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On Harsanyi dividends and asymmetric values

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Abstract

The concept of dividend in transferable utility games was introduced by Harsanyi [1959]. It offers a unifying framework for studying various valuation concepts, from the Shapley value (symmetric as well as weighted) to the different notions of values introduced by Weber [1988]. Using the decomposition of the characteristic function used by Shapley [1953] to prove uniqueness of his value, the idea of Harsanyi was to associate to each coalition a dividend to be distributed among its members to define an allocation. Many authors have contributed to that question. Here, we offer a synthesis of their work, with a particular attention to restrictions on dividend distributions, starting with the seminal contributions of Vasil'ev [1978], Hammer, Peled and Sorensen [1977] and Derks, Haller and Peters [2000], until the recent paper of van den Brink, van der Laan and Vasil'ev [2014].

Keywords: Harsanyi dividends, Weber set, weighted Shapley values, core

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1. Introduction

We begin with a chronological enumeration of the concepts that will be formally defined and interrelated in the present paper.

The notion of game with transferable utility, defined by a set of players and a characteristic function, was introduced by von Neumann and Morgenstern in 1944. The characteristic function associates to each subset (coalition) of players a real number measuring what that coalition can do at best in terms of some commodity-money.

The concept of *value* of a transferable utility game was introduced by Shapley in 1953. His initial idea was to define what a player may reasonably expect from playing a game. However, by requiring players' evaluations to be consistent in order to achieve efficiency (an exact distribution of the social output) and symmetry (equal treatment of equals), the Shapley value is more of a normative tool.

The concept of *dividend* was introduced by Harsanyi in 1959. His idea is to associate to each coalition a dividend (positive or negative) that can be distributed among its members to define an allocation of the social surplus. The dividends are identified with the coefficients of the decomposition of the characteristic function used by Shapley to prove uniqueness of his value. The set of allocations that results from all possible distributions of the dividends is an object that has been studied in the 70's independently by Vasil'ev in papers published in Russian, and by Hammer, Peled and Sorensen in a paper published in a Belgian operations research journal. While the latter used the name "*selectope*", here we shall retain the term "*Harsanyi set*". Derks, Haller and Peters popularized that concept in a paper published in *International Journal of Game Theory* in 2000. At that time, they were not aware of the contributions of Vasil'ev. These became known with the publication in 2002 by Vasil'ev and van der Laan of a paper containing all the results known by that time. Since then, a number of papers have been published, in particular by Derks, van der Laan and Vasil'ev [2006, 2010].

In 1971, Shapley has characterized geometrically the core of a convex game using the concept of marginal contribution vector that associates allocations to players' orderings: for a given ordering, each player receives his marginal contribution, following the ordering. He shows that the core of a convex game is the non-empty and bounded polyhedral convex set whose vertices are precisely these marginal contribution vectors.

In 1988, Weber has introduced the notion of *probabilistic values* that allocates to each player his expected marginal contribution computed with respect to a probability distribution independent of the game's data. *Quasi-values* are probabilistic values obtained by considering probability distributions ensuring efficiency and the Shapley value is the unique efficient and symmetric quasi-value. Weber also defines the concept of *random order value* as the expected marginal contribution vector, computed with respect to a given probability distribution over players' orderings. He shows that random order values are quasi-value, the Shapley value being the random order value corresponding to the uniform probability distribution. Moreover, he shows that the core is a subset of the set of random order values, equivalently defined as the convex hull of the marginal contribution vectors. This set is known as the *Weber set* and, following Shapley's characterization of the core, the two sets coincide on the class of convex games, and only for these games as shown by Ichiishi [1981].

In his doctoral thesis, Shapley [1953a] did also consider the possibility for symmetric players to be treated differently. The *asymmetric* version of the value is obtained by introducing *exogenous* weights in order to cover asymmetries that are not included in the underlying game. The weighted Shapley value has been axiomatized later, in particular by Kalai and Samet [1987] without explicit reference to weights, by Hart and Mas-Colell [1989] using a generalized potential function and by Dehez [2011] in a cost sharing context along the lines suggested by Shapley [1981]. The value associated to positive weights is obtained as a weighted division of dividends. As a consequence, weighted values are Harsanyi payoffs. Alternatively, it can be defined as the expected marginal contribution vector corresponding to a probability distribution over players' orderings derived from weights. The *Shapley set* is the set of all weighted values obtained by considering all possible weight systems, including zero weights. Monderer, Samet and Shapley [1992] show the Shapley set contains the core, an inclusion "*somewhat surprising in light of the difference in concept behind these solutions*" to quote the authors. Weighted values being random order values, the Shapley set is contained in the Weber set and the three solution sets coincide when applied to convex games.

The Harsanyi set turns out to be the largest solution set. It includes the Weber set. Hammer et al. [1977] and Vasil'ev [1981] show that the Harsanyi set and the core coincide if and only if they apply to almost positive games – games whose dividends of multiplayer coalitions are non-negative. Consequently, positive games being convex, the four solution sets – core, Shapley, Weber and Harsanyi sets – coincide when applied to almost positive games.

All these solutions have been characterized axiomatically. They can also be characterized starting with the Harsanyi set and imposing restrictions on dividend distributions. A natural restriction is *monotonicity*. It requires that the share of a player in the dividend of a coalition does not increase if the coalition is enlarged. Even if such restrictions reduce considerably the set of possible dividend distributions, it is not sufficient to generate a particular solution set. Billot and Thisse [2005] claim that the set of Harsanyi payoff vectors resulting from monotonic dividend distributions coincides with the core if (and only if) the game is convex. We show that this is actually true only for 3-player games! Assuming a monotonic dividend distribution, individual rationality obtains in 3-player superadditive games and in 4-player convex games. Beyond four players, we show that there is no hope. Under a stronger monotonicity condition, Vasil'ev [1988] shows that Harsanyi payoff vectors are random order values. Under the assumption that the distributions of dividends within coalitions are compatible with Bayesian updating, Derks et al. [2000] show that the resulting Harsanyi subset coincides with the Shapley set.

The paper is organized as follows. Section 2 introduces transferable utility games. Solution concepts are then defined and interrelated in Section 3, with a particular attention to weighted Shapley values and the case where some players are assigned a zero weight. In Section 4, we review the axiomatic characterizations of the Shapley, Weber and Harsanyi sets. We then look at the characterization of the Weber and Shapley sets by way of restrictions on dividend distributions. We finally consider the subsets of Harsanyi payoffs that result from graph structures on the set of players. The last section offers concluding remarks and an appendix gathers intermediary results.

2. Transferable Utility Games

2.1 *Characteristic functions*

Cooperative games cover situations in which a group of individuals cooperate on a common project with the objective of maximizing the resulting collective gain. It is assumed that *utility is transferable* through some commodity-money, allowing for transfers (side-payments) between players. A cooperative game with transferable utility is defined by a player set N and a *characteristic function* v that associates to each *coalition* $S \subset N$ a real number $v(S)$ that represents its (potential) worth defined as the gain that it can realize without the participation of the others. In particular, $v(i)$ is what player i could obtain alone and the value of the game $v(N)$ is the maximum amount that the "grand coalition" is able to generate. By convention, we set $v(\varnothing) = 0$.

Notation: Set inclusion is denoted by $\subset$ and *strict* inclusion by $\subsetneq$. For sa given subset S, $S \setminus i$ denotes the subset obtained by *subtracting* i from S. Upper-case letters are used to denote sets and the corresponding lower-case letters to denote their sizes: $s = |S|, t = |T|, \ldots$ Coalitions $\{i,j,k,\ldots\}$ are sometimes written as $ijk\ldots$ For any given set T, we denote by $\mathcal{C}(T) = \{S \subset T \mid S \neq \varnothing\}$ the collection of non-empty subsets of T and by $\mathcal{C}_i(T) = \{S \subset T \mid i \in S\}$ the collection of subsets of T containing player i. Given a vector $x \in \mathbb{R}^n$ and a subset $S \subset N$, it will be convenient to write $x(S) = \sum_{i \in S} x_i$, $x_S = (x_i \mid i \in S)$ and $x = (x_S, x_{N \setminus S})$ with the convention $x(\varnothing) = 0$. Vectors are compared following the sequence $x \geq y$, $x > y$ and $x >> y$. In some instances, the summation sign Σ will be used without reference to a set when there is no ambiguity. For a given finite set A, we denote by $\Delta(A)$ the set of all probability distributions on A.

Two games (N, v') and (N, v'') on a common player set are said to be *strategically equivalent* if there exists $a > 0$ and $b \in \mathbb{R}^n$ such that $v''(S) = a\, v'(S) + b(S)$. This defines an *equivalence relation*. In particular, a game (N, v) and its 0-normalization (N, v_0) defined by

$$v_0(S) = v(S) - \sum_{i \in S} v(i)$$

are strategically equivalent. A game can be restricted to a subset of its player set. Formally, given a game (N,v) and a subset $R \subset N,$ the restriction to R is the game (R,v_R) defined by $v_R(S)=v(S)$ for all $S \subset R.$ Given two games (N,v') and (N,v'') on a common set of players, the game *sum* (N,v) is simply defined by $v(S)=v(S')+v(S'')$ for all $S \subset N.$

2.2 *Superadditivity and monotonicity*

A game (N,v) is *superadditive* if getting together is beneficial, or at least harmless:

$$v(S)+v(T) \leq v(S \cup T) \text{ for all } S \text{ and } T \text{ such that } S \cap T = \varnothing.$$

A game (N,v) is *monotonic* if $S \subset T \Rightarrow v(S) \leq v(T).$ It implies that the largest surplus is generated by the grand coalition. There is no direct relation between superadditivity and monotonicity except for the following lemma.[1]

Lemma 1. Consider a game (N,v) such that $v(i) \geq 0$ for all $i \in N.$ Superadditivity then implies that the characteristic function v is monotonic and positive valued.

A game (N,v) is *0-monotonic* if its 0-normalization (N,v_0) is a monotonic game. As a consequence of Lemma 1, the 0-normalization of a superadditive game is a monotonic and positive-valued game and thereby, superadditive games are 0-monotonic.

0-monotonicity implies that the inequalities $\sum_{i \in S} v(i) \leq v(S)$ hold for all $S \subset N.$ A coalition S is said to be *essential* if the inequality is strict. If equality holds, S is said to be inessential. Obviously, for 0-monotonic games, if a coalition is inessential, so are all its subcoalitions. A game is said to be essential if the grand coalition is essential and a game whose coalitions are all inessential is an *additive* game.

Superadditivity is a natural assumption that is satisfied in most economic and social situations. It ensures that allocating *exactly* the value of a game among the players if efficient: no partition of players can form and generate a total gain larger than the value of the game. This is not ensured by 0-monotonicity for games with more than 3 players.

2.3 *Harsanyi dividends*

We denote by $G(N)$ the set of all set functions on the finite set N. It is a *vector space* that is formally equivalent to $\mathbb{R}^{2^n-1}$. Shapley [1953b] shows that the collection of *unanimity games* (N,u_T) defined for all $T \in \mathcal{C}(N)$ by $u_T(S) \in \{0,1\}$ and $u_T(S)=1$ if and only if $T \subset S$ forms a basis of the vector space $G(N)$: for any given set function v on N, there exists a unique 2^n-1 dimensional vector $\alpha(N,v)=(\alpha_T(N,v) \mid T \in \mathcal{C}(N))$ such that:

$$v(S)=\sum_{T \in \mathcal{C}(N)} \alpha_T(N,v)\, u_T(S) = \sum_{T \in \mathcal{C}(S)} \alpha_T(N,v). \tag{1}$$

[1] Proofs of lemmas are in Appendix.

Following Harsanyi [1959, 1963], α_T is interpreted as the *dividend* accruing to coalition *T*. By (1), $v(N)$ is the sum of the dividends of all coalitions. Hence, an allocation of $v(N)$ can be obtained by distributing the dividends of every coalition among its members.[2] Dividends can be defined recursively, starting with $\alpha_\varnothing = 0,$ as follows:

$$\alpha_T = v(T) - \sum_{S \subsetneq T} \alpha_S \quad \text{for all } T \subset N \tag{2}$$

i.e.

$$\alpha_{\{i\}} = v(i)$$

$$\alpha_{\{ij\}} = v(ij) - v(i) - v(j)$$

$$\alpha_{\{ijk\}} = v(ijk) - v(ij) - v(ik) - v(jk) + v(i) + v(j) + v(k), \ldots$$

Alternatively, the collection of α_T's is the *unique* solution of the linear system (1):

$$\alpha_T(N,v) = \sum_{S \in \mathcal{C}(T)} (-1)^{t-s} v(S) \quad (T \subset N, T \neq \varnothing). \tag{3}$$

Additivity of the dividends follows from (2):

$$\alpha_T(N, v' + v'') = \alpha_T(N, v') + \alpha_T(N, v'') \ \text{ for all } T \subset N.$$

Two games (N, v') and (N, v'') are *disjoint* if no dividend are simultaneously different from zero: $\alpha_T(N,v') \cdot \alpha_T(N,v'')$ for all $T \subset N.$

Remark 1. The dividends associated to the unanimity games (N, u_S) are given by:

$$\alpha_T(N, u_S) = 1 \quad \text{if } T = S,$$
$$= 0 \quad \text{otherwise.}$$

Remark 2. The dividends associated to an additive game are all zero except for single players. The dividends of 0-monotonic games associated to inessential *multi-player* coalitions are equal to zero. Furthermore, the dividends associated to the 0-normalization (N, v_0) of an arbitrary game (N, v) are unchanged except for singletons:

$$\alpha_{\{i\}}(N, v_0) = 0 \qquad \text{for all } i \in N,$$

$$\alpha_T(N, v_0) = \alpha_T(N, v) \qquad \text{for all } T \subset N, t \geq 2.$$

2.4 *Positive games*

Dividends can be negative or positive. A game is (totally) *positive* if its dividends are all non-negative.[3] The term *almost positive* is used for games whose dividends of *multi-player* coalitions are non-negative. Equivalently, a game is almost positive if its 0-normalization is

[2] To keep notation simple, the dependence of dividends on the game will sometimes be omitted.

[3] Totally positive games were introduced and systematically studied by Vasil'ev [1975, 1981].

positive. Obviously, 2-player 0-monotonic games are almost positive and unanimity games are positive.

Remark 3. Positive games are monotonic. This can easily be seen from (1).

2.5 *Marginal contributions*

The *marginal contribution* of player i to coalition S is defined by $v(S)-v(S\setminus i)$. It is the value added of player i to coalition S and it is obviously zero for all coalitions of which he is not a member. For 0-monotonic games, marginal contributions are bounded below by individual worth. Indeed, for all $S\subset N$ such that $i\in S$, we have:

$$v(S)-\sum_{j\in S}v(j)\geq v(S\setminus i)-\sum_{\substack{j\in S\\ j\neq i}}v(j)\ \Rightarrow v(S)-v(S\setminus i)\geq v(i). \tag{4}$$

Two players i and j are *symmetric* in a game (N,v) if they contribute equally to all coalitions to which they belong: $v(S)-v(S\setminus i)=v(S)-v(S\setminus j)$ for all $S\subset N$ such that $i,j\in S$. A player i is *null* in a game (N,v) if he never contributes: $v(S)-v(S\setminus i)=0$ for all $S\subset N$.

Player i is *necessary* for player j in a game (N,v) if the marginal contributions of j are zero in all coalition not containing i: $v(S)=v(S\setminus j)$ for all $S\not\ni i$. In particular, $v(j)=0$. Using (2), van den Brink et al. [2014] prove the following Lemma.

Lemma 2. If player i is necessary for player j in a game (N,v), then $\alpha_T(N,v)=0$ for all $T\subset N\setminus i$ such that $j\in T$.

Remark 4. A player is null *if and only if* the dividends associated to coalitions containing that player are all equal to zero. This is an immediate consequence of (2). See also Lemma 2.

Given a player set N, we denote by Π_N the set of all players' orderings. The *marginal contribution vector* $\mu^\pi(N,v)$ associated to the players' ordering $\pi=(i_1,...,i_n)\in\Pi_N$ is the vector of dimension n defined by:

$$\mu^\pi_{i_1}(N,v)=v(i_1)-v(\varnothing)=v(i_1)$$

$$\mu^\pi_{i_k}(N,v)=v(i_1,...,i_k)-v(i_1,...,i_{k-1})\quad (k=2,...,n)$$

i.e.

$$\mu^\pi_i(N,v)=v(\pi^i)-v(\pi^i\setminus i)\quad (i=1,...,n). \tag{5}$$

Here, π^i denotes the set of players preceding i in π, i included. There are n! marginal contribution vectors, not necessarily all distinct.

Looking at strategically equivalent games, if $v''(S)=a\,v'(S)+b(S)$ for some $a>0$ and $b\in\mathbb{R}^n$, the following identities prevail:

$$\mu^\pi_i(N,v'')=a\ \mu^\pi_i(N,v')+b_i\ \ (i=1,...,n) \tag{6}$$

2.6 *Convex games*

A game (N,v) is *convex* (or *supermodular*) if $v(S)+v(T)\le v(S\cup T)+v(S\cap T)$ for all S and $T\subset N$. Obviously, convexity implies superadditivity and a game that is strategically equivalent to a convex game is itself convex. It is easily verified that unanimity games are convex. As a consequence, positive games are convex as positive linear combinations of convex games, and almost positive games are convex as well by strategic equivalence.

Shapley [1971] shows that a game is convex *if and only if* players' marginal contributions do not decrease with coalition size:

$$i\in S\subset T \;\Rightarrow\; v(S)-v(S\setminus i)\le v(T)-v(T\setminus i).$$

Hence convexity means *increasing returns to size* and marginal contributions are *maximal* at the grand coalition *N*.

For any given player set *N*, the set of superadditive games, the set of monotonic games, the set of 0-monotonic games and the set of convex games are *convex cones*, denoted *SG*(*N*), *MG*(*N*), $MG_0(N)$ and *CG*(*N*) respectively. They are indeed closed under addition and positive scalar multiplication.[4] The set of positive games is a convex cone as well. It is denoted by $G^+(N)$. Following Remark 3, we have the following sequences of inclusions:

$$G^+(N)\subset CG(N)\subset SG(N)\subset MG_0(N) \text{ and } G^+(N)\subset MG(N).$$

An arbitrary game (N,v) can be decomposed in a difference between two positive (and thereby convex) games. Indeed (1) can be written as

$$v(S)=\sum_{T\in\mathcal{C}(N)}\alpha_T(N,v)\,u_T(S)=v^+(S)-v^-(S) \text{ for all } S\subset N$$

where

$$v^+(S)=\sum_{T:\alpha_T>0}\alpha_T(N,v)\,u_T(S) \text{ and } v^-(S)=\sum_{T:\alpha_T<0}-\alpha_T(N,v)\,u_T(S). \tag{7}$$

The dividends associated to these two games are given by:

$$\alpha_T(N,v^+)=Max\big(0,\alpha_T(N,v)\big),$$
$$\alpha_T(N,v^-)=-Min\big(0,\alpha_T(N,v)\big).$$

Convex games form an interesting class of games because solution concepts tend to agree when applied to convex games.[5] Moreover, many interesting economic situations can be modeled as convex games, like production games with increasing returns, bankruptcy games (Aumann and Maschler [1985]) and airport games (Littlechild and Owen [1973]). Positive games are convex and monotonic games with interesting properties and applications, like

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[4] A nonempty set *X* is a convex cone if and only if $x,y\in X$ and $\alpha\ge 0 \;\Rightarrow\; \alpha x+y\in X$.

[5] See Maschler et al. [1972].

river games (Ambec and Sprumont [2002]), queuing games (Maniquet [2003]) and liability games (Dehez and Ferey [2013]).

3. Values and Solution Sets

3.1 *Basic properties*

Given a game (N,v), the problem is to allocate $v(N)$ between the *n* players. Given a player set *N*, a *value* is a mapping that associates a payoff vector $\varphi(N,v)\in\mathbb{R}^n$ to any game (N,v).

A *solution set* is a mapping Φ that associates a *subset* $\Phi(N,v)$ of payoff vectors to any game (N,v). Basic properties that a solution set should ideally possess are the following:

- *Non-emptiness*: $\Phi(N,v)\neq\varnothing$ for all game (N,v).
- *Efficiency*: $x\in\Phi(N,v) \;\Rightarrow\; x(N)=v(N)$.
- *Individual rationality*: $x\in\Phi(N,v) \;\Rightarrow\; x_i\geq v(i)$ for all $i\in N$.
- *Covariance*: $x\in\Phi(N,v) \;\Rightarrow\; ax+b\in\Phi(N,a\,v+b)$ for all $a>0$ and $b\in\mathbb{R}^n$.
- *Convexity*: $\Phi(N,v)$ is a convex set.

Non-emptiness implies restrictions on the class of games on which the solution applies. As such, efficiency is an accounting identity. It does not necessarily imply full efficiency except for superadditive games. Indeed, there may exist a partition $(S_1,...,S_k)$ of the grand coalition such that $\sum v(S_h)>v(N)$. Individually rationality is a minimal requirement to be imposed on allocations: no player will ever accept to take part in a collective project if his remuneration falls short of what he could secure by himself. A solution is covariant if, once it has been applied to a game, it can be extended to all strategically equivalent games. Convexity is a natural requirement in a world where utility is transferable. Corresponding properties apply to values. Ideally, a value should be covariant and define an efficient and individually rational allocation.

3.2 *Imputations*

Imputations are efficient and individually rational allocations. This defines the *imputation set*:

$$I(N,v)=\{x\in\mathbb{R}^n \mid x(N)=v(N),\;\; x(i)\geq v(i) \text{ for all } i\in N\}.$$

The class of games (N,v) satisfying the inequality $\sum v(i)\leq v(N)$ is the largest class of games on which the imputation set is a well-defined solution. It includes 0-monotonic games. If the game is essential, $I(N,v)$ it is a regular simplex of dimension *n*–1. If instead the game is inessential, $I(N,v)$ is reduced to the singleton $(v(1),...,v(n))$. The imputation set is the largest solution set satisfying the above requirements.

Remark 5. For 0-monotonic games, marginal contributions as defined by (5) are imputations. Indeed, for a given players' ordering, adding the *n*! vectors results in $v(N)$ and, according to (4), marginal contributions are bounded below by individual worth's. Efficiency and individual rationality then follow.

3.3 *Stable allocations: the core*

The *core* is the set of imputations that no coalition can improve upon:

$$C(N,v) = \left\{ x \in \mathbb{R}^n \;\middle|\; x(N) = v(N),\;\; x(S) \geq v(S) \;\text{ for all } S \subset N \right\}. \tag{8}$$

The core extends rationality from individuals to coalitions: given an allocation, a coalition that receives less than what it could secure for itself is in a position to object. In this sense, core allocations are "stable".[6]

The core is a *polytope* i.e. a bounded polyhedral convex set. It is indeed bounded and results from the intersection of finitely many closed half spaces. It is a subset of the imputation set and it may be empty. The largest class of games on which the core is a well-defined solution is the class of *balanced games*.[7] Superadditivity is neither necessary nor sufficient for a game to have a non-empty core. However, for games with a non-empty core, no partition of the grand coalition can do better. Core allocations are therefore are fully efficient.

Remark 6. It can be easily verified that, if *i* and *j* are symmetric players, allocation obtained by exchanging x_i and x_j in a core allocation *x* are also core allocations. Hence, if non-empty, the core contains allocations that give to players *i* and *j* equal amounts. Furthermore, the core allocates zero to null players.

Shapley [1971] shows that convex games are balanced and that the core of a convex game is the polytope whose vertices are the marginal contribution vectors as defined by (5). Ichiishi [1981] shows that this is actually a *necessary and sufficient* condition for convexity.

3.4 *Quasi-values and random order values: the Weber set*

The concept of value was introduced by Shapley as a measure of what a player may expect from playing a game and the Shapley value belongs to the family of probabilistic values introduced later by Weber [1988]. Consider a collection $q = (q^S \mid S \subset N, S \neq \varnothing)$ of $2^n - 1$ non-negative vectors in $\mathbb{R}^n$ such that for all $S \subset N$, $q_i^S = 0$ for all $i \notin S$. The resulting object can be written as a $n \times (2^n - 1)$ matrix and we denote by Q_N the set of such matrices. For a given player *i,* Weber interpretes $(q_i^S \mid S \subset N, S \neq \varnothing)$ as a probability distribution over coalitions that may be objective, as the result of some random mechanism, or subjective. The *probabilistic value* associated a probability matrix $q \in Q_N$ is then defined for each player as his expected marginal contribution:

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[6] The term "core" was introduced by Gillies [1953, 1959] in connection to von Neumann-Morgenstern stable sets. It was later introduced as an independent solution concept by Shapley. The core has been axiomatized by Peleg [1986] using a reduced game property.

[7] See Bondareva [1963] and Shapley [1967]. For a complete account, see Kannai [1992].

$$PV_i(N,v,q) = \sum_{S\in\mathcal{C}(N)} q_i^S \left(v(S) - v(S\setminus i)\right) \quad (i = 1,...,n). \tag{9}$$

A probabilistic value does not necessarily define an efficient payoff vector because probability distributions are unrelated. *Quasi-values* instead are efficient probabilistic value obtained from probability distributions matrices $q \in Q_N$ satisfying

$$\sum_{i\in N} q_i^N = 1 \text{ and } \sum_{i\in S} q_i^S = \sum_{i\in N\setminus S} q_i^{S\cup i} \text{ for all } S \in \mathcal{C}(N), S \neq N. \tag{10}$$

Weber [1988] indeed proves that the probabilistic values defined in (9) satisfy efficiency under (10).[8] By requiring consistency of the probability distributions, quasi-values can be given a normative content. Vasil'ev and van der Laan [2002, Lemma 4.2] prove that (10) is equivalent to the following condition:

$$\sum_{i\in S} \sum_{T:\, S\subset T} q_i^T = 1 \text{ for all } S \in \mathcal{C}(N). \tag{11}$$

Let $Q_N^* \subset Q_N$ be the subset of probability distributions matrices satisfying (10) or (11), a polytope whose vertices have been characterized in terms of players' permutations by Vasil'ev [2003, 2007].[9] He proves that the n! vertices of Q_N^* are the matrices q^π $(\pi \in \Pi_N)$ defined by:

$$\left(q^\pi\right)_i^S = 1 \quad \text{if } S = \pi^i,$$
$$= 0 \quad \text{otherwise,}$$

where π^i denotes the set of players preceding i in π, i included.

The probabilistic values associated to the vertices of Q_N^* are then the corresponding marginal contribution vectors:

$$PV_i(N,v,q^\pi) = \sum_{S\in\mathcal{C}(N)} (q^\pi)_i^S \left(v(S) - v(S\setminus i)\right) = v(\pi^i) - v(\pi^i \setminus i) \text{ for all } \pi \in \Pi_N. \tag{12}$$

Random order values are average marginal contribution vectors computed with respect to some probability distribution on players' orderings.

For a given game (N,v), the *random order value* associated to the probability distribution $p \in \Delta(\Pi_N)$ is given by:

$$RV_i(N,v,p) = \sum_{\pi\in\Pi_N} p(\pi)\, \mu_i^\pi(N,v) \quad (i = 1,...,n).$$

The following proposition establishing the equivalence between quasi-values and random order values was proved by Weber [1988].

Proposition 1. A solution is a quasi-value *if and only if* it is a random-order value

[8] See also Derks [2005].

[9] See also Vasil'ev and van der Laan [2002].

Proof. Consider a game (N,v). We have to show that the sets $\{PV(N,V,q) \mid q \in Q_N^*\}$ and $\{RV(N,V,p) \mid p \in \Delta(\Pi_N)\}$ coincide. Following (12), we know that $Q_N^* = co\{q^\pi \mid \pi \in \Pi_N\}$ and $PV(N,v,q^\pi) = \mu^\pi(N,v)$ Probabilistic values $PV(N,V,q)$ being linear in *q*, we have successfully: [10]

$$\begin{aligned}\left\{PV(N,v,q) \mid q \in Q_N^*\right\} &= \left\{PV(N,v,\sum_{\pi\in\Pi_N}\eta_\pi q^\pi) \mid \eta \in \Delta(\Pi_N)\right\}\\ &= \left\{\sum_{\pi\in\Pi_N}\eta_\pi PV(N,V,q^\pi) \mid \eta \in \Delta(\Pi_N)\right\}\\ &= co\left\{PV(N,v,q^\pi) \mid \pi \in \Pi_N\right\} = co\left\{\mu^\pi(N,v) \mid \pi \in \Pi_N\right\}\\ &= \left\{RV(N,v,p) \mid p \in \Delta(\Pi_N)\right\}.\end{aligned}$$

In particular, $RV_i(N,v,p) = \sum_{S\subset N} q_i^S \left(v(S) - v(S\setminus i)\right)$ where the probability distributions (q_i^S) defined by $q_i^S = \sum_{\pi:\pi^i=S} p(\pi)$ satisfy (10). □

The set of all quasi-values – or alternatively the set of all random order values – is known as the *Weber set*. Marginal contribution vectors being imputations, the Weber set is a well-defined solution on the class of superadditive games.

The Shapley value is a particular quasi-value. Weber [1988] proves that it is the *unique symmetric* quasi-value. It is defined by probabilities that depend only on coalitions' sizes:

$$q_i^S = \frac{(s-1)!(n-s)!}{n!}$$

i.e.

$$SV_i(N,v) = \frac{1}{n!}\sum_{S\subset N}(s-1)!(n-s)!\left(v(S)-v(S\setminus i)\right). \tag{13}$$

These probabilities correspond to the following two-step random mechanism:[11] first, a coalition size between 1 and *n* is picked up at random and then each player receives his marginal contribution to a coalition picked up at random among the coalitions of the predetermined size of which he is a member. All sizes have the same probability, namely $1/n$, and the probability of picking up a coalition of size *s* containing a given player is given by $(s-1)!(n-s)!/(n-1)!$ As a random order value, the Shapley value corresponds to *uniformly* drawn random orders i.e. $p(\pi) = 1/n!$ for all π:

$$SV_i(N,v) = \frac{1}{n!}\sum_{\pi\in\Pi_N}\mu_i^\pi(N,v) \qquad (i = 1,\dots,n). \tag{14}$$

[10] The convex hull of a set *A*, denoted *co*[*A*], is the smallest convex set containing *A*. See Rockafellar [1970].

[11] A random allocation mechanism is "fair" if it treats *ex ante* all players equally.

The Shapley value is the average marginal contribution vector and can then be seen as resulting from another two-step random mechanism: first, players are ordered randomly and they then receive their marginal contribution, depending on their position in the order that has been picked up. To show that (13) and (14) are equivalent, consider the coalition π^i defined as the subset of players preceding *i* in π and including player *i*. Then (13) can be written as:

$$SV_i(N,v) = \frac{1}{n!}\sum_{\pi\in\Pi_N}\left(v(\pi^i) - v(\pi^i \setminus i)\right).$$

For a coalition $S \subset N$, there are $(s-1)!(n-s)!$ orderings such that $\pi^i = S$. Hence, we have:

$$SV_i(N,v) = \frac{1}{n!}\sum_{S\in\mathcal{C}_i(N)}(s-1)!(n-s)!\left(v(S) - v(S \setminus i)\right) \quad i = 1,\ldots,n.$$

The Shapley value defined by (13) or (14) is a well-defined single-valued solution on the class of all games, hence including superadditive games. Covariance follows from (6) and superadditivity ensures that marginal contribution vectors are imputations. Hence, the Shapley value defines an imputation that is not necessarily stable, independently of the core being empty or not. However, as an average of marginal contribution vectors, it defines a core allocation when applied to a convex game. Furthermore, in view of the geometric characterization of the core of a convex game, the Shapley value occupies a central position within the core. It generally differs from the barycenter of the core introduced as a solution concept by Gonzáles-Díaz and Sánchez-Rodríguez [2007]. It also differs from the simple average of core's vertices except for the particular case of convex games with distinct marginal contribution vectors.[12]

3.5 *Dividend distributions: the Harsanyi set*

A distribution of the Harsanyi dividends can be summarized by a matrix λ of dimension $n \times (2^n - 1)$ whose columns are the non-negative vectors λ^T $(T \subset N, T \neq \emptyset)$ that satisfying

$$\sum_{i\in N}\lambda_i^T = 1 \text{ and } \lambda_i^T = 0 \text{ for all } i \notin T.$$

λ^T specifies how the dividend α_T is allocated within coalition *T*. In particular, $\lambda_i^{\{i\}} = 1$ for all *i* and λ^N can be any vector in the unit simplex Δ_n. We denote by M_n the set of all distribution matrices in the case of *n* players. For a given game (N,v), the *Harsanyi payoff vector* $h(N,v,\lambda)$ derived from a distribution matrix $\lambda \in M_n$ is given by the inner product $h(N,v,\lambda) = \lambda \cdot \alpha(N,v)$:

$$h_i(N,v,\lambda) = \sum_{T\in\mathcal{C}(N)}\lambda_i^T\alpha_T(N,v) = v(i) + \sum_{T\in\mathcal{C}(N)\setminus\{i\}}\lambda_i^T\alpha_T(N,v) \qquad (i = 1,\ldots,n). \qquad (15)$$

[12] This characterizes what Shapley [1971] calls *strictly* convex games, games with increasing marginal contributions.

It is an allocation:

$$\sum_{i\in N} h_i(N,v,\lambda) = \sum_{i\in N}\sum_{T\in\mathcal{C}(N)} \lambda_i^T \alpha_T(N,v) = \sum_{T\in\mathcal{C}(N)} \alpha_T(N,v) \sum_{i\in N}\lambda_i^T = v(N).$$

We call $h(N,v,\lambda)$ a *Harsanyi value* and the set of all H-payoff vectors obtained by considering all distributions of dividends defines the *Harsanyi set*: [13]

$$H(N,v) = \left\{ x\in\mathbb{R}^n \;\middle|\; x = h(N,v,\lambda) \text{ for some } \lambda\in M_n \right\}.$$

Following Remark 2, the Harsanyi set of an inessential game reduces to a single allocation, namely $\left(v(1),\ldots,v(n)\right)$. The Harsanyi set associated to the unanimity game (N,u_N) is the unit simplex: $H(N,u_N)=\Delta_n$.

For a given subset *T*, the dividend α_T is allocated between the members of *T* and, depending on its sign, players in *T* receive a positive or a negative amount. Hence, the Harsanyi set can alternatively be written as:

$$H(N,v) = \sum_{T\in\mathcal{C}(N)} \{x\in\mathbb{R}^n \mid x(T)=\alpha_T,\, x(N\backslash T)=0,\, sign(x_i)=sign(\alpha_T) \text{ for all } i\in N\}. \quad (16)$$

It is obviously a non-empty and convex set. It is covariant. Indeed, if $v''(S) = a\,v'(S)+b(S)$ for some $a>0$ and $b\in\mathbb{R}^n$, we have:

$$\begin{aligned}
\alpha_{\{i\}}(N,v'') &= v''(i) = a\,v'(i)+b_i = a\,\alpha_{\{i\}}(N,v')+b_i \\
\alpha_{\{ij\}}(N,v'') &= v''(ij)-v''(i)-v''(j) = a\,(v'(ij)+b_i+b_j)-(a\,v'(i)+b_i)-(a\,v'(j)+b_j) \\
&= a\,\alpha_{\{ij\}}(N,v'),\ldots
\end{aligned}$$

i.e. the additive term affects only the coefficient associated to singletons. Therefore, we have:

$$\begin{aligned}
h_i(N,v'',\lambda) &= \sum_{T\in\mathcal{C}(N)} \lambda_i^T \alpha_T(N,v'') = a\,\alpha_{\{i\}}(N,v')+b_i + a\sum_{T\in\mathcal{C}(N)\backslash\{i\}} \lambda_i^T\alpha_T(N,v') \\
&= a\,h_i(N,v',\lambda)+b_i \quad (i=1,\ldots,n).
\end{aligned}$$

H-payoff vectors are not necessarily imputations and it may even be that the Harsanyi set contains no imputation at all.[14] However, for almost positive games, H-payoff vectors are imputations, an immediate consequence of (15). The following proposition is due to Derks et al. [2000].

Proposition 2. Marginal contribution vectors are Harsanyi payoff vectors.

[13] The Harsanyi set was introduced as a solution concept by Vasile'v [1978, 1981] and by Hammers et al. [1977], independently. The later used the term *selectope*.

[14] Derks, van der Laan and Vasil'ev [2010] give the exemple of a game that fails to be superadditive, whose Harsanyi set has no intersection with the imputation set.

Proof. Consider a game (N,v), an arbitrary ordering $\pi \in \Pi_N$ and the distribution matrix $\lambda \in M_n$ defined by

$$\lambda_i^T = 1 \quad \text{if } i \in T \text{ and } T \subset \pi^i,$$
$$= 0 \quad \text{otherwise.}$$

where π^i is the set of players preceding *i* in π and including *i*. For any given coalition *T*, λ gives a positive share *only* to the player in T that has the highest rank in π. Consequently, $\sum_{i \in T} \lambda_i^T = 1$ for all $T \subset N$ and $\lambda \in \Lambda_n$. The corresponding H-payoff vector is then given by:

$$h_i(N,v,\lambda) = \sum_{T \in \mathcal{C}(N)} \lambda_i^T \alpha_T = \sum_{T \in \mathcal{C}_i(\pi^i)} \alpha_T = \sum_{T \subset \pi^i} \alpha_T - \sum_{T \subset \pi^i \setminus i} \alpha_T = v(\pi^i) - v(\pi^i \setminus i).$$

Hence, $h(N,v,\lambda)$ is the marginal contribution vector associated to the ordering π. □

Hence, following Remark 5, the *Harsanyi imputation set* defined by:

$$HI(N,v) = H(N,v) \cap I(N,v).$$

is a solution set that satisfies the five basic properties when applied to 0-monotonic games.[15]

3.6 *Weighted Shapley values*

The Shapley value relies on symmetry: equal amounts are allocated to symmetric players. Shapley [1953b] derives (13) from the following formula

$$SV_i(N,v) = \sum_{T \in \mathcal{C}_i(N)} \frac{1}{t} \alpha_T(N,v) \quad (i = 1,...,n) \tag{17}$$

i.e. the Shapley value is the H-payoff vector associated to the *uniform distribution* of dividends within each coalition: $\lambda_i^T = 1/t$ for all $i \in T$ and $\lambda_i^T = 0$ for all $i \notin T$. Dropping symmetry opens the possibility for symmetric players to be treated differently. Shapley [1953a] also introduces an *asymmetric* version of the value obtained by introducing exogenous weights in order to cover asymmetries that are not included in the underlying game. Weighted games are denoted by (N,v,w) where (N,v) is a transferable utility game and $w = (w_1,...,w_n) \in \mathbb{R}_+^n \setminus 0$ are individual weights.

We denote by $SV(N,v,w)$ the weighted Shapley value associated to the game (N,v,w). It is the Harsanyi payoff vector associated to the dividends' distribution derived from *w*:

$$SV_i(N,v,w) = \sum_{T \in \mathcal{C}_i(N)} \frac{w_i}{w(T)} \alpha_T(N,v). \tag{18}$$

[15] Notice that for 2-player games, the Harsanyi set coincides with the imputation set. Vasil'ev [1981] provides necessary and sufficient conditions for nonemptiness of the Harsanyi imputation set. See also Derks et al. [2010].

That definition is actually valid only if *at most one* of the w_i's is equal to zero. To show this, consider an arbitrary player, say player 1. With positive weights, (18) can be decomposed as follows:

$$SV_1(N,v,w)=\alpha_{\{1\}}+\sum_{T\in\mathcal{C}_1(N)\setminus\{1\}}\frac{w_1}{w(T)}\alpha_T(N,v)=v(1)+\sum_{T\in\mathcal{C}_1(N)\setminus\{1\}}\frac{1}{1+\sum_{i\in T\setminus 1}(w_i/w_1)}\alpha_T(N,v).$$

Assuming $w_i>0$ for all $i\neq 1$ and letting $w_1\to 0,$ we obtain a well-defined limit, namely $v(1)$. When there are more than three players and at least two of them are assigned a zero weight, there may be a continuum of values depending on the relative speeds of convergence.

The set of all weighted values is obtained by considering all positive weights and all possible limits of sequences of positive weights. More precisely, considering normalized weights in Δ_n and a sequence of positive weights (w^k) converging to some boundary point $w\in\partial\Delta_n,$ the resulting limit is given by

$$SV_i(N,v,w)=v(i)+\lim_{w^k\to w}\sum_{T\in\mathcal{C}_i(N)\setminus\{i\}}\frac{1}{1+\sum_{j\in T\setminus i}(w_j^k/w_i^k)}\alpha_T(N,v).$$

It exists and it coincides with $v(i)$ if $w_j^k/w_i^k\to\infty$ for all $j\neq i.$ We denote by $WS(N,v)$ the resulting set of all weighted values.

Applying (18) to the unanimity game $(N,u_N),$ we get:

$$SV_i(N,u_N,w)=\frac{w_i}{w(N)}\quad (i=1,...,n). \tag{19}$$

It is a well defined expression for all $w\in\mathbb{R}_+^n\setminus 0$ and it equals $1/n$ in the symmetric case.

Weighted Shapley values $SV(N,v,w)$ can alternatively be obtained as random order values $RV(N,v,p_w)$ where p_w is a probability distribution on players' orderings depending on w. For an arbitrary players' ordering π, the marginal contributions of player i in the unanimity game (N,u_N) is defined by:

$$\begin{aligned}\mu_i^\pi(N,u_N)\ &=1\quad \text{if (and only if) } i \text{ comes } \textit{last} \text{ in } \pi,\\ &=0\quad \text{otherwise.}\end{aligned}$$

Hence, (19) corresponds to the random order values associated to distributions such that $w_i/w(N)$ is the probability that player i comes *last* in an arbitrary ordering, i.e. $p\in\Delta(\Pi_N)$ should satisfy

$$RV_i(N,u_N,p)=\sum_{\pi^{-i}\in\Pi_{N\setminus i}}p(\pi^{-i},i)=\frac{w_i}{w(N)}\ \text{ for all } i\in N.$$

Let's assume for a moment that weights are positive and natural numbers, w_i being interpreted as the number of players of type *i*. We then compute the probability that a given ordering comes out through a sequence of $w(N)$ independent drawings, knowing that each time a player is drawn, he is removed and only the last player of a given type to be removed is placed in the ordering. The number of possible sequences is given by $w(N)!\big/\prod w_i!$ and they all have the same probability of occurrence. To illustrate the process, let's take $n = 3$ and $w = (1,2,3)$. Then, $w(N) = 6$ and there are 60 drawing sequences. For instance, the sequence (3,2,3,3,1,2) leads to the ordering (3,1,2) where player 2 comes last.

Player j comes out last in a given ordering if and only if he is the last to be drawn. This occurs with probability

$$\frac{\big(w(N)-1\big)!}{(w_j-1)!\prod_{i\neq j} w_i!}\,\frac{\prod w_i!}{w(N)!}=\frac{w_j}{w(N)}.$$

The probability that player *k* comes next to last knowing that player *j* came last is given by:

$$\frac{\big(w(N\setminus j)-1\big)!}{(w_k-1)!\prod_{i\neq j,k} w_i!}\,\frac{\prod_{i\neq j} w_i!}{w(N\setminus j)!}=\frac{w_k}{w(N\setminus j)}.$$

Using this argument repeatedly until the second position, the probability that the ordering $\pi=(i_1,\ldots,i_n)$ comes out is given by:

$$p_w(\pi)=\frac{w_{i_2}}{w_{i_1}+w_{i_2}}\cdots\frac{w_{i_{n-1}}}{w_{i_1}+\ldots+w_{i_{n-1}}}\,\frac{w_{i_n}}{w_{i_1}+\ldots+w_{i_n}}=\prod_{k=2}^{n}\frac{w_{i_k}}{\sum_{j=1}^{k} w_{i_j}}$$

or

$$p_w(\pi)=\prod_{k=2}^{n}\frac{1}{1+\sum_{j=1}^{k-1}(w_{i_j}/w_{i_k})}. \tag{20}$$

This formula is then extended to the case where weights are real numbers.[16] If a player is assigned a zero weight, weighted values are obtained as limit of sequences of positively weighted values. If there is a *single* zero weight player, say player *i*, the limit distribution is still uniquely defined: player *i* is *first* with probability 1 and receives his individual worth $v(i)$. When two players or more are assigned a zero weight, a continuum of values may be associated to the same normalized weight system. Considering converging sequences of positive weights, the resulting value may depend on their relative speeds of convergence.

[16] I am grateful to Gerard van der Laan for suggesting this procedure. I initially used the sequence of *n* drawings where, each time a player is drawn, all players of the same type are removed. It leads to a probability distribution where $w_i/\Sigma w_j$ is the probability that player *i* comes *first*. This is appropriate for cost games and duals of surplus sharing games. The distribution (20) is then obtained by considering the reverse order; see Dehez [2011].

The probability distributions p_w are homogeneous of degree zero in *w*. Weights may therefore be normalized. For a given set *N* of players and weights $w \in \Delta_n$, the set of *all* weighted values is obtained from probability distributions in the set

$$F_N(w) = \left\{ p \in \Delta(\Pi_N) \middle| \ p(\pi) = \lim_{w^k \to w} p_{w^k}(\pi) \text{ for some converging sequence } (w^k) \subset \operatorname{int} \Delta_n \right\}.$$

In view of (20), this is a well-defined set: for any positive sequence (w^k) converging to $w \in \Delta_n$, the associated sequence of distributions p_{w^k} converges to a distribution $p_w \in \Delta(\Pi_N)$. The *Shapley set* is then defined as the set of all weighted values:

$$WS(N,v) = \left\{ x \in \mathbb{R}^n \,\middle|\, x = RV(N,v,p) \text{ for some } p \in F_N(w) \text{ and } w \in \Delta_n \right\}. \tag{21}$$

Remark 7. There is a one-to-one relationship between the set of *positively* weighted values and the (relative) *interior* of Δ_n: any positively weighted value is associated to a unique *normalized* weight system, and vice-versa.

Let us denote by $Z = \{ i \in N \mid w_i = 0 \}$ the set of zero weight players.[17] Given a game (N,v), let (Z, v_Z) be its restriction to *Z* and define the game $(N \backslash Z, \hat{v})$ by $\hat{v}(S) = v(Z \cup S) - v(Z)$. It is the game that concerns the subset of non-zero weight players, once $v(Z)$ has been distributed to zero weight players. The following proposition establishes that, in order to compute weighted values, positive-weight players and zero weight players can be treated separately.

Proposition 3. The values of the weighted game (N,v,w) consists of the allocations $x = (x_Z, x_{N\backslash Z})$ where $x_Z \in W(Z, v_Z)$ and $x_{N\backslash Z} = SV(N \backslash Z, \hat{v}, w_{N\backslash Z})$.

Proof. Inspecting (20), we observe that the distributions in $F_N(w)$ assign a zero probability to orderings in which a non-zero weight player is followed by a zero weight player. Hence, only orderings of the form $\pi = (\pi', \pi'') \in (\Pi_Z \times \Pi_{N\backslash Z})$ do actually matter and the distributions $p_w \in F_N(w)$ are of the form

$$\begin{aligned} p_w(\pi) &= p_0(\pi')\, p_{w_{N\backslash Z}}(\pi'') \quad \text{for all } \pi = (\pi', \pi'') \in \Pi_Z \times \Pi_{N\backslash Z}, \\ &= 0 \qquad\qquad\qquad\quad \text{otherwise.} \end{aligned}$$

where p_0 is an arbitrary probability distribution on Π_Z. The corresponding allocation is then given by:

$$x_i = \sum_{(\pi', \pi'') \in \Pi_Z \times \Pi_{N\backslash Z}} p_0(\pi')\ p_{w_{N\backslash Z}}(\pi'')\ \mu_i^{(\pi', \pi'')}(N,v).$$

By definition of the marginal contribution vectors, $\mu_i^{(\pi', \pi'')}$ can be decomposed as follows:

$$\mu_i^{(\pi', \pi'')}(N,v) = (\mu_i^{\pi'}(Z, v_Z),\ \mu_i^{\pi''}(N \backslash Z, \hat{v})).$$

[17] We omit the dependence of *Z* on *w*.

Hence, for a player $i \in Z$, we have:

$$x_i = \sum_{\pi' \in \Pi_Z} p_0(\pi')\, \mu_i^{\pi'}(Z, v_Z) \sum_{\pi'' \in \Pi_{N\backslash Z}} p_{w_{N\backslash Z}}(\pi'') = \sum_{\pi' \in \Pi_Z} p_0(\pi')\, \mu_i^{\pi'}(Z, v_Z)$$

i.e. $x_Z = RV(Z, v_Z, p_0)$. Hence, the probability distribution p_0 being arbitrary, we can conclude that $x_Z \in W(Z, v_Z)$.

Consider now the game $(N\backslash Z, \hat{v})$. For a player $i \in N\backslash Z$ and an arbitrary ordering $(\pi', \pi'') \in (\Pi_Z \times \Pi_{N\backslash Z})$ and we have:

$$x_i = \sum_{\pi'' \in \Pi_{N\backslash Z}} p_{w_{N\backslash Z}}(\pi'')\, \mu_i^{\pi''}(N\backslash Z, \hat{v}) \sum_{\pi' \in \Pi_Z} p_0(\pi') = \sum_{\pi'' \in \Pi_{N\backslash Z}} p_{w_{N\backslash Z}}(\pi'')\, \mu_i^{\pi''}(N\backslash Z, \tilde{v})$$

i.e. $x_{N\backslash Z}$ is the weighted value of the game $(N\backslash Z, \hat{v}, w_{N\backslash Z})$. □

Remark 8. In practice, when more than one player are assigned a zero weight, it would be natural to treat them equally by considering converging sequences such that the ratios of their weights are equal to 1. The distribution is then given by $p_0(\pi) = 1/z!$ for all $\pi \in \Pi_Z$ and the resulting allocation is the symmetric Shapley value of the game (Z, v_Z).[18]

The Shapley set is clearly a non-empty subset of the Weber set and, as a solution, it is covariant. It is however not a convex set in general, as was observed by Monderer, Samet and Shapley [1992], except for the 2-player games or for convex games, as we shall see later.

Remark 9. Owen [1968] has been the first to notice that weighted values are not necessarily monotonic with respect to weights: an increase in the weight assigned to a player may indeed result in a decrease of his payoff. Weights being interpreted as measures of players' relative importance (Shapley talks about bargaining abilities), this is an embarrassing fact. It is however no surprise in view of (18), knowing that dividends may be negative.[19] Monotonicity clearly holds for almost positive games. Monderer et al. [1992] have shown that it actually holds for (and *only* for) convex games. This can be explained intuitively by the link that exists between a characteristic of convex games and the probability distribution over orderings induced by the weights. Increasing the weight of a player means increasing his probability of arriving late and we know that marginal contributions are increasing with coalition size in convex games. Hence, increasing the weight of a player naturally increases his expected payoff.

[18] See Dehez and Tellone [2013] for an application of the weighted Shapley value with zero weight players.

[19] Owen [1968] suggests to interpret weights as reflecting slowness to reach a decision. An alternative definition of weighted value has been proposed by Haeringer [2006] in which an increase in the weight of a player leads to an increase in his share in positive dividends and a decrease of his share in negative dividends.

3.7 *Relation between solutions*

The question is now to see how the core, the Weber set, the Shapley set and the Harsanyi set are interrelated.

Proposition 4. The following sequence of inclusions holds for 0-monotonic games:

$$C(N,v) \subset WS(N,v) \subset W(N,v) \subset HI(N,v) \subset H(N,v).$$

When applied to convex games, the first three solutions coincide.

Proof. Weber [1988] has shown that the core is a subset of the Weber set and that, when applied to convex games, the two solutions coincide. Actually this coincidence is a *necessary and sufficient* condition for convexity. We have already seen that weighted values are random order values.[20] Monderer, Samet and Shapley [1992] have shown that the core is a subset of the set of weighted values. By Proposition 2, random order values are convex combinations of Harsanyi imputation vectors. Hence, the Weber set is a subset of the Harsanyi set. The sequence of inclusions then follows from the convexity of the Harsanyi set. □

Proposition 5. All solutions coincide on the set of almost positive games:

$$C(N,v) = WS(N,v) = W(N,v) = HI(N,v) = H(N,v).$$

This is a corollary of the following proposition due to Hammer et al. [1977] and Vasil'ev [1978].

Proposition 6. The core and the Harsanyi set coincide on the set of almost positive games.

Proof. Let (N,v) be an almost positive (and thereby convex) game. Looking at its 0-normalization, we have:

$$v_0(S) = \sum_{T \in \mathcal{C}(N)} \alpha_T(N,v_0)\, u_T(S)$$

where the $\alpha_T\,'s$ are all non-negative. For any given $T \subset N,$ the core of the game $(N,\alpha_T u_T)$ is given by:

$$C(N,\alpha_T u_T) = \left\{ x \in \mathbb{R}^n_+ \;\middle|\; x(T) = \alpha_T \text{ and } x_i = 0 \text{ for all } i \notin T \right\}.$$

Indeed $u_T(i) = 0$ for all $T \neq \{i\},$ $u_{\{i\}}(i) = 1$ and $u_{\{i\}}(j) = 0$ for all $j \neq i.$ The core is additive on the class of convex games. This follows from the following two lemmas:

Lemma 3. The core is a superadditive solution (Peleg [1986]).

Lemma 4. The Weber set is a subadditive solution (Dragan, Potters and Tijs [1989]).

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[20] In fact, the Shapley set is in general a dimensionally small subset of the Weber set.

Hence, we have:

$$C(N,v_0)=\sum_{T\in\mathcal{C}(N)} C(N,\alpha_T u_T)=\sum_{T\in\mathcal{C}(N)} \{x\in\mathbb{R}^n_+ \mid x(T)=\alpha_T \text{ and } x(N\backslash T)=0\}$$

where the right-hand side is the Harsanyi set of the game (N,v_0). □

Actually, Hammer et al. [1977] and Vasil'ev [1981] prove that the core and the Harsanyi set coincide *only if* they apply to almost positive games. Knowing that core allocations (if any) are H-payoff vectors, another way to prove Proposition 6 consists in showing that the reverse inclusion $H(N,v)\subset C(N,v)$ holds for almost positive games. Indeed, using (15), we have:

$$\begin{aligned}\sum_{i\in S} h_i(N,v,\lambda) &= \sum_{i\in S}\sum_{T\in\mathcal{C}(N)} \lambda_i^T\alpha_T = \sum_{i\in S}\sum_{T\in\mathcal{C}(S)} \lambda_i^T\alpha_T + \sum_{i\in S}\sum_{\substack{T\subset\mathcal{C}(N)\\ T\not\subset S}} \lambda_i^T\alpha_T \\ &= \sum_{T\in\mathcal{C}(S)}\alpha_T\sum_{i\in T}\lambda_i^T + \sum_{i\in S}\sum_{\substack{T\subset\mathcal{C}(N)\\ T\not\subset S}} \lambda_i^T\alpha_T = v(S) + \sum_{i\in S}\sum_{\substack{T\subset\mathcal{C}(N)\\ T\not\subset S}} \lambda_i^T\alpha_T \geq v(S).\end{aligned}$$

At this stage, we can conclude that the core, the Shapley set, the Weber set and the Harsanyi imputation set all satisfy the five basic properties when applied to convex games. In particular, the Shapley set is a convex set in this case.

By Proposition 6, $C(N,v^+)=H(N,v^+)$ and $C(N,v^-)=H(N,v^-)$. Furthermore, Derks et al. [2000] proves that $H(N,v)=C(N,v^+)-C(N,v^-)$.

Remark 10. Looking at the Shapley set and assuming convexity, there is a homeomorphism between the *relative interior* of the unit simplex and the relative interior of the core. This homeomorphism cannot be extended to non-negative weights and boundary core allocations, except if the game is strictly convex.

4. Characterizing Solutions

There are different ways to characterize values and solutions. We will consider two ways: by axioms and by restrictions on dividend distributions.

4.1 *Characterization by axioms*

Given a player set *N*, consider the following properties applying to a value $\varphi: G(N)\to\mathbb{R}^n$:

- *Efficiency*: $\sum\varphi_i(N,v)=v(N)$.
- *Weak positivity*: $v\in G^+(N) \Rightarrow \varphi(N,v)\in\mathbb{R}^n_+$.
- *Strong positivity*: $v\in MG(N) \Rightarrow \varphi(N,v)\in\mathbb{R}^n_+$.
- *Null player*: i null in $(N,v) \Rightarrow \varphi_i(N,v)=0$.
- *Additivity*: $\varphi(N,v_1+v_2)=\varphi(N,v_1)+\varphi(N,v_2)$.

These are usual properties. The following proposition is due to Vasil'ev [1982, 2006].[21] We give here a simple proof.

Proposition 7. For any given player set N, a value φ satisfies efficiency, weak positivity, null player and additivity *if and only if* φ is a Harsanyi value.

Proof. H-payoffs vectors are efficient and the H-payoff of a null player is obviously zero. Weak positivity follows from the definition of H-payoffs (15). Additivity follows from dividends' additivity.

Consider the unanimity game $(N,\beta u_T)$ where $T \subset N$ and $\beta \in \mathbb{R}$. Because unanimity games are positive and players outside T are null players, an application φ satisfying efficiency, weak positivity and null player must be such that:

$$\sum \varphi_i(N,\beta u_T) = \beta,$$
$$\varphi_i(N,\beta u_T) \geq 0 \text{ for all } i \in N,$$
$$\varphi_i(N,\beta u_T) = 0 \text{ for all } i \notin T.$$

Hence, $\varphi(N,\beta u_T)$ is a H-payoff of the game $(N,\beta u_T)$: $\varphi(N,\beta u_T) = h(N,\beta u_T,\lambda)$ for some $\lambda \in M_n$. More precisely, there exists $\lambda \in M_n$ such that $\varphi_i(N,\beta u_T) = \lambda_i^T \beta$ for all $i \in N$. Now consider an arbitrary characteristic function v on N and its decomposition $v = \sum \alpha_T u_T$ in terms of dividends. Following (7), v can be written as $v = v^+ - v^-$ where (N,v^+) and (N,v^-) are positive games that decompose as $v^+ = \sum_{\alpha_T > 0} \alpha_T u_T$ and $v^- = \sum_{\alpha_T < 0} -\alpha_T u_T$. By additivity, we have:

$$\varphi(N,v^+) = \varphi(N,v^+ - v^-) + \varphi(N,v^-) \iff \varphi(N,v) = \varphi(N,v^+) - \varphi(N,v^-).$$

As a consequence, $\varphi(N,v)$ is a H-payoff vector of the game (N,v):

$$\varphi_i(N,v) = \sum_{T:\alpha_T>0} \varphi_i(N,\alpha_T u_T) - \sum_{T:\alpha_T<0} \varphi_i(N,-\alpha_T u_T) = \sum_{T \subset N} \lambda_i^T \alpha_T \text{ for all } i \in N. \ \square$$

Notice that the Harsanyi set, as a solution, is not additive.[22] Indeed, considering for instance the unanimity game (N,u_N), we have:

$$H(N,u_N + (-u_N)) = \{0\},$$
$$H(N,u_N) + H(N,-u_N) = \Delta_n - \Delta_n \neq \{0\}.$$

Vasil'ev [1981] provides an axiomatization of the Harsanyi set that requires convexity and a restricted notion of additivity applying to disjoint games.[23]

---

[21] Vasil'ev also requires homogeneity although only additivity is actually needed.

[22] The author is grateful to a referee for pointing out the non-additivity of the Harsanyi set.

[23] See also Vasil'ev and van der Laan [2002].

Weber [1988] proved that strengthening the positivity axiom results in the Weber set.[24]

Proposition 8. For any given player set *N*, a value φ satisfies efficiency, strong positivity, null player and additivity *if and only if* φ is a random order value.

To obtain weighted Shapley values, a specific axiom is needed. Derks et al. [2000] use the following axiom:[25]

Consistency: $i \in S \subset T \;\Rightarrow\; \varphi_i(N, \varphi(N, u_T)(S)\, u_S) = \varphi_i(N, u_T).$

Proposition 9. For any given player set *N*, a value φ satisfies efficiency, consistency, null player and additivity *if and only if* φ is a weighted Shapley value.

It is easy to check that consistency is satisfied by the weighted value when associated to positive weights. Indeed, we have:

$$i \in T \;\Rightarrow\; SV(N, u_T, w) = \frac{w_i}{w(T)}$$

and

$$i \in S \subset T \;\Rightarrow\; SV(N, \varphi(N, u_T)(S)\, u_T, w) = \frac{w_i}{w(S)} \frac{w(S)}{w(T)} = \frac{w_i}{w(T)}.$$

4.2 Characterization by restrictions on dividend distributions

Harsanyi payoff vectors are defined by distribution matrices in M_n without any further restrictions. A natural question concerns the identification of restrictions on distribution matrices such that the resulting set of H-payoff vectors corresponds to particular solution sets.

Derks et al [2000] suggest to link distributions within a coalition to distributions within its sub-coalitions by requiring that a player's share in the dividend of a coalition does not increase if the coalition is enlarged:

$$i \in S \subset T \;\Rightarrow\; \lambda_i^S \geq \lambda_i^T. \qquad (22)$$

It means that if a player leaves a coalition, that should not reduce the share of those remaining in the coalition. This monotonicity property imposes strong restrictions on distribution matrices. In particular, if the share of a player in a coalition is zero, his share must be equally zero for all larger coalitions.

We denote by $H^m(N, v)$ the subset of Harsanyi payoffs vectors derived from monotonic distribution matrices. The following proposition is due to Derks et al. [2000].

---

[24] See also Derks et al. [2000].

[25] Consistency is a weaker version of the axiom of partnership consistency used by Kalai and Samet [1987] to axiomatize the weighted Shapley value.

Proposition 3. Distribution matrices associated to random order values are monotonic.

Proof. Let us fix a player set N and a non-monotonic distribution matrix $\lambda \in M_n$ i.e. there exist a player i and coalitions S and T in N such that $i \in S \subsetneq T$ and $\lambda_i^S < \lambda_i^T$. Consider the game (N,v) defined by $v = u_S + u_{T\setminus S} - u_T$. Following Remark 1, its dividends are all zero except for coalitions S, T and $T \setminus S$:

$$\alpha_S(N,v) = \alpha_{T\setminus S}(N,v) = 1,$$
$$\alpha_T(N,v) = -1.$$

The H-payoff of player i is therefore given by $h_i(N,v,\lambda) = \lambda_i^S - \lambda_i^T < 0$. Hence, (N,v) being positive valued and monotonic, $h(N,v,\lambda)$ cannot be a random order value. □

We therefore have the following inclusion: $W(N,v) \subset H^m(N,v)$. There may however be H-payoff vectors derived from monotonic distribution matrices that are not random order value. To have equality, a stronger monotonicity requirement is needed. Vasil'ev [1988], Dragan [1994] and Derks et al. [2006] prove the following proposition.

Proposition 11. H-payoff vectors are random order values *if and only if* they are derived from distribution matrices λ satisfying the following inequalities

$$\sum_{S:S\supset T} (-1)^{s-t} \lambda_i^S \geq 0 \text{ for all } T \subset N \text{ and } i \in T. \tag{23}$$

Probabilities (q_i^S) and dividend distributions λ are then related by a Möbius transform:

$$q_i^S = \sum_{T:T\supset S} (-1)^{t-s} \lambda_i^T \quad \text{for all } S \subset N \text{ and } i \in S,$$

$$\lambda_i^S = \sum_{T:T\supset S} q_i^T \quad \text{for all } S \subset N \text{ and } i \in S,$$

and $H^{sm}(N,v) = W(N,v)$ where $H^{sm}(N,v)$ denotes the set of Harsanyi payoff vectors derived from distribution matrices satisfying the strong monotonicity condition (23).

An even stronger restriction consists in assuming that the distribution vectors λ^S are consistent in the Bayesian sense:

$$i \in S \subset T \;\Rightarrow\; \lambda_i^S = \frac{\lambda_i^T}{\lambda^T(S)} \quad \text{if } \lambda^T(S) > 0. \tag{24}$$

In the case where $\lambda^T(S) = 0$, λ^S is any distribution on Δ_n satisfying $\lambda_i^S = 0$ for all $i \notin S$. It is easily verified that distribution matrices satisfying (24) also satisfy the monotonicity properties (22) and (23). The following proposition can be found in Derks et al. [2000] or Billot and Thisse [2005].

Proposition 12. The weighted Shapley values associated to the weight system $w \in \mathbb{R}^n_+ \setminus 0$ coincides with the set of H-payoff vectors of the form $h(N,v,\lambda)$ where λ is a distribution matrix satisfying (24) for $\lambda^N = w/w(N)$.

Proof. Let's fix some $w \in \Delta_n$. If $w \gg 0$, the equivalence follows for the distribution matrix λ satisfying (24) with $\lambda^N = w$:

$$h_i(N,v,\lambda) = \sum_{T \subset N} \lambda_i^T \alpha_T = \sum_{T \in \mathcal{C}_i(N)} \frac{\lambda_i^N}{\lambda^N(T)} \alpha_T = \sum_{T \in \mathcal{C}_i(N)} \frac{w_i}{w(T)} \alpha_T = SV_i(N,v,w).$$

Assume now that the set Z of zero weight players is non-empty. We first observe that, for all $S \subset N$ such that $S \cap (N \setminus Z) \neq \varnothing$, (24) implies $\lambda_i^S = 0$ for all $i \in Z \cap S$. Hence, we can treat zero weight and non-zero weight players separately. Consider first non-zero weight players and the distribution matrix $\hat{\lambda}$ derived from the distribution vector $\lambda^{N\setminus Z} = w_{N\setminus Z}$ using (24). Applying the above argument for the player set $N\setminus Z$, we obtain $h(N\setminus Z, \hat{v}, \hat{\lambda}) = WS(N\setminus Z, \hat{v}, w_{N\setminus Z})$ where $(N \setminus Z, \hat{v})$ is the game defined by $\hat{v}(S) = v(Z \cup S) - v(Z)$ introduced within the proof of Proposition 3. Consider now zero weight players and an arbitrary distribution λ^Z on Z. The distribution vectors λ^T for subsets $T \subset Z$ are then obtained applying (24). Considering all distributions on Z generates the Harsanyi set $H(Z, v_Z)$ which coincides with $W(Z, v_Z)$ by Proposition 8.

Hence, $H^b(N,v) = WS(N,v)$ where $H^b(N,v)$ denotes the set of H-payoff vectors derived from distribution matrices satisfying (24). □

4.3 *Implications of monotonic dividend distributions*

Monotonicity of dividend distribution can be considered as a natural restriction. Can we characterize $H^m(N,v)$, the set of H-payoff vectors derived from distribution matrices satisfying (24) ?

We already know that $W(N,v) \subset H^m(N,v)$. Furthermore, the core and the Harsanyi set coincide if (and only if) they apply to almost positive games, in which case $C(N,v) = H^m(N,v) = H(N,v)$: core allocations can be written as H-payoff vectors derived from monotonic dividend distributions.

What about a larger class of games? Billot and Thisse [2005] claim that $H^m(N,v)$ coincides with the core if (and only if) the game is convex. Actually, this is true only for 2 and 3-player games!

Consider a 3-player game and let x be the H-payoff vector corresponding to a monotonic dividend distribution matrix λ. Using (22), individual rationality results from superadditivity:

$$
\begin{aligned}
x_1 - v(1) &= \lambda_1^{12}(v(12)-v(1)-v(2)) + \lambda_1^{13}(v(13)-v(1)-v(3)) \\
&\quad + \lambda_1^{123}(v(123)-v(12)-v(13)-v(23)+v(1)+v(2)+v(3)) \\
&= (\lambda_1^{12}-\lambda_1^{123})(v(12)-v(1)-v(2)) + (\lambda_1^{13}-\lambda_1^{123})(v(13)-v(1)-v(3)) \\
&\quad + \lambda_1^{123}(v(123)-v(23)-v(1)) \\
&\geq \lambda_1^{123}(v(123)-v(23)-v(1)) \geq 0.
\end{aligned} \tag{25}
$$

Considering 2-player coalitions, superadditivity implies the following inequality:

$$
\begin{aligned}
(x_1 + x_2) - v(12) &= \lambda_1^{13}(v(13)-v(1)-v(3)) + \lambda_2^{23}(v(23)-v(2)-v(3)) \\
&\quad + (\lambda_1^{123}+\lambda_2^{123})(v(123)-v(12)-v(13)-v(23)+v(1)+v(2)+v(3)) \\
&\geq \lambda_1^{123}(v(123)-v(12)-v(23)+v(2)) \\
&\quad + \lambda_2^{123}(v(123)-v(12)-v(13)+v(1)).
\end{aligned}
$$

where the last part is non-negative under convexity. The argument applies identically to all single players and 2-player coalitions, confirming that *x* is a core allocation under convexity.

For more than three players, convexity is not sufficient to ensure that H-payoff vectors are core allocations under monotonicity. Consider the 4-player convex game defined by $v(S) = s - 1.$ Its dividends are given by:

$$
\begin{aligned}
&\alpha_{\{i\}} = 0, \\
&\alpha_{\{i,j\}} = 1, \\
&\alpha_{\{i,j,k\}} = -1, \\
&\alpha_{\{1,2,3,4\}} = 1.
\end{aligned}
$$

The H-payoffs associated to the monotonic matrix given in Table 1 are $(0.4, 0.7, 0.8, 1.1),$ an allocation that does not belong to the core: the coalition {1,2,3} indeed obtains only 1.9.

	12	**13**	**14**	**23**	**24**	**34**	**123**	**124**	**134**	**234**	**1234**
1	0.5	0.4	0.3	0	0	0	0.4	0.3	0.3	0	0.2
2	0.5	0	0	0.5	0.3	0	0.3	0.3	0	0.2	0.2
3	0	0.6	0	0.5	0	0.3	0.3	0	0.3	0.2	0.2
4	0	0	0.7	0	0.7	0.7	0	0.4	0.4	0.6	0.4

Table 1

For 3-player games, (25) tells us that superadditivity is enough to ensure that monotonic dividend distributions define imputations. For 4-player games, adding convexity ensures individual rationality. To simplify and without loss of generality, let's assume $v(i)=0$ for all i. Then, by Lemma 1, $v(S)\geq 0$ for all S. The H-payoff of player 1 for an arbitrary distribution matrix λ is given by:

$$\begin{aligned} x_1 &= \lambda_1^{12}v(12)+\lambda_1^{13}v(13)+\lambda_1^{14}v(14)+\lambda_1^{123}(v(123)-v(12)-v(13)-v(23)) \\ &\quad +\lambda_1^{124}(v(124)-v(12)-v(14)-v(24))+\lambda_1^{134}(v(134)-v(13)-v(14)-v(34)) \\ &\quad +\lambda_1^{1234}(v(1234)-v(123)-v(124)-v(134)-v(234) \\ &\qquad +v(12)+v(13)+v(14)+v(23)+v(24)+v(34)). \end{aligned}$$

Using the monotonicity condition (22), we obtain:

$$\begin{aligned} x_1 &\geq \lambda_1^{123}(v(123)-v(13)-v(23))+\lambda_1^{124}(v(124)-v(12)-v(24)) \\ &\quad +\lambda_2^{134}(v(134)-v(14)-v(34)) \\ &\quad +\lambda_1^{1234}(v(1234)-v(123)-v(124)-v(134)-v(234) \\ &\qquad +v(12)+v(13)+v(14)+v(23)+v(24)+v(34)). \end{aligned}$$

Rearranging the above expression, we get:

$$\begin{aligned} x_1 &\geq (\lambda_1^{123}-\lambda_1^{1234})(v(123)-v(13)-v(23))+(\lambda_1^{124}-\lambda_1^{1234})(v(124)-v(12)-v(24)) \\ &\quad +(\lambda_1^{134}-\lambda_1^{1234})(v(134)-v(14)-v(34))+\lambda_1^{1234}(v(1234)-v(234)) \end{aligned}$$

Combining (22) and convexity, we finally get $x_1 \geq \lambda_1^{1234}(v(1234)-v(234))\geq 0$.

Convexity is needed for the result to hold. Indeed, consider the following 4-player game and associated dividends.

$v(i)=0$		$\alpha_{\{i\}}=0$
$v(i,j)=1$		$\alpha_{\{i,j\}}=1$
$v(1,i,j)=1$	$\rightarrow$	$\alpha_{\{i,j,k\}}=1-3=-2$
$v(2,3,4)=2$		$\alpha_{\{2,3,4\}}=2-3=-1$
$v(1,2,3,4)=2$		$\alpha_{\{1,2,3,4\}}=2-5+6=3$

This game is superadditive but not convex. The distribution matrix given in Table 2, while satisfying the conditions of monotonicity, leads to an allocation that violates individual rationality. Player 1 is indeed allocated a negative amount: $x_1 = 2.1-4.2+1.65=-0.45<0$.

	12	13	14	23	24	34	123	124	134	234	1234
1	0.7	0.7	0.7	0	0	0	0.7	0.7	0.7	0	0.55
2	0.3	0	0	0.7	0.3	0	0.15	0.15	0	0.15	0.15
3	0	0.3	0	0.3	0	0.3	0.15	0	0.15	0.15	0.15
4	0	0	0.3	0	0.7	0.7	0	0.15	0.15	0.7	0.15

Table 2

Beyond four players, convexity does not ensure individual rationality under monotonicity. Indeed, consider the following convex game:

$$
\begin{array}{lcl}
v(i)=0 & & \alpha_{\{i\}}=0 \\
v(i,j)=1 & & \alpha_{\{i,j\}}=1 \\
v(1,i,j)=2 & & \alpha_{\{1,i,j\}}=-1 \\
v(i,j,k)=3 & \rightarrow & \alpha_{\{i,j,k\}}=0 \\
v(1,i,j,k)=4 & & \alpha_{\{1,i,j,k\}}=1 \\
v(2,3,4,5)=5 & & \alpha_{\{2,3,4,5\}}=-1 \\
v(1,2,3,4,5)=6 & & \alpha_{\{1,2,3,4,5\}}=-1
\end{array}
$$

Again here, player 1 is allocated a negative amount (– 0.2) on the basis of the monotonic distribution matrix given by Table 3.[26]

	12	13	14	15	123	124	125	134	135	145	1234	1235	1245	1345	12345
1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0	0	0	0	0
2	0.9	0	0	0	0.45	0.45	0.45	0	0	0	0.33	0.33	0.33	0	0.25
3	0	0.9	0	0	0.45	0	0	0.45	0.45	0	0.33	0.33	0	0.33	0.25
4	0	0	0.9	0	0	0.45	0	0.45	0	0.45	0.33	0	0.33	0.33	0.25
5	0	0	0	0.9	0	0	0.45	0	0.45	0.45	0	0.33	0.33	0.33	0.25

Table 3

[26] We only reproduce the shares of player 1. Shares can easily be allocated to the other players so as to satisfy monotonicity.

4.4 *Graph structures and restrictions on dividend distributions*

Given a game, additional data can be used to place restrictions on dividend distribution. This question has been studied by van den Brink, van der Laan and Vasil'ev [2014] assuming that players are ordered following a *directed graph*. Given a finite set *N*, a directed graph $D \subset N \times N$ is a set of pairs (i, j) such that $(i,i) \notin D.$ The set of directed graphs is denoted by $\mathscr{D}$. Nodes are players and $(i, j) \in D$ means that *i* "precedes" *j*. For a given game, the payoff of a player then depends not only upon the characteristic function but also on his position on the graph.[27]

More specifically, van den Brink et al. suggest that, for any given pair of players, the shares in the dividends of all coalitions containing them must be larger or equal for the player that precedes the other:

$$(i, j) \in G \ \Rightarrow\ \lambda_i^S \geq \lambda_j^S \ \text{ for all } S \text{ containing } i \text{ and } j. \qquad (26)$$

Given a game (N,v) and a graph $D \in \mathscr{D}$, this condition leads to a subset of the Harsanyi set that we denote by $H^g(N,v,D)$.

The authors consider only positive games, in which case $H^g(N,v,D)$ is a subset of the core by Proposition 6, that they call *Harsanyi constrained core*. Core allocations are then H-payoff vectors derived from Bayesian consistent distributions and $(i, j) \in D$ is equivalent to $\lambda_i^N \geq \lambda_j^N$. Equivalently, core allocations are weighted values and $(i, j) \in D$ is equivalent to $w_i \geq w_j$. They prove the following two propositions, the second one applying exclusively to positive games.

Proposition 13. For any given game (N,v) and graph $D \in \mathscr{D}$, the Shapley value $SV(N,v)$ is an element of $H^g(N,v,D)$. Furthermore, the Shapley value is the *unique* element of $H^g(N,v,D)$ if and only if *D* is the *complete* directed graph $\bar{D} = \{(i, j) \in N \times N \mid i \neq j\}$.

Proof. The Shapley value is defined by $\lambda_i^S = 1/s$ for all $i \in S$ and $S \subset N$ and therefore (26) is verified for all graph $D \in \mathscr{D}$. If $D = \bar{D}$, $\lambda_i^S = \lambda_j^S$ for all $i, j \in S$ and $S \subset N$ and $\lambda_i^S = 1/s$ for all $i \in S$ and $S \subset N$. Hence, $H^g(N,v,\bar{D}) = \{SV(N,v)\}$. Next, consider a graph *D* such that $(j,k) \notin D$ for some *j* and *k*, $j \neq k$, and the unanimity game $(N,u_{\{j,k\}})$. From Remark 1, the allocation $\bar{x} \in \mathbb{R}^n$ defined by $\bar{x}_k = 1$ and $\bar{x}_i = 0$ for all $i \neq k$ belongs to $H^g(N,u_{\{j,k\}},D)$. It differs from the Shapley value $SV(N,u_{\{j,k\}})$ which allocated 1/2 to *j* and *k*. □

Proposition 14. For any *positive* game $(N,v) \in G^+(N)$ and graph $D \in \mathscr{D}$, we have:

$$H^g(N,v,D) = C(N,v) \ \text{ if and only if } \ D = \varnothing.$$

---

[27] Another instance of graph-driven restrictions on distributions is given by van den Brink, van der Laan and Pruzhansky [2011] who consider games with communication graphs à la Myerson [1977]. Their idea is to link the dividend distribution to the "power" of players as measured for instance by the size of their neighborhood.

Proof. When $D=\varnothing$, $H^g(N,v,D)=H(N,v)$ and $H(N,v)=C(N,v)$ by Proposition 6. When instead $D\neq\varnothing$, there exist *j* and *k* such that $(j,k)\in D$ and the allocation $\overline{x}$ defined in the proof of Proposition 13 does not belong to $H^g(N,u_{\{j,k\}},D)$. However, it belongs to $C(N,u_{\{j,k\}})$. □

van den Brink et al. [2014] also characterize axiomatically the Harsanyi constrained core on the class $G^+(N)$ of positive games. A solution associates a subset $\Phi(N,v,D)$ to any game $(N,v)\in G^+(N)$ and directed graph $D\in\mathcal{D}$. They show that the Harsanyi constrained core is *maximal* among the solutions satisfying efficiency, null player property, additivity, weak positivity, together with the additional property of *structural monotonicity* defined by:

> For all game $(N,v)\in G^+(N)$ and directed graph $D\in\mathcal{D}$, allocations $x\in\Phi(N,v,D)$ are such that $x_i\geq x_j$ if $(i,j)\in D$ and *i* is necessary to *j* in (N,v).

4.5 *An illustration: liability games*

Liability games have been introduced in Dehez and Ferey [2013].[28] They cover situations where damage has been caused to a victim by several tortfeasors. The causality question is solved once the damage $v(S)$ that the members of any coalition *S would have caused* is known, their *potential* damage. The problem is to divide the actual damage $v(N)$ between the *n* tortfeasors. In this framework, the symmetric Shapley value stands as a benchmark from which a judge may deviate if he considers that some tortfeasors are faultier than others. Furthermore, the core has an interesting interpretation: core allocations of a liability game are *fair judgments* in the sense that they satisfy the following two (equivalent) conditions:

- no coalition of players contributes *less than its potential damage*
  $x(S)\geq v(S)$ for all $S\subset N$,
- no coalition of players contributes *more than its additional damage*
  $x(S)\leq v(N)-v(N\backslash S)$ for all $S\subset N$.

Here we will consider the sequential case, usually considered as a difficult one in the legal literature. Following the natural order 1,2,…,*n*, each player is responsible for an additional damage, d_i for player *i*. The associated game (N,v) is then given by:

$$
\begin{aligned}
v(S)&=0 && \text{if } 1\notin S,\\
v(S)&=d_1 && \text{if } 1\in S \text{ and } 2\notin S,\\
v(S)&=d_1+d_2 && \text{if } 1,2\in S \text{ and } 3\notin S.
\end{aligned}
$$

and so on… Defining $T_i=\{1,...,i\}$ as the set of successive players, starting with 1 and ending with *i*, the characteristic function can be written as:

[28] The legal aspects are documented in Dehez and Ferey [2016].

$$v(S)=\sum_{i\in N} d_i\, u_{T_i}(S)$$

and Harsanyi dividends are given by:

$$\begin{aligned}\alpha_T(N,v) &= d_i \quad \text{if } T=T_i \\ &= 0 \quad \text{otherwise}\end{aligned} \qquad (27)$$

Hence, sequential liability games are positive (and thereby convex) and all solutions coincide with the core.[29] In the 3-player case, the vector of dividends is given by $(d_1,0,0,d_2,0,0,d_3)$.

The problem is to divide the total damage $v(N)=d_1+...+d_n$ among the n players. The resulting allocation specifies the compensation that each player must pay to the victim. Using (8), it can be verified that the core of a liability game can be written in terms of the T_i's:

$$C(N,v)=\left\{x\in\mathbb{R}^n_+ \mid x(N)=d(N) \text{ and } x(T_i)\geq v(T_i) \text{ for all } i\in N\right\}.$$

We observe that the allocation that imposes to the first player to pay the entire damage as well as the allocation $x=d$ that imposes to players to pay each his additional damage are core allocations. Sequential liability games being convex, the core is the polytope whose vertices are the 2^{n-1} distinct marginal contribution vectors.

In the 3-player case, there are four distinct marginal contribution vectors:

$$\begin{aligned}&\mu^{(1,2,3)}=(d_1,d_2,d_3),\\ &\mu^{(1,3,2)}=\mu^{(3,1,2)}=(d_1,d_2+d_3,0),\\ &\mu^{(2,1,3)}=(d_1+d_2,0,d_3),\\ &\mu^{(2,3,1)}=\mu^{(3,2,1)}=(d_1+d_2+d_3,0,0).\end{aligned}$$

The core and the Shapley set coincide. Hence, fair judgments can be defined equivalently as weighted Shapley values with normalized weights w or Harsanyi payoff vectors with distribution matrix λ satisfying the Bayesian consistency condition (24) such that $\lambda^N=w$. In the context of Tort Law, weights can be used to reflect differences in the degree of misconduct or negligence. Given (27), the H-payoff vector associated to $\lambda^N>>0$ defines the following apportionment rule:

$$SV_i(N,v,w)=\sum_{j=i}^{n}\frac{\lambda_i^N}{\lambda^N(T_j)}\,d_j. \qquad (28)$$

[29] Liability games are dual of airport games, known to be positive games.

In the 3-player case, for positive weights, (28) reduces to:

$$x_1 = d_1 + \lambda_1^{12} d_2 + \lambda_1^{123} d_3 = d_1 + \frac{\lambda_1^{123}}{\lambda^{123}(12)} d_2 + \lambda_1^{123} d_3,$$

$$x_2 = \lambda_2^{12} d_2 + \lambda_2^{123} d_3 = \frac{\lambda_2^{123}}{\lambda^{123}(12)} d_2 + \lambda_2^{123} d_3,$$

$$x_3 = \lambda_3^{123} d_3.$$

This triangular formula shows the one-to-one relationship that exists under convexity between the relative interior of the core and the relative interior of the unit simplex: to each interior core allocation is associated one and only one weight vector in $\lambda^N \in \operatorname{int} \Delta_n$ and vice versa. For boundary core allocations, some players may be exempted and there is indeterminacy if there exists $j,\ 1 < j < n,$ such that $\lambda^N(T_i) = 0$ for $i = 1, 2, ..., j.$ If it is the case, then any distribution λ^S on T_i for $i = 1, 2, ..., j$ is possible, for all $S \subsetneq N.$

As a matter of illustration, consider the case where n = 4. If $\lambda^{1234} = (0, 0, 0, 1),$ we have

$$\lambda^{12} = (a, 1-a, 0, 0) \text{ and } \lambda^{123} = (b, c, 1-b-c, 0)$$

for some $a, b, c \in [0,1]$ such that $b + c \leq 1.$ The choice a = 1/2 and b = c = 1/3 corresponds to the Shapley value restricted to the player set {1,2,3}. In the case n = 3, if $\lambda^{123}(12) = 0,$ we have

$$\lambda^{123} = (0,0,1) \text{ and } \lambda^{12} = (a, 1-a, 0)$$

for some $a \in [0,1].$ It corresponds to the allocation $(d_1 + a\, d_2, (1-a) d_2, d_3).$ The choice of a = 1/2 corresponds to the Shapley value restricted to the player set {1,2}. Instead, the allocations (d_1, d_2, d_3) and $(d_1 + d_2, 0, d_3)$ corresponds to a = 0 and a = 1 respectively. The allocation $(d_1 + d_2 + d_3, 0, 0)$ that exempts players 2 and 3 is associated to the weight vector $\lambda^{123} = (1, 0, 0).$ The allocation $(d_1, d_2 + d_3, 0)$ that exempts player 3 is associated to the weight vector $\lambda^{123} = (0, 1, 0).$ In this way, we have covered the four vertices of the core.

The symmetric Shapley value is given by:

$$SV_i(N, v) = \sum_{j=i}^{n} \frac{1}{j} d_j.$$

In the 3-player case, we get the following allocation:

$$x_1 = d_1 + \frac{1}{2} d_2 + \frac{1}{3} d_3,$$

$$x_2 = \frac{1}{2} d_2 + \frac{1}{3} d_3,$$

$$x_3 = \frac{1}{3} d_3.$$

It is important to observe that, *as a rule*, weighted values (or Harsanyi payoffs) are such that no one is liable for damage caused *downstream* in the sequence: what player i contributes depends only on $(d_i,...,d_n)$. This is a characteristic of the core.

5. Concluding Remarks

Other solution concepts could be considered, for instance the *nucleolus* introduced by Schmeidler [1969]. When the core is non-empty, it is a core selection and we know that it is then a particular H-payoff vector. Can it be characterized in terms of dividend distributions? The answer is negative: sequential liability games offer a counter-example. As shown in Dehez and Ferey [2013], the *nucleolus* of a 3-player sequential liability game is given by

$$\begin{aligned}\varphi(N,v) &= \left(d_1+\frac{d_2}{2}+\frac{d_3}{4},\frac{d_2}{2}+\frac{d_3}{4},\frac{d_3}{2}\right) && \text{if } d_3 \le 2d_2, \\ &= \left(d_1+\frac{d_2+d_3}{3},\frac{d_2+d_3}{3},\frac{d_2+d_3}{3}\right) && \text{if } d_3 \ge 2d_2.\end{aligned}$$

In the first case, it is the average of core's vertices which coincides with the H-payoff associated to $\lambda_1^{12}=1/2$ and $\lambda_1^{123}=\lambda_2^{123}=1/4.$ In the second case, it is the *equal loss* allocation to which it is not possible to associate an admissible distribution matrix. Furthermore, as an apportionment rule, the nucleolus violates the "downstream" condition: in the second case, what player 3 contributes depends upon damage caused by player 2.

Among the questions that remain open, there is the identification of restrictions on dividend distributions such that the resulting H-payoff vectors are imputations. Following Proposition 11 and individual rationality of random order values for 0-monotonic games, we know that it is the case under strong monotonicity for 0-monotonic games. There may be some weaker restrictions. At this stage, we have only learned that monotonicity is not sufficient even in the case of convex games. Another question concerns the set of H-payoffs resulting from restrictions on dividend distributions associated to graphs, possibly combined with monotonicity and/or convexity.

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Appendix

Proof of Lemma 1. Consider a coalition $S \subset N$ and a player $i \notin S$. By superadditivity, we have $v(S \cup i) \geq v(S) + v(i)$. Hence, if $v(i) \geq 0$ for all $i \in N$, adding a player to a coalition does not decrease its worth. This extends to the addition of any number of players. Positivity of v follows using the above inequality, starting with $S = \{j\}, j = 1, ..., n$. □

Proof of Lemma 2. Consider a coalition $T \subset N \setminus i$ not containing j. We already know that $\alpha_T = v(j) = 0$ if $T = \{j\}$. Assume now that $\alpha_S = 0$ for all $S \subset T$ such that $j \in S$. We proceed by induction using (2). Because i is necessary for j in (N, v), we have:

$$\alpha_T = v(T) - \sum_{S \subset T} \alpha_S = v(T) - \sum_{S \subset T \setminus j} \alpha_S = v(T) - v(T \setminus j) = 0. \quad \square$$

Proof of Lemma 3. Given a player set N, consider two set functions $v_1, v_2 \in G(N)$ and an allocation $x \in C(N, v_1) + C(N, v_2)$. Hence there exist x^1 and x^2 such that $x = x^1 + x^2$, $x^1(S) \geq v_1(S)$ and $x^2(S) \geq v_2(S)$ for all $S \subset N$. Consequently, we have:

$$x(S) = x^1(S) + x^2(S) \geq (v_1 + v_2)(S) \text{ for all } S \subset N$$

i.e. $x \in C(N, v_1 + v_2)$. □

Proof of Lemma 4. Given a player set N, consider two set functions $v_1, v_2 \in G(N)$ and the corresponding marginal contribution vectors μ^1 and μ^2 as defined by (5). By definition of convex hull (Rockafellar [1970]), we have:

$$\begin{aligned} W(N, v_1) + W(N, v_2) &= co\{\mu^1(\pi) \mid \pi \in \Pi_N\} + co\{\mu^2(\pi) \mid \pi \in \Pi_N\} \\ &= co\left(\{\mu^1(\pi) \mid \pi \in \Pi_N\} + \{\mu^2(\pi) \mid \pi \in \Pi_N\}\right). \end{aligned}$$

The marginal contribution vectors associated to the game $(N, v^1 + v^2)$ are the sum of the marginal contribution vectors. Hence, $W(N, v^1 + v^2) = co\{\mu^1(\pi) + \mu^2(\pi) \mid \pi \in \Pi_N\}$ where

$$\{\mu^1(\pi) + \mu^2(\pi) \mid \pi \in \Pi_N\} \subset \{\mu^1(\pi) \mid \pi \in \Pi_N\} + \{\mu^2(\pi) \mid \pi \in \Pi_N\}.$$

Consequently, by the definition of the convex hull, we have:

$$co\{\mu^1(\pi) + \mu^2(\pi) \mid \pi \in \Pi_N\} \subset co\left(\{\mu^1(\pi) \mid \pi \in \Pi_N\} + \{\mu^2(\pi) \mid \pi \in \Pi_N\}\right). \quad \square$$

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