Optimal regime switching and threshold effects: theory and application to a resource extraction problem under irreversibility*

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May 22, 2012

Abstract

We consider a general control problem with two types of optimal regime switch. The first one concerns technological and/or institutional regimes indexed by a finite number of discrete parameter values, and the second features ecological-like regimes relying on given threshold values for given state variables. We propose a general optimal control framework allowing to derive the first-order optimality conditions and in particular to characterize the geometry of the shadow prices at optimal switching times (if any). We apply this new optimal control material to address the problem of the optimal management of natural resources under ecological irreversibility, and with the possibility to switch to a backstop technology.

Key words: Multi-stage optimal control, threshold effects, irreversibility, non-renewable resources, backstop technology

JEL classification: Q30, Q53, C61, O33.

*We'd like to thank Hippolyte d'Albis, Antoine Bommier, Jean-Michel Grandmont, Cuong Le Van, Katheline Schubert, Georges Zaccour and participants in conferences and seminars in Montreal, Paris, Toulouse and Zurich for invaluable comments and suggestions. Boucekkine acknowledges the financial support of a Belgian ARC on sustainability. The usual disclaimer applies.

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1 Introduction

While regime switching modelling is a quite old and known tool in economics and econometrics (see Hamilton, 1990, for a seminal presentation, and Zampolli, 2006, for a recent application to monetary policy), the economic and mathematical literatures of optimal regime switching are probably much less known. In the latter context, regime switching is a decision variable taken by the economic agents or the whole economy on the basis of explicit or implicit trade-offs, and not the result of random (Markovian or non-Markovian) processes. Typically, regimes refer to institutional and/or technological states of the world. For example, an economy starting with a given technology might find it optimal to switch to a newly available technology or to stick to the old one. Similarly, an economy initially in autarky might decide to switch to full or partial financial liberalization letting international assets flow in and out. More institutional examples can be easily picked: an economy initially out of international agreements (like the Kyoto Protocol) might decide to join them or remain out forever. In all cases, the corresponding switching decision is shaped by the inherent nontrivial trade-offs under scrutiny. For example, in the first technological example, the superiority of a newly available technology does not necessarily lead to immediate switching to this technology (or immediate adoption) because of the associated obsolescence and learning costs (see Parente, 1994, or Boucekkine et al., 2004). In the second and more institutional example, switching to full financial liberalization is beneficial because it brings more resources to the economy, but it also induces a clear cost through the external debt burden (see Makris, 2001).

Since the mid 80s, a substantial optimal control literature has been devoted to handle the class of optimal switching problems described above. This literature is mostly concerned with deterministic setting. We shall consider the same framework in this paper.[1] Tomiyama (1985) and Amit (1986) are, to our knowledge, the earliest contributors to the related optimal control literature. Interestingly enough, these two authors reformulate the optimal switching problem as an optimal timing problem, therefore introducing the time of switch as an explicit decision variable. Immediate switching and sticking to the initial regime (or never switching) correspond to the corner solutions of the problem in this setting. The authors have accordingly developed the maximum principle fitting the general optimal switching problems considered, with a special attention to the geometric properties of the shadow prices at the switching times (if any). Extensions of the maximum principle

[1]Optimal switching problems under uncertainty can be found in the literature, see for example Pommeret and Schubert (2009).

developed to any finite number of switching dates (corresponding to any finite number of successive technological and/or institutional regimes) have been already provided (see for example Saglam, 2011). Finally, dynamic games settings encompassing optimal switching problems à la Tomiyama have been recently considered (see Boucekkine et al., 2011).

A common feature of all the problems studied in the literature outlined above is that the technological and/or institutional regimes are exogenously given. Precisely, these regimes are differentiated through a finite number of discrete parameter values. For example, a newly available technological regime may exhibit a higher productivity parameter but a larger abatement cost parameter compared to the currently used technology. Switching or not to the new technology and the related optimal timing strategy depend on the induced efficiency/environmental friendliness tradeoff. In this paper, we encompass another type of regime switching problems related to the notion of **irreversibility**. To fix the ideas, let us give an immediate example of what's meant by irreversibility and the corresponding optimal switching problem. Consider the standard pollution problem where Nature absorbs part of the pollution stock, giving rise to the so-called rate of natural decay of pollution. Irreversibility comes to the story here as follows: when the pollution stock exceeds a certain threshold value, the natural decay rate goes down permanently, that's in an irreversible way. Accordingly, we get here another regime switch, from reversible to irreversible pollution, a kind of ecological transition governed by the law of motion of pollution and an associated threshold value. This is in contrast to the optimal regime switching problems à la Tomiyama where the successive regimes are indexed by a finite number of discrete parameter values and not by threshold values on state variables. Optimal switching problems featuring irreversibility in the sense given just above have been already studied by Tahvonen and Withagen (1996), and more recently by Prieur et al. (2011). In particular, it is shown that similarly to the optimal regime switching problems à la Tomiyama, one can formulate them as optimal timing problems, the date at which the threshold value is reached (if any) being again an explicit control variable.

In this paper, we consider the general problem where both types of regime switch co-exist. It is easy to understand why such problems are highly relevant from the economic point of view. Consider the ecological problem with irreversibility described above and allow the economy to also decide about whether to switch to a newly available technology, cleaner but less productive than the pre-existing one. Clearly enough, the two switching problems, the ecological and the technological optimal switching problems, will "interact". Indeed, the possibility to choose (at a certain optimal date) a cleaner technology might decisively shape the decision to go or not for an ecological switch. To our knowledge, the unique paper considering this type of problems is Boucekkine et al. (2012). The authors show that the interaction of two types of switching problems may generate a wide variety of relationships between pollution and capital (as a proxy for output), mostly inconsistent with the environmental Kuznets curve.

This paper goes much beyond the previous contribution and proposes a general appraisal of optimal switching problems involving both types of regimes: technological and/or institutional regimes indexed by a finite number of discrete parameter values, and ecological-like regimes which rely on given threshold values for given state variables. We propose a general optimal control framework allowing to derive the first-order optimality conditions and in particular to characterize the geometry of the shadow prices at optimal switching times (if any). We do this using standard optimal control techniques and we do obtain a clear-cut characterization of how the optimal solutions look like in this sophisticated control framework.[2]

A second contribution of this paper is to apply this new optimal control material to address the problem of the optimal management of natural resources under ecological irreversibility and the possibility to switch to a backstop technology. For that purpose, we extend the classical exhaustible-resource/stock-pollution model, studied by Tahvonen (1997), by introducing the two types of regime switchings discussed so far. We first have irreversible pollution as in Tahvonen and Withagen (1996) and Prieur (2009). The economy originally extracts and consumes a non-renewable resource, say fossil fuels. Consumption causes emissions to be released and contributes to the accumulation of a stock of pollutant. Beyond a critical pollutant concentration, the decay rate of pollution abruptly and permanently vanishes. This ecological switch is triggered by a threshold level defined on the state variable. The economy has also a backstop technology available. This technology can supply a fixed share of the overall energy consumption at a constant unit cost. Under perfect substitution, adopting the backstop allows for the replacement of the polluting energy source with the clean renewable source, like solar or wind power. In the same vein as Valente (2011), we model backstop adoption as an optimal switching problem. This is a means to capture the second type of regime switch linked to technological change. So, the economy has to decide whether it switches to the backstop and **when**. Our work is therefore a natural extension of Prieur et al. (2011) who study the optimal management of exhaustible resources under irreversible pollution but do not deal with the

[2] An alternative would have been to use more complex techniques belonging to hybrid control theory as in Shaikh and Caines (2007) commonly used in engineering. Given the specificity of the general hybrid problems involved in economics (see next section), our approach seems more natural and more accessible to the economists community.

backstop technology adoption issue.

In this framework, there naturally exist multiple optimality candidates that correspond to every possible combination of ecological and technological regimes. Our first aim is to discuss the existence conditions of both irreversible solutions (that's, those characterized by the occurrence of an ecological switch) and reversible solutions (those entailing a technological switch). This analysis is conducted in the simplest case where the natural resource is abundant. The next important question is whether the optimal policy is reversible or not. This point is addressed is the case of a scarce resource, notably by numerical solution of a calibrated version of the model. For the sake of comparison, we take the plausible parameters values used by Prieur et al. (2011). This allows us to stress the impact of technological change on the nature of the optimal policy. Our analysis notably emphasizes the role of the initial resource stock and of the unit cost of the backstop. The main conclusion is that with the opportunity to adopt a backstop technology, irreversible policies, that's the policies leading to the irreversible ecological regime, become more worthwhile in the sense that they do exist for lower initial resource stocks compared to Prieur et al. (2011). It also appears that when the cost of the backstop is low enough, irreversible policies yield the optimal solution as soon as they exist. At first glance, this result seems to be surprising but upon reflexion, it is very intuitive. For the optimal policy to be irreversible, the endowment in resource should be high enough so that the economy is able to compensate (irreversible) pollution damages by consuming a lot. Now, adopting the backstop after the crossing of the irreversibility threshold is a means to keep environmental damages under control and explains why irreversible solutions are more likely.

The paper is organized as follows. Section 2 presents a general framework that encompasses the class of economic problems featuring both technological and/or institutional regimes and ecological-like regimes. Section 3 develops the optimal control techniques suitable for dealing with these problems. Section 4 is devoted to the application of the theory to the optimal management of exhaustible resources under ecological irreversibility and backstop adoption. Section 5 concludes.

2 The problem

We consider the following formal model with two types of state variables, x_1 and x_2; x_1 and x_2 could be in any space R^k_+, $1 \leq k \leq K$. To simplify the exposition, we set $k = 1$. Much more crucially, we shall distinguish between the two state variables in a fundamental way: while the law of motion of

x_1 depends on the i-regime on which the economy lays, $i = 1, ..., I$, with $2 \leq I < \infty$, which is exogenously determined (that's, independently of the state and control variables of the problem), the law of motion of variable x_2 depends on the j-regime on which the economy lays, $j = 1, ..., J$, with $2 \leq J < \infty$, the j-regimes being determined by an increasing sequence of threshold values, $\bar{x}_2^j$, $j = 0, ..., J-1$, with $\bar{x}_2^0 = 0$: the economy is in a j-regime if $\bar{x}_2^{j-1} \leq x_2 < \bar{x}_2^j$, for $j = 1, .., J-1$, and is in regime J if $x_2 \geq \bar{x}_2^{J-1}$.

The control set U is defined as follows:

$$U = \{u_m \in R, 1 \leq m \leq M < \infty\} \cup \{t_{1i}^+ \in R_+, 1 \leq i \leq I-1\} \cup \{t_{1i}^- \in R_+, 2 \leq i \leq I\}$$

$$\cup \{t_{2j}^+ \in R_+, 1 \leq j \leq J-1\} \cup \{t_{2j}^- \in R_+, 2 \leq j \leq J\}.$$

We distinguish between two types of control variables, the optimal timing variables, t_{1i}^k and t_{2j}^k, $k \in \{+, -\}$ and the non-timing variables, u_m. In particular, t_{1i}^+, $i = 1, ..., I-1$, refers to the time at which the economy switches from the i-regime i to the i-regime $i+1$. Similarly, t_{2j}^+, $j = 1, ..., J-1$, refers to the time at which the economy switches from the j-regime j to the j-regime $j+1$. By definition of regimes j, we have: $x_2 \left(t_{2j}^+\right) = \bar{x}_2^j$, $x_2(t) \in [\bar{x}_2^{j-1},\ \bar{x}_2^j]$ in a neighborhood to the left of t_{2j}^+, and $x_2(t) \in [\bar{x}_2^j,\ \bar{x}_2^{j+1}]$ in a neighborhood to the right of t_{2j}^+ . *A priori*, backward switchings are also possible: t_{2i}^- (Resp. t_{2j}^-) denotes the time at which the economy switches from regime i (Resp. j) to $i-1$ (Resp. $j-1$). Again by definition of regimes j, we have: $x_2 \left(t_{2j}^-\right) = \bar{x}_2^{j-1}$, $x_2(t) \in [\bar{x}_2^{j-1}\ \bar{x}_2^j]$ in a neighborhood to the right of t_{2j}^-, and $x_2(t) \in [\bar{x}_2^{j-2}\ \bar{x}_2^{j-1}]$ in a neighborhood to the left of t_{2j}^- . Some comments are in order here. First of all, our framework encompasses either irreversible or reversible regime changes. Problems with irreversible choices result in smaller control sets since in these cases the timing variables t_{1i}^- and t_{2j}^- are irrelevant. In our Section 4, we examine the irreversible case: a problem with irreversible pollution and with technology adoption in which switching back is not allowed.

Second, by definition of the j-regimes, there is no option for the economy to jump directly from regime j to regime $j+2$, $j \in \{1, ..., J-2\}$, because this implies a discrete jump in the state variable x_2; we shall assume as in the standard optimal control theory case that all the optimal trajectories of the state variables should be continuous and piecewise differentiable. In contrast, jumping from regime i to $i+2$, $i = \{1, ..., I-2\}$ is possibly optimal since it is not related to a jump in a state variable. In the example considered in Section 4, regime i is linked to the realization of a given vector of exogenous variables, say $Z \in R^n$, $n \geq 1$, taking I discrete values $\{Z_1, Z_2, ..., Z_I\}$: the economy is in regime i if $Z = Z_i$, which happens by definition if $t \in [t_{1i-1}^+,\ t_{1i}^+[$. Therefore, it is possible for the economy to optimally jump from regime i, given by Z_i, to regime $i+2$ given by Z_{i+2}. In such a case, one gets the corner solution: $t_{1i}^{+} = t_{1,i+1}^{+}$. Third, a fundamental property of the problem is that while the timing variables t_{2j}^{+}, $j = 1, ..., J-1$, are ordered by construction, and that the timing variables t_{1i}^{+}, $i = 1, ..., I-1$, could be also naturally ordered once the underlying sequence of exogenous realizations $\{Z_1, Z_2, ..., Z_I\}$ is ordered, there is no natural ordering at all between the two sets of timing variables, t_{2j}^{+} and t_{1i}^{+}: a switch through x_1 may precede, coincide or follow a switch through x_2. This is the main complication of our setting. It goes without saying that allowing for reversibility complicates massively the analysis. In the following, we will assume irreversibility, the control set is then reduced to:

$$U' = \{u_m \in R, 1 \leq m \leq M < \infty\} \cup \{t_{1i}^{+} \in R_+, 1 \leq i \leq I-1\} \cup \{t_{2j}^{+} \in R_+, 1 \leq j \leq J-1\},$$

which can be written without any risk of confusion under irreversibility as:

$$U' = \{u_m \in R, 1 \leq m \leq M < \infty\} \cup \{t_{1i} \in R_+, 1 \leq i \leq I-1\} \cup \{t_{2j} \in R_+, 1 \leq j \leq J-1\}.$$

State equations

We now specify precisely the general optimal control problem under study. We start with the law of motions of the state variables. Naturally enough, the laws of motion for x_1 and x_2 depend on which state the economy is laying. By construction, the state of the economy is determined here by the couples (i, j), $i = 1, ..., I$ and $j = 1, ..., J$, describing on which regime x_1 and x_2 respectively are laying. We shall introduce the following definitions to ease the exposition.

Definition 2.1
- *The economy is said **in state** (i, j) if x_1 is in regime i, $i = 1, ..., I$, and x_2 is in regime j, $i = 1, ..., J$.*
- *We say that the economy i-**switches** (Resp. j-**switches**) at time $t \geq 0$ if and only if $t = t_{1i}$ (Resp. $t = t_{2j}$).*

Because the dynamics of x_1 and x_2 are generally inter-dependent, the general laws of motion, and concretely the state functions, should account for the realization of both the i-regimes and j-regimes. Suppose the economy is in state (i, j) on a given time interval (with nonzero measure), then the law of motions of both variables will be written as follows:

$$\dot{x}_1 = G_1^{ij}(u_1, ..., u_M, x_1, x_2; Z_i)$$

and

$$\dot{x}_2 = G_2^{ij}\left(u_1, ..., u_M, x_1, x_2; Z_i\right).$$

Functions G_k^{ij}, $i = 1, ..., I$, $j = 1, ..., J$ and $k = 1, 2$ are some given smooth functions, we shall be more precise about smoothness later. For example in the minimal case where $I = J = 2$, 4 state functions are to be specified to describe the 4 states on which the economy can lay depending on the occurrence of i-regimes and j-regimes. Importantly enough, we shall make the following assumption on the economy's initial position.

Hypothesis 2.1 *The economy is initially in state* $(1, 1)$*, that it is initially* $Z(0) = Z_1$ *and* $x_2(0) \in [0\ \bar{x}_2^1[$*.*

Also from now on, we will remove Z_i from the list of inputs of functions G_k^{ij} as the upper-index i of these functions is enough to reflect the dependence of these functions on i. More important, it is crucial to note that not all the states are necessarily optimal. For example switching the i-regimes may not be optimal while switching the j-regimes may be so. In such a case, only the states $(1, j)$, $j = 1, ..., J$, may be optimal. We shall be more precise about optimality hereafter.

Optimality

We postulate that the objective function to be maximized in the control set U' takes the form:

$$V(x_0^1, x_0^2) = \int_0^\infty F\left(u_1, ..., u_M, x_1, x_2\right)\ e^{-\delta t} dt,$$

where $x_0^1 = x_1(0)$ and $x_0^2 = x_2(0)$, and $F(.)$ is another smooth function. The exact statement of the state equations depends on the ordering of the possible i-switches timings with respect to those of the possible j-switches. In other terms, the state equations depend on how the controls t_{1i} situate, compared to the controls t_{2j}. No ordering can be *a priori* excluded. If all the timing variables are strictly ordered, then the time support, R_+ is divided in $I + J - 1$ time sub-intervals. This is the maximal number of sub-intervals to be considered, the lower bound being 1, namely when no i-switch nor j-switch is undertaken (that's the starting state $(1, 1)$ is optimal for all t). The technique developed to solve the problem relies on a recursive scheme on the successive sub-problems induced by the restriction of the problem to these sub-intervals. This technique has been first put forward by Tomiyama (1985) and Amit (1986), and more recently applied to economics by Boucekkine et al. (2004). In all the latter papers, only i-switches are considered. Boucekkine et al. (2012) is the first application to problems involving both i-switches and j-switches. As mentioned in the introduction, these authors consider a specific linear-quadratic environmental problem with one possible technological switching and one possible ecological switching. A variant of it is re-examined in the examples section. This paper generalizes their approach.

Before developing more in detail this approach, some technical assumptions are needed.

Hypothesis 2.2

- *Functions $F(.)$ and G_k^{ij}, i, j, k are smooth and are all assumed to be of class C^1.*
- *When the economy is in state (i, j), the optimal control sub-problem, restriction of the optimal control problem to state (i, j) for given timing variables, that is with restricted control set $U'' = \{u_1, u_2, ..., u_M\}$, verifies the Arrow-Kurz sufficiency conditions.*

The assumption will not only ensure that the optimal control problems and sub-problems encountered along the way are well-behaved, it also allows to use safely some envelope properties requiring the differentiability of the value-function. In particular, the second part of the hypothesis ensures the latter property: if Arrow-Kurz conditions are met, the optimal control problems and sub-problems to be handled are concave enough for the involved value-functions to be differentiable (see Theorem 9, page 213, in Seierstad and Sydsaeter, 1987). The definition of the sub-problems in the statement of Hypothesis (2.2) will be clear when the multi-stage optimal control used will be detailed in the next section.

Finally, as mentioned above, we do not depart from the traditional functional spaces in control theory: we keep on seeking for solutions where the control $u(t)$ in piecewise continuous on R_+ and the state variables, $x_i(t)$, $i = 1, 2$, are continuous, piecewise differentiable on R_+.

3 Theory

To ease the exposition, we shall start with the detailed analysis of the minimal case $I = J = 2$, which can be considered as the first generalization of the work of Boucekkine et al. (2012).

3.1 The case $I = J = 2$

We consider to this end the following formal model with two states, x_1 and x_2, and three controls $u = u_1$, $t_1 = t_{11}$ and $t_2 = t_{21}$. In particular, we have

$x_2(t_2) = \bar{x}_2^1 = \bar{x}_2$, where $\bar{x}_2$ is the threshold value of state variable x_2. We set $M = 1$ for simplicity. As mentioned above, the exact statement of the state equations depends on the position of t_1 with respect to t_2. We shall write the first-order conditions corresponding to all possible orderings, including the special case $t_1 = t_2$. We also consider the possibility to have either immediate i-switch or no switch at all in last place.

3.1.1 The case $t_1 < t_2$

In this case, we have the following laws of motion:

$$\dot{x}_1 = \begin{cases} G_1^{11}(u, x_1, x_2) & \text{if } 0 \le t < t_1 \\ G_1^{21}(u, x_1, x_2) & \text{if } t_1 \le t < t_2 \\ G_1^{22}(u, x_1, x_2) & \text{if } t \ge t_2 \end{cases}$$

and

$$\dot{x}_2 = \begin{cases} G_2^{11}(u, x_1, x_2) & \text{if } 0 \le t < t_1 \\ G_2^{21}(u, x_1, x_2) & \text{if } t_1 \le t < t_2 \\ G_2^{22}(u, x_1, x_2) & \text{if } t \ge t_2 \end{cases}$$

with both $x_1(0) = x_0^1$ and $x_2(0) = x_0^2$ given. The objective function to be maximized with respect to u, t_1 and t_2, is:

$$V(x_0^1, x_0^2) = \int_0^\infty F(u, x_1, x_2)\, e^{-\delta t} dt.$$

As in Boucekkine et al. (2004) or Tahvonen and Withagen (1996), a natural approach is to decompose the problem into several sub-problems for given timing variables, to solve each of them (which is usually very simple although not always analytically conclusive), and then to identify the optimal timings. With one timing variable, two sub-problems are involved. In our case, three would result from the decomposition. Indeed since:

$$V(x_0^1, x_0^2) = \int_0^{t_1} F(u, x_1, x_2)\, e^{-\delta t} dt + \int_{t_1}^{t_2} F(u, x_1, x_2)\, e^{-\delta t} dt + \int_{t_2}^{\infty} F(u, x_1, x_2)\, e^{-\delta t} dt,$$

under $t_1 < t_2$, one can view the problem sequentially, starting with the regime arising at the last switch, in the spirit of Tomiyama (1985). Precisely, one has to follow the following recursive scheme in our case:

- *Third interval sub-problem*: the problem in this regime is:

$$\max_{\{u\}} V_3 = \int_{t_2}^{\infty} [F(u, x_1, x_2)]\, e^{-\delta t} dt$$

subject to,

$$\begin{cases} \dot{x}_1 = G_1^{22}(u, x_1, x_2) \\ \dot{x}_2 = G_2^{22}(u, x_1, x_2) \end{cases}$$

where t_2 and the initial conditions $x_1(t_2) = x_2^1$ and $x_2(t_2) = \bar{x}_2$ are fixed. The associated hamiltonian is:

$$H_3 = F\ e^{-\delta t} + \lambda_1^{22}\ G_1^{22} + \lambda_2^{22}\ G_2^{22},$$

where λ_k^{ij} is the co-state variable associated with the state variable k when the economy is in state (i, j). The resulting value-function is of the form $V_3^*(t_2, x_2^1)$.

- *Second interval sub-problem*: in the next interval, the maximization problem is:

$$\max_{\{u, t_2, x_2^1\}} V_2 = \int_{t_1}^{t_2} F(u, x_1, x_2) e^{-\delta t} dt + V_3^*(t_2, x_2^1)$$

subject to,

$$\begin{cases} \dot{x}_1 = G_1^{21}(u, x_1, x_2) \\ \dot{x}_2 = G_2^{21}(u, x_1, x_2) \end{cases},$$

where t_1, $x_1(t_1) = x_1^1$ and $x_2(t_1) = x_1^2$ are given, and t_2 and $x_1(t_2) = x_2^1$ are free. The corresponding hamiltonian is

$$H_2 = F\ e^{-\delta t} + \lambda_1^{21}\ G_1^{21} + \lambda_2^{21}\ G_2^{21},$$

and the value-function obtained is of the form $V_2^*(t_1, x_1^1, x_1^2)$.

- *First interval sub-problem*: This sub-problem considers the interval $[0\ t_1]$:

$$\max_{\{u, t_1, x_1^1, x_1^2\}} V_1 = V = \int_{0}^{t_1} F(u, x_1, x_2) e^{-\delta t} dt + V_2^*(t_1, x_1^1, x_1^2)$$

subject to,

$$\left\{ \begin{array}{l} \dot{x}_1 = G_1^{11}\,(u, x_1, x_2) \\ \dot{x}_2 = G_2^{11}\,(u, x_1, x_2) \end{array} \right. ,$$

with $x_1(0)$ and $x_2(0)$ given, and with free t_1, x_1^1, and x_1^2. H_1, the hamiltonian of the the sub-problem, is given by:

$$H_1 = F\ e^{-\delta t} + \lambda_1^{11}\ G_1^{11} + \lambda_2^{11}\ G_2^{11},$$

and it is obvious that $V_1^* = V^*$.

The recursive scheme above works exactly as a dynamic programming device as first noticed by Tomiyama (1985): one would use exactly the same scheme to handle a dynamic optimization problem in discrete time over three periods. Here the Bellman principle applies on the three intervals involved by the double timing problem instead of discrete periods of time. Also notice that given our smoothness and concavity assumptions, each optimal control sub-problem is well-behaved, we will not spend space on writing the corresponding standard Pontryagin conditions. Rather, we will focus on two aspects: uncovering the optimality conditions with respect to the timing variables and the so-called matching conditions.[3] Matching conditions refer to how the hamiltonians, H_n, $n = 1, ..., I + J - 1 = 3$, and the co-state variables λ_{ij}^k, $i, j, k = 1, 2$, behave at the optimal junction times, here t_1^* and t_2^*. This is solved by the following theorem.

Theorem 3.1 *Under Hypothesis (2.2), one gets at the optimal junction times t_1^* and t_2^*:*

$$H_2^*(t_2^*) = H_3^*(t_2^*), \tag{1}$$

$$\lambda_1^{22*}(t_2^*) = \lambda_1^{21*}(t_2^*), \tag{2}$$

$$H_1^*(t_1^*) = H_2^*(t_1^*), \tag{3}$$

$$\lambda_1^{21*}(t_1^*) = \lambda_1^{11*}(t_1^*), \tag{4}$$

and

$$\lambda_2^{21*}(t_1^*) = \lambda_2^{11*}(t_1^*). \tag{5}$$

[3] As we will see later one, there is a non-empty intersection between these two types of conditions

A detailed proof is in the Appendix. A few comments are in order here. First of all, one can read the 5 optimality conditions above as continuity or matching conditions at the junction points. In this respect, conditions (1) and (3) impose the continuity of the hamiltonian at the optimal junction times while the other conditions ensure the continuity of co-state variables at these times. Interestingly enough, one can observe that while at the i-switching time, both co-state variables are optimally continuous, only the one associated with x_1 is necessarily continuous at the j-switching time. This points at the major difference between the two switching types: in the latter, the state variable (here x_2) is fixed at the j-switching time, equal to the threshold value, while at the i-switching time, both state variables can be freely chosen. This generally implies discontinuity of the co-state variables associated with the state variables of the x_2 type at j-switching times in contrast to i-switching times.

Second, one can interpret the matching conditions (1)-(3) as first-order optimal timing conditions for t_2 and t_1 respectively. Generally speaking, the matching condition for timing t_n may be therefore written as:

$$H_n^*(t_n) - H_{n+1}^*(t_n) = 0,$$

for $n = 1, ..., I + J - 2$. This condition is quite common in the literature of multi-stage technological switching (see Saglam, 2011, for example), described here by switches of type i. Importantly enough, we show here that it applies also to switches of type j (here, at the switching time t_2). The explicit mathematical proof that the conditions above are indeed the first-order optimality conditions for interior t_i's maximizers is given in the Appendix using standard envelope properties on the successive sub-problems' value functions. Keeping the discussion non-technical, one may simply interpret the difference $H_n^*(t_n) - H_{n+1}^*(t_n)$, $n = 1, ..., I + J - 2$, following Makris (2001), as the marginal gain from extending the regime inherent to the time interval $[t_{n-1},\ t_n]$, with $t_0 = 0$, at the expense of the regime associated with interval $[t_n,\ t_{n+1}]$. Because there are no direct switching costs (unlike in the original works on multi-stage optimal control, more operation-research oriented as in Amit, 1986), the marginal switching cost is nil. Therefore, the matching conditions with respect to hamiltonians at the optimal switching times do equalize marginal benefits and costs of delaying the switching times. Hence they do feature first-order necessary conditions with respect to the latters.

We consider now briefly two remaining cases of interest with two interior switchings. As we we will see, they derive quite trivially from the analysis of the benchmark case.

3.1.2 Other orderings of timings

If j-switching precedes i-switching, that is if $0 < t_2 < t_1 < \infty$, then Theorem 1 can be easily reformulated using the same techniques exposed in the Appendix, after decomposition of the optimal control problem into the corresponding three sub-problems and proper definition of the associated hamiltonians and co-sate variables. Precisely, one gets the following optimal timing and matching conditions:

Proposition 3.1 *Under Hypothesis (2.2), one gets at the optimal junction times t_1^* and t_2^*:*

$$H_1^*(t_2^*) = H_2^*(t_2^*), \tag{6}$$

$$\lambda_1^{11*}(t_2^*) = \lambda_1^{12*}(t_2^*), \tag{7}$$

$$H_2^*(t_1^*) = H_3^*(t_1^*), \tag{8}$$

$$\lambda_1^{12*}(t_1^*) = \lambda_1^{22*}(t_1^*), \tag{9}$$

and

$$\lambda_2^{12*}(t_1^*) = \lambda_2^{22*}(t_1^*). \tag{10}$$

Again the hamiltonians are continuous at the optimal junction times while the co-state variable associated to x_1 is also continuous at all optimal switching times, and the co-state variable associated to x_2 is only continuous at the i-switching time.

The last two (interior) switching cases follow the same logic though the list of corresponding first-order timing and matching conditions is shorter because only two successive regimes are optimal, not three: one before $t_1 = t_2 = t^s$ and one after the simultaneous i-switch and j-switch. Similarly to the previous section, denote by H_1 and H_2 the hamiltonians corresponding to the sub-problems on the intervals $[0,\ t^s]$ and $[t^s,\ \infty)$ respectively. Precisely, denoting the hamiltonians by:

$$H_1 = F\ e^{-\delta t} + \lambda_1^{11}\ G_1^{11} + \lambda_2^{11}\ G_2^{11},$$

and

$$H_2 = F\ e^{-\delta t} + \lambda_1^{22}\ G_1^{22} + \lambda_2^{22}\ G_2^{22},$$

one gets the following proposition, which can be seen as the counterpart of Theorem (3.1).

Proposition 3.2 *Under Hypothesis (2.2), one gets at the optimal junction time $t^s = t_1 = t_2$:*

$$H_1^*(t^s) = H_2^*(t^s), \tag{11}$$

and

$$\lambda_1^{11*}(t^s) = \lambda_1^{22*}(t^s). \tag{12}$$

The proof follows exactly the same technique used to demonstrate Theorem (3.1), we skip it to save space. Naturally enough, we obtain the same qualitative results as in the previous case with distinct switching times: the hamiltonians are continuous at the optimal junction time, the co-state variable of x_1 is also continuous at that time in contrast to the co-state variable associated to x_2 which may jump. Hence despite we assume that the i-switch and j-switch coincide, the difference between the two co-state variables remain. Again, this is simply due to the fact that at $t = t^s$, the state variable x_2 is fixed, equal to the threshold value posited (that is $x_2(t^s) = \bar{x}_2$) while the state variable x_1 is free at that time.

3.1.3 Corner solutions

We now briefly discuss the existence of corner solutions. In the case $I = J = 2$, four corner solutions are possible: (i) no i-switch: $t_1 = \infty$; (ii) no j-switch: $t_2 = \infty$; (iii) immediate i-switch: $t_1 = 0$; (iv) immediate j-switch: $t_2 = 0$. The cases where there is a simultaneous and immediate i and j switch or no switch at all in both regimes are special corner situations covered by the latter categorization. Corner regimes are important and should be accounted for rigorously. They are considered in all the papers in multi-stage optimal control either for i-switching type (see Tomiayama, 1985, or more recently Saglam, 2011) or for j-switching type (see in particular, Tahvonen and Withagen, 1996).

Corner solutions can be handled in two ways. One way is to solve for the optimal control problems induced by the corner solutions in order to determine the associated value-function and to compare it with the one(s) implied by the alternative interior solution(s). If this strategy is followed, there is no need to derive necessary conditions for the occurrence of corner solutions. A more analytical approach would, in contrast, attempt to establishing this set of conditions. It is quite trivial to identify these conditions. For example, suppose we seek for necessary conditions for the corner case $t_1^* = 0$ and $t_2^* > 0$. From Section 3.1.1, we know that interior optimal timings are given by the condition

$$H_n^*(t_n) - H_{n+1}^*(t_n) = 0,$$

for $i = 1, ..., I + J - 2$, which amounts to equalize marginal benefits and costs of delaying the switching times. The same type of arguments could be used to visualize easily the kind of necessary conditions inherent to corner switching times: in our working example, $t_1^* = 0$ implies $H_1^*(0) - H_2^*(0) \leq 0$ while $t_2^* > 0$ implies $H_2^*(t_2^*) - H_3^*(t_2^*) = 0$.[4] The former inequality simply means that the marginal benefits from delaying immediate i-switching are equal to or lower than its costs.

Symmetrically, it is also straightforward to identify necessary conditions for the never-switching case. Consistently with the timing $t_1 < t_2$ of Section 3.1.1, let us do the job for the corner case $t_1^* > 0$ and $t_2^* = \infty$. The necessary conditions should be:

$$H_1^*(t_1) - H_2^*(t_1) = 0,$$

and

$$\lim_{t_2 \to +\infty} H_2^*(t_2) - H_3^*(t_2) \geq 0.$$

Again the interpretation is simple: the economy would never j-switch if the marginal benefits of delaying such a switch are asymptotically equal to or larger than its costs.[5] While the corner conditions exhibited here above are rather elementary, they are not likely to be decisive in practice, not only because they are only necessary conditions, but also because they may not yield a clear-cut analytical characterization (of the corner solution) if the problems are highly non-linear. Interestingly enough, Boucekkine et al. (2004, 2011) find them useful in models with linear production functions and log-linear or linear utility functions. Outside this class of problems, the analytical usefulness of these conditions may be questioned, and the first approach, more computational, explained at the beginning of this section, may be largely preferred.

3.2 The general case

We now consider the general case with any finite numbers of i and j-regimes ($2 \leq I, J < \infty$). The working introductory case $I = J = 2$ has been detailed enough to catch the main intuitions behind the optimality conditions (as stated in Theorem 1) and to extract the relevant methodological conclusions.

[4] Needless to say, in this case $H_2^*(t_2^*) - H_3^*(t_2^*) = 0$ has to be solved for t_2^* with t_1^* fixed to 0.

[5] The proof is slightly less trivial that in the symmetrical corner case with immediate switching as it requires the application of the envelope theorem. This is ensured here under Hypothesis (2.2). See also Tahvonen and Withagen, 1996, for an earlier analysis of such a case.

While the exact optimal timing and matching conditions strongly depend on the ordering between the i-switching times t_{1i} and the j-switching times t_{2j}, the following necessary conditions always apply in the case of interior optimal switching times, independently of the latter ordering.

Proposition 3.3 *Suppose Hypothesis (2.2) holds. Also suppose* $0 < t_{11} < ... < t_{1,I-1} < \infty$ *and* $0 < t_{21} < ... < t_{2,J-1} < \infty$ *are optimal then.*

- *for any switching time,* t_n*,* $n = 1, ..., I + J - 2$*, whatever the switching type, we have:*
$$H_n^*(t_n) - H_{n+1}^*(t_n) = 0,$$
where H_n *(Resp.* H_{n+1}*) is the hamiltonian of the optimal control sub-problem, restricted to the state where the economy is laying before (Resp. after) the switching time* t_n*.*

- *For any switching time,* t_n*,* $n = 1, ..., I + J - 2$*, whatever the switching type, we have:*
$$\lambda_1^{ij}(t_n) = \lambda_1^{i'j'}(t_n),$$
provided economy switches from state (i, j) *to state* (i', j') *at* t_n*,* λ_1^{ij} *(Resp.* $\lambda_1^{i'j'}$*) is* x_1*'s co-state variable when the economy is in state* (i, j) *(Resp.* (i', j')*).*

- *For any switching time,* t_{1i}*,* $i = 1, ..., I - 1$*, we have:*
$$\lambda_2^{ij}(t_{1i}) = \lambda_2^{i',j}(t_{1i}),$$
with, λ_2^{ij} *the co-state variable associated with* x_2 *when the economy is in state* (i, j)*.*

Proposition (3.3) is a kind of summarized wisdom of the lessons extracted from the detailed analysis of case $I = J = 2$. The proof can simply be done by elementary adaptation of the proof of Theorem 1. The first conditions on hamiltonians at switching times (whatever the type of switching) generalize conditions (1) and (3) in Theorem 1. The second conditions on continuity of the co-state variables of variable x_1 at the switching times, again whatever the switching type, are a generalization of conditions (2) and (4) of Theorem 1. Finally the third conditions on the continuity of the co-state variables associated with variable x_2 at the switching times of i-type only are a broader view of condition (5) of Theorem 1.

We now move to the application.

4 Technological switching under irreversible pollution

In this section, we revisit the classical problem in resource economics of the optimal management of non-renewable resources. As an application to the theory developed in the preceding section, we will analyze this problem when the economy is submitted to the potential irreversibility of pollution and has a backstop technology available.

The basic model is extensively based on a recent study by Prieur et al. (2011) who look at the features of the optimal extraction/emission policy under irreversibility and exhaustibility. So, Prieur et al. (2011) already deal with the second type of switch discussed so far that is, the one related to a threshold effect. Here we extend their analysis with the introduction of a technological switch, which will be related to the adoption of the backstop technology. To be more precise, things work as follows.

The economy extracts and consumes an exhaustible resource, say fossil fuels, x at rate $e \geqq 0$. The initial resource stock x_0 is given. Extraction is costless but is responsible for GHG emissions, with a one-to-one relationship. The economy has also a backstop technology available that can satisfy a given amount $s \geqq 0$ of the total energy consumption, y. The kind of backstop technology we have in mind includes the renewable and non-polluting energy sources such as the solar energy and the wind power. The backstop is produced at a fixed unit cost $c > 0$.

Utility is defined over the consumption of fossil fuels and the backstop, $U(y)$, with $y = e + s$. In the same vein as Tahvonen and Withagen (1996) and Tahvonen (1997) we make:[6]

Hypothesis 4.1 *The utility function is such that:* $U(0) = 0$, $U''(.) < 0$, $0 < U'(0) < \infty$ *and there exists* $\bar{y}$ *such that* $U'(\bar{y}) = 0$.

Rewriting the rate of extraction as $e = y - s$, the law of motion of the stock of non renewable resource is:

$$\dot{x} = -(y - s) \text{ with } x(0) = x_0 \text{ given,} \tag{13}$$

Emissions contribute to the accumulation of a pollution stock, z. The initial pollution stock is given: z_0. As in Prieur et al. (2011), we assume

[6]Tahvonen and Withagen (1996) are the first to provide a rigorous analysis of the impact of irreversibility on the optimal control of pollution. They use a quadratic decay function but do not consider neither resource scarcity nor the backstop technology. Tahvonen (1997) examines in detail the classical exhaustible-resource/stock-pollution model, with a backstop technology but without irreversible pollution.

that pollution would turn irreversible if the stock passes a critical threshold $\bar{z}$, with $\bar{z} > z_0$. This irreversibility threshold corresponds to the first possible regime switch that affects the law of motion of the pollution stock. Hence, the dynamics of the stock of the pollutant are defined piece-wise:

$$\dot{z} = \begin{cases} y - s - \alpha z & \text{if } z \leq \bar{z} \\ y - s & \text{else} \end{cases} \tag{14}$$

The natural decay rate α is constant and positive as long as accumulated emissions are not too high, that is, as long as the stock remains below or is at the irreversibility threshold $\bar{z}$. Once the threshold is surpassed, natural decay abruptly vanishes. Thus, pollution becomes irreversible. Hereafter, the domain where $z \leq \bar{z}$ is called the reversible region whereas whenever pollution is higher than $\bar{z}$, the economy is said to be in the irreversible region. Let $0 \leqq t^z \leqq \infty$ be the instant when the irreversible region is entered.

Pollution is damaging to the economy. For any level z, pollution damage is denoted by $D(z)$.

Hypothesis 4.2 *The damage function is such that:* $D(0) = 0$*,* $D'(z) > 0$*,* $D''(z) > 0$ *for all* $z > 0$*,* $D'(0) = 0$ *and* $\lim_{z\to\infty} D'(z) = \infty$*.*

The social welfare function thus reads

$$V = \int_0^\infty \left[U(y) - D(z) - cs\right] e^{-\delta t} dt, \tag{15}$$

with δ the discount rate.

The second possible regime switch is related to the adoption of the backstop technology. Hereafter, we will assume that the economy initially does not use the backstop, which is in the line of the related literature (see for instance Tahvonen, 1997). However, we depart from this literature in the following sense. In our framework, s is not a control variable. This means that once the economy finds it worthwhile to use the backstop, it cannot choose the exact amount to provide. Rather we consider that the provision of energy by the backstop is fixed. So, in the same vein as Valente (2011), we treat the issue of the backstop adoption as an optimal timing problem (see also for related problems Boucekkine et al., 2004, 2011, 2012) and the control variable is the instant $0 \leqq t^s \leqq \infty$ when the backstop is introduced and used at a given intensity. This is consistent with the kind of technological change studied in the section devoted to the theory. This is of course a simplifying assumption but it allows us to convey the idea that the energy consumption from the backstop is limited by nature. Here, our s can be seen as the maximum share of energy consumption that can be provided by solar and wind equipments, due to technical constraints. For instance, Europe has engaged in provisioning 20 % of the total energy consumption thanks to renewables at the horizon 2020. In our framework, given that the maximum of energy consumption from all sources is $\bar{y}$, this boils down to defining s as a share of $\bar{y}$.

Remark. The resource stock corresponds to the first type of state variable considered in the theoretical section, that is the one whose law of motion is influenced only by the technological regime, whereas pollution is the second type of state variable whose evolution is determined by both the technological and the ecological regime. Also note that the two possible switchings are irreversible.

Let us start the analysis with the case where the economy does not face resource scarcity, that is $x_0 = \infty$.

4.1 The case of an abundant resource

In this section, we don't pay attention to the dynamics of the resource (13), which boils down to considering that $x_0 = \infty$ or put differently, that the co-state of the resource, often called the scarcity rent, is generically equal to zero. We further assume that the economy originally is in the regime with no use of the backstop and with reversible pollution (indexed by 11). But any optimality candidate, that is any solution to the necessary optimality conditions, can experience four different regimes: the initial one, a regime without backstop and with irreversible pollution (12), with a backstop and reversible pollution (21) and finally with the backstop and irreversible pollution (22).

Let us state the set of necessary optimality conditions in the most general regime 21. Consider the problem of maximizing social welfare with respect to $\{y\}$ given $\dot{z} = y - s - \alpha z$, over any non-degenerate period of time. The hamiltonian in current value is $H = U(y) - D(z) - \lambda(y - s - \alpha z)$, with λ the co-state variable of z. For an interior solution, $y > 0$, the necessary optimality conditions include:

$$\begin{aligned} &U'(y) = \lambda \\ &\dot{\lambda} = (\alpha + \delta)\lambda - D'(z) \\ &\dot{z} = y - s - \alpha z \end{aligned} \tag{16}$$

The corresponding conditions for regime 11 are obtained by putting s equal to zero whereas those of irreversible regimes can be derived by setting $\alpha = 0$ with $s > 0$ (22) or $s = 0$ (12).

We also have the necessary conditions of Theorem 3.1, related to the two kinds of timing decisions. First consider a switch to the backstop technology in any ecological regime at instant t^s. From what has been established before, both the state variable and its co-state must be continuous at t^s. From (16), this implies that $y^{1j}(t^s) = y^{2j}(t^s) = y^j(t^s)$ for $j = 1, 2$. Energy consumption is continuous at the switching time but of course the part originating from fossil fuels jumps downward with the adoption of the backstop: $e^{2j}(t^s) = e^{1j}(t^s) - s < e^{1j}(t^s)$. Making use of the continuity of consumption, the continuity of the Hamiltonian in turn yields the following condition:

$$U'(y^j(t^s)) = c. \tag{17}$$

The economy adopts the backstop technology when the marginal gain of doing so is equal to the marginal cost. At first glance, this condition does not seem to be directly affected by the proximity of the irreversibility threshold. But of course, the evolution of energy consumption, the value $y^j(t^s)$ and the switching time will ultimately depend on the pollution level, which is itself based on the particular ecological regime.

As for the ecological switch, we know that the co-state is not continuous is general since the pollution stock that triggers the switch is given and equal to $\bar{z}$. The Hamiltonian is still continuous at date t^z, this leads to the following optimality condition:[7]

$$U(y^{21}(t^z)) - U'(y^{21}(t^z))(y^{21}(t^z) - s - \alpha\bar{z}) = U(y^{22}(t^z)) - U'(y^{22}(t^z))(y^{22}(t^z) - s). \tag{18}$$

Under hypothesis 4.1, the function $U(y) - U'(y)(y - s)$ is monotonically increasing in y (recall that by definition $s \leqq y$). This implies that the entrance in the irreversible region is accompanied by an upward discontinuity in y – and e – as already noticed by Prieur et al. (2011). The economy compensates for the loss of benefits (from pollution decay) by an increase in energy consumption.

It is also possible that the two switches occur simultaneously $t^s = t^z = t^1$ in which case the necessary condition reads:

$$U(y^{11}(t^1)) - U'(y^{11}(t^1))(y^{11}(t^1) - \alpha\bar{z}) = U(y^{22}(t^1)) - U'(y^{22}(t^1))(y^{22}(t^1) - s) - cs, \tag{19}$$

and it is not clear whether the consumption of fossil fuels falls or rises as a consequence of the occurrence of the double switch. Indeed, the two underlined mechanisms push in opposite direction. On the one hand, entering the

[7]The condition is written in the case where the ecological switch occurs once the backstop has been adopted. The corresponding condition, for a regime switch in the absence of backstop, is the same, with $s = 0$.

irreversible region is an incentive for the economy to increase the consumption of energy – and of fossil fuels given s (irreversibility effect). On the other, adopting the backstop allows the economy to substitute part of the consumption of fossil fuels with the consumption brought by the backstop (backstop effect). Overall energy consumption increases because if the technological switch has not occurred up to date t_1, one necessarily has $U'(y^{22}(t^1)) \leq c$.

Following Prieur et al. (2011)'s approach, we can define a set of conditions under which we have an interesting problem in the sense that it allows for multiple solutions with different features. In particular, a solution is said to be reversible when it remains in the reversible region whereas it is irreversible when the irreversibility threshold is exceeded in finite time. To start with, ignore the issue of irreversibility and assume that regime 21 is terminal then the resulting dynamics

$$\begin{aligned} \dot{y} &= \tfrac{1}{U''(y)}((\alpha+\delta)U'(y) - D'(z)) \\ \dot{z} &= y - s - \alpha z \end{aligned} \tag{20}$$

lead the economy to a saddle point $(y^{21}_\infty, z^{21}_\infty)$ uniquely defined by

$$U'(\alpha z^{21}_\infty + s) = \frac{D'(z^{21}_\infty)}{\alpha+\delta} \text{ and } y^{21}_\infty = s + \alpha z^{21}_\infty.$$

In addition, it is obvious that the fixed point associated with the initial regime $(y^{11}_\infty, z^{11}_\infty)$ is such that $y^{11}_\infty < y^{21}_\infty$ and $z^{11}_\infty > z^{21}_\infty$. This implies that adopting the backstop at some point in time allows the economy to reach a steady state with lower pollution, high energy consumption but lower fossil fuels consumption.

Some straightforward conclusions can be drawn from the comparison between $\bar{z}$, z^{11}_∞ and z^{21}_∞. If the pollution threshold is high enough $\bar{z} = \bar{z}_h > z^{11}_\infty$ then the solution will necessarily be reversible, including or not a switch to the backstop technology (see the phase diagram depicting the dynamics of regimes 11 and 12 in Figure 1). Originating in the reversible region and in regime 11 (the most favorable for the occurrence of the ecological switch since the economy relies on polluting resources), it is clear that the dynamics, represented by the arrows, cannot drive the economy to a level $y^{11}(t^z)$ that would allow for the upward jump to $y^{12}(t^z)$. By contrast, with a low threshold, reversible and irreversible solutions are both feasible but reversible policies will necessarily feature a final stage at the threshold. Suppose that the economy is in regime 21. Then, when $\bar{z} = \bar{z}_l < z^{21}_\infty$ the steady states above cannot be achieved and the only possible reversible policy is the solution where the pure state constraint $z \leq \bar{z}$ is binding for a non degenerate

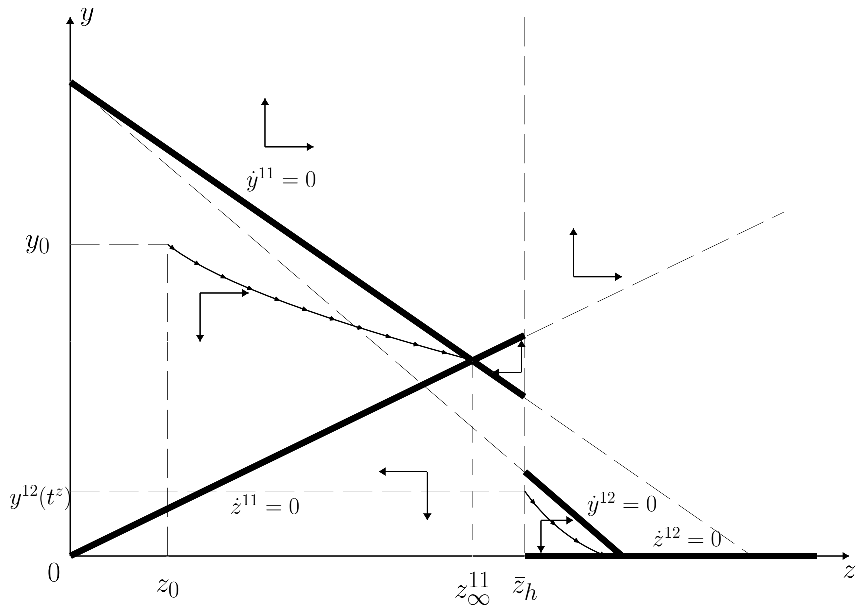

Figure 1: High irreversibility threshold and reversible solutions.

period of time. (see the phase diagram given by Figure 2).[8] In the remainder of the analysis, we will focus on the most interesting case where adopting the backstop may be a means to avoid the situation when the pollution turns irreversible. This translates into the following ordering: $z_\infty^{21} < \bar{z} < z_\infty^{11}$, which is equivalent to:

$$U'^{-1}\left(\frac{D'(\bar{z})}{\alpha+\delta}\right) - s < \alpha\bar{z} < U'^{-1}\left(\frac{D'(\bar{z})}{\alpha+\delta}\right). \tag{21}$$

and one may note that the second part of the inequality implies that $\bar{y} > \alpha\bar{z}$ the maximum rate of energy and fossil fuels consumption that is the maximum rate of emissions allowed is higher than the maximum rate of decay. This is also necessary for the existence of irreversible optimality candidates.

In this case there are two options available to the economy. Either it adopts the backstop technology before the threshold is hit, which produces

[8] Prieur et al. (2011) provide a comprehensive study of solutions featuring a stage at the threshold. Here, we do not have to bother with this kind of solution since in our model there exists an alternative reversible solution that consists in adopting the backstop before the threshold is reached. That's why we do not consider the pure state constraint.

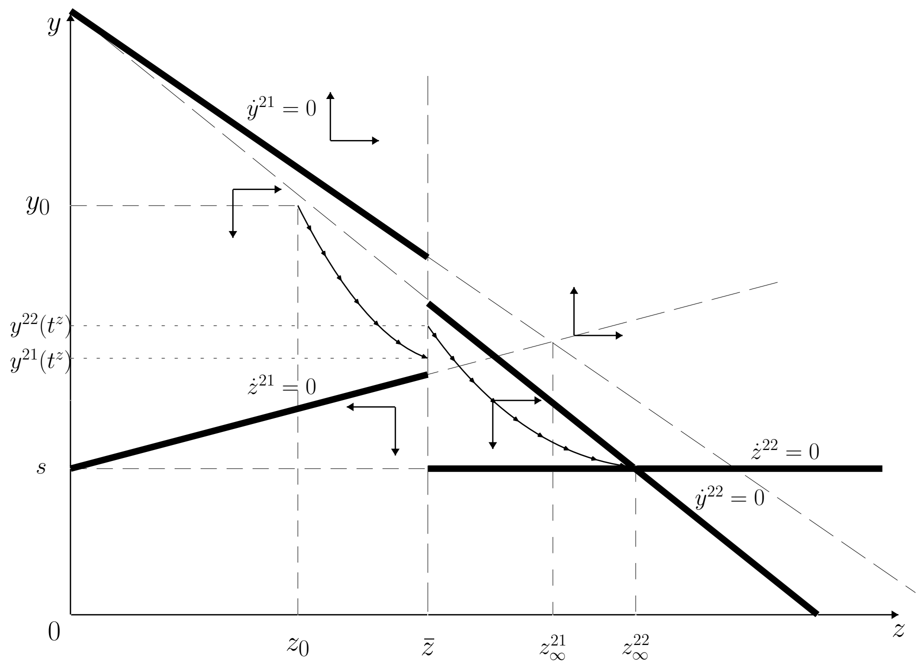

Figure 2: Low irreversibility threshold and irreversible solutions.

a reversible policy. This corresponds to the timing $0 < t^s < t^z = \infty$ with the sequence of regimes 11 and 21. Or it enters the irreversible region before the technological switch. This may be observed when the optimal timing is $0 < t^z < t^s \leqq \infty$ with sequence 11 then 12 and possibly 22 as a terminal regime.

A last necessary condition is required in order for irreversible solutions to exist. Consider the combination of regime 11 and 12. Suppose the economy is at the threshold, then it should at least not be refrained, from a welfare point of view, from passing the threshold. This condition identified by Tahvonen and Withagen (1996) requires the marginal utility of the first unit of consumption $U'(0)$ to be larger than total discounted marginal damages: $D'(\bar{z})/\delta$. To see this, focus now on the problem with irreversibility of decay. If an economy with abundant resource were to exceed the threshold in finite time, it would achieve in the long run a steady state with the extraction rate $y(= e) = 0$ and a level of pollution given by $U'(0) = D'(z^{12}_{\infty})/\delta$. If $U'(0) \leq D'(\bar{z})/\delta$ then we obtain a contradiction because $z^{12}_{\infty} \leq \bar{z}$. So, it must hold that $U'(0) > D'(\bar{z})/\delta$. If we further require $U'(s) > D'(\bar{z})/\delta$, then the previous condition is satisfied because $U'(s) < U'(0)$ and it is also possible to have the combination 11 - 12 - 22 that is, to adopt the backstop after the situation as turned irreversible. Indeed the steady state of regime 22 is feasible. This case makes sense because adopting the backstop once the solution has turned irreversible allows the economy to keep pollution damage under control.

Regarding reversible solutions (given by the timing 11 21), a first necessary existence condition is $c < U'(0)$. Suppose that $c \geqq U'(0)$, then the marginal benefit from adopting the backstop is always less than the marginal cost, which precludes the occurrence of a technological switch. In the same spirit one has to impose $c < U'(y_\infty^{11})$, which is stronger than the first condition. Suppose the economy starts in regime 11 from $z_0 < z_\infty^{11}$ and that this regime is permanent. There is no opportunity to switch either technically or ecologically. Then the optimal policy consists in choosing initial consumption $y(0)$ so that the system converges to the saddle point along the stable branch. Now assume that $c \geqq U'(y_\infty^{11})$. Given that the optimal consumption path is monotonically decreasing and $U'(y)$ is decreasing in y, this implies that $U'(y^{11}(t)) < c$ for all $t < \infty$. So, we can claim that the adoption of the backstop in finite time is implausible. This condition conveys the idea that the unit cost of the non polluting technology should be low enough to make the reversible policies attractive. The third necessary condition involves both the unit cost of the backstop and the level of energy it supplies, s: $c < U'(y_\infty^{21})$. It generalizes the preceding one and also states that, given s, the adoption of the backstop is worthwhile for low c. A quick look at the phase diagram depicted in figure 3 confirms that the trajectory that originates in regime 11 and converges to $(y_\infty^{21}, z_\infty^{21})$ necessarily ends with a period during which consumption decreases whereas pollution increases. This in turn implies that the consumption level $U'^{-1}(c)$ that triggers the technological switch must be higher than the long run value y_∞^{21}.

Figure 3 gives an illustration of the shape of reversible and irreversible policies in the (z, y) plan. It depicts a reversible policy (green) when the technological switch occurs at a pollution level $z^{21}(t^s)$ below z_∞^{21} (with $t^z = \infty$). It should be clear that energy consumption satisfies $y^{i1}(t^s) > y_\infty^{21}$: Once the switch has occurred, pollution monotonically increases and consumption of fossil fuels decreases till the convergence to the steady state. The other trajectory represents the evolution of both y and z when the threshold is hit in finite time (with $t^s = \infty$). Along an irreversible path (purple), consumption of fossil fuels is monotonically decreasing except at the instant of the switch when y, and therefore e, jumps upward whereas the pollution stock is monotonically increasing till the convergence toward z_∞^{12}.

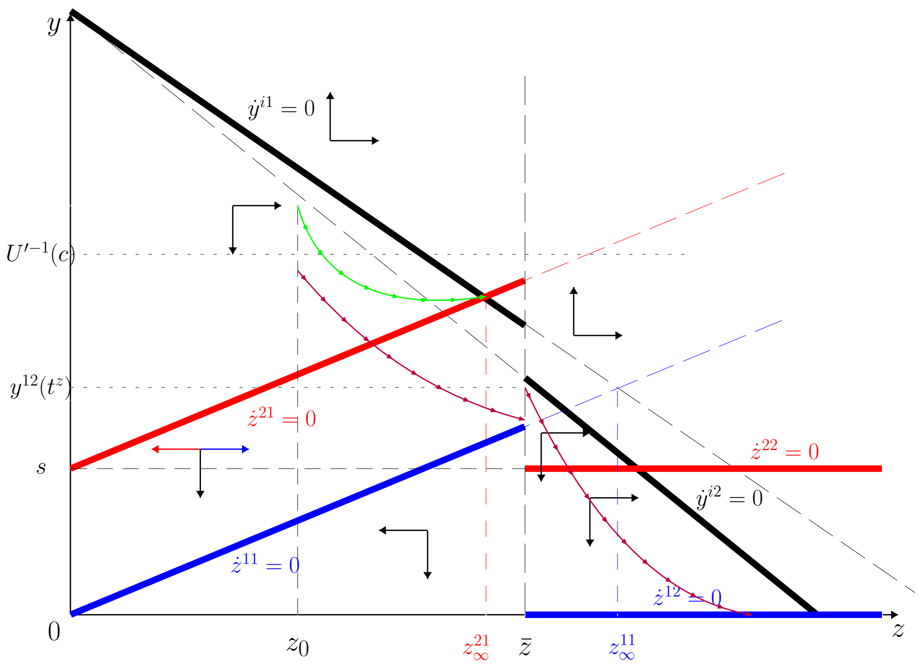

Figure 3: Reversible vs. irreversible solutions.

Under the set of conditions discussed above, our general problem with an abundant resource may feature multiple optimality candidates: Both a reversible and an irreversible policy, originating from the same initial condition z_0, can satisfy the necessary optimality conditions. It proves however difficult to investigate more deeply the conditions under which the two kinds of trajectories depicted in figure 3 exist. The issue of multiplicity in turn raises the question of the nature of the optimal solution. It is clear that to answer this question we have no option but to compute the present value associated with each candidate and determine which one yields the highest value. There is of course little chance to conduct this analysis analytically.

That is why, in the next section, we will proceed by means of a calibration of our model. The numerical analysis will allow us not only to address the issues of multiplicity and optimality but also to compare our results with those of Prieur et al. (2011), who analyze a similar problem but do not consider the possible adoption of a backstop technology. Since they have a non-renewable resource in their study, from now on, we will work in the more general framework where the economy is submitted to the resource constraint (13).

4.2 The case of a scarce resource

The numerical analysis is conducted using the same functional forms and parameters values as Prieur et al. (2011). They use a linear quadratic model

$$\begin{cases} U(y) = \theta y(2\bar{y} - y),\ \theta > 0 \\ D(z) = \frac{\gamma z^2}{2},\ \gamma > 0 \end{cases}, \tag{22}$$

and rely on a calibration exercise developed by Karp and Zhang (2012) to fix the following set of baseline parameters:[9]

$$\delta = 0.05,\ \alpha = 0.0083,\ \gamma = 0.0022,\ \theta = 26.992,\ \bar{y} = 16.206, z_0 = 781 \text{ and } \bar{z} = 1200. \tag{23}$$

The initial stock of exhaustible resource x_0 is not set to a particular value because there is a substantial uncertainty surrounding the exact level of available fossil fuels in the ground. In what follows, the variable will be critical in determining the number of optimality candidates and the nature of the optimal solution.

Regarding the parameters that are specific to the backstop technology, we consider in our benchmark that the non polluting technology supplies 20% of the maximum level of energy consumption, $s = 0.2\bar{y}$, and we assume $c = 300$. These assumptions allow for a phase of simultaneous use of the two sources of energy (before exhaustion) because they satisfy $U'(s) \neq c$.

Table (1) presents the existence results in the benchmark case.[10] Assuming that the economy initially does not use the backstop technology, our problem can a priori exhibits nine optimality candidates corresponding to all the possible combinations of regimes. The first, second and sixth columns are the optimality candidates identified in Prieur et al. (2011), who do not deal with the backstop technology. They show that for a low enough initial

[9] Here we refer the reader to Prieur et al. (2011) for a justification of the choice of the parameter values. Simply notice that in their study $U(y)$ represents the utility or benefit from emissions. In our framework, this function is defined in terms of energy consumption. In particular, $\bar{y}$ represents the maximum level of energy consumption whereas in their model, it yields the maximum level of CO_2 emissions. This obviously means that we have to be careful with the calibration of $U(y)$. In the simple numerical exercise to follow, we however decided to keep the same values as in the scenario where the authors allow for the highest $\bar{y}$, at least for the sake of comparison.

[10] Tables and figures are relegated in the appendix. The values reported in row are the x_0 critical for the existence of the optimality candidates (in column). The inequality "$\leq$" means that the critical x_0 is an upper bound whereas "$\geq$" and "$>$" refer to a lower bound for existence. A cell with a "no" indicates that the corresponding optimality candidate doesn't exist. All the figures depict the present values, for particular solutions, as functions of x_0.

resource stock ($x_0 \leq 1211, 5$), the unique solution is reversible (regime 11 is permanent) and thus yields the optimum. For larger $x_0 (\geq 2141.1)$, they obtain two kinds of optimality candidates, one is reversible (with the pollution stock staying at the threshold for a while) and the others are irreversible.[11] The optimum remains the reversible solution up to a critical $x_0 \approx 2400$ from which the optimal solution becomes irreversible. In the remainder of the section, we investigate how the opportunity to adopt a backstop technology affects their conclusions.

For low enough $x_0 (\leq 1145.5)$, there exist two optimality candidates, both are fully reversible, i.e., are such that $z(t) \leq \bar{z}$ for all t (where the equality can hold for no more than one instant of time). Fully reversible policies with adoption exist for a smaller interval of values of the initial resource stock x_0 than their counterpart without adoption. In addition, they always yield lower present values than fully reversible policies with no adoption (see figure 5, left). For higher $x_0 (> 1211.5)$, the unique reversible policy does not entail a switch to the backstop technology and exhibits a stage at the threshold. So, we don't have an optimality candidate featuring both adoption and a period of time spent at the threshold (these situations correspond to timings 11 $\bar{z}$ 11 21 and 11 21 $\bar{z}$ 21). The reason why adopting the backstop technology before staying for a while at the threshold is not optimal is the following. For the adoption of the backstop to be optimal, the economy has to incur a substantial reduction of its consumption of fossil fuels to a level $\tilde{e}$ such that $U'(\tilde{e} + s) = c$. For our parameters, this level is lower than the level of extraction compatible with the pollution stock staying at $\bar{z}$ for a period of time: $\tilde{e} < \alpha \bar{z}$. Along candidate 11 21 $\bar{z}$ 21, the adoption of the backstop should then be followed by a phase during which extraction increases. But this is not optimal. If the economy were to adopt the backstop, then it would not increase its rate of extraction immediately after it has paid c to reduce it. Regarding the timing 11 $\bar{z}$ 11 21, it appears that the level of extraction that triggers adoption, $\hat{e}$ such that $U'(\hat{e}) = c$, is now higher than $\alpha \bar{z}$, the

[11] When the initial stock of resource is large, it is necessary that the economy stays at the threshold for a non degenerated period of time for a reversible solution to exist. In other words, the pure state constraint $z \leq \bar{z}$ is binding for a period of time. They also have an irreversible candidate for which the economy spends a period of time at the threshold before entering the irreversible region. But, the authors have shown that this kind of candidate is always dominated by the path along which the economy directly enters the irreversible region upon arrival to the threshold (this corresponds to 11 12). Therefore this candidate is irrelevant for the optimality analysis and we won't consider it in the subsequent analysis. Finally, we don't pay attention to irreversible policies that do not exhaust the resource since it has been observed by Prieur et al. (2011) that they exist only for very large and not relevant values of the initial endowment in exhaustible resources, which is also the case here.

extraction level which prevails during the phase at the threshold. Thus, this path would require extraction to follow a U-shape pattern, which is not optimal either.

As for irreversible solutions, the opportunity to adopt the backstop makes irreversible policies more likely. In other words, solutions with timing 11 12 22 exist for lower x_0 than irreversible candidates without adoption, given by the timing 11 12 (2094.1 vs. 2141.1). For $x_0 \geq 2141.1$, we have three optimality candidates: a reversible solution, corresponding to timing 11 $\bar{z}$ 11, and two irreversible, with and without adoption. We never obtain irreversible solutions along which the backstop is adopted before the irreversible region is entered (this would coincide with timing 11 21 22). By contrast with what has been observed for reversible policies, irreversible candidates with the adoption of the backstop in finite time outperform the ones without adoption (see figure 4) for intermediate values of x_0 (the present values converge to the same level for high enough x_0). Figure 4 also reveals that the optimal policy turns irreversible for a lower value of x_0 than the one obtained by Prieur et al. (2011), this critical level being given by the intersection between the curves representing the present values of, on one hand, the reversible candidate, and on the other, the irreversible solution with adoption of the backstop.

Hence, two conclusions can be drawn from the analysis of the benchmark. First, reversible policies that involve a switch to the backstop technology exist for a smaller range of values of x_0 than – and are always dominated by – the reversible policies with no adoption. So, it does not pay to adopt the backstop technology in a reversible world, that is, in a world where the initial stock of the polluting and exhaustible resource is so low that it is physically impossible to cross the threshold in finite time. Second, our analysis stresses that the opportunity to adopt a backstop translates into the existence of irreversible policies for a larger range of values of the initial resource stock. Upon reflexion, this conclusion seems quite natural because adopting the backstop after the crossing of the irreversibility threshold is a means to keep environmental damages, due to the ever increasing pollution stock, under control. So, it makes irreversible policies more worthwhile. Regarding the welfare comparison, the previous argument also explains why irreversible policies with a backstop yield a higher present value than irreversible policies without adoption and become optimal for a lower level of the initial stock of resource.

Let us now see whether these findings are robust to a change in the unit cost of the backstop and in the level of energy consumption brought by the backstop. The results of the sensitivity analysis are summarized in table 2.

Consider a low unit cost: $c = 100$. In this case, our first conclusion does not hold anymore. Fully reversible policies, with backstop adoption in finite time, exist for larger x_0 than the ones for which reversible policies with no technological switch exist. We also observe that the ranking between fully reversible policies is reversed (see figure 5, left). This means that the cost of the backstop should be low enough for the optimal policy to entail a switch to the non polluting technology. Another difference with the benchmark scenario is that reversible policy with adoption exists for any $x_0 > 0$. Indeed, for large enough $x_0 (> 1294.1)$, the economy will stay at threshold for a non degenerate period of time (which corresponds to the timing 11 $\bar{z}$ 11 21).[12] We can however note that for $x_0 > 1211.5$, reversible policies with a stage at the threshold yield higher values when they are not accompanied by the adoption of the backstop at some point in time (figure 6, left).

Our second conclusion regarding irreversible candidates is reinforced. For this low c, we can observe that irreversible candidates with adoption exist for lower levels of the initial stock of resource than in the benchmark. We can still conclude that as far as irreversible policies are concerned, adopting the backstop is always better than sticking to the carbon economy (see figure 5, right). Furthermore, it turns out that once this irreversible candidate exists, it yields the optimal solution.[13] Thus by contrast with the benchmark case and with Prieur et al. (2011)'s findings, reversible policies with a stage at the threshold are always dominated by irreversible policies with adoption (see figure 6, right). This also means that the optimum is irreversible for a much lower level of the initial stock of fossil fuels, $x_0 \approx 1815.8$ GtC, than for a high c.

Things work differently when we set the unit cost to the highest possible one: $c = 699.9$. This level is defined by the following equality: $U'(s) = c$ and therefore corresponds to the limit case where adoption occurs at the instant when the resource is exhausted and extraction ceases. In this scenario the first conclusion drawn from the benchmark is valid but the second one is not because for a given x_0, the irreversible policy with adoption yields a lower present value than the irreversible policy without (figure 5, right). The previous logic discussed above now is cancelled by the fact that the backstop is too costly. Therefore, whatever the ecological regime, it is never optimal to switch to the backstop technology and the conclusion is qualitatively the same as in Prieur et al. (2011) i.e. irreversible solutions with adoption don't matter in the analysis of the optimum.

[12] This combination of regimes is possible here since it does not involve a phase during which the extraction rate increases: $\alpha \bar{z} > \tilde{e}$.

[13] In the range $[1114.7, 1210.8]$, there exists a last irreversible candidate with timing 11 22: The economy adopts the backstop at the instant when the threshold is exceeded. But, this solution is always dominated in terms of welfare.

In sum, the sensitivity analysis reveals that varying c has the following impact on the existence and ordering between optimality candidates: The lower c the higher (resp. lower) the critical bound x_0 for the existence of fully reversible (resp. irreversible) policies with adoption. In addition, for low enough or high enough c there exist reversible policies with adoption whatever x_0 while for intermediate value of c, x_0 must be small enough.

The last variation around the benchmark consists in increasing the level of the backstop to $s = 0.4\bar{y}$. As can be seen from table 2 and figure 5, left and right, conclusions remain qualitatively the same as in the benchmark scenario. We however observe that a higher s makes irreversible candidates with adoption optimal for lower x_0 than in the benchmark. Hence the impact of an increase in s on the occurrence and optimality of irreversible solutions is similar to the one of a decrease in c.

5 Conclusion

In this paper, we have examined a quite general optimal control problem involving two types of regime switch, which in our view represent a large part of regime switch problems occurring in economics. In particular, our setting includes most of the technological and/or institutional regime switch problems considered in the economic literature. Typically, these regimes are indexed by a finite number of discrete parameter values. We also incorporate another type of regime switch problems borrowed from the environmental economics literature where regimes rely on given threshold values for given state variables consistently with the irreversible pollution specification. We have proposed a general optimal control framework allowing to derive the first-order optimality conditions and in particular to characterize the geometry of the shadow prices at optimal switching times (if any). We have also applied our methodology to solve the problem of the optimal management of natural resources under ecological irreversibility, and with the possibility to switch to a backstop technology.

While the methodology presented cannot be fully analytical (because the inherent problem necessarily displays multiple potential optimal solutions) and should be complemented with an adequate numerical assessment, we do believe that it paves the way to handle a much wider class of problems, beyond environmental economics. Indeed, it is easy to figure out other control problems involving technological-like switches under threshold effects: for example, technology adoptions problems should be ideally treated together with a human capital accumulation engine, the former being potentially sub-optimal if the stock of human capital is below a threshold value. We can think of many more applications in other areas like economy demography and unified growth theory where the notion of minimal population size for economic take-off is invoked (see for example, Galor, 2005).

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Appendix

A Proof of Theorem 3.1

This appendix is devoted to give the proof of Theorem 1, the main theorem of the paper. The proof use standard calculus of variations techniques in a sequence of three control sub-problems as explained in the main text.

Third interval sub-problem: The corresponding control sub-problem is:

$$\max_{\{u\}} V_3 = \int_{t_2}^{\infty} [F(u, x_1, x_2)]\, e^{-\delta t} dt \tag{24}$$

subject to,

$$\begin{cases} \dot{x}_1 = G_1^{22}\,(u, x_1, x_2) \\ \dot{x}_2 = G_2^{22}\,(u, x_1, x_2) \end{cases}$$

where t_2 and the initial conditions $x_1(t_2)$ and $x_2(t_2)$ are fixed. $x_1(t_2)$ will be made free in the next stages while $x_2(t_2)$ is fixed by construction, equal to $\bar{x}_2$. Problem (24) is standard and can be solved trivially using the usual Pontrygin method. We don't develop the method for this trivial problem. In particular, and to fix the notation, one would rely on the hamiltonian of the problem, H_3, given by

$$H_3 = F\, e^{-\delta t} + \lambda_1^{22}\, G_1^{22} + \lambda_2^{22}\, G_2^{22}.$$

Problem (24) yields straightforward first-order necessary conditions (including the appropriate transversality conditions which ultimately depend on the shape of function F and G_k^{ij}, $i, j, k = 1, 2$, and possible sign constraint on state variables). Let's denote by upperscript $*$ the paths identified by these conditions. A **crucial** property of the corresponding value function is that **it does not depend on $x_2(t_2)$ as a free variable since the latter is fixed equal to $\bar{x}_2$**. Henceforth, the value-function V_3^* is only a function of t_2 and $x_1(t_2)$. Denote x_2^1 the latter, one can write: $V_3^*(t_2, x_2^1)$. Moreover, one also have trivially the following envelope conditions under Hypothesis (2.2):

$$\frac{\partial V_3^*}{\partial t_2} = -H_3^*(t_2), \tag{25}$$

and

$$\frac{\partial V_3^*}{\partial x_2^1} = \lambda_1^{22*}(t_2). \tag{26}$$

Second interval sub-problem: The corresponding control sub-problem is:

$$\max_{\{u,t_2,x_2^1\}} V_2 = \int_{t_1}^{t_2} [F(u,x_1,x_2)]\, e^{-\delta t} dt + V_3^*(t_2,x_2^1) \qquad (27)$$

subject to,

$$\begin{cases} \dot{x}_1 = G_1^{21}(u,x_1,x_2) \\ \dot{x}_2 = G_2^{21}(u,x_1,x_2) \end{cases},$$

where t_1, $x_1(t_1) = x_1^1$ and $x_2(t_1) = x_1^2$ are given, and t_2 and $x_1(t_2) = x_2^1$ are free. This problem is much less trivial than the previous one. We shall develop the calculus of variations method required to extract all the first-order conditions. Denote by H_2 the corresponding hamiltonian, that is:

$$H_2 = F\, e^{-\delta t} + \lambda_1^{21}\, G_1^{21} + \lambda_2^{21}\, G_2^{21}.$$

One can readily write the value-function in terms of the hamiltonian:

$$V_2 = \int_{t_1}^{t_2} \left[H_2 - \lambda_1^{21}\, \dot{x}_1 - \lambda_2^{21}\, \dot{x}_2\right] dt + V_3^*(t_2,x_2^1).$$

Standard integrations by parts yield for $k = 1,2$:

$$\int_{t_1}^{t_2} \lambda_k^{21}\, \dot{x}_k\, dt = \lambda_k^{21}(t_2)x_k(t_2) - \lambda_k^{21}(t_1)x_k(t_1) - \int_{t_1}^{t_2} \dot{\lambda}_k^{21}\, x_k\, dt,$$

which allows to rewrite V_2 as:

$$V_2 = \begin{array}{l} \int_{t_1}^{t_2} \left[H_2 + \dot{\lambda}_1^{21}\, x_1 + \dot{\lambda}_2^{21}\, x_2\right] dt + V_3^*(t_2,x_2^1) - \lambda_1^{21}(t_2)x_1(t_2) + \lambda_1^{21}(t_1)x_1(t_1) \\ -\lambda_2^{21}(t_2)x_2(t_2) + \lambda_2^{21}(t_1)x_2(t_1) \end{array}$$

First-order variation of V_2 with respect to the state and control variables' paths, for fixed t_1, $x_1(t_1) = x_1^1$ and $x_2(t_1) = x_1^2$ but free t_2 and x_2^1, yields:

$$\partial V_2 = \begin{array}{l} \int_{t_1}^{t_2} \left[\frac{\partial H_2}{\partial x_1}\, \partial x_1 + \frac{\partial H_2}{\partial x_2}\, \partial x_2 + \frac{\partial H_2}{\partial u}\, \partial u + \dot{\lambda}_1^{21}\, \partial x_1 + \dot{\lambda}_2^{21}\, \partial x_2\right] dt \\ + \left(H_2(t_2) + \dot{\lambda}_1^{21}(t_2)\, x_2^1 + \dot{\lambda}_2^{21}(t_2)\, \bar{x}_2\right)\, \partial t_2 \\ + \frac{\partial V_3^*(t_2,x_2^1)}{\partial t_2}\, \partial t_2 + \frac{\partial V_3^*(t_2,x_2^1)}{\partial x_2^1}\, \partial x_2^1 - \dot{\lambda}_1^{21}(t_2)x_2^1\, \partial t_2 - \lambda_1^{21}(t_2)\, \partial x_2^1 - \dot{\lambda}_2^{21}(t_2)\bar{x}_2\, \partial t_2. \end{array}$$

Rearranging terms, one gets:

$$\partial V_2 = \begin{array}{l} \int\limits_{t_1}^{t_2} \left[\left(\frac{\partial H_2}{\partial x_1} + \dot{\lambda}_1^{21} \right) \partial x_1 + \left(\frac{\partial H_2}{\partial x_2} + \dot{\lambda}_2^{21} \right) \partial x_2 + \frac{\partial H_2}{\partial u} \partial u \right] dt \\ + \left(H_2(t_2) + \frac{\partial V_3^*(t_2, x_2^1)}{\partial t_2} \right) \partial t_2 + \left(\frac{\partial V_3^*(t_2, x_2^1)}{\partial x_2^1} - \lambda_1^{21}(t_2) \right) \partial x_2^1. \end{array}$$

A trajectory is (locally) optimal if any (local) departure from it decreases the value function, that is $\partial V_2 \leq 0$ for any $\partial x_1(t)$ and $\partial x_2(t)$, $t \in (t_1\ t_2)$, for any $\partial u(t)$, $t \in [t_1\ t_2]$, and for any ∂t_2 and ∂x_2^1, which gives the following necessary conditions for an interior maximizer:

$$\begin{cases} \frac{\partial H_2}{\partial u} = 0 \\ \frac{\partial H_2}{\partial x_1} + \dot{\lambda}_1^{21} = 0 \\ \frac{\partial H_2}{\partial x_2} + \dot{\lambda}_2^{21} = 0 \\ H_2(t_2) + \frac{\partial V_3^*(t_2, x_2^1)}{\partial t_2} = 0 \\ \frac{\partial V_3^*(t_2, x_2^1)}{\partial x_2^1} - \lambda_1^{21}(t_2) = 0 \end{cases} \tag{28}$$

The first three equations are the standard Pontryagin conditions, the last two may be interpreted as optimality conditions with respect to the switching time, t_2, and the free state value, x_2^1. Together with conditions (25)-(26) obtained from the first sub-problem, one gets the first two optimality conditions of Theorem (3.1), that is:

$$H_2^*(t_2) = H_3^*(t_2),$$

and

$$\lambda_1^{21*}(t_2) = \lambda_1^{22*}(t_2).$$

Notice finally that by construction of the second sub-problem, its value function V_2 depends on the fixed initial conditions and t_1, we thus write it $V_2^*(t_1, x_1^1, x_1^2)$. Again one can write the following envelope properties thanks to Hypothesis (2.2):

$$\frac{\partial V_2^*}{\partial t_1} = -H_2^*(t_1), \tag{29}$$

$$\frac{\partial V_2^*}{\partial x_1^1} = \lambda_1^{21*}(t_1), \tag{30}$$

and

$$\frac{\partial V_2^*}{\partial x_1^2} = \lambda_2^{21*}(t_1), \tag{31}$$

First interval sub-problem: The corresponding control sub-problem is:

$$\max_{\{u,t_1,x_1^1,x_1^2\}} V_1 = V = \int_0^{t_1} [F(u,x_1,x_2)]\, e^{-\delta t} dt + V_2^*(t_1,x_1^1,x_1^2) \tag{32}$$

subject to,

$$\begin{cases} \dot{x}_1 = G_1^{11}\,(u,x_1,x_2) \\ \dot{x}_2 = G_2^{11}\,(u,x_1,x_2) \end{cases},$$

with $x_1(0)$ and $x_2(0)$ given, and with free t_1, x_1^1, and x_1^2. Again the problem is not trivial. One may use the same technique of calculus of variations seen above for the second sub-problem. Indeed, denoting H_1 the hamiltonian of the the sub-problem, given by:

$$H_1 = F\; e^{-\delta t} + \lambda_1^{11}\; G_1^{11} + \lambda_2^{11}\; G_2^{11},$$

one can replicate the same steps just above, and obtain the following first variation of the value function ∂V_1:

$$\begin{aligned} \partial V_1 = \; & \int_0^{t_1} \left[\left(\tfrac{\partial H_1}{\partial x_1} + \dot{\lambda}_1^{11} \right) \partial x_1 + \left(\tfrac{\partial H_1}{\partial x_2} + \dot{\lambda}_2^{11} \right) \partial x_2 + \tfrac{\partial H_1}{\partial u}\; \delta u \right] dt + \left(H_1(t_1) + \tfrac{\partial V_2^*(t_2,x_1^1,x_1^2)}{\partial t_1} \right) \partial t_1 \\ & + \left(\tfrac{\partial V_2^*(t_2,x_1^1,x_1^2)}{\partial x_1^1} - \lambda_1^{11}(t_1) \right) \partial x_1^1 + \left(\tfrac{\partial V_2^*(t_2,x_1^1,x_1^2)}{\partial x_1^2} - \lambda_2^{11}(t_1) \right) \partial x_1^2 \end{aligned}$$

Again, for $\partial V_1 \leq 0$ for any (local) variations of the free variables, on should impose the following first-order necessary conditions (for a local maximizer):

$$\begin{cases} \frac{\partial H_1}{\partial u} = 0 \\ \frac{\partial H_1}{\partial x_1} + \dot{\lambda}_1^{11} = 0 \\ \frac{\partial H_1}{\partial x_2} + \dot{\lambda}_2^{11} = 0 \\ H_1(t_1) + \frac{\partial V_2^*(t_2,x_1^1,x_1^2)}{\partial t_1} = 0 \\ \frac{\partial V_2^*(t_2,x_1^1,x_1^2)}{\partial x_1^1} - \lambda_1^{11}(t_1) = 0 \\ \frac{\partial V_2^*(t_2,x_1^1,x_1^2)}{\partial x_1^2} - \lambda_2^{11}(t_1) = 0 \end{cases} \tag{33}$$

The three last conditions could be interpreted as optimality conditions with respect to t_1, x_1^1 and x_1^2 respectively. Coupled with the envelope conditions (29) to (31), one gets the last three matching conditions stated in Theorem (3.1):

$$H_1^*(t_1) = H_2^*(t_1),$$

$$\lambda_1^{11*}(t_1) = \lambda_1^{21*}(t_1),$$

and

$$\lambda_2^{11*}(t_1) = \lambda_2^{21*}(t_1),$$

which ends the proof of the theorem. $\square$

B Numerical example

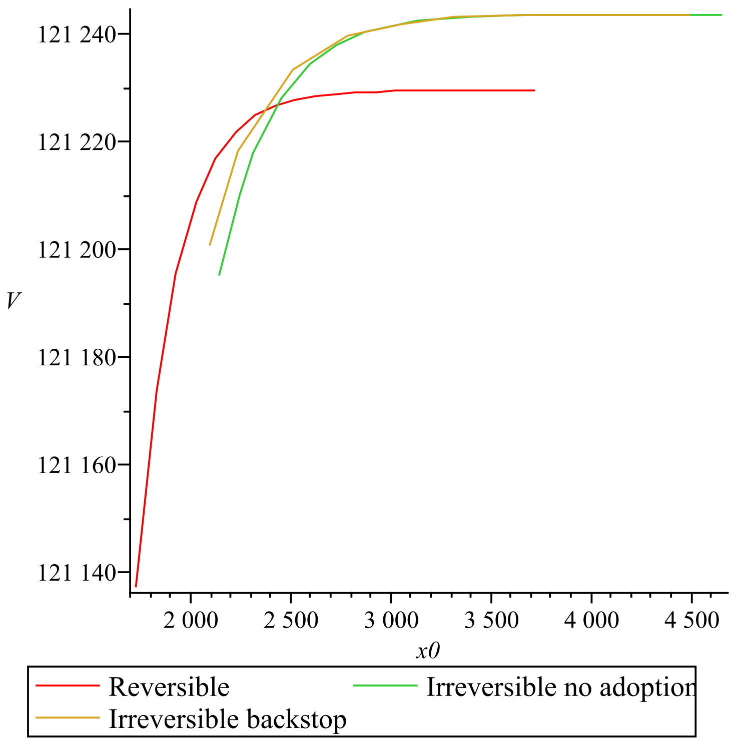

Figure 4: Benchmark: Multiplicity and optimality.

Table 1: Benchmark $s = 0.2\bar{y}$, $c = 300$

11	11 $\bar{z}$ 11	11 21	11 21 $\bar{z}$ 21	11 $\bar{z}$ 11 21	11 12	11 12 22	11 21 22	11 22
$x_0 \leq$ 1211,5	>1211,5	≤1145.5	no	no	≥2141.1	≥2094.1	no	no

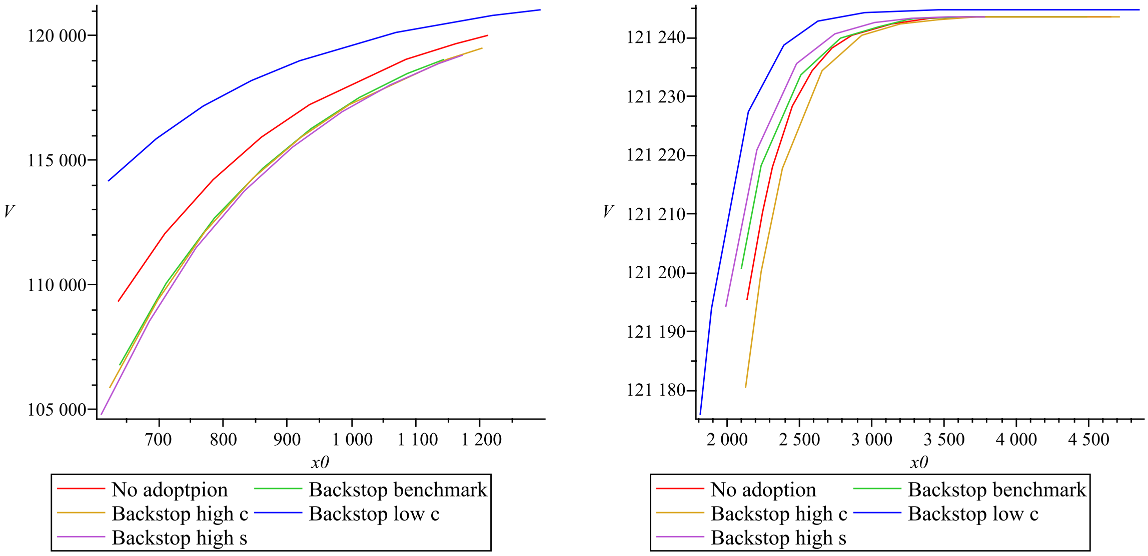

Figure 5: Reversible policies (left). Irreversible policies (right)

Table 2: Sensitivity analysis

	11 21	11 21 $\bar{z}$ 21	11 $\bar{z}$ 11 21	11 12 22	11 21 22	11 22
$c = 699.9$:	x_0 ≤1203.6	no	>1203.6	≥2132	no	no
$c = 100$:	x_0 ≤1294.1	no	>1294.1	≥1815.8	no	[1114.7, 1210.8]
$s = 0.4\bar{y}$:	x_0 ≤1172.9	no	no	≥1995.5	no	[1095.9, 1102.4]

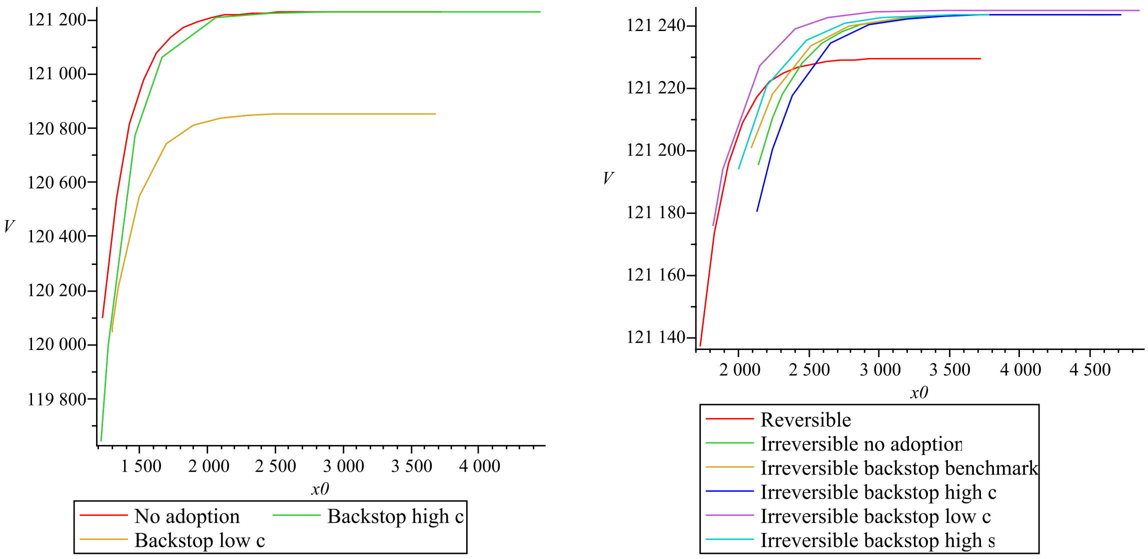

Figure 6: Comparison between present values