INT. J. CONTROL, 1979, VOL. 30, NO. 2, 235–243

Properties of the system matrix of a generalized state-space system†

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For an irreducible polynomial system matrix $P(s)=\begin{bmatrix} T(s) & -U(s) \\ V(s) & W(s) \end{bmatrix}$, Rosenbrock (1970, p. 111) has shown that the *polar* structure of the associated transfer function $R(s)=V(s)T^{-1}(s)U(s)$ at any finite frequency is isomorphic to the zero structure of $T(s)$ at that frequency, while the *zero* structure of $R(s)$ at any finite frequency is isomorphic to that of $P(s)$ at the same frequency. In this paper we obtain the appropriate extensions for the structure *at infinite frequencies* in the particular case of systems for which $T(s)=sE-A$ (with E possibly singular), $U(s)=B$, $V(s)=C$, and $W(s)=D$, under a strengthened irreducibility condition. We term such systems *generalized state-space systems*, and note that *any* rational $R(s)$ may be realized in this form. We also demonstrate in this case that a minimal basis (in the sense of Forney (1975) for the left or right null space of $P(s)$ directly generates one with the same minimal indices for the corresponding null space of $R(s)$, and vice versa. These results also enable us to identify the pole-zero excess of $R(s)$ as being equal to the sum of the minimal indices of its null spaces. Connections with Kronecker's theory of matrix pencils are made.

1. Introduction

Given the rational matrix

$$R(s)=V(s)T^{-1}(s)U(s)+W(s) \tag{1}$$

where T. U, V, W are polynomial matrices that constitute a realization of $R(s)$, it is of interest, as the pioneering work of Rosenbrock (1970) has shown, to study the properties of the associated system matrix

$$P(s)=\begin{bmatrix} T(s) & -U(s) \\ V(s) & W(s) \end{bmatrix} \tag{2}$$

and to relate them to properties of $R(s)$.

It is by now well-known (see Rosenbrock (1970, p. 111)) that if $P(s)$ is *irreducible*, i.e. $[T(s) \quad -U(s)]$ and $[T'(s) \quad V'(s)]'$ have full row and column rank respectively for all finite s, then, with the classical Smith–McMillan

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Received 16 October 1978.

† This work was supported by the Air Force Office of Scientific Research, Air Force Systems Command, under Contract AF44–620–74–C–0068, and in part by the Joint Services Electronics Program under Contract N00014–75–C–0601. Work of the second author was also supported by the National Science Foundation under Contract ENG–75–03423.

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definition of pole-zero structure of a rational matrix (see, for example, Rosenbrock (1970) or McMilland (1952)) :

(i) the *polar* structure of $R(s)$ at its finite poles is isomorphic to the zero structure of the ' denominator ' matrix $T(s)$ at its finite zeros ;

(ii) the *zero* structure of $R(s)$ at its finite zeros is isomorphic to the zero structure of the system matrix $P(s)$ at its finite zeros. (3)

Here we shall develop certain important extensions of these results for an interesting class of systems that we shall call *generalized state-space systems*, namely with

$$T(s)=sE-A, \quad U(s)=B, \quad V(s)=C, \quad W(s)=D \tag{4}$$

where E is permitted to be singular. In the special case where E is non-singular, and assumed without loss of generality to be the identity, we have the familiar regular state-space systems with proper transfer functions, i.e. with $R(\infty)$ finite. Several aspects of generalized state-space systems are explored in Verghese (1978) and Verghese *et al.* (1980). In particular, it is demonstrated there (see Rosenbrock (1974) also) and we shall show again here that an *arbitrary* rational matrix may be realized in this form.

For generalized state-space systems, (under a condition that we shall term strong irreducibility), we shall prove the following extensions of (3) :

(i) the polar structure of $R(s)$ at its *infinite* poles is isomorphic to the zero structure of the denominator $sE-A$ at its *infinite* zeros (Theorem 1 i) ;

(ii) the zero structure of $R(s)$ at its *infinite* zeros is isomorphic to the zero structure of the system matrix $P(s)$ at its *infinite* zeros (Theorem 1 ii) ;

(iii) the left and right *null-space* structures of $R(s)$ are directly related, in a sense that will become clear shortly, to the corresponding null-space structures of $P(s)$ (Theorem 2). (5)

The results will also be used to prove (Theorem 3) an important relation expressing the difference between the total number of poles and zeros of an arbitrary rational matrix in terms of certain indices associated with its right and left null spaces.

The poles and zeros at infinity of a rational matrix $R(s)$ are of interest in several problems of system theory, such as system inversion, asymptotic root-locus determination, singular estimation and control, and also in circuit theory. The null spaces of rational matrices likewise arise in several contexts ; some applications may be found, for example, in the paper of Forney (1975), where the null spaces of polynomial matrices in particular are discussed. It is therefore important to relate these elements of $R(s)$ to the structure of associated system matrices. Perhaps as important is the fact that for generalized state-space systems both the denominator $T(s)=sE-A$ and the system matrix $P(s)=sG-H$, where

$$G=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad H=\begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \tag{6}$$

are in the form of matrix *pencils*, whose zero structure and null-space structures have been well understood since the work of Kronecker, as reviewed by Gantmacher (1960, Chap. 12) in considerable detail. Furthermore, several algorithms for determining the structure of matrix pencils have been and continue to be proposed; see, for example, Van Dooren (1977).

2. Background

2.1. *Pole-zero structure at infinity*

The classical definition of the pole-zero structure at infinity of a rational matrix $Q(s)$ simply gives it as the pole-zero structure at $s=0$ of the matrix $Q(s^{-1})$ (or uses some other first-order conformal mapping of the complex plane to bring the point at infinity to a finite point); see McMillan (1952) and Rosenbrock (1970) for example.

The following lemma on the infinite zeros of a general pencil $sK-L$ is important for the sequel:

Lemma 1

Use a constant non-singular transformation on the left of $sK-L$ to bring it to the form

$$s\begin{bmatrix} K_1 \\ 0 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \tag{7}$$

where K_1 has full row rank. (This operation preserves the pole-zero structure of $sK-L$.) Then the zero structure of $sK-L$ at infinity is isomorphic to the zero structure of

$$\begin{bmatrix} K_1 - L_1 s \\ -L_2 \end{bmatrix} \tag{8}$$

at $s=0$.

Proof

The zero structure of the matrix in (7) at $s=\infty$ is isomorphic to that of

$$\begin{bmatrix} s^{-1}K_1 - L_1 \\ -L_2 \end{bmatrix} \tag{9}$$

at $s=0$. An irreducible realization of the latter matrix is given by

$$P_1(s) = \left[\begin{array}{c|c} sI & K_1 - L_1 s \\ \hline -I & 0 \\ 0 & -L_2 \end{array}\right] \tag{10}$$

and by (3 ii) $P_1(s)$ has the same finite zero structure as (9). Now $P_1(s)$ is easily seen to have the same finite zeros as (8), which completes the proof. ■

Corollary 1

$sK-L$ has no zeros at infinity if and only if

$$\operatorname{rank}\begin{bmatrix} K_1 \\ -L_2 \end{bmatrix} = \text{normal rank } (sK-L) \tag{11}$$

Proof

The normal rank of $sK-L$ equals the normal rank of (8), and the latter matrix loses rank at $s=0$ if and only if $sK-L$ has an infinite zero. ■

Other proofs of the above lemma and its corollary are possible, see Verghese (1978).

2.2. *Minimal bases for rational vector spaces*

We briefly summarize certain key ideas from the paper of Forney (1975). A polynomial basis may be constructed for any finite-dimensional space of rational vectors. The degree of a polynomial vector is defined to be the highest degree among its entries. The columns of a polynomial matrix $M(s)$ are said to form a *minimal basis* for the space spanned by them if

(i) $M(s)$ has full column rank for all finite s, and

(ii) $M(s)$ is ' column-reduced ', i.e. denoting the degree of the ith column of $M(s)$ by k_i, and constructing the constant matrix M_h whose ith column is the (vector) coefficient of s^{k_i} in the ith column of $M(s)$, we have M_h to be of full column rank. (12)

M_h in (12 ii) is termed the *highest-order column coefficient* matrix.

The set of degrees of the vectors in any minimal basis for a given space is known to be invariant, and these degrees are termed the *minimal indices* of the space.

The following lemma will be useful in our discussion of bases for the null-spaces of given matrices.

Lemma 2

Let $[X_1 \;\; X_2]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}=0$ and let X_1, $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ have full column rank, then Y_2 must also have full column rank.

Proof

If there exists some $v\neq 0$ such that $Y_2v=0$, we must have $Y_1v\neq 0$, but then $X_1(Y_1v)=0$, which is impossible. ■

3. Main results

A generalized state-space system, of the form (4), is termed *strongly irreducible* if $[sE-A \;\; -B]$ and $[sE'-A' \; C']'$ have full row and column rank respectively for *all* s, including $s=\infty$, i.e. if they have *no zeros*, finite or infinite. While irreducibility at finite s is a familiar notion, irreducibility at infinity is more recent; see Verghese (1978), Verghese *et al.* (1980) for more detailed discussion of its dynamical significance. For now we are interested only in its algebraic aspects.

Theorem 1

For a strongly irreducible generalized state-space system (1), (2), (4),

(i) the polar structure of $R(s)$ at $s=\infty$ is isomorphic to the zero structure of $(sE-A)$ at $s=\infty$;

(ii) the zero structure of $R(s)$ at $s=\infty$ is isomorphic to the zero structure of

$$P_g(s) \triangleq \begin{bmatrix} sE-A & -B \\ C & D \end{bmatrix} \tag{13}$$

at $s=\infty$.

Proof

A preliminary constant non-singular transformation of the first (block) row of (13) brings it to the form

$$\left[\begin{array}{c|c} sE_1-A_1 & -B_1 \\ -A_2 & -B_2 \\ \hline C & D \end{array}\right] \tag{14}$$

with E_1 of full row rank.

The pole-zero structures of $sE-A$, $[sE-A \quad -B]$, $[sE'-A' \quad C']'$, and $P_g(s)$ at all frequencies are preserved by this transformation. The strong irreducibility of $P_g(s)$ implies, by Corollary 1 that

$$\begin{bmatrix} E_1 & 0 \\ -A_2 & -B_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_1 \\ -A_2 \\ C \end{bmatrix} \tag{15}$$

have full row and column rank respectively.

Now note that since $R(s)=C(sE-A)^{-1}B+D$, an irreducible realization of $R(1/s)$ is given by

$$P_R(s)=\left[\begin{array}{c|c} E_1-sA_1 & -sB_1 \\ -A_2 & -B_2 \\ \hline C & D \end{array}\right] \tag{16}$$

so that by (3 ii) we have

(i) the polar structure of $R(1/s)$ at $s=0$ is isomorphic to the zero structure of $\begin{bmatrix} E_1 - sA_1 \\ -A_2 \end{bmatrix}$ at $s=0$, and

(ii) the zero structure of $R(1/s)$ at $s=0$ is isomorphic to the zero structure of $P_R(s)$ at $s=0$.

The theorem follows immediately on applying Lemma 1 to (i) and (ii) above. ■

Theorem 2

(i) Given a strongly irreducible generalized state-space realization of a transfer function $R(s)$, let $N(s)$ be a minimal basis for the right null space of its system matrix $P_g(s)$, so that

$$\begin{bmatrix} sE-A & -B \\ C & D \end{bmatrix}\begin{bmatrix} N_1(s) \\ N_2(s) \end{bmatrix} = 0 \tag{17}$$

Then $N_2(s)$ is a minimal basis for the right null space of $R(s)$, and has the same minimal indices as $N(s)$.

(ii) Conversely, let $N_2(s)$ be a minimal basis for the right null space of $R(s)$. Then with

$$N_1(s) = (sE-A)^{-1}BN_2(s) \tag{18}$$

we have that $N(s) = [N'_1(s) \;\; N'_2(s)]'$ is a minimal basis for the right null space of $P_g(s)$, and has the same minimal indices as $N_2(s)$.

Proof

(i) By non-singular row operations on (17) we obtain

$$\begin{bmatrix} sE-A & -B \\ 0 & R(s) \end{bmatrix}\begin{bmatrix} N_1(s) \\ N_2(s) \end{bmatrix} = 0 \tag{19}$$

which shows firstly that the right null space of $R(s)$ has the same dimension as that of $P_g(s)$, and secondly that $N_2(s)$ lies in this space. To show that $N_2(s)$ is indeed a *minimal* basis for this space, apply Lemma 2 to (17) and use the fact that $[sE' - A' \;\; C']'$ and $N(s)$ have full column rank for all finite s to prove that $N_2(s)$ also has full column rank for all finite s, thus satisfying condition (12 i). For condition (12 ii), first bring $P_g(s)$ to the form (14). Denoting the highest column coefficient matrix of $N(s)$ by N_h (where N_h has full column rank since $N(s)$ is a minimal basis), we see from (17) that

$$\left[\begin{array}{c|c} E_1 & 0 \\ -A_2 & -B_2 \\ \hline C & D \end{array}\right]\begin{bmatrix} N_{h_1} \\ \hline N_{h_2} \end{bmatrix} = 0 \tag{20}$$

Using Lemma 2 again, and the strong irreducibility of $P_g(s)$, cf. (15), we have that N_{h_2} has full column rank. Thus N_{h_2} must be the highest column coefficient matrix of $N_2(s)$ and $N_2(s)$ must be column-reduced, with the same column degrees as $N(s)$.

(ii) For the converse, we first show that $N_1(s)$ in (18) must be polynomial. It is well known, see Rosenbrock (1970, p. 71) for example, that since $[sE'-A' \;\; C']'$ has no finite zeros, it has a polynomial left inverse. Premultiplying (17) by this left inverse shows that $N_1(s)$ is polynomial. Since $N_2(s)$ has full column rank for all s, so has $N(s)$, which is condition (12 i). With N_h defined as before, we again obtain (20), and the strong irreducibility of $P_g(s)$ shows that no column of N_{h_2} can be zero. It follows that N_{h_2} must be the highest column coefficient matrix of $N_2(s)$. Since $N_2(s)$ is column-reduced, N_{h_2} has full column rank, hence N_h has full rank too, which is condition (12 ii). Also $N(s)$ must have the same column degrees as $N_2(s)$. ■

A dual theorem evidently holds for the left null spaces of $P_g(s)$ and $R(s)$.

The following theorem†, whose proof we merely outline for lack of space, demonstrates an important consequence of the preceding two theorems.

Theorem 3

Let $\delta_p(R)$ and $\delta_z(R)$ denote the total number of poles and zeros (finite *and* infinite) respectively of an arbitrary rational matrix $R(s)$, and let $\alpha(R)$ denote the sum of the minimal indices of the left and right null spaces of $R(s)$. Then

$$\delta_p(R)=\delta_z(R)+\alpha(R) \tag{21}$$

Proof

Let $P_g(s)$ be as before. Since it is in the form of a matrix pencil, its poles, zeros and minimal indices are the same as those of its Kronecker canonical form (see Gantmacher (1960)). From the Kronecker form, denoted by $K(s)$, it is easy to show that

$$\delta_p(K)=\delta_z(K)+\alpha(K)$$

whence

$$\delta_p(P_g)=\delta_z(P_g)+\alpha(P_g) \tag{22}$$

Now

$$\begin{aligned}\delta_p(P_g)&=\operatorname{rank} E=\delta_p(sE-A)\\&=\delta_z(sE-A)\end{aligned}$$

where the last equality follows from the fact that a non-singular matrix has many poles as zeros. From Theorem 1 we have

$$\delta_z(sE-A)=\delta_p(R) \tag{23 a}$$

and

$$\delta_z(P_g)=\delta_z(R) \tag{23 b}$$

† First obtained, in a slightly different way, by Van Dooren, in earlier unpublished research.

while from Theorem 2 and its dual

$$\alpha(P_g) = \alpha(R) \tag{23 c}$$

Substituting (23) in (22) proves the result. ■

There are other routes to Theorem 3, see, for example, Verghese (1978) and Kung and Kailath (1979).

We conclude this section with a cautionary remark to the reader pursuing the connections of this paper with the Kronecker pencil theory: a 'kth order infinite elementary divisor' of a pencil, in the terminology of that theory, corresponds to a $(k-1)$th order zero at infinity in our sense.

4. Concluding remarks

For the special case of irreducible regular state-space realizations of proper transfer functions the paper of Thorp (1973) contains the seeds of our Theorem 1 (ii) and Theorem 2 (i). We know of no other work in this vein.

For irreducible regular state-space systems realizing arbitrary rational transfer functions, i.e. with $T(s) = sI - A$, $U(s) = B$, $V(s) = C$ and $W(s) = D(s)$, a *polynomial* matrix, it may be shown quite straightforwardly that the results of our Theorems 1 (ii) and 2 continue to hold (Instead of Theorem 1 (i), we have that $R(s)$ and $D(s)$ have the same polar structures at infinity.)

Extension of our results to arbitrary polynomial systems of the form (2) appear to be considerably harder. It is possible to extend Theorem 2 to such systems, under more complicated conditions whose dynamical interpretation and significance is not yet clear. Appropriate extensions of Theorem 1 are not known.

In conclusion, we note that it is easy in principle to obtain a strongly irreducible generalized state-space realization of an arbitrary $R(s)$ by using standard procedures for regular state-space realization of proper transfer functions. Thus let

$$R(s) = \bar{R}(s) + D(s) \tag{24}$$

where

$$\bar{R}(\infty) = 0 \quad \text{(i.e. } \bar{R}(s) \text{ is 'strictly proper')}$$

and

$$D(s) \text{ is a polynomial matrix}$$

We can then realize $\bar{R}(s)$ and $s^{-1}D(s^{-1})$ 'minimally' (i.e. such that the corresponding system matrices are irreducible) in the form

$$\left.\begin{aligned} \bar{R}(s) &= C(sI - A)^{-1}B \\ s^{-1}D(s^{-1}) &= \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} \end{aligned}\right\} \tag{25}$$

since both are strictly proper. Then a realization of $R(s)$ in the desired form is

$$\left[\begin{array}{cc|c} sI - A & & -B \\ & I - \tilde{A}s & -\tilde{B} \\ \hline C & \tilde{C} & 0 \end{array}\right] \tag{26}$$

(We refer the reader to Verghese *et al.* (1980) for further details.) A consequence of our results is that the pencil (26) may be used to obtain structural information on $R(s)$. The process of obtaining a pencil that contains this information on $R(s)$ may be termed *linearization* of $R(s)$.

Acknowledgments

The authors would like to thank Professor S. Kung of University of Southern California for several useful discussions.

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