Krylov Techniques for Model Reduction of Second-Order Systems

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Abstract

The purpose of this paper is to present a Krylov technique for model reduction of second-order systems. These can be seen as linear systems and classical Krylov techniques can be applied to such systems but they do not generally give a reduced order system that preserves the second order structure of the original system. The Krylov technique that is presented in this paper provides always a second-order reduced transfer function but the price to pay is that its Mc Millan degree is $2k$ instead of k in order to satisfy $2k$ interpolation conditions. A generalization to tangential interpolation is briefly sketched.

1 Introduction

Linear Time Invariant systems can be represented as follows.

$$\left\{ \begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + Bu(t), \\ y(t) & = & Cx(t), \end{array} \right. \tag{1}$$

where $u(t)$ is the input, $y(t)$ is the output and $x(t)$ is the state of the system.

If the pencil $sE - A$ is regular, one can associate to the system (1) the transfer function

$$T(s) \doteq C(sE - A)^{-1}B,$$

that links the inputs to the outputs in the Laplace domain.

In this paper, we consider second-order systems of the following form

$$\begin{array}{rcl} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{bmatrix} & = & \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix} \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, \end{array} \tag{2}$$

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where $M, D, K \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{p\times n}$ and $B \in \mathbb{C}^{n\times m}$. Such a system corresponds to the following transfer function

$$H(s) \doteq C(Ms^2 + Ds + K)^{-1}B. \quad (3)$$

The dimension n of the matrices M, C and K is called the order of the second-order transfer function $H(s)$. It should be pointed that if the order of $H(s)$ is equal to n, the Mc Millan degree of $H(s)$ is generically equal to $2n$.

By defining

$$\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \doteq x(t) \quad , \quad \mathcal{E} \doteq \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}$$
$$\mathcal{A} \doteq \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix} \quad , \quad \mathcal{B} \doteq \begin{bmatrix} 0 \\ B \end{bmatrix} \quad , \quad \mathcal{C} \doteq \begin{bmatrix} C & 0 \end{bmatrix}, \quad (4)$$

we can rewrite (2) as follows:

$$\begin{cases} \mathcal{E}\dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t), \\ y(t) &= \mathcal{C}x(t). \end{cases}$$

We see that second-order systems are particular linear systems with the property that the state can be divided into two parts. The first part is the *position* $q(t)$. The second part, called the *velocity* $\dot{q}(t)$, is the derivative of the position of the system.

Second-order systems, also called mechanical systems or structural dynamics systems arise naturally in engineering. Often, the Mc Millan degree $2n$ is too large for being able to solve various control or simulation problems. It then makes sense to construct a transfer function $\hat{T}(s)$ of smaller Mc Millan degree $2k$ that approximates the original transfer function $T(s)$. If one applies a classical model reduction technique to a second-order system, there no guarantee at all that the resulting reduced system would be a second-order system. Because the second-order structure of the system has an important physical meaning, it makes sense to develop model reduction techniques that preserve the second-order structure of the system. Most model reduction techniques can be cast into two categories [1].

On the one hand, SVD-based model reduction methods such as balanced truncation [12] and optimal Hankel norm approximation [8] are very popular for systems of relatively small Mc Millan degree (roughly speaking with $n < 1000$). They require $O(n^3)$ computations. So, they are not applicable for large scale systems of very high Mc Millan degree (say $n > 1000$) but they guarantee a global error bound between the original and the reduced system and they are fully automatic once the global error tolerance or the Mc Millan degree of the reduced system is specified.

On the other hand, interpolation based model reduction methods (also called Krylov methods) such as Multipoint Padé [4] can be applied to dynamical systems of very high dimension ($n > 1000$) but they are not fully automatic and the quality of the reduced system is not guaranteed. Even worse, stability may be lost. From a practical point of view, they have proved to be very efficient for instance in circuit simulation and have been widely used.

At present time, a lot of work has been put in deriving SVD-like or Krylov-like methods that preserve the second-order structure of the original system. For instance, in [10], new pairs of *second-order* gramians have been introduced. This has recently been extended in [3] in order to give a new second-order structure preserving *balanced truncation* technique.

The first Krylov-like second-order structure preserving model reduction technique that appeared in the literature is the method presented in [11]. It constructs a second-order system of Mc Millan degree $2k$ that interpolates the original transfer function at the point $s = 0$ up to the $2k$ first derivatives. The following question has been mentioned in [2]: "How do we introduce the shifting strategy to generalize the frequency response analysis around an arbitrary expansion point s_0". The purpose of this paper is to provide an algorithm that constructs a second-order reduced system of order k (i.e. of Mc Millan degree $2k$) that satisfies $2k$ interpolation conditions with respect to the original system, where the interpolation points are arbitrarily chosen in the complex plane. It is important to notice that similar results have independently been obtained by Freund [5]. A natural extension of our second-order structure preserving Krylov technique for the MIMO case corresponding to tangential interpolation is also briefly introduced.

This paper is organized as follows. In section 2, the main results concerning Krylov techniques for model reduction are summarized. In section 3, a second-order structure preserving projection framework is derived. In section 4, our new second-order Krylov technique is developed. Concluding remarks are given in section 5.

2 A Review on Krylov Techniques

First, some words about the notation. We say that a rational matrix function $R(s)$ is $O(\lambda - s)^k$ in s with $k \in \mathbb{Z}$ if its Taylor expansion about the point λ can be written as follows :

$$R(s) = O(\lambda - s)^k \iff R(s) = \sum_{i=k}^{+\infty} R_i (\lambda - s)^i, \tag{5}$$

where the coefficients R_i are constant matrices. If $R_k \neq 0$, then we say that $R(s) = \Theta(\lambda - s)^k$. As a consequence, if $R(s) = \Theta(\lambda - s)^k$ and k is strictly negative, then λ is a pole of $R(s)$ and if k is strictly positive, then λ is a zero of $R(s)$. Analogously, we say that $R(s)$ is $O(s^{-1})^k$ if the following condition is satisfied :

$$R(s) = O(s^{-1})^k \iff R(s) = \sum_{i=k}^{+\infty} R_i s^{-i}, \tag{6}$$

where the coefficients R_i are constant matrices. It should be stressed that, in general, $R(s)$ being $O(s)^{-k}$ is not equivalent to $R(s)$ being $O(s^{-1})^k$.

Definition 2.1 *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$. A Krylov subspace of order k of the pair (A, B), written $\mathcal{K}_k(A, B)$, is the image of the matrix* $\begin{bmatrix} B & AB & \dots & A^{k-1}B \end{bmatrix}$.

If the matrix E of (1) is invertible, one can expand the transfer function $T(s)$ at infinity :

$$T(s) = \sum_{k=0}^{+\infty} C(E^{-1}A)^k E^{-1} B s^{-k-1} \doteq \sum_{k=0}^{\infty} M_k^{(\infty)} s^{-k-1},$$

where the coefficients $M_k^{(\infty)} \doteq C(E^{-1}A)^k E^{-1}B$ are called the Laurent coefficients of $T(s)$. If one wants to find a reduced transfer function, $\hat{T}(s) \doteq \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} \doteq \sum_{k=0}^{\infty} \hat{M}_k^{(\infty)} s^{-k-1}$, that approximates as well as possible the original transfer function for large frequency $s \to \infty$, it makes sense to choose $\hat{T}(s)$ such that for $0 \leq k \leq K$,

$$\hat{M}_k^{(\infty)} = M_k^{(\infty)}.$$

It is equivalent to say that

$$T(s) - \hat{T}(s) = O(s^{-1})^K.$$

If one wants to obtain a good approximation in the low frequency domain, one might prefer to construct a transfer function

$$\hat{T}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} = \sum_{k=0}^{\infty} \hat{M}_k^{(\lambda)} (\lambda - s)^k,$$

such that

$$\hat{M}_k^{(\lambda)} = M_k^{(\lambda)} \qquad for\ 1 \leq k \leq K, \tag{7}$$

with

$$\begin{aligned} M_k^{(\lambda)} &\doteq C\left[(\lambda E - A)^{-1}E\right]^k (\lambda E - A)^{-1}B, \\ \hat{M}_k^{(\lambda)} &\doteq \hat{C}\left[(\lambda\hat{E} - \hat{A})^{-1}\hat{E}\right]^k (\lambda\hat{E} - \hat{A})^{-1}\hat{B}. \end{aligned}$$

Equation (7) can be rewritten more compactly as follows

$$T(s) - \hat{T}(s) = O(\lambda - s)^K.$$

More generally, one can prefer to choose a transfer function $\hat{T}(s)$ that interpolates $T(s)$ at several points in the complex plane, up to several orders. The main results concerning this problem are summarized in the following Theorem :

Theorem 2.1 *Let the original system be*

$$T(s) \doteq C(sE - A)^{-1}B, \tag{8}$$

and the reduced system be

$$\hat{T}(s) \doteq CV\left(Z^T(sE - A)V\right)^{-1} Z^T B. \tag{9}$$

If

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{b_k}}((\sigma_k E - A)^{-1}E, (\sigma_k E - A)^{-1}B) \subseteq Im(V) \tag{10}$$

and

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{c_k}}((\sigma_k E - A)^{-T} E^T, (\sigma_k E - A)^{-T} C^T) \subseteq Im(Z) \tag{11}$$

where the interpolation points σ_k are chosen such that the matrices $\sigma_k E - A$ are invertible $\forall k \in \{1, \ldots, K\}$ then the moments of the systems (1) and (9) at the points σ_k satisfy

$$M_{j_k}^{(\sigma_k)} = \hat{M}_{j_k}^{(\sigma_k)} \tag{12}$$

for $j_k = 1, 2, \ldots, J_{b_k} + J_{c_k}$ and $k = 1, 2, \ldots, K$, provided these moments exist, i.e. provided the matrices $\sigma_k \hat{E} - \hat{A}$ are invertible.

For a proof, see [4] or [9]. □

One could want to apply such a Krylov technique to the state space realization (4) as follows. In order to find a transfer function of Mc Millan degree k, $\hat{H}(s)$, that interpolates $H(s)$ at $2k$ interpolation points λ_1 up to λ_{2k} (we assume for simplicity that the interpolation points are finite, distinct and not poles of $H(s)$), i.e.

$$H(s) - \hat{H}(s) = O(\lambda_i - s) \qquad for\ 1 \leq i \leq 2k, \tag{13}$$

proceed as follows

Algorithm 2.1 *1. Construct Z and V such that*

$$\begin{aligned} V &= \begin{bmatrix} (\lambda_1 \mathcal{E} - \mathcal{A})^{-1}\mathcal{B} & \ldots & (\lambda_k \mathcal{E} - \mathcal{A})^{-1}\mathcal{B} \end{bmatrix} M \\ Z^T &= N \begin{bmatrix} \mathcal{C}(\lambda_{k+1}\mathcal{E} - \mathcal{A})^{-1} \\ \vdots \\ \mathcal{C}(\lambda_{2k}\mathcal{E} - \mathcal{A})^{-1} \end{bmatrix}, \end{aligned}$$

where the matrices $M, N \in \mathbb{C}^{k \times k}$ are invertible and are chosen for computational reasons (they can be any $k \times k$ invertible matrix).

2. Construct the matrices

$$\hat{\mathcal{C}} \doteq \mathcal{C}V \quad , \quad \hat{\mathcal{A}} \doteq Z^T \mathcal{A} V \quad , \quad \hat{\mathcal{E}} \doteq Z^T \mathcal{E} V \quad , \quad \hat{\mathcal{B}} \doteq Z^T \mathcal{B}.$$

3. Define the reduced transfer function

$$\hat{H}(s) \doteq \hat{\mathcal{C}}(s\hat{\mathcal{E}} - \hat{\mathcal{A}})^{-1}\hat{\mathcal{B}}.$$

From the preceding theorem, $\hat{H}(s)$ generically satisfies the interpolation conditions (13). Moreover, it can be shown that, generically, there exists no transfer function of smaller Mc Millan degree that satisfies the interpolation constraints. The problem is that $\hat{H}(s)$ is not a second-order transfer function anymore. The purpose of this paper is to develop a general procedure of constructing a reduced transfer function that satisfies the interpolation constraints AND preserves the second-order structure of the original system. The price to pay is that, in order to satisfy $2k$ interpolation constraints, the resulting reduced transfer function will not be of Mc Millan degree k but of Mc Millan degree $2k$.

3 Second-order structure preserving model reduction

The following result is hidden in [3].

Lemma 3.1 *Let $(\mathcal{C}, \mathcal{E}, \mathcal{A}, \mathcal{B})$ be the state space realization defined in* (4). *If one projects such a state space realization with $2n \times 2k$ bloc diagonal matrices*

$$\mathcal{Z} \doteq \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \quad , \quad \mathcal{V} \doteq \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix},$$

where $Z_1, V_1, Z_2, V_2 \in \mathbb{C}^{n\times k}$, then the reduced transfer function

$$\hat{T}(s) \doteq \mathcal{C}\mathcal{V}\left(\mathcal{Z}^T(s\mathcal{E} - \mathcal{A})\mathcal{V}\right)^{-1}\mathcal{Z}^T\mathcal{B}$$

is a second-order transfer function, provided the matrices $Z_1^T V_1$ and $Z_1^T V_2$ are invertible.

Proof :
First, let us verify that if the matrices M_1, M_2 are invertible, then the state space realizations $(\mathcal{C}M_2, M_1\mathcal{E}M_2, M_1\mathcal{A}M_2, M_1\mathcal{B})$ and $(\mathcal{C}, \mathcal{E}, \mathcal{A}, \mathcal{B})$ give rise to the same transfer function. Indeed,

$$\begin{aligned} \mathcal{C}M_2(sM_1\mathcal{E}M_2 - M_1\mathcal{A}M_2)^{-1}M_1\mathcal{B} &= \mathcal{C}M_2\left(M_1(s\mathcal{E} - \mathcal{A})M_2\right)^{-1}M_1\mathcal{B} \\ &= \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}. \end{aligned}$$

From the invertibility of $Z_1^T V_1$ and $Z_1^T V_2$, there exist $k \times k$ invertible matrices L_1, L_2, R_1 and R_2 such that

$$L_1 Z_1^T V_1 R_1 = I_k \quad , \quad L_1 Z_1^T V_2 R_2 = I_k. \tag{14}$$

A possible choice is

$$R_1 = L_2 = I_k \quad , \quad L_1 = (Z_1^T V_1)^{-1} \quad , \quad R_2 = (L_1 Z_1^T V_2)^{-1},$$

but other choices may be preferable for computational efficiency. Define the following $2k \times 2k$ invertible matrices

$$L \doteq \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \quad , \quad R \doteq \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}.$$

From the preceding results,

$$\begin{aligned} \hat{T}(s) &= \hat{T}(s) \doteq \mathcal{C}\mathcal{V}R\left(L\mathcal{Z}^T(s\mathcal{E} - \mathcal{A})\mathcal{V}R\right)^{-1}L\mathcal{Z}^T\mathcal{B} \\ &= CV_1R_1\left(s^2L_2Z_2^TMV_2R_2 + sL_2Z_2^TDV_2R_2 + L_2Z_2^TKV_1R_1\right)^{-1}L_2Z_2^TB. \end{aligned}$$

This is clearly a second-order transfer function. □

4 Second-Order Krylov Techniques

4.1 Second-Order Structure Preserving Krylov

By combining the results of sections 2 and 3, one obtains the following Theorem.

Theorem 4.1 *Let* $H(s) \doteq D(Ms^2 + Cs + K)^{-1}B = \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$*, with*

$$\mathcal{E} \doteq \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \quad \mathcal{A} \doteq \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \quad \mathcal{B} \doteq \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad \mathcal{C} \doteq \begin{bmatrix} C & 0 \end{bmatrix},$$

be a second-order transfer function of Mc Millan degree $2n$ $(M, C, K) \in \mathbb{C}^{n\times n}$*). Let* $Z, V \in \mathbb{C}^{2n\times k}$ *be defined as*

$$V \doteq \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad , \quad Z \doteq \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

with V_1, V_2, Z_1 *and* $Z_2 \in \mathbb{C}^{n\times k}$*. Let us define the* $2n \times 2k$ *projecting matrices*

$$\mathcal{V} \doteq \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \quad , \quad \mathcal{Z} \doteq \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}.$$

Define the second-order transfer function $\hat{H}(s)$ *of order* k *by*

$$\begin{aligned} \hat{H}(s) &\doteq \mathcal{C}\mathcal{V}\left(\mathcal{Z}^T(s\mathcal{E} - \mathcal{A})\mathcal{V}\right)^{-1}\mathcal{Z}^T\mathcal{B} \\ &\doteq \hat{\mathcal{C}}(s\hat{\mathcal{E}} - \hat{\mathcal{A}})^{-1}\hat{\mathcal{B}}. \end{aligned} \tag{15}$$

If

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{b_k}}((\sigma_k\mathcal{E} - \mathcal{A})^{-1}\mathcal{E}, (\sigma_k\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}) \subseteq Im(V) \tag{16}$$

and

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{c_k}}((\sigma_k\mathcal{E} - \mathcal{A})^{-T}\mathcal{E}^T, (\sigma_k\mathcal{E} - \mathcal{A})^{-T}\mathcal{C}^T) \subseteq Im(Z) \tag{17}$$

where the interpolation points σ_k *are chosen such that the matrices* $\sigma_k\mathcal{E} - \mathcal{A}$ *are invertible* $\forall k \in \{1, \ldots, K\}$ *then, if the matrices* $Z_1^T V_1, Z_1^T V_2$ *are invertible and the matrix* Z_2 *is full rank,*

$$T(s) - \hat{T}(s) = O(s - \sigma_k)^{J_{b_k} + J_{c_k}} \tag{18}$$

for the finite points σ_k*, provided these moments exist, i.e. provided the matrices* $\sigma_k\hat{\mathcal{E}} - \hat{\mathcal{A}}$ *are invertible and*

$$T(s) - \hat{H}(s) = O(s^{-1})^{J_{b_k} + J_{c_k}} \tag{19}$$

if $\sigma_k = \infty$*, provided* $\hat{\mathcal{E}}$ *is invertible.*

Proof :
The second-order structure of $\hat{H}(s)$ follows from Lemma 3.1. It is clear that

$$Im(V) \subset Im(\mathcal{V}) \quad , \quad Im(Z) \subset Im(\mathcal{Z}).$$

The interpolation conditions are then satisfied because of Theorem 2.1. □

This gives rise to the following procedure. In order to find a *second-order* transfer function of order k (i.e. of Mc Millan degree $2k$), $\hat{H}(s)$, that interpolates $T(s)$ at $2k$ interpolation points λ_1 up to λ_{2k} (we assume for simplicity that the interpolation points are finite, distinct and not poles of $T(s)$), i.e.

$$T(s) - \hat{H}(s) = O(\lambda_i - s) \qquad \textit{for } 1 \leq i \leq 2k, \tag{20}$$

proceed as follows

Algorithm 4.1 *1. Construct Z and V such that*

$$\begin{aligned} V &= \begin{bmatrix} (\lambda_1\mathcal{E}-\mathcal{A})^{-1}\mathcal{B} & \dots & (\lambda_k\mathcal{E}-\mathcal{A})^{-1}\mathcal{B} \end{bmatrix} M \\ Z^T &= N \begin{bmatrix} \mathcal{C}(\lambda_{k+1}\mathcal{E}-\mathcal{A})^{-1} \\ \vdots \\ \mathcal{C}(\lambda_{2k}\mathcal{E}-\mathcal{A})^{-1} \end{bmatrix}, \end{aligned}$$

where the matrices $M, N \in \mathbb{C}^{k\times k}$ are invertible and are chosen for computational reasons (M and N can be any $k\times k$ invertible matrix).

2. Let V_1 and $V_2 \in \mathbb{C}^{n\times k}$ be the first n rows and the last n rows of V respectively. Let Z_1 and $Z_2 \in \mathbb{C}^{n\times k}$ be the first n rows and the last n rows of Z respectively. Construct

$$\mathcal{V} \doteq \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix} \quad , \quad \mathcal{Z} \doteq \begin{bmatrix} Z_1 & \\ & Z_2 \end{bmatrix}.$$

3. Construct the matrices

$$\hat{\mathcal{C}} \doteq \mathcal{C}\mathcal{V} \quad , \quad \hat{\mathcal{A}} \doteq \mathcal{Z}^T\mathcal{A}\mathcal{V} \quad , \quad \hat{\mathcal{E}} \doteq \mathcal{Z}^T\mathcal{E}\mathcal{V} \quad , \quad \hat{\mathcal{B}} \doteq \mathcal{Z}^T\mathcal{B}.$$

4. Define the reduced transfer function

$$\hat{H}(s) \doteq \hat{\mathcal{C}}(s\hat{\mathcal{E}} - \hat{\mathcal{A}})^{-1}\hat{\mathcal{B}}.$$

From Theorem 4.1, $\hat{H}(s)$ is a second-order transfer function of order k (i.e. of Mc Millan degree $2k$) that satisfies the interpolation conditions (20).

4.2 A comparison with a previous Krylov technique for second-order systems

To simplify the notation, the mass matrix M is assume to be equal to the identity matrix. $T(s)$ can be rewritten as follows :

$$T(s) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} sI & -I \\ K & sI+D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B \end{bmatrix} \doteq \mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}. \tag{21}$$

By defining $\mathcal{A}^k\mathcal{B} \doteq \begin{bmatrix} Q_{p,k} \\ Q_{v,k} \end{bmatrix}$, we can rewrite the right Krylov subspaces corresponding to interpolation at infinity as follows

$$\begin{bmatrix} \mathcal{B} & \mathcal{A}\mathcal{B} & \mathcal{A}^2\mathcal{B} & \dots \end{bmatrix} = \begin{bmatrix} 0 & B & -CB & \dots \\ B & -CB & -KB + C^2B & \dots \end{bmatrix} \quad (22)$$

$$= \begin{bmatrix} 0 & Q_{v,0} & Q_{v,1} & \dots \\ Q_{v,0} & Q_{v,1} & Q_{v,2} & \dots \end{bmatrix}. \quad (23)$$

This implies that if the image of $\begin{bmatrix} Q_{v,0} & \dots & Q_{v,k} \end{bmatrix}$ belongs to the image of V, then the image of $\begin{bmatrix} \mathcal{B}, \dots, \mathcal{A}^k\mathcal{B} \end{bmatrix}$ belongs to $\begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix}$.

In other words, if we write

$$\mathcal{K}_k(\mathcal{A}, \mathcal{B}) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix},$$

then

$$Im(V_1) \in Im(V_2).$$

So by projecting with

$$\mathcal{V} \doteq \begin{bmatrix} V_2 & 0 \\ 0 & V_2 \end{bmatrix}$$

and $\mathcal{Z} \doteq \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$ such that $Z^TV_2 = I$, we get an interpolating second-order transfer function of the form

$$\hat{T}(s) = CV_2 \left(Z^T(s^2M + sD + K)^{-1}V_2 \right) Z^TB. \quad (24)$$

Hence if we want to construct a second-order system such that its k first Taylor coefficients at infinity are the same as the ones of the original second-order, we just have to project with V containing the image of $Q_{v,0}, \dots, Q_{v,k-1}$ and Z such that $Z^TV = I$. All that we have discussed in this subsection is explained in [11].

Because they wanted to construct the reduced system by only projecting as in (24), they did not realize how to tackle the case for interpolation at finite points in the complex plane as we did in Theorem 4.1.

4.3 Second-Order Structure Preserving Tangential Interpolation

Let us briefly show how to generalize this result in the MIMO to tangential interpolation by replacing the Krylov subspaces into *generalized* Krylov subspaces as done in [6] and [7]. One might want to construct a second-order transfer function $\hat{T}(s)$ of order k that satisfies the following interpolation conditions with respect to the second-order transfer function $T(s)$ of order n:

$$x_i \left(T(s) - \hat{T}(s) \right) = O(\lambda_i - s) \quad , \quad \left(T(s) - \hat{T}(s) \right) x_{i+k} = O(\lambda_{i+k} - s), \quad (25)$$

where $x_1, \dots, x_k \in \mathbb{C}^{1\times p}$ and $x_{k+1}, \dots, x_{2k} \in \mathbb{C}^{m\times 1}$. This can be done by computing *generalized* Krylov subspaces as follows :

Algorithm 4.2 *1. Construct Z and V such that*

$$\begin{aligned} V &= \left[\begin{array}{ccc} (\lambda_{k+1}\mathcal{E}-\mathcal{A})^{-1}\mathcal{B}x_{k+1} & \ldots & (\lambda_{2k}\mathcal{E}-\mathcal{A})^{-1}\mathcal{B}x_{2k} \end{array} \right] M \\ Z^T &= N \left[\begin{array}{c} x_1\mathcal{C}(\lambda_1\mathcal{E}-\mathcal{A})^{-1} \\ \vdots \\ x_k\mathcal{C}(\lambda_k\mathcal{E}-\mathcal{A})^{-1} \end{array} \right], \end{aligned}$$

where the matrices $M, N \in \mathbb{C}^{k\times k}$ are invertible and are chosen for computational reasons (M and N can be any $k\times k$ invertible matrix).

2. Let V_1 and $V_2 \in \mathbb{C}^{n\times k}$ be the first n rows and the last n rows of V respectively. Let Z_1 and $Z_2 \in \mathbb{C}^{n\times k}$ be the first n rows and the last n rows of Z respectively. Construct

$$\mathcal{V} \doteq \left[\begin{array}{cc} V_1 & \\ & V_2 \end{array} \right] \quad , \quad \mathcal{Z} \doteq \left[\begin{array}{cc} Z_1 & \\ & Z_2 \end{array} \right].$$

3. Construct the matrices

$$\hat{\mathcal{C}} \doteq \mathcal{C}\mathcal{V} \quad , \quad \hat{\mathcal{A}} \doteq \mathcal{Z}^T\mathcal{A}\mathcal{V} \quad , \quad \hat{\mathcal{E}} \doteq \mathcal{Z}^T\mathcal{E}\mathcal{V} \quad , \quad \hat{\mathcal{B}} \doteq \mathcal{Z}^T\mathcal{B}.$$

4. Define the reduced transfer function

$$\hat{T}(s) \doteq \hat{\mathcal{C}}(s\hat{\mathcal{E}}-\hat{\mathcal{A}})^{-1}\hat{\mathcal{B}}.$$

It can be shown that $\hat{T}(s)$ is a second-order transfer function of order k (i.e. of Mc Millan degree $2k$) that satisfies the interpolation conditions (25).

5 Concluding Remarks

As we have shown in this paper, it is possible to use a Krylov technique while preserving the second-order structure, but there is a price to pay. Generically, imposing $2k$ interpolation conditions and the second order structure results in a reduced transfer function of order $2k$ instead of k if the second order structure was not imposed. It is possible to extend such a technique to systems of order larger than 2. Further work would be to consider Krylov techniques for periodic systems and more generally Krylov techniques for interconnected systems. The idea is always the same: First look at the linearized state space system and look at the form of the projecting matrices needed to keep the model structure. Then, compute the Krylov subspaces corresponding to the interpolation conditions. Finally, construct projecting matrices with appropriate form and such that their images contain the Krylov subspaces. These ideas will be further pursued in another paper.

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