \ell_0\}$ (resp. zeros $\{\beta_j \mid j = 1, \cdots, k_0\}$) are outside Γ, they can be dislocated recursively one after the other by such first degree sections $C_i(\lambda)$ as described in (4-7). This then yields :

$$\left[\frac{\not\alpha_{\ell_0}}{\gamma_{\ell_0}}\right] \cdots \left[\frac{\not\alpha_2}{\gamma_2}\right] \cdot \left[\frac{\not\alpha_1}{\gamma_1}\right] \cdot \left(\frac{\beta_1, \beta_2, \cdots, \beta_k}{\not\alpha_1, \cdots, \not\alpha_{\ell_0}, \alpha_{\ell_0+1}, \cdots, \alpha_\ell}\right) = \left(\frac{\beta_1, \beta_2, \cdots, \beta_k}{\gamma_1, \cdots, \gamma_{\ell_0}, \alpha_{\ell_0+1}, \cdots, \alpha_\ell}\right), \tag{8}$$

$$\left[\frac{\delta_{\ell_0}}{\not\beta_{\ell_0}}\right] \cdots \left[\frac{\delta_2}{\not\beta_2}\right] \cdot \left[\frac{\delta_1}{\not\beta_1}\right] \cdot \left(\frac{\not\beta_1, \cdots, \not\beta_{k_0}, \beta_{k_0+1}, \cdots, \beta_k}{\alpha_1, \alpha_2, \cdots, \alpha_\ell}\right) = \left(\frac{\delta_1, \cdots, \delta_{k_0}, \beta_{k_0+1}, \cdots, \beta_k}{\alpha_1, \alpha_2, \cdots, \alpha_\ell}\right). \tag{9}$$

Considering the product of the regular factors $C_i(\lambda)$ as $R_1^{-1}(\lambda)$ and the right hand side as $R_2(\lambda)$, this certainly satisfies the imposed conditions since (1) $R_1(\lambda)$ is constructed to be regular, (2) the poles (resp. zeros) of $R_2(\lambda)$ and $R_1^{-1}(\lambda)$ are in Γ by construction and (3) the degree of $R_1(\lambda)$ equals the number of poles (resp. zeros) to be moved inside Γ. Notice that the latter also sheds some light on condition (3) of *minimal* degree. Indeed, any solution $R_1(\lambda)$ to (1-2) has degree *at least* equal to (3) as is easily seen from the above discussion. The above recursive scheme thus allows us to generate factorizations of the type described in section 1, and this for any given set Γ.

The major disadvantage of the above "transfer function"-approach is its complexity. First one computes the poles (and zeros) and the partial fraction expansion of $R(\lambda)$. From the coefficient matrices of this expansion one constructs the $C_i(\lambda)$ factors and after each pole/zero cancelation with a factor $C_i(\lambda)$, the expansion has to be updated. The most appealing methods for calculating poles and zeros use state space models [5]. One could then as well try to solve the problem using this parametrization, which is now done in the next two sections.

3 Dislocating poles in state space

Since we assumed (without loss of generality) that $R(\lambda)$ has no infinite poles, it has a realization quadruple $\{A, B, C, D\}$, i.e. $R(\lambda) = C(\lambda I - A)^{-1}B + D$, which we denote as :

$$R(\lambda) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{10}$$

Let also

$$R_1^{-1}(\lambda) \sim \left[\begin{array}{c|c} F & G \\ \hline H & J \end{array}\right]. \tag{11}$$

In order to construct a quadruple for the product $R_2(\lambda) = R_1^{-1}(\lambda).R(\lambda)$ we make use of the following lemma, which can be found implicitly in the literature ([2], see also [12]).

Lemma 1 : *Let* $\left[\begin{array}{c|c} F & G \\ \hline H & J \end{array}\right]$ *and* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *be realizations of two transfer matrices, then a realization of the product of the two corresponding transfer matrices (in that order) is given by the constant matrix product :*

$$\left[\begin{array}{cc|c} F & 0 & G \\ 0 & I & 0 \\ \hline H & 0 & J \end{array}\right] \cdot \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & A & B \\ \hline 0 & C & D \end{array}\right] = \left[\begin{array}{cc|c} F & GC & GD \\ 0 & A & B \\ \hline H & JC & JD \end{array}\right]. \tag{12}$$

We now derive necessary and sufficient conditions for canceling the poles of a transfer function $R(\lambda)$. Let us choose a *minimal* realization (10) for $R(\lambda)$ where A is in Schur form and where the eigenvalues of A outside Γ are all grouped in the top left corner A_{11}. Let us assume that the blocks A_{ii} have dimensions $n_i \times n_i$ for $i = 1, 2$ (where $n = n_1 + n_2$) :

$$R(\lambda) \sim \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{13}$$

(such a realization can always be obtained by updating (10) with a *unitary* state space transformation) [13]. Then the canceling factor $R_1^{-1}(\lambda)$ will thus have degree n_1 according to condition (3). Let its *minimal* realization be given by (11) where F has order n_1. Following Lemma 1, a (non-minimal) realization for $R_2(\lambda)$ is then given by :

$$\left[\begin{array}{ccc|c} F & & & G \\ & I_{n_1} & 0 & \\ & 0 & I_{n_2} & \\ \hline H & & & J \end{array}\right] \cdot \left[\begin{array}{ccc|c} I_{n_1} & & & \\ & A_{11} & A_{12} & B_1 \\ & 0 & A_{22} & B_2 \\ \hline & C_1 & C_2 & D \end{array}\right] = \left[\begin{array}{ccc|c} F & GC_1 & GC_2 & GD \\ & A_{11} & A_{12} & B_1 \\ & & A_{22} & B_2 \\ \hline H & JC_1 & JC_2 & JD \end{array}\right]. \tag{14}$$

This has all its poles inside Γ iff the eigenvalues of A_{11} are either unobservable or uncontrollable, since A_{22} is Γ-stable by assumption and F is chosen to be Γ-stable. Since we assumed (13) to be a minimal realization, the eigenvalues of A_{11} are controllable in (13) and hence also in (14). Let $\mathcal{X}$ be the invariant subspace of (14) corresponding to the eigenvalues of A_{11}. This space is uniquely defined since the spectrum of A_{11} is disjoint from the rest of the poles of (14). A basis for $\mathcal{X}$ is easily seen to be :

$$\mathcal{X} = < \left[\begin{array}{c} X \\ I_{n_1} \\ 0 \end{array}\right] >, \tag{15}$$

where X is the (unique) solution of the Sylvester equation :

$$XA_{11} - FX = GC_1. \tag{16}$$

Since this space must be unobservable one has $[H, JC_1, JC_2]\mathcal{X} = 0$ or :

$$HX + JC_1 = 0. \tag{17}$$

A reduced realization for $R_2(\lambda)$ is then obtained using the state space transformation

$$T^{-1} \left[\begin{array}{ccc} I_{n_1} & -X & 0 \\ & I_{n_1} & 0 \\ & & I_{n_2} \end{array}\right], \; T = \left[\begin{array}{ccc} I_{n_1} & X & 0 \\ & I_{n_1} & 0 \\ & & I_{n_2} \end{array}\right]. \tag{18}$$

Performed on (14) this yields, because of (16)(17) :

$$R_2(\lambda) \sim \left[\begin{array}{ccc|c} F & 0 & GC_2 - XA_{12} & GD - XB_1 \\ 0 & A_{11} & A_{12} & B_1 \\ 0 & 0 & A_{22} & B_2 \\ \hline H & 0 & JC_2 & JD \end{array}\right] \sim \left[\begin{array}{cc|c} F & GC_2 - XA_{12} & GD - XB_1 \\ 0 & A_{22} & B_2 \\ \hline H & JC_2 & JD \end{array}\right], \tag{19}$$

which again has state space dimension $n_1 + n_2$. Summarizing we thus proved the following theorem (see [12] for more details).

Theorem 1 : *Let $R(\lambda)$ be a $p \times m$ rational matrix with a minimal realization of order $n = n_1 + n_2$ as given in (13), where $\Lambda(A_{22}) \subset \Gamma$ and $\Lambda(A_{11}) \subset \Gamma_c$, the complement of Γ. Then $R_1(\lambda)$ and $R_2(\lambda)$, realized by*

$$R_1^{-1}(\lambda) \sim \left[\begin{array}{c|c} F & G \\ \hline H & J \end{array}\right], \; R_2(\lambda) \sim \left[\begin{array}{cc|c} F & GC_2 - XA_{12} & GD - XB_1 \\ 0 & A_{22} & B_2 \\ \hline H & JC_2 & JD \end{array}\right], \tag{20}$$

satisfy the requested conditions of the factorization (1-3) iff (i) $\{F,G,H,J\}$ represents a regular transfer matrix, (ii) $\Lambda(F)\subset\Gamma$, and (iii) the following equation is satisfied :

$$\left[\begin{array}{c|c} F & G \\ \hline H & J \end{array}\right]\cdot\left[\begin{array}{c} X \\ \hline C_1 \end{array}\right] = \left[\begin{array}{c} X \\ \hline 0 \end{array}\right]\cdot A_{11}. \tag{21}$$

The question now of course remains how to find matrices F,G,H,J and X satisfying the conditions of Theorem 1. We know from section 2 that such a solution must exist, but we would like a *constructive* proof *not* relying on this section. For this, one shows that the regularity of the system $\{F,G,H,J\}$ implies X to be invertible. Indeed, from (21) it follows that $\left[\begin{array}{c} X \\ \hline C_1 \end{array}\right]$ and X must have the same rank and hence the same kernel, say $\mathcal{N}$. But then (21) also implies that $A_{11}\mathcal{N}\subset\mathcal{N}$, and hence $\mathcal{N}$ must be an unobservable subspace of (A_{11},C_1) which contradicts the assumptions (for a more detailed proof, we refer to [12]).

Lemma 2 : *Let (A_{11},C_1) be observable and let $\{F,G,H,J\}$ satisfy (21). Then $\{F,G,H,J\}$ represents a regular system only if X in (21) is regular. Moreover, the eigenvalues of A_{11} – i.e. the poles of $R(\lambda)$ – to be canceled are the zeros of the regular system $\{F,G,H,J\}$.*

Since X is invertible it can be "absorbed" into the quadruple $\{F,G,H,J\}$ as a state space transformation. Putting

$$\{\hat F,\hat G,\hat H,\hat J\} = \{X^{-1}FX,\ X^{-1}G,\ HX,\ J\}, \tag{22}$$

it follows indeed from (21) that we are looking for a system $\{\hat F,\hat G,\hat H,\hat J\}$ satisfying :

$$\left[\begin{array}{c|c} \hat F & \hat G \\ \hline \hat H & \hat J \end{array}\right]\left[\begin{array}{c} I \\ \hline C_1 \end{array}\right] = \left[\begin{array}{c} A_{11} \\ \hline 0 \end{array}\right], \tag{23}$$

and this is easily solved via the following procedure :

Algorithm 1

Step 1 : Determine $\hat F,\hat G$, with $\Lambda(\hat F)\subset\Gamma$ by solving the pole placement problem : $\hat F = A_{11}-\hat G C_1$. This has always a solution since (A_{11},C_1) is observable.

Step 2 : Determine $[\ \hat H \mid \hat J\ ]^T$ as any basis for the null space of $[\ I_{n_1} \mid C_1^T\ ]$, i.e. $[\ \hat H \mid \hat J\ ] = M[-C_1 \mid I\,]$ for an arbitrary *invertible* M.

Since the regularity of the system $\{\hat F,\hat G,\hat H,\hat J\}$ follows from the invertibility of $\hat J = M$, all conditions of Theorem 1 are satisfied. We have thus derived here a constructive proof that the undesired *poles* of a transfer function $R(\lambda)$ can be canceled by a regular transfer function $R_1^{-1}(\lambda)$ whose degree equals the number of poles to be canceled (here n_1), whose zeros will be those unwanted poles and whose poles can be chosen arbitrarily in Γ.

4 Dislocating zeros in state space

Here we consider the case of canceling the undesired zeros of $R(\lambda)$ that are outside a specified set Γ by the poles of $R_1^{-1}(\lambda)$. As before, let (11) be a *minimal* realization of the factor $R_1^{-1}(\lambda)$ and let n_1 be its state space dimension. Then according to Lemma 1, the product $R_2(\lambda) = R_1^{-1}(\lambda)\cdot R(\lambda)$ is realized by :

$$R_2(\lambda)\sim\left[\begin{array}{cc|c} F & GC & GD \\ 0 & A & B \\ \hline H & JC & JD \end{array}\right]. \tag{24}$$

We would like the poles of $R_1^{-1}(\lambda)$ – i.e. the spectrum of F – to cancel with the undesired zeros of $R(\lambda)$, which are assumed to be n_1 in number. Since the (F,H)-pair is observable in

(11) it is also observable in (24) and the spectrum of F in (24) must thus be uncontrollable. Since (A, B) is controllable in (24) the controllable subspace $\mathcal{X}$ of the realization (24) must be of the form [3] :

$$\mathcal{X} = < \begin{bmatrix} X \\ I_n \end{bmatrix} > , \tag{25}$$

where X satisfies [3] :

$$XA - FX = GC \; ; \quad XB = GD. \tag{26}$$

The state space transformation $T = \begin{bmatrix} I_{n_1} & X \\ 0 & I_n \end{bmatrix}$ applied to (24) then yields :

$$R_2(\lambda) \sim \left[\begin{array}{cc|c} F & 0 & 0 \\ 0 & A & B \\ \hline H & JC + HX & JD \end{array}\right], \tag{27}$$

where we used (26) for the zero blocks in the top row of (27). Condition (26) is only a *necessary* condition for the pole-zero cancelation to occur (see [12] for an example where (26) is met and no pole-zero cancelation occurs). Additional conditions have to be imposed on X in order to enforce a zero dislocation. In order to do so one has to explicitate first what zeros of $R(\lambda)$ have to be dislocated as was e.g. done in (13) for the poles. A similar decomposition for the zeros is now derived using the following lemmas, proven in [11] :

Lemma 3 : *Let U be any invertible transformation such that*

$$\left[\begin{array}{c|c} \lambda I - A & B \\ \hline -C & D \end{array}\right] \cdot U = \left[\begin{array}{c|c} \lambda \hat{E} - \hat{A} & \lambda \hat{B} - \hat{F} \\ \hline 0 & \hat{D} \end{array}\right], \tag{28}$$

where has linearly independent columns. Then the generalized eigenvalues of $\lambda \hat{E} - \hat{A}$ are the zeros of $R(\lambda) = C(\lambda I - A)^{-1}B + D$.

Notice that, since $[-C \mid D] \cdot U = [\, 0 \mid \hat{D} \,]$, $rank.\hat{D}$ equals $rank.([-C \mid D\,])$ and the number of columns $\hat{m}$ of $\hat{D}$ and m of D may thus differ. We also remark that in practice one uses unitary matrices U, which yields a numerically stable construction of what was called the *zero pencil* $(\lambda \hat{E} - \hat{A})$ in [11]. The use of unitary matrices is also maintained in the following lemma, leading to the separation between two parts of the spectrum of $\lambda \hat{E} - \hat{A}$.

Lemma 4 : *Let $\lambda \hat{E} - \hat{A}$ be an arbitrary singular pencil with spectrum $\Lambda(\hat{E}, \hat{A})$ – i.e. $\Lambda(\hat{E}, \hat{A})$ is the set of generalized eigenvalues of $(\lambda \hat{E} - \hat{A})$. Then for any complementary sets Γ and Γ_c of the extended complex plane separating $\Lambda(\hat{E}, \hat{A})$ in two disjoint parts Λ_Γ and Λ_{Γ_c}, there exist unitary transformations Q and Z such that :*

$$Q^*(\lambda \hat{E} - \hat{A})Z = \begin{bmatrix} \lambda \hat{E}_{11} - \hat{A}_{11} & \lambda \hat{E}_{12} - \hat{A}_{12} & \lambda \hat{E}_{13} - \hat{A}_{13} \\ 0 & \lambda \hat{E}_{22} - \hat{A}_{22} & \lambda \hat{E}_{23} - \hat{A}_{23} \\ 0 & 0 & \lambda \hat{E}_{33} - \hat{A}_{33} \end{bmatrix}, \tag{29}$$

whereby (i) $\Lambda(\hat{E}_{11}, \hat{A}_{11}) = \Lambda_\Gamma$; $\Lambda(\hat{E}_{22}, \hat{A}_{22}) = \Lambda_{\Gamma_c}$; $\Lambda(\hat{E}_{33}, \hat{A}_{33}) = \emptyset$, (ii) $\lambda \hat{E}_{11} - \hat{A}_{11}$ is right invertible for $\lambda \in \Gamma_c$, (iii) $\lambda \hat{E}_{22} - \hat{A}_{22}$ is invertible for $\lambda \in \Gamma$, and (iv) $\lambda \hat{E}_{33} - \hat{A}_{33}$ is left invertible.

This essentially says that there exists a (generalized) block Schur decomposition (29) with the generalized eigenvalues of $\lambda \hat{E} - \hat{A}$ inside Γ gathered in $\lambda \hat{E}_{11} - \hat{A}_{11}$, and the remaining ones gathered in $\lambda \hat{E}_{22} - \hat{A}_{22}$. This leads to the following theorem (proven in [12]).

Theorem 2 : *One can always update a minimal realization (10) by a unitary state space transformation Q such that its zero pencil (28) is automatically in generalized Schur form (29), i.e. :*

$$\left[\begin{array}{c|c} Q^*(\lambda I - A)Q & Q^*B \\ \hline -CQ & D \end{array}\right] = \left[\begin{array}{ccc|c} \lambda I_{n_1} - A_{11} & -A_{12} & -A_{13} & B_1 \\ -A_{21} & \lambda I_{n_2} - A_{22} & -A_{23} & B_2 \\ -A_{31} & -A_{32} & \lambda I_{n_3} - A_{33} & B_3 \\ \hline -C_1 & -C_2 & -C_3 & D \end{array}\right] \tag{30}$$

and

$$\left[\begin{array}{ccc|c} \lambda I_{n_1} - A_{11} & -A_{12} & -A_{13} & B_1 \\ -A_{21} & \lambda I_{n_2} - A_{22} & -A_{23} & B_2 \\ -A_{31} & -A_{32} & \lambda I_{n_3} - A_{33} & B_3 \\ \hline -C_1 & -C_2 & -C_3 & D \end{array}\right] V$$

$$= \underbrace{}_{\hat{n}_1}\left[\begin{array}{ccc|c} \lambda \hat{E}_{11} - \hat{A}_{11} & \lambda \hat{E}_{12} - \hat{A}_{12} & \lambda \hat{E}_{13} - \hat{A}_{13} & \lambda \hat{F}_1 - \hat{B}_1 \\ 0 & \lambda \hat{E}_{22} - \hat{A}_{22} & \lambda \hat{E}_{23} - \hat{A}_{23} & \lambda \hat{F}_2 - \hat{B}_2 \\ 0 & 0 & \lambda \hat{E}_{33} - \hat{A}_{33} & \lambda \hat{F}_3 - \hat{B}_3 \\ \hline 0 & 0 & 0 & \hat{D} \end{array}\right] \begin{array}{l} \}n_1 \\ \}n_2 \\ \}n_3 \\ \}p \end{array} \tag{31}$$

$$\underbrace{\qquad}_{\hat{n}_1} \quad \underbrace{\qquad}_{\hat{n}_2} \quad \underbrace{\qquad}_{\hat{n}_3} \quad \underbrace{\qquad}_{\hat{m}}$$

where $\lambda \hat{E}_{11} - \hat{A}_{11}$, $\lambda \hat{E}_{22} - \hat{A}_{22}$ and $\lambda \hat{E}_{33} - \hat{A}_{33}$ satisfy conditions (i)(ii)(iii)(iv) of Lemma 4 (hence $\hat{n}_1 \geq n_1$, $\hat{n}_2 = n_2$, $\hat{n}_3 < n_3$).

Without loss of generality we can thus assume that our state space realization is in a form satisfying (31) where $\lambda \hat{E}_{22} - \hat{A}_{22}$ contains the zeros of $R(\lambda)$ to be dislocated. This form is to be considered an analogue to (13), now isolating the *zeros* to be canceled in a separate block $\lambda \hat{E}_{22} - \hat{A}_{22}$. The fact that now there is a third block $\lambda \hat{E}_{33} - \hat{A}_{33}$ is due to the possible singularity of the pencil (30-31) - and hence of $R(\lambda)$ (see [11]). Indeed, when $R(\lambda)$ happens to be right invertible, so will (30-31), and the block $\lambda \hat{E}_{33} - \hat{A}_{33}$ vanishes [11]. We are now ready to formulate a theorem on zero dislocation analogously to Theorem 1.

Theorem 3 : *Let $R(\lambda)$ be a $p \times m$ rational matrix with a minimal realization of order $n = n_1 + n_2 + n_3$ as given in (30-31), where $\Lambda(\hat{E}_{11}, \hat{A}_{11}) \subset \Gamma$ and $\Lambda(\hat{E}_{22}, \hat{A}_{22}) \subset \Gamma_c$, the complement of Γ and $\Lambda(\hat{E}_{33}, A_{33}) = \emptyset$. Then $R_1(\lambda)$ and $R_2(\lambda)$, realized by*

$$R_1^{-1}(\lambda) \sim \left[\begin{array}{c|c} F & G \\ \hline H & J \end{array}\right], \; R_2(\lambda) \sim \left[\begin{array}{c|c} \lambda I - A & B \\ \hline -JC - HX & JD \end{array}\right], \tag{32}$$

satisfy the requested conditions of the factorization (1-3) iff (i) $\{F, G, H, J\}$ represents a regular transfer matrix, (ii) the zeros of $\{F, G, H, J\}$ lie in Γ, and (iii) the following equation is satisfied for the $n_2 \times (n_1 + n_2 + n_3)$ matrix $X = [\,0 \;\; X_2 \;\; X_3\,]$:

$$[\,X \quad G\,] \cdot \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = F \cdot [\,X \quad 0\,], \tag{33}$$

with X_2 invertible

For the construction of a solution to the above theorem, we can again "absorb" the invertible factor X_2 into the quadruple $\{F, G, H, J\}$ as a state space transformation. Putting

$$\{\hat{F}, \hat{G}, \hat{H}, \hat{J}\} = \{X_2^{-1} F X_2,\; X_2^{-1} G,\; H X_2,\; J\}, \tag{34}$$

it then follows that we are looking for a system $\{\hat{F}, \hat{G}, \hat{H}, \hat{J}\}$ satisfying :

$$[\,I \quad Y_3 \quad \hat{G}\,] \cdot \left[\begin{array}{cc|c} \lambda \hat{E}_{22} - \hat{A}_{22} & \lambda \hat{E}_{23} - \hat{A}_{23} & \lambda \hat{F}_2 - \hat{B}_2 \\ 0 & \lambda \hat{E}_{33} - \hat{A}_{33} & \lambda \hat{F}_3 - \hat{B}_3 \\ \hline 0 & 0 & \hat{D} \end{array}\right] = [\,\lambda I - \hat{F}\,] \cdot [\,\hat{E}_{22} \quad \hat{Y}_3 \quad \hat{Y}_4\,] \tag{35}$$

and this is easily solved via the following procedure :

Algorithm 2

Step 1 : Put $\hat{F} = \hat{A}_{22} \hat{E}_{22}^{-1}$

Step 2 : Solve $Y_3(\lambda \hat{E}_{33} - \hat{A}_{33}) - (\lambda I - \hat{F})\hat{Y}_3 = -(\lambda \hat{E}_{23} - \hat{A}_{23})$ for Y_3 and $\hat{Y}_3$ (see [10][8])

Step 3 : Put $\hat{Y}_4 = \hat{F}_2 + Y_3 \hat{F}_3$ and solve $\hat{G}$ from $\hat{G}\hat{D} = \hat{B}_2 + Y_3 \hat{B}_3 - \hat{F}\hat{Y}_4$

Step 4 : Choose $\hat{H}$, $\hat{J}$ such that the zeros of $\{\hat{F}, \hat{G}, \hat{H}, \hat{J}\}$ lie in Γ. For *finite* zeros this is a pole placement problem since then $\hat{J}$ is invertible. Choose then K such that $\Lambda(\hat{F} + \hat{G}K) \subset \Gamma$ and then solve for $[\,\hat{H} \;\; \hat{J}\,]^T$ as any basis for the null space of $[\,I_{n_2} \mid K^T\,]$, i.e. $[\,\hat{H} \mid \hat{J}\,] = M[-K \mid I\,]$ for an arbitrary *invertible* M.

5 Conclusion

In this paper a method was presented to perform factorizations involving polynomial matrices, with constraints on the poles and/or zeros of the factors. The method uses a state-space approach in contrast to recursive solutions using transfer function techniques [1], [4], [9]. The main advantage of the present method is its algorithmic simplicity : first the poles/zeros of the original matrix are computed via a (generalized) Schur decomposition of the corresponding state space model. From thereon, one merely has to solve a pole placement problem in order to find the realizations of the two factors (this is just a set of linear equations to solve !). A similar "block"-approach was also recently used in [6] for a related problem. In contrast to the recursive transfer function approach, all poles/zeros can be dislocated simultaneously, which is useful when *real* factorizations are requested (this is more involved in the transfer function approach [9]). On the other hand, it is shown in [12] that the state space method can also be implemented in a recursive fashion, which is useful for other types of *rational* factorizations (inner-outer factorization, all-pass extraction, ...).

Notice that the strength of the present approach is the use of the Schur decomposition as a starting point for the construction of the factors. For problems as coprime factorization and GCD extraction, it could be argued that no eigenvalue computation is required since these problems can be solved in a finite number of operations. But the use of eigenvalue methods here can be compared with the solution of Sylvester and Lyapunov equations (these are also sets of linear equations !) where again Schur methods prove to be attractive from a computational point of view (see [7], [13], [8]).

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