Joint spectral characteristics

Algorithms, applications, and conjectures on switched dynamics

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MFO
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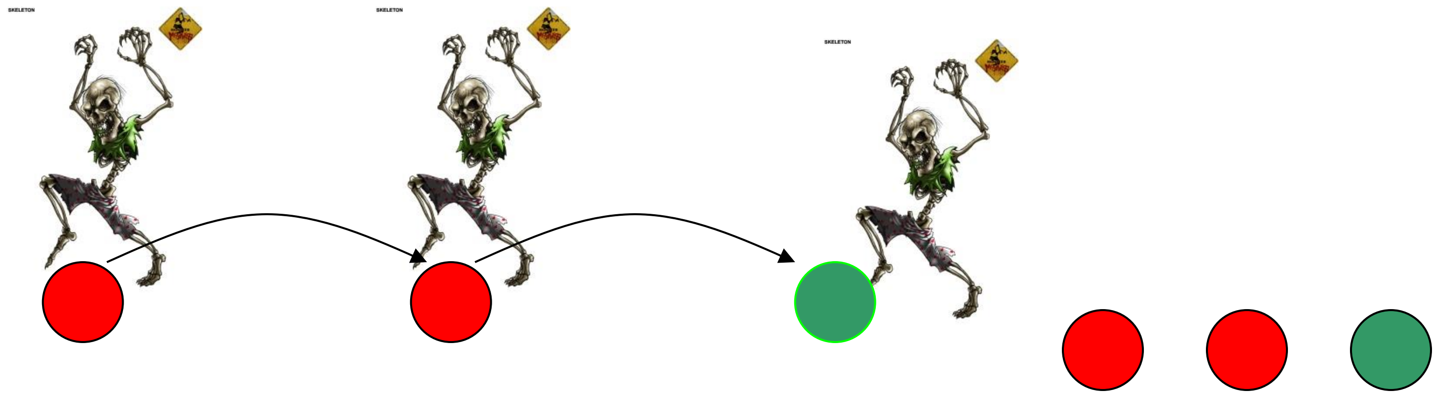
Trackable graphs



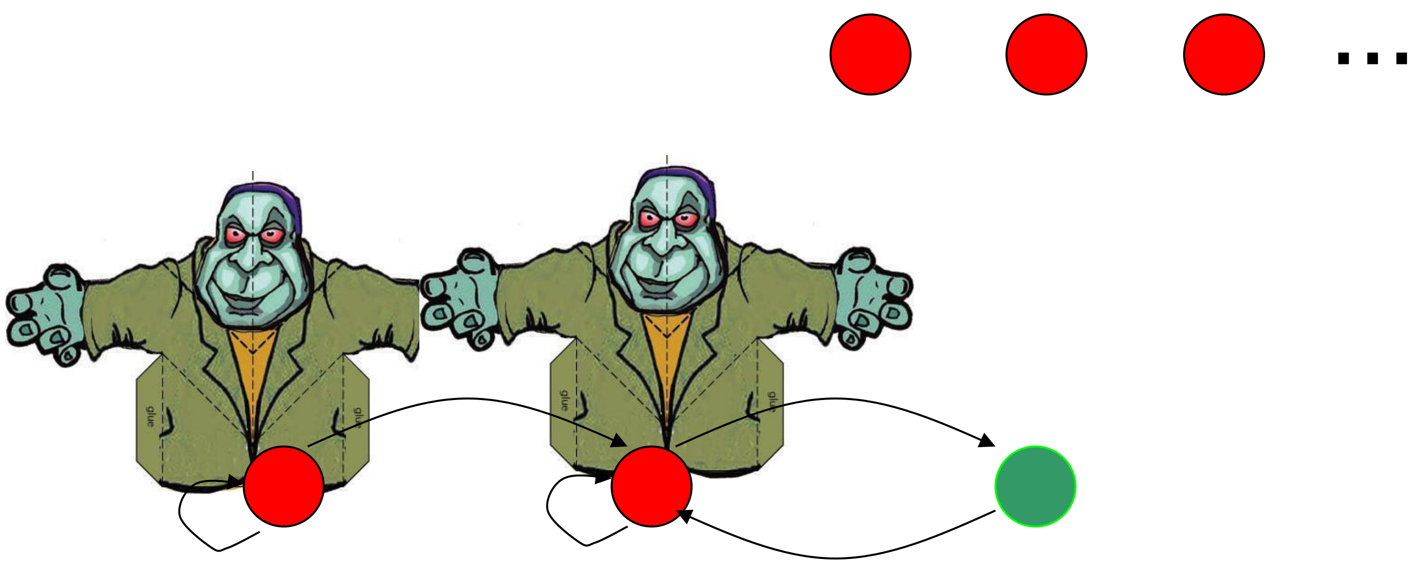
Let $N(t)$ be te worst possible number of trajectories compatible with an observation of length t
A network is trackable if $N(t)$ grows subexponentially

[Crespi et al. 05]

Here: number of possibilities asymptotically zero $N(t) \approx 0$

➔ Trackable

Trackable graphs

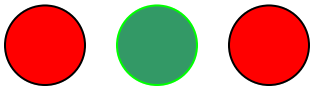


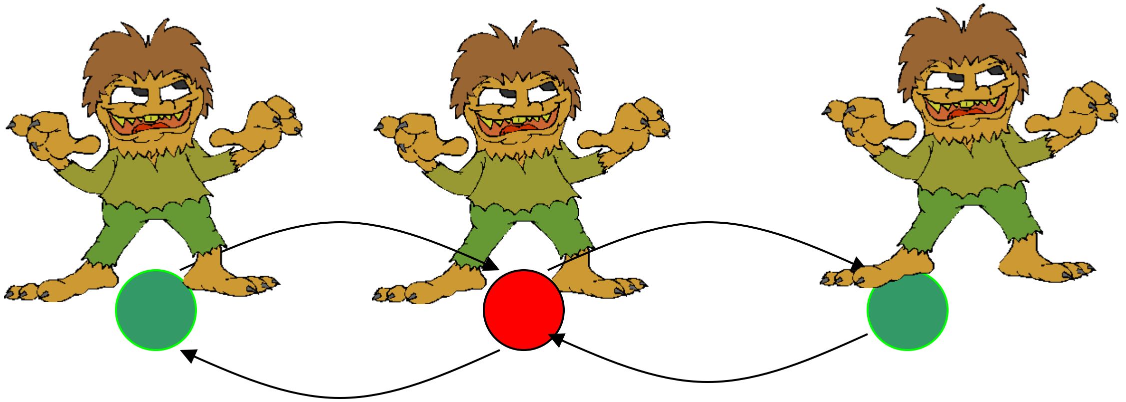
Worst case : RRRRRR… ➔ $N(t) \approx t$

Polynomial number of possibilities

➔ Trackable

Trackable graphs





Worst case : RGRGRG…➔ $N(t) \approx 2^{t/2}$

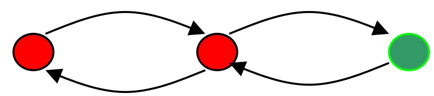
Exponential number of possibilities

➔ Not trackable

Trackability : the formal problem

We are given

- A graph G(V,E) :

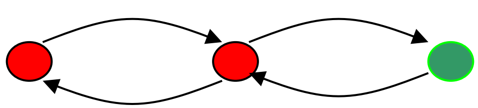


$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- A set of possible observations : defining a partition of the nodes

$$\begin{cases} R = \{1,2\} \\ G = \{3\} \end{cases}$$

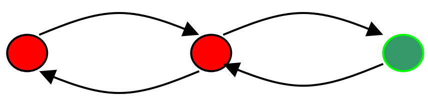
For each possible color, we define the corresponding matrix by erasing the incompatible columns from A :



$$A_r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Trackability : the formal problem

To a given observation, associate the corresponding product:



$$A_r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_r A_g A_r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The number of possible trajectories is given by the sum of the entries of the matrix

$$\mathbf{x}_{t+1} = \begin{array}{l} \mathbf{A}_0\, \mathbf{x}_t \\ \mathbf{A}_1\, \mathbf{x}_t \end{array}$$

Outline

- Joint spectral characteristics
- Path-complete methods for switching systems stability
- Applications:
  - Trackability
  - WCNs and packet dropouts
  - Switching delays
- Conclusion and perspectives

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Switching systems

$$\mathbf{x}_{t+1} = \begin{matrix} \mathbf{A}_0\, \mathbf{x}_t \\ \mathbf{A}_1\, \mathbf{x}_t \end{matrix}$$

Point-to-point Given x_0 and x_*, is there a product (say, $A_0\ A_0\ A_1\ A_0\ \ldots\ A_1$) for which $x_* = A_0\ A_0\ A_1\ A_0\ \ldots\ A_1\ x_0$?

Mortality Is there a product that gives the zero matrix?

Boundedness Is the set of all products $\{A_0, A_1, A_0A_0, A_0A_1, \ldots\}$ bounded?

Global convergence to the origin Do all products of the type $A_0\ A_0\ A_1\ A_0\ \ldots\ A_1$ converge to zero?

Switching systems

$$x_{t+1} = \begin{array}{l} A_0\, x_t \\ A_1\, x_t \end{array}$$

Global convergence to the origin Do all products of the type $A_0\, A_0\, A_1\, A_0 \ldots A_1$ converge to zero?

The spectral radius of a matrix A controls the growth or decay of powers of A

$$\rho(A) = \lim_{t \to \infty} ||A^t||^{1/t}$$

The powers of A converge to zero iff $\rho(A) < 1$

The joint spectral radius of a set of matrices Σ is given by

$$\rho(\Sigma) = \lim_{t \to \infty} \max_{A_i \in \Sigma} ||A_1 A_2 \ldots A_t||^{1/t}$$

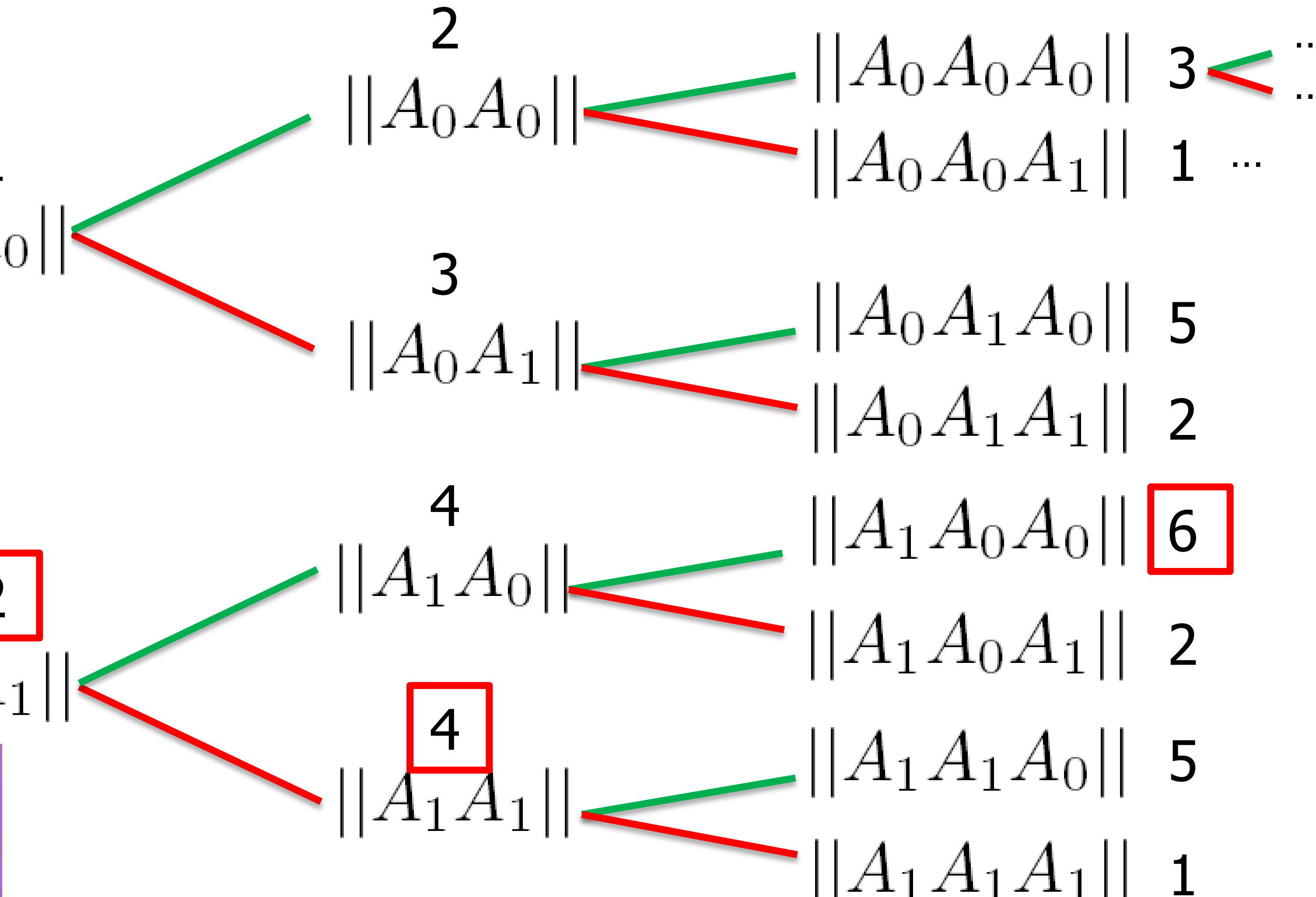
All products of matrices in Σ converge to zero iff $\rho(\Sigma) < 1$


[Rota, Strang, 1960]

The joint spectral characteristics

$$\boxed{\rho(\Sigma) = \lim_{t\to\infty} \left[\max_{A_i \in \Sigma} ||A_1 A_2 \dots A_t||\right]^{1/t}}$$

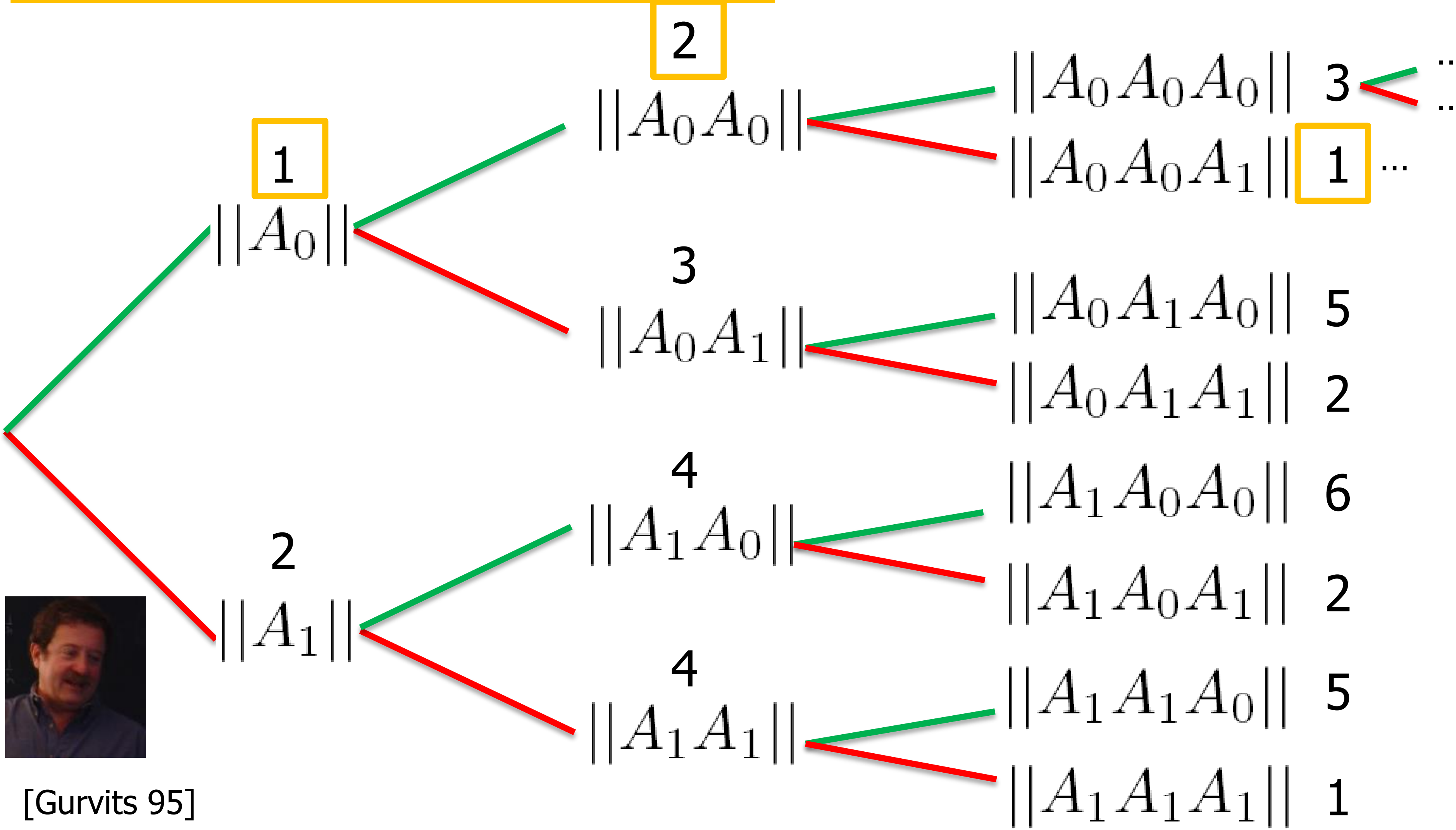
The joint spectral radius



The joint spectral characteristics

$$\boxed{\check{\rho}(\Sigma) = \lim_{t\to\infty}\left[\min_{A_i\in\Sigma} ||A_1A_2\dots A_t||\right]^{1/t}}$$

The joint spectral subradius



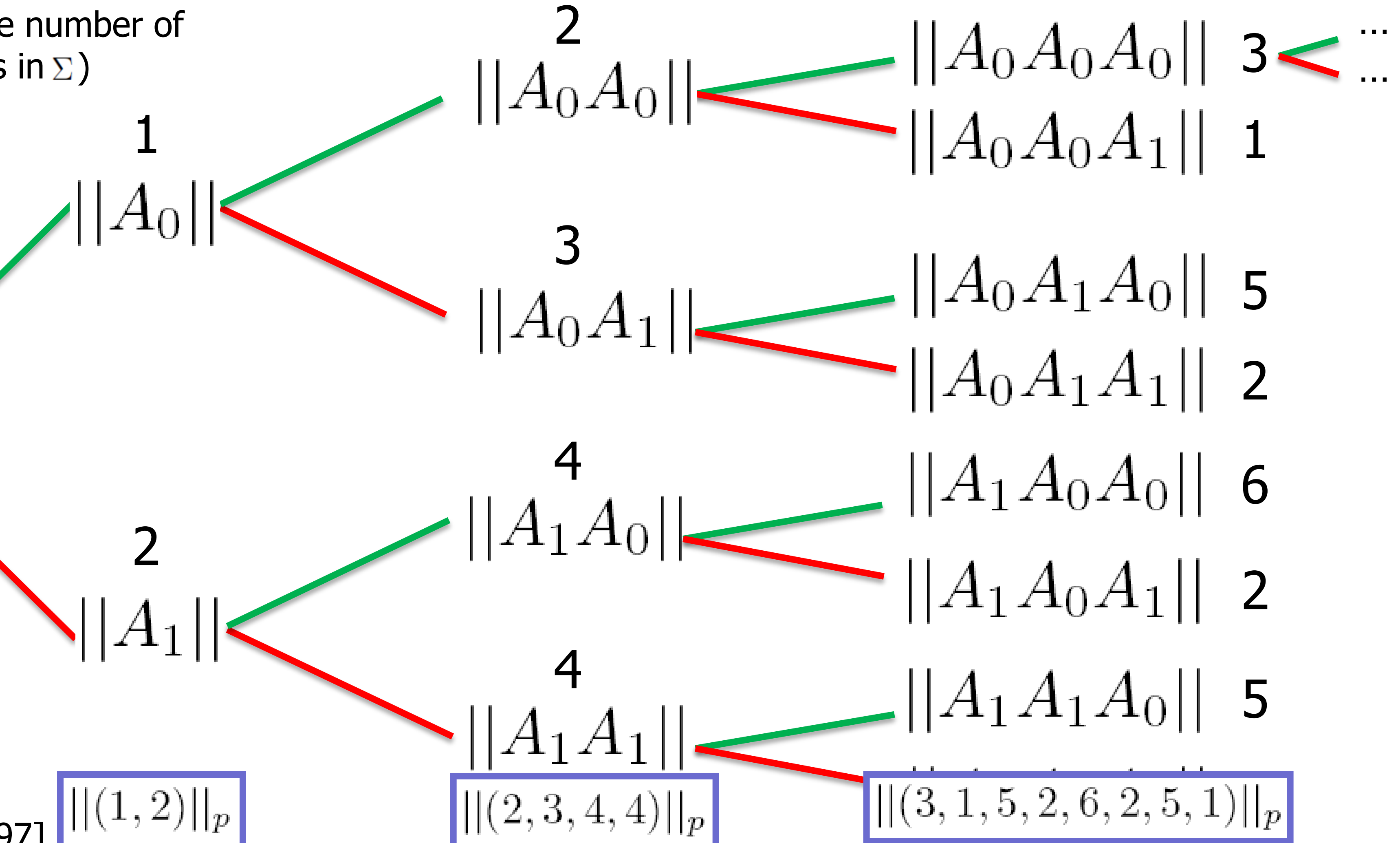
[Gurvits 95]

The joint spectral characteristics

$$\rho_p(\Sigma) = \lim_{t\to\infty}\left[\, m^{-t} \sum_{A_i\in\Sigma^t} \|A_1A_2\ldots A_t\|^p \,\right]^{1/(pt)}$$

The p-radius

(m is the number of matrices in Σ)



[Protasov 97]

The joint spectral characteristics

$$\bar{\rho}(\Sigma) = \lim_{t\to\infty} \left[\prod_{A_i\in\Sigma^t} \|A_1 A_2 \dots A_t\| \right]^{1/(tm^t)}$$

The Lyapunov Exponent

(m is the number of matrices in Σ)

1 $||A_0||$

2 $||A_0A_0||$

$||A_0A_0A_0||$ 3 …
…

$||A_0A_0A_1||$ 1 …

3 $||A_0A_1||$

$||A_0A_1A_0||$ 5

$||A_0A_1A_1||$ 2

4 $||A_1A_0||$

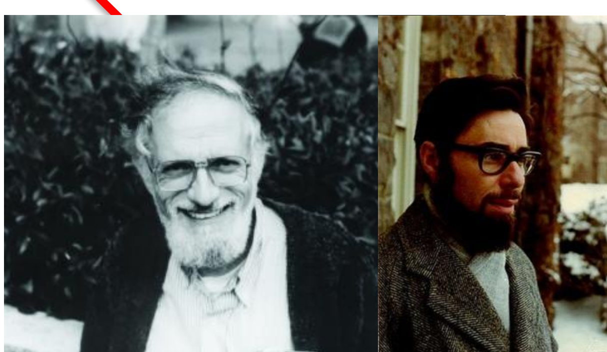
$||A_1A_0A_0||$ 6

$||A_1A_0A_1||$ 2

4 $||A_1A_1||$

$||A_1A_1A_0||$ 5

$||A_1A_1A_1||$ 1



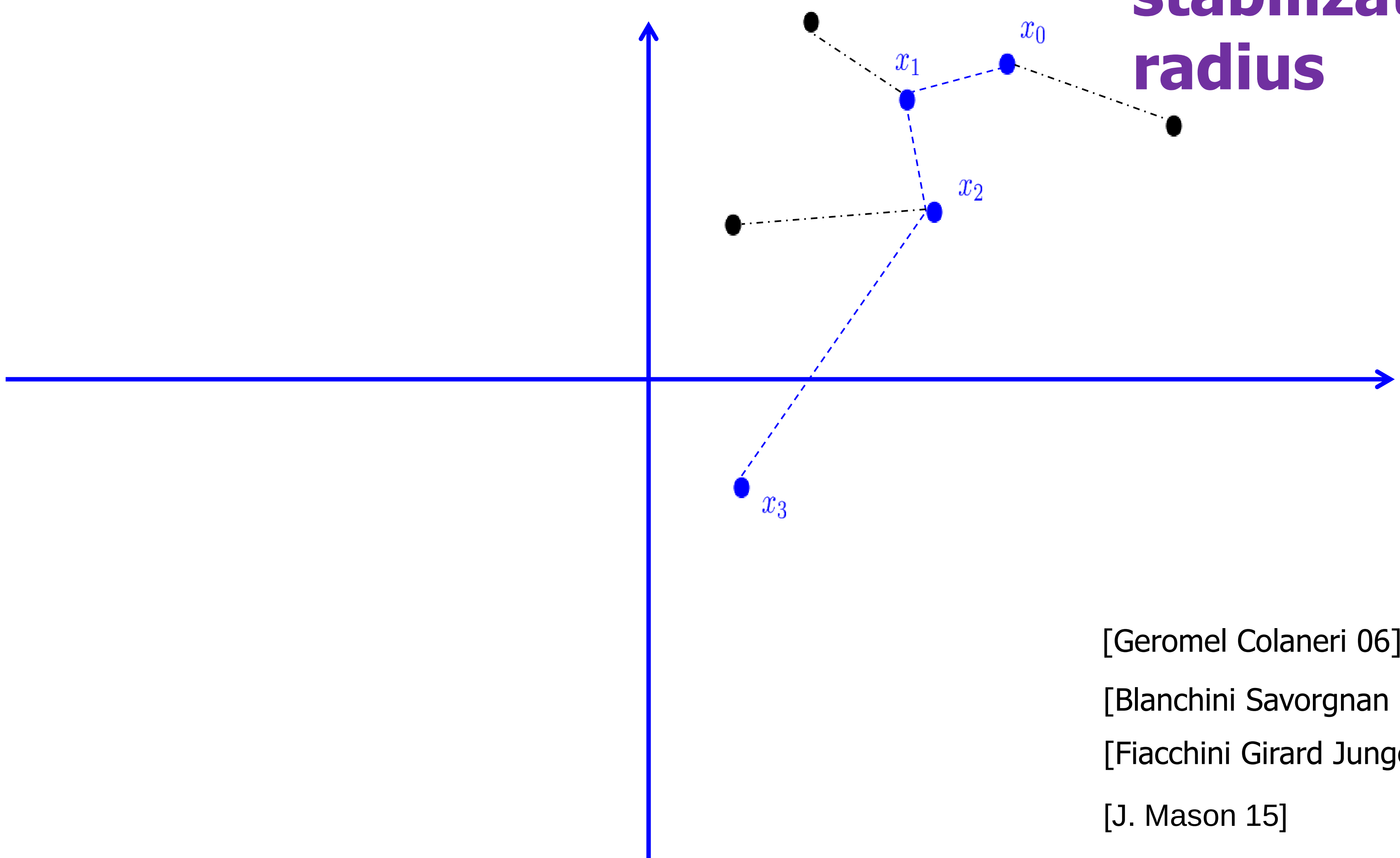
[Furstenberg Kesten, 1960]

The joint spectral characteristics

$$\tilde{\rho}_x(\Sigma) = \inf\{\lambda \geq 0 : \exists \sigma(0), \sigma(1), \ldots\ldots, \exists M > 0 \text{ s.t. } |x_{\sigma,x}(t)| \leq M\lambda^t |x|, \forall t \geq 0\}$$

$$\tilde{\rho}(\Sigma) = \sup_{x \in \mathbb{R}^n} \tilde{\rho}_x(\Sigma)$$

The feedback stabilization radius



[Geromel Colaneri 06]

[Blanchini Savorgnan 08]

[Fiacchini Girard Jungers 15]

[J. Mason 15]

The joint spectral characteristics

$$\tilde{\rho}_x(\Sigma) = \inf\{\lambda \geq 0 : \exists \sigma(0), \sigma(1), \ldots.., \exists M > 0 \text{ s.t. } |x_{\sigma,x}(t)| \leq M\lambda^t |x|, \forall t \geq 0\}$$

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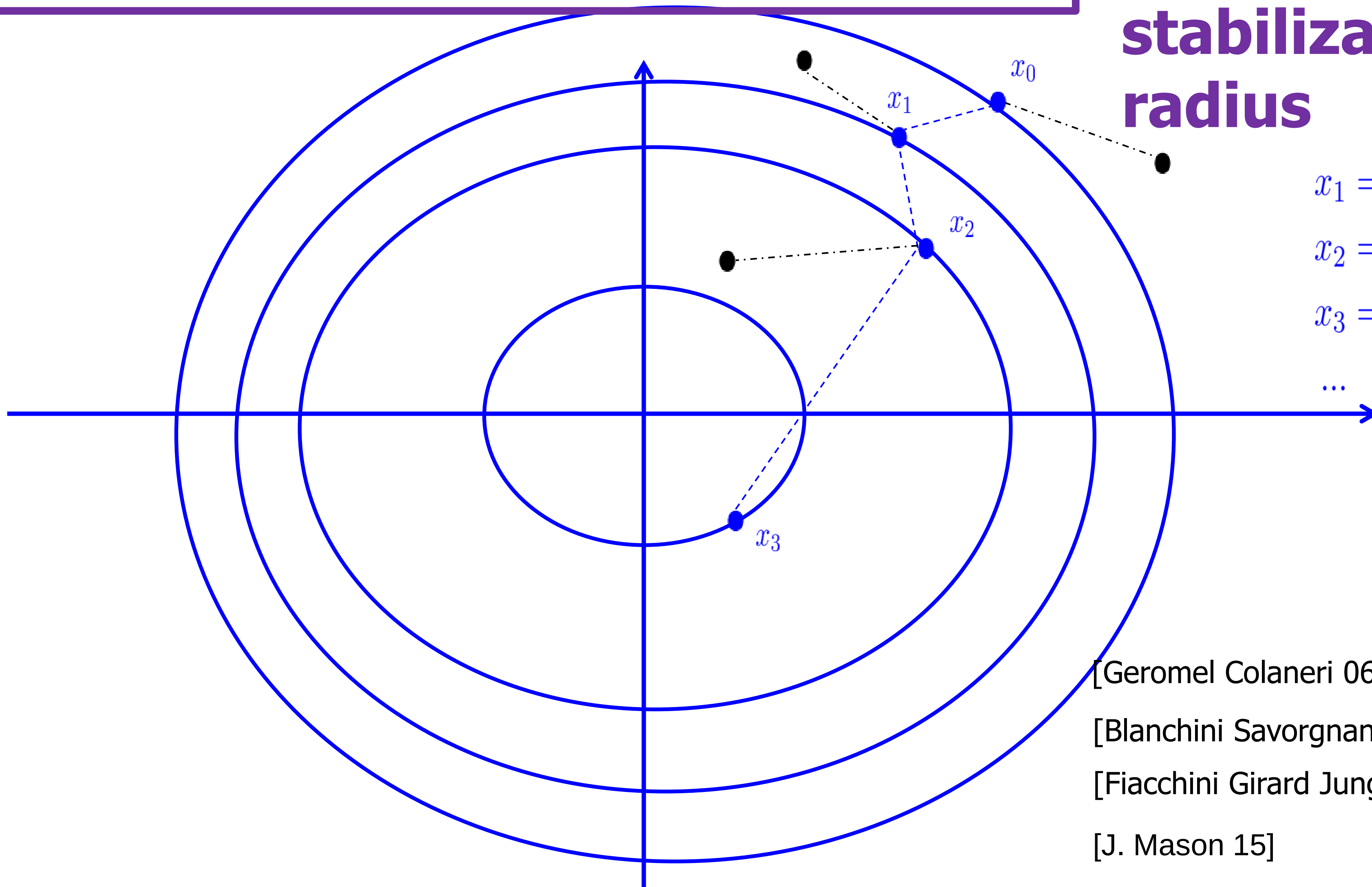


$x_1 = A_{\sigma(0)} x_0$

$x_2 = A_{\sigma(1)} x_1$

$x_3 = A_{\sigma(2)} x_2$

...

[Geromel Colaneri 06]

[Blanchini Savorgnan 08]

[Fiacchini Girard Jungers 15]

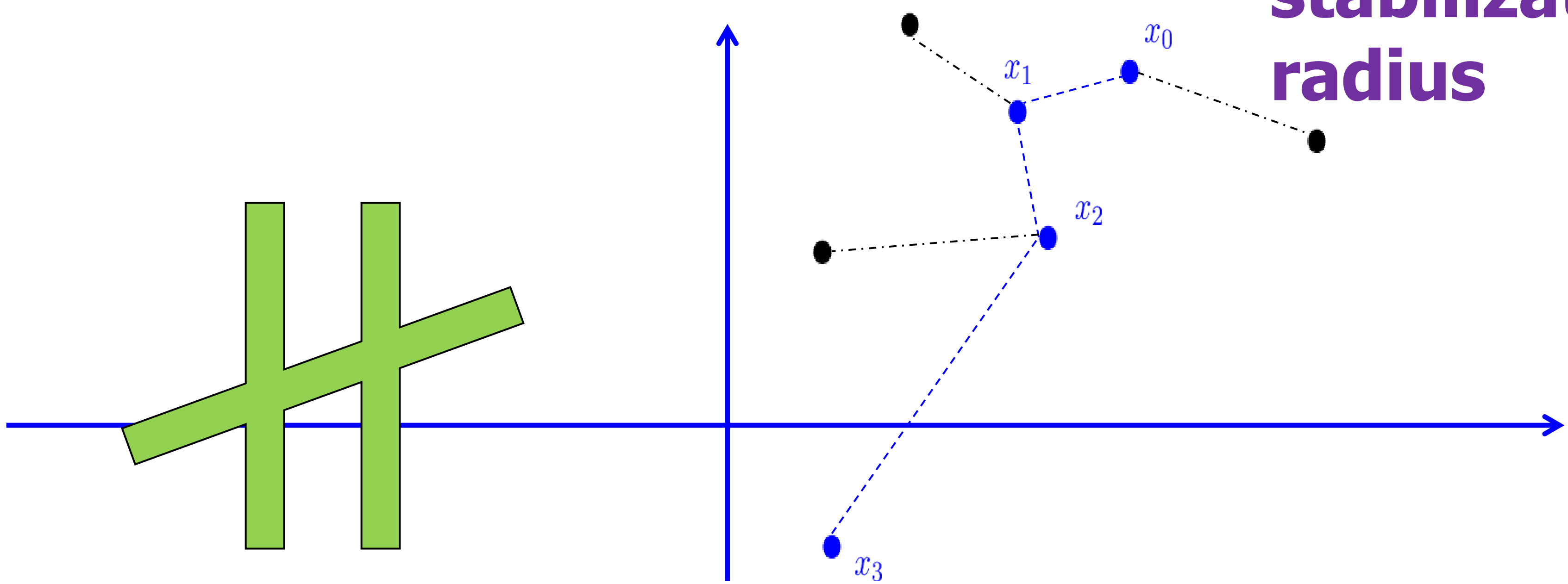
[J. Mason 15]

The joint spectral characteristics

$$\tilde{\rho}_x(\Sigma) = \inf\{\lambda \geq 0 : \exists \sigma(0), \sigma(1), \ldots.., \exists M > 0 \text{ s.t. } |x_{\sigma,x}(t)| \leq M\lambda^t |x|, \forall t \geq 0\}$$

$$\tilde{\rho}(\Sigma) = \sup_{x \in \mathbb{R}^n} \tilde{\rho}_x(\Sigma)$$

The feedback stabilization radius



Alternative definition: suppose you can observe x(t) at every step, and apply the switching you want, as a function of the x(t)

[Geromel Colaneri 06]

[Blanchini Savorgnan 08]

[Fiacchini Girard Jungers 15]

[J. Mason 15]

The joint spectral characteristics

$$\tilde{\rho}_x(\Sigma) = \inf\{\lambda \geq 0 : \exists\sigma(0), \sigma(1), \ldots\ldots, \exists M > 0 \text{ s.t. } |x_{\sigma,x}(t)| \leq M\lambda^t |x|, \forall t \geq 0\}$$

$$\tilde{\rho}(\Sigma) = \sup_{x \in \mathbb{R}^n} \tilde{\rho}_x(\Sigma)$$

The feedback stabilization radius

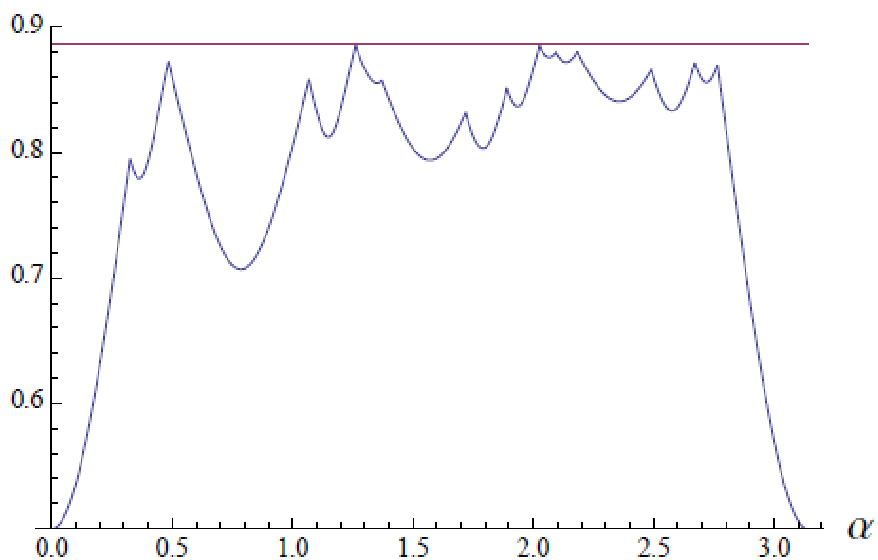


Figure 1: The function $F(\alpha) = \min_i \left(|A[i]z_\alpha|^{1/t_i}\right)$, where $z_\alpha = (\cos\alpha, \sin\alpha)^T$ and t_i is the length of the matrix product $A[i]$. Its maximum is an upper bound on the feedback stabilization radius. This maximum is approximately equal to 0.886.

The joint spectral characteristics

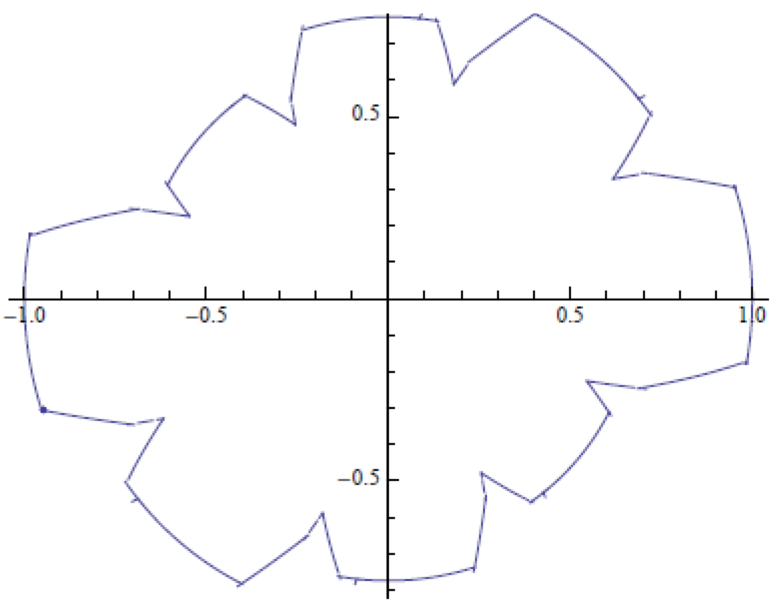
$$\tilde{\rho}_x(\Sigma) = \inf\{\lambda \geq 0 : \exists \sigma(0), \sigma(1), \ldots\ldots, \exists M > 0 \text{ s.t. } |x_{\sigma,x}(t)| \leq M\lambda^t |x|, \forall t \geq 0\}$$

$$\tilde{\rho}(\Sigma) = \sup_{x \in \mathbb{R}^n} \tilde{\rho}_x(\Sigma)$$

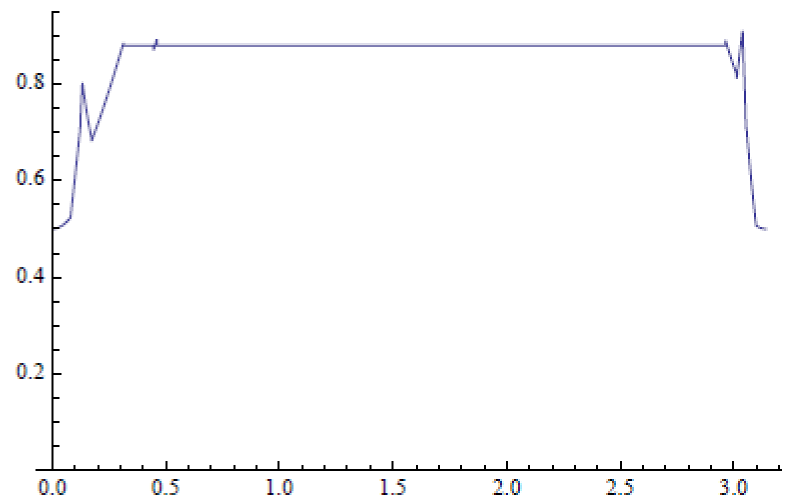
The feedback stabilization radius

Proposition 5. *Suppose Assumption 1 holds. Then for any $\lambda > \tilde{\rho}(\mathcal{M})$ the function $V_\lambda : \mathbb{R}^d \to \mathbb{R}_+$*

$$V_\lambda(x) = \sup_{t \geq 0} \inf_{\sigma(\cdot)} \frac{|x_{\sigma,x}(t)|}{\lambda^t} \tag{5}$$



(a) The level set $\hat{V}_\lambda^{-1}(1)$



(b) Ratio $\min\{\hat{V}_\lambda(A_1 x), \hat{V}_\lambda(A_2 x)\}/\hat{V}_\lambda(x)$ for x belonging to the first two quadrants.

The joint spectral characteristics

$$\rho(\Sigma) = \lim_{t\to\infty} \left[\max_{A_i\in\Sigma} ||A_1 A_2 \ldots A_t||\right]^{1/t}$$

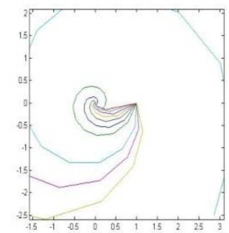
The joint spectral radius addresses the **stability** problem

$$\check{\rho}(\Sigma) = \lim_{t\to\infty} \left[\min_{A_i\in\Sigma} ||A_1 A_2 \ldots A_t||\right]^{1/t}$$

The joint spectral subradius addresses the **stabilizability** problem

$$\rho_p(\Sigma) = \lim_{t\to\infty} \left[m^{-t} \sum_{A_i\in\Sigma^t} ||A_1 A_2 \ldots A_t||^p \right]^{1/(pt)}$$

The p-radius addresses the… **p-weak stability**



[J. Protasov 10]

$$\bar{\rho}(\Sigma) = \lim_{t\to\infty} \left[\prod_{A_i\in\Sigma^t} ||A_1 A_2 \ldots A_t|| \right]^{1/(tm^t)}$$

The Lyapunov exponent addresses the **stability with probability one**

(Cfr. Oseledets Theorem)

$$\tilde{\rho}_x(\Sigma) = \inf\{\lambda \geq 0 : \exists \sigma(0), \sigma(1), \ldots.., \exists M > 0 \text{ s.t. } |x_{\sigma,x}(t)| \leq M\lambda^t |x|, \forall t \geq 0\}$$

$$\tilde{\rho}(\Sigma) = \sup_{x\in\mathbb{R}^n} \tilde{\rho}_x(\Sigma)$$

The feedback stabilization radius addresses the **feedback stabilizability**

[J. Mason 16]

[Fiacchini Girard Jungers 15]

The joint spectral characteristics: Mission Impossible?

Theorem Computing or approximating ρ is NP-hard

Theorem The problem $\rho>1$ is algorithmically undecidable

Conjecture The problem $\rho<1$ is algorithmically undecidable

Theorem Even the question « $|\check{\rho} - r| \leq a + b\check{\rho}$?» is algorithmically undecidable for all (nontrivial) a and b

Theorem The same is true for the Lyapunov exponent

Theorem The p-radius is NP-hard to approximate

Theorem The feedback stabilization radius is turing-uncomputable

See [Blondel Tsitsiklis 97,
Blondel Tsitsiklis 00,
J. Protasov 09
J. Mason 15]

Algorithmic complexity

	Arbitrary approximation	Arbitrary approximation in polynomial time	Arbitrary approximation for positive matrices	Decidability
Joint Spectral Radius	✓	✓	✓	?
Joint Spectral Subradius	✗	✗	✓	✗
Lyapunov Exponent	✗	✗	✓	✗
p-radius	**Depends on p**	**Depends on p**	✓	?
Feedback st. radius	✗	✗	✓	✗

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- Joint spectral characteristics
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- Applications:
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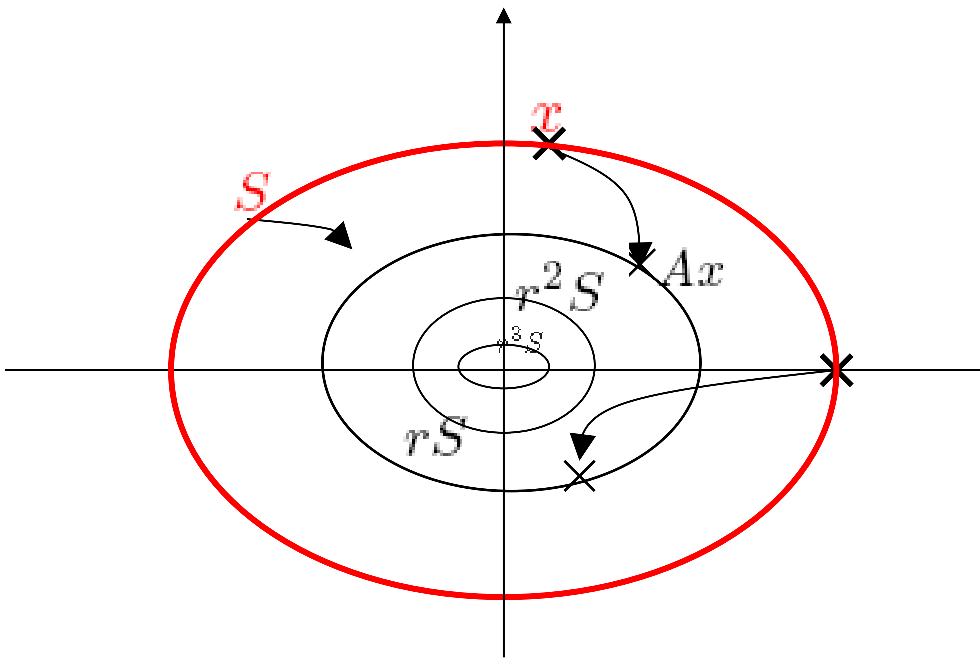
LMI methods

- The CQLF method

$$
\begin{array}{lll}
\inf_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A^T P A & \preceq & r^2 P, \quad \forall A \in \Sigma \\
P & \succeq & 0.
\end{array}
$$

$$\Leftrightarrow \frac{|Ax|_P}{|x|_P} \leq r$$



$$\rho \leq r$$

SDP methods

- Theorem For all $\epsilon > 0$ there exists a norm such that

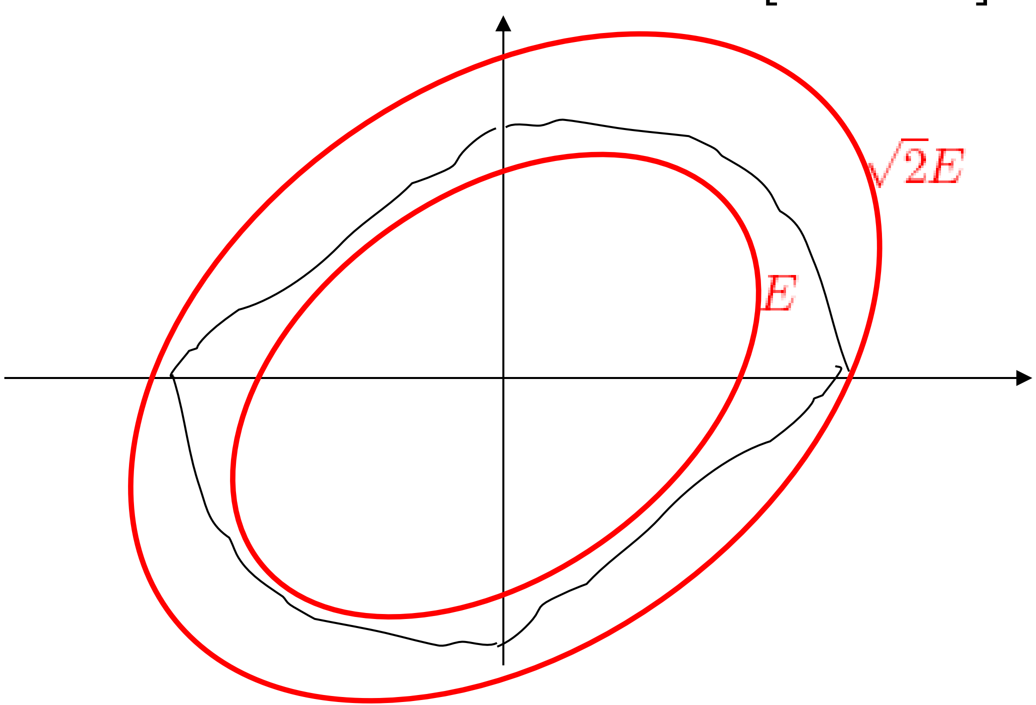
$$\forall A \in \Sigma, \forall x, |Ax| \leq (\rho + \epsilon)|x|$$

[Rota Strang, 60]

- John's ellipsoid Theorem: Let K be a compact convex set with nonempty interior symmetric about the origin. Then there is an ellipsoid E such that $E \subset K \subset \sqrt{n}E$

[John 1948]

•So we can approximate the unit ball of an extremal norm with an ellipsoid

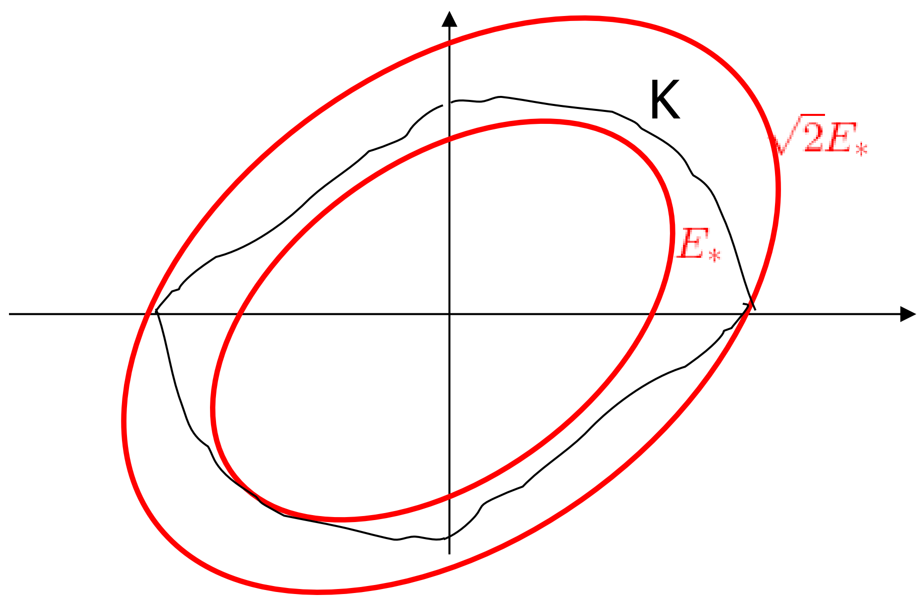


SDP methods

- Theorem The best ellipsoidal norm $||\cdot||_{E_*}$ approximates the joint spectral radius up to a factor $\sqrt{n}$ [Ando Shih 98]

$$\rho \leq \max ||A||_{E_*} \leq \sqrt{n}\rho$$

$$\frac{1}{\sqrt{n}}\rho* \leq \rho \leq \rho*$$



$$\frac{1}{\sqrt[2d]{n}}\rho* \leq \rho \leq \rho*$$

$\rho < 1/n^{\frac{1}{2d}} \Rightarrow$ There exists a Lyap. function of degree d

One can improve this method by lifting techniques [Nesterov Blondel 05]
[Parrilo Jadbabaie 08]

Algorithm that approximates the joint spectral radius of arbitrary sets of m (nXn)-matrices up to an arbitrary accuracy ϵ in $\mathcal{O}(n^{m\frac{1}{\epsilon}})$ operations

PTAS

Yet another LMI method

- A strange semidefinite program

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P & \succeq & 0.
\end{array}
$$

$$\Longrightarrow \qquad \rho \leq r$$

[Goebel, Hu, Teel 06]

- But also…

[Daafouz Bernussou 01]

[Bliman Ferrari-Trecate 03]

[Lee and Dullerud 06] …

Yet another LMI method

- An even stranger program:

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P A_1 & \preceq & r^2 P, \\
(A_2 A_1)^T P (A_2 A_1) & \preceq & r^4 P, \\
(A_2^2)^T P (A_2^2) & \preceq & r^4 P, \\
P & \succeq & 0.
\end{array}
$$

$$\rho \leq r$$

[Ahmadi, J., Parrilo, Roozbehani10]

Yet another LMI method

- Questions:
  - Can we characterize all the LMIs that work, in a unified framework?
  - Which LMIs are better than others?
  - How to prove that an LMI works?
  - Can we provide converse Lyapunov theorems for more methods?

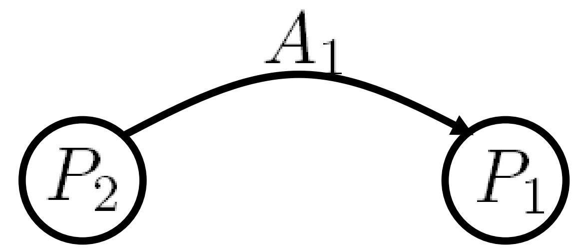
$$\frac{1}{\sqrt[2d]{n}}\rho* \leq \rho \leq \rho*$$

$\rho < 1/n^{\frac{1}{2d}} \Rightarrow$ There exists a Lyap. function of degree d

From an LMI to an automaton

- Automata representation Given a set of LMIs, construct an automaton like this:

$$A_1^T P_1 A_1 \preceq r^2 P_2$$



$$\begin{array}{lcl}
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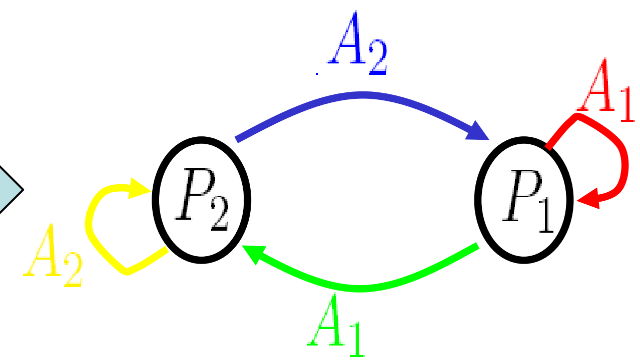
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}$$

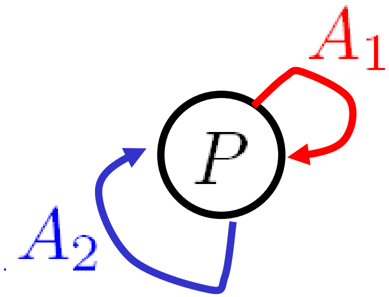


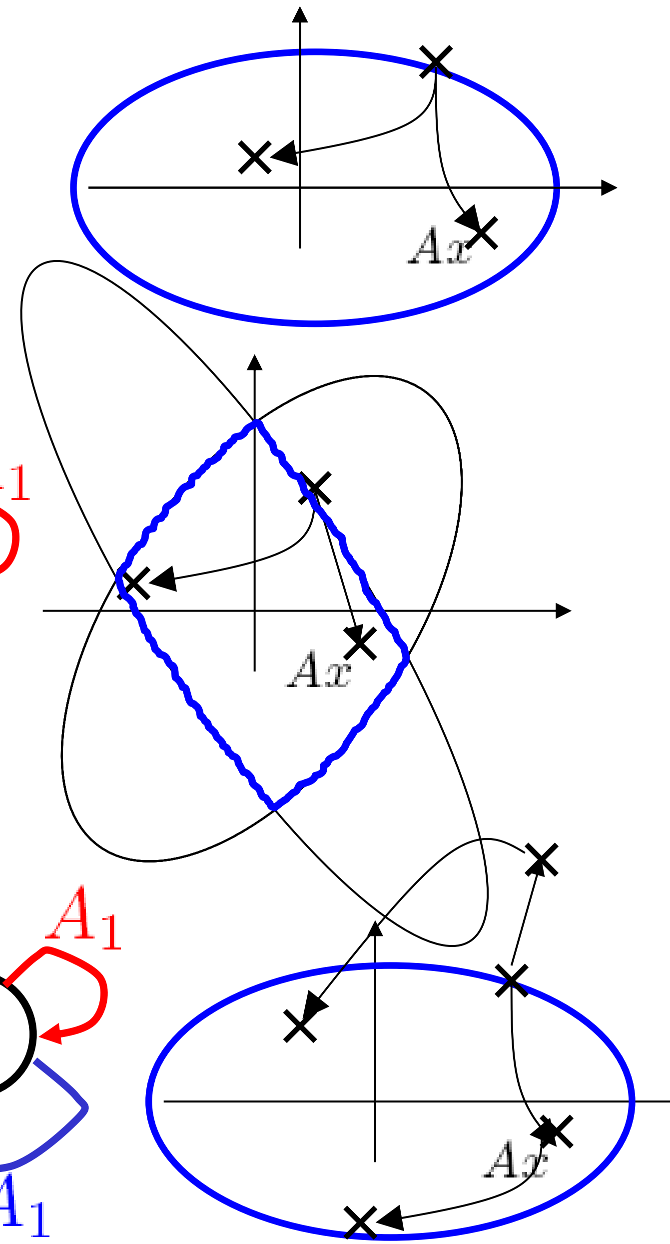
- Definition A labeled graph (with label set A) is path-complete if for any word on the alphabet A, there exists a path in the graph that generates the corresponding word.
- Theorem If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability.

[Ahmadi J. Parrilo Roozbehani 14]

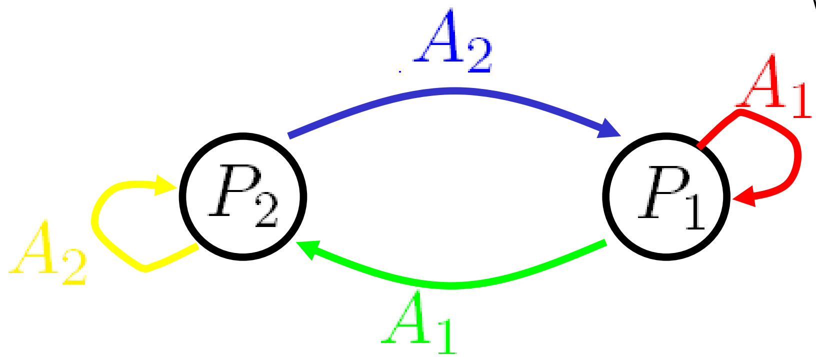
Some examples

- Examples:
  - CQLF





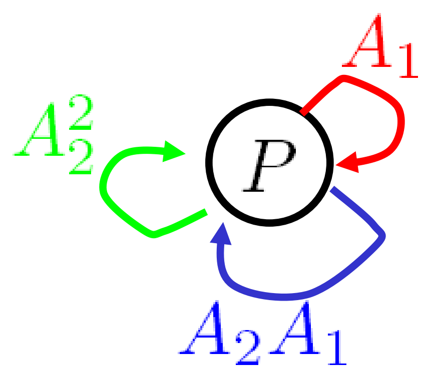
  - Example 1



This type of graph gives a max-of-quadratics Lyapunov function (i.e. intersection of ellipsoids)

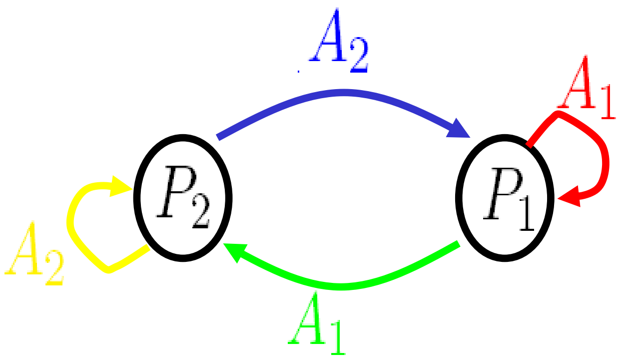
  - Example 2

This type of graph gives a common Lyapunov function for a generating set of words



An obvious question: are there other valid criteria?

- Theorem



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A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}
$$

Path complete ??? Sufficient condition for stability

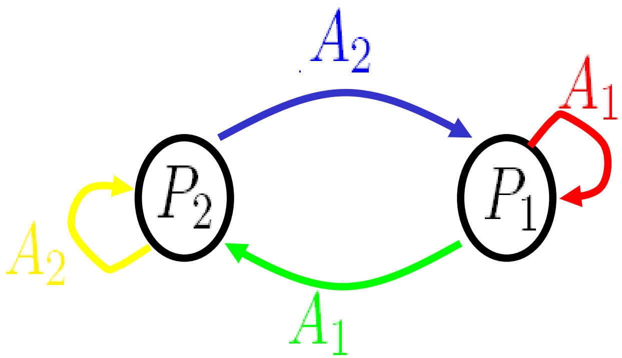
If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability.

- Are all valid sets of equations coming from path-complete graphs?
- …or are there even more valid LMI criteria?

Are there other valid criteria?

- Theorem Non path-complete sets of LMIs are not sufficient for stability.

[J. Ahmadi Parrilo Roozbehani 15]



$$
\begin{array}{lcl}
\min_{r\in\mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}
$$

Path complete

???

Sufficient condition for stability

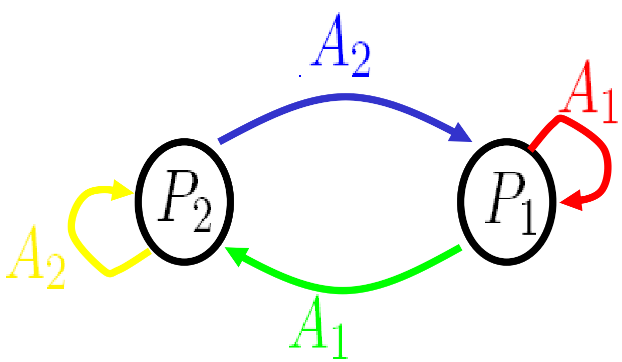
- Corollary

  It is PSPACE complete to recognize sets of equations that are a sufficient condition for stability

- These results are not limited to LMIs, but apply to other families of conic inequalities

So what now?

After all, what are all these results useful for?



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Optimize on optimization problems!

This framework is generalizable to harder problems

- Constrained switching systems
- Controller design for switching systems
- Automatically optimized abstractions of cyber-physical systems
- …

So what now?

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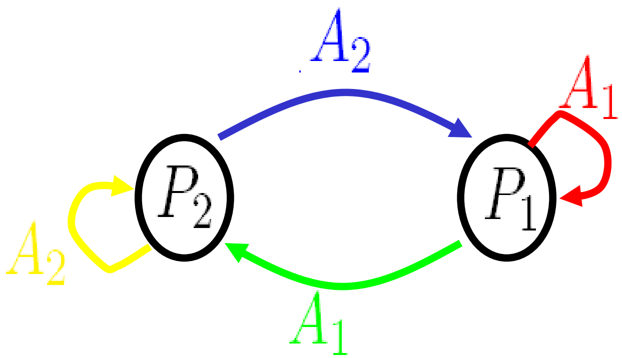

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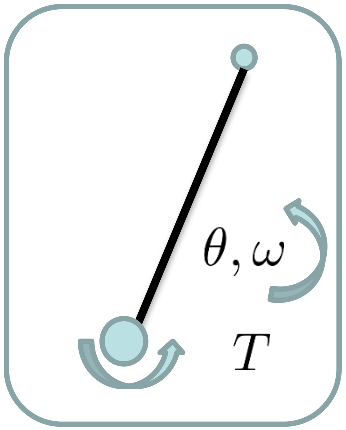
Optimize on optimization problems!

This framework is generalizable to harder problems

- **Constrained switching systems**
- Controller design for switching systems
- Automatically optimized abstractions of cyber-physical systems
- …

We begin with an example ☺

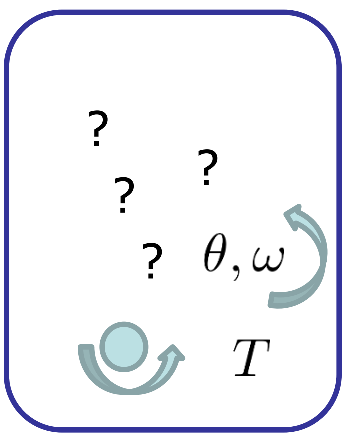
- Take an inverted pendulum…



Linearized around "up" :

$$x_{k+1} = (A + BK)x_k = A_1 x_k$$
$$u_k = K x_k$$

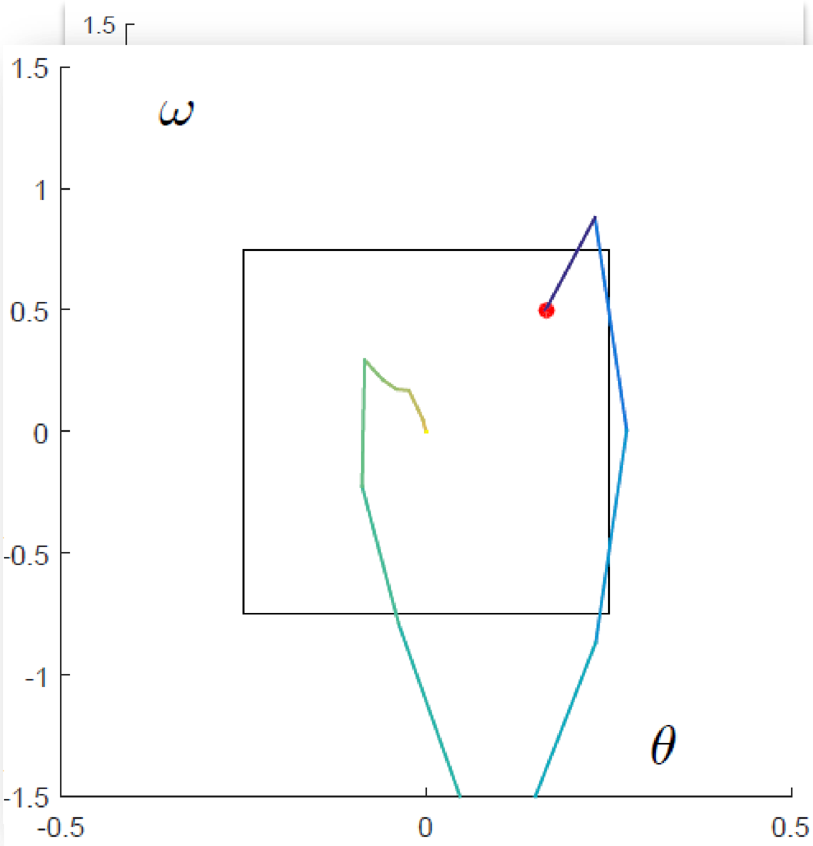
- "Close the eye" of the controller…

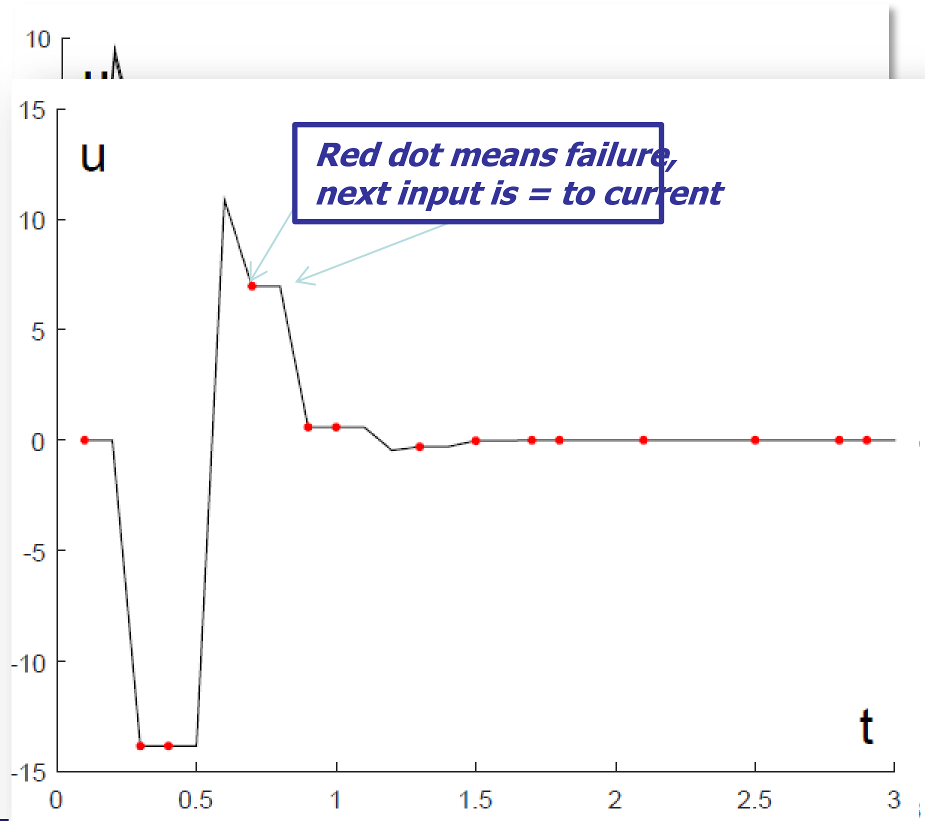


Linearized around "up" :

$$x_{k+1} = A x_k + B u_k$$
$$u_k = u_{\text{last updated}}$$

Some plots:





thing goes well

When there is **at most 2 consecutive** failures…
Already pretty bad. **But is this stable?**

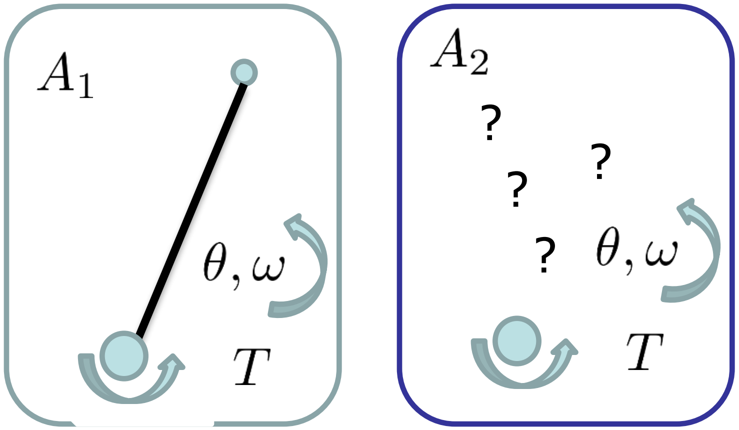
Switching systems

State update	$x_{t+1} = A_{\sigma(t)} x_t$
Modes of the system	$A_{\sigma(t)} \in \mathbf{A} = \{A_1, \ldots, A_N\}$
Switching signal	$t \to \sigma(t) : \{0, 1, 2, \ldots\} \mapsto \{1, \ldots, N\}$

Switching systems: Dropouts

Controlled plant : "$x_{t+1} = A_1 x_t$"

Stable if the controller fails never more than 2 times in a row



$$A_1 \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} A + BK & 0 \\ K & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

$$A_2 \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

Switching systems: Dropouts

Controlled plant : "$x_{t+1} = A_1 x_t$"

Stable if the controller fails never more than 2 times in a row

Switching system with 2 modes.

$$x_{t+1} = \begin{cases} \mathbf{A_1} x_t, & \text{if controller works} \\ \mathbf{A_2} x_t, & \text{if controller fails} \end{cases}$$

Constrained switching sequences.

$\cdots \mathbf{A_1 A_1 A_1 A_1} x_0$

This is possible!

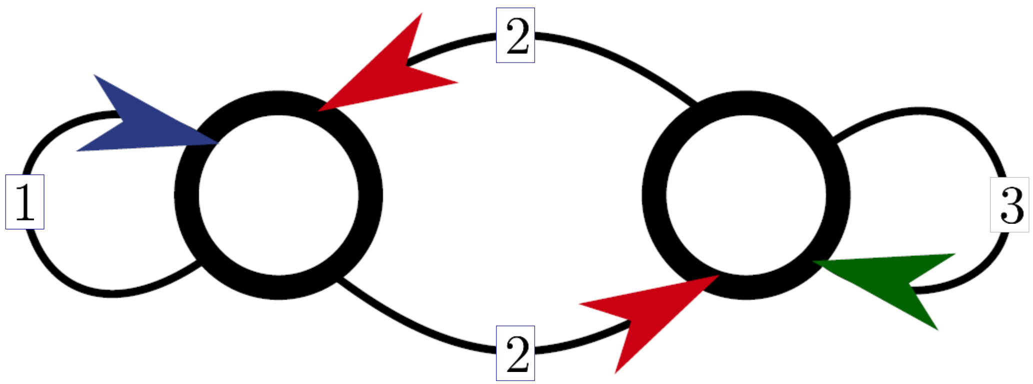
$\cdots \mathbf{A_2 A_2 A_1 A_1} x_0$

$\cdots \mathbf{A_2 A_2 A_2 A_1} x_0$

This is *not* possible!

Switching rules through graphs

$$\Theta(V,E) : \text{the graph},$$
$$V : \text{set of vertices},$$
$$E : \text{set of labeled directed edges},$$
$$(v,w,\ell) \in E,\ v,w \in V, \ell \in \{1,\dots,N\}$$



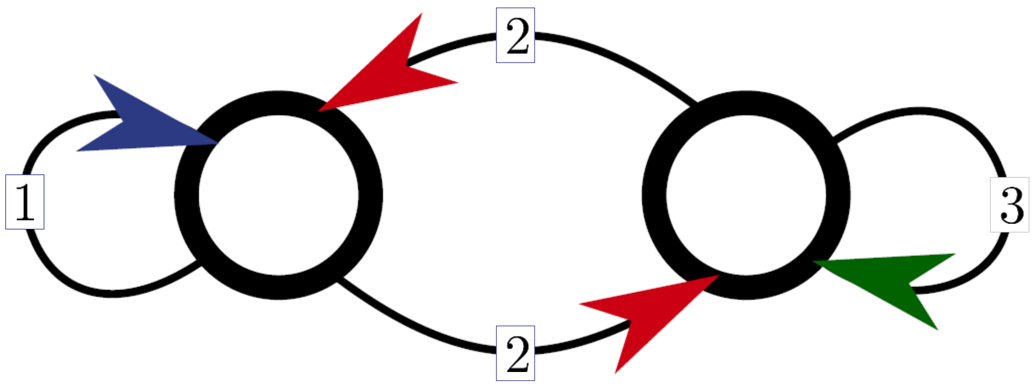
Paths and switching sequences

Paths of the graphs.

$$p = \{(v_0, v_1, \ell(1)), (v_1, v_2, \ell(2)), \ldots, (v_{T-1}, v_T, \ell(T))\}$$

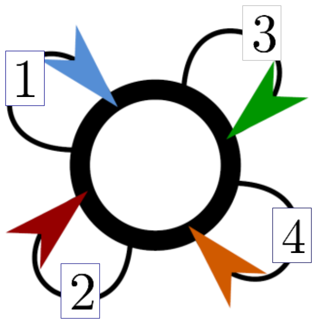
Paths map to trajectories.

$$A_p = A_{\ell(T)} \cdots A_{\ell(1)} \qquad x_0 \underset{p\in\Theta, |p|=t}{\rightarrow} A_p x_0 = x_t$$

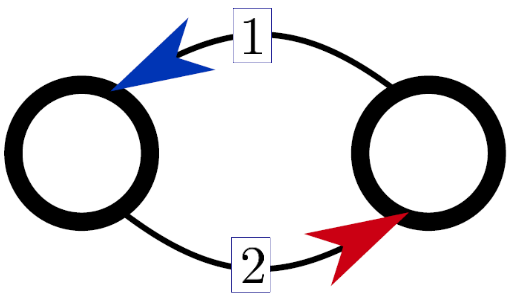


Defines rules on the switching sequences of the syste

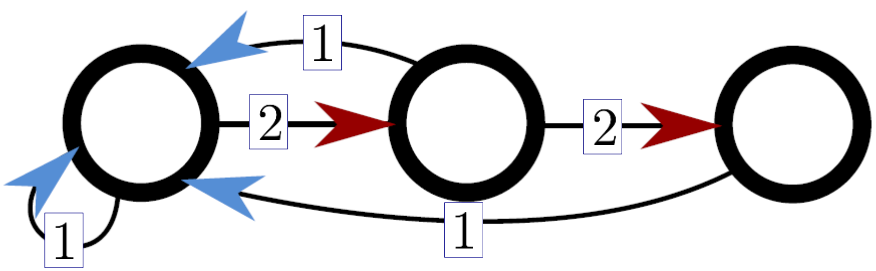
A graph for maximum dwell time



Arbitrary switching on 4 modes.
Any sequence is OK, take the loops you need



Periodic system on 2 modes.



Maximum dwell time on mode 2.
Cannot have …1,2,2,2…

Stability and boundedness

Given a constrained switching system $S = (\Theta, \mathbf{A})$

Dead-beat Stab.

$$\exists T \geq 1 : \forall x_0 \in \mathbb{R}^n, \forall p \in \Theta : |p| = T,$$
$$\rightarrow A_p x_0 = x_T = 0.$$

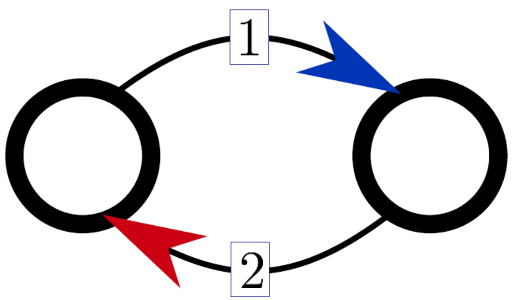
Exponential Stab.

$$\exists \lambda < 1, K \geq 1 : \forall x_0 \in \mathbb{R}^n, \forall t, \forall p \in \Theta : |p| = t,$$
$$\rightarrow \|A_p x_0\| \leq K\lambda^t \|x_0\|.$$

Boundedness

$$\exists K \geq 1 : \forall x_0 \in \mathbb{R}^n, \forall t, \forall p \in \Theta,$$
$$\rightarrow \|A_p x_0\| \leq K \|x_0\|.$$

Failure of contractive norms



Scalar, $A_1 = 2$, $A_2 = 1/8$

We are stable ☺

$$x_{2T} = \frac{x_0}{4^T}$$

$$\hat{\rho}(S) = \left(\tfrac{1}{4}\right)^{1/2} = 1/2 < 1.$$

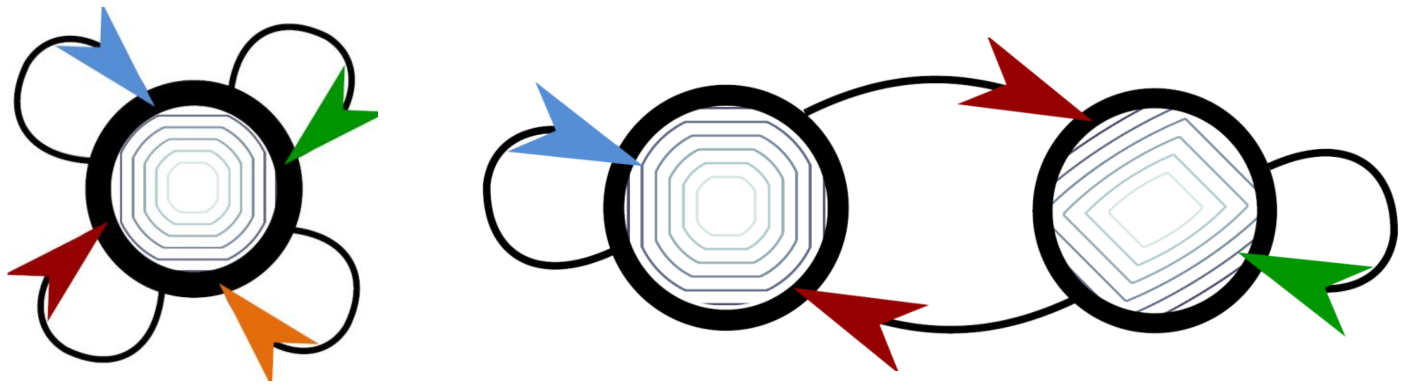
A **common norm** would need to satisfy $||2x|| \leq ||x||$

Multinorms for stability

JSR defined through sets of norms.

Theorem:

$$\hat{\rho}(S) = \begin{cases} \inf \gamma \\ s.t.\ \exists\{\|\cdot\|_v,\ v \in V\} :\ \forall x,\ \forall (v,w,\ell) \in E, \\ \|A_\ell x\|_w \leq \gamma \|x\|_v \end{cases}$$



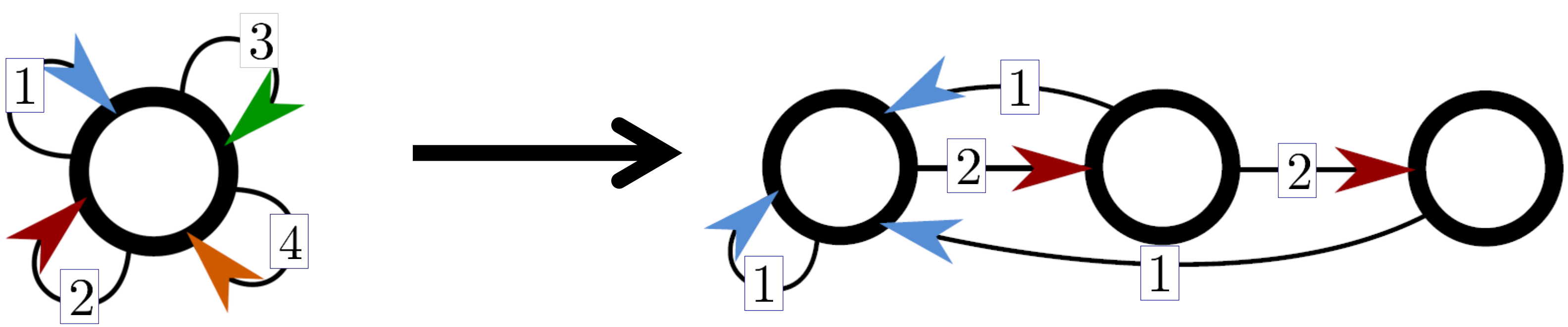
- **Direct generalization of the arbitrary switching case**
- **Stability if and only if Multiple Lyapunov Function**

The approximation problem

- Computing the JSR is hard (≤ 1 is undecidable)
- Approximation for **arbitrary switching** systems, **bounded time complexity** achieved by **approximating contractive norms.**

Given S, a constrained system and $r > 0$, a desired accuracy le
output γ satisfying

$$\hat{\rho}(S) \leq \gamma \leq (1+r)\hat{\rho}(S)$$



Approx. using "contractive norms".

Approx. using "contractive multinorms".

Norms VS quadratic norms

Constrained JSR as an infimum over **multinorms.**

$$\hat{\rho}(S) = \left\{ \begin{array}{l} \inf \gamma \\ \quad s.t.\, \exists\{\|\cdot\|_v,\ v \in V\} :\ \forall x,\ \forall (v,w,\ell) \in E, \\ \qquad\qquad \|A_\ell x\|_w \leq \gamma \|x\|_v \end{array} \right.$$

How bad can this be? (Approximate with quadratic norms)

$$\gamma_Q(S) = \left\{ \begin{array}{l} \inf \gamma \\ \quad s.t.\, \exists\{Q_v \succ 0,\ v \in V\} :\ \forall (v,w,\ell) \in E, \\ \qquad\qquad A_\ell^\top Q_w A_\ell \leq \gamma^2 Q_v \end{array} \right.$$

Fixed accuracy bounds

John's Ellipsoid Theorem

For all norm $\|\cdot\|_K$, there is a quadratic norm $\|\cdot\|_Q$ such that

$$\|\cdot\|_Q \leq \|\cdot\|_K \leq \sqrt{n}\|\cdot\|_Q.$$

Accuracy when using quadratics

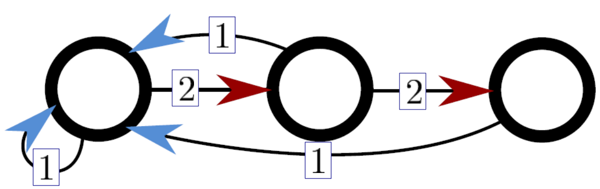
$$\gamma_Q(S) = \left\{ \begin{array}{c} \inf \gamma \\ s.t. \exists \{Q_v \succ 0,\ v \in V\} : \forall (v, w, \ell) \in E, \\ A_\ell^\top Q_w A_\ell \preceq \gamma^2 Q_v \end{array} \right.$$

$$\Rightarrow \hat{\rho}(S) \leq \gamma_Q(S) \leq \sqrt{n}\hat{\rho}(S)$$

(Another cool bound!)

$$\gamma_Q(S) = \begin{cases} \inf \gamma \\ s.t. \, \exists \{Q_v \succ 0, \; v \in V\} : \forall (v, w, \ell) \in E, \\ A_\ell^\top Q_w A_\ell \preceq \gamma^2 Q_v \end{cases}$$

$$\Rightarrow \hat{\rho}(S) \leq \gamma_Q(S) \leq \sqrt{\text{Spectral Radius of the adjacency matrix of } \Theta} \; \hat{\rho}(S)$$



$$\rho \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \simeq 1.84.$$

Its better than what you'll get for any *n*!

[Legat, Jungers, Parillo] –
Generating unstable trajectories for Switched Systems via Dual Sum-Of-Squares techniques –
Accepted HSCC2016

Perspective

L2-Gains

Approximation of the L2-gain for control-systems?

Stabilization?

Can we use the framework to obtain stabilizing switching sequences?

Generalization?

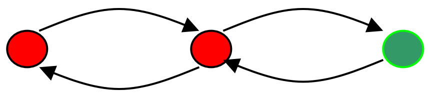
More general systems? Control?
Switching affine, State-Dependent Switching , Continuous-time,...?

Outline

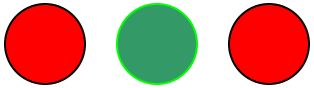
- Joint spectral characteristics
- Path-complete methods for switching systems stability
- Applications:
  - Trackable graphs
  - WCNs and packet dropouts
  - Switching delays
- Conclusion and perspectives

Trackable graphs

To a given observation, associate the corresponding product:



$$A_r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



$$A_r A_g A_r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The number of possible trajectories is given by the sum of the entries of the matrix

Trackable graphs

The maximal total number of possibilities is

$$N(t) = \max\left\{ \|A\|_1 \ : A \in \Sigma^t \right\}$$

We are interested in the asymptotic worst case :

$$\lim_{t\to\infty} N(t)^{1/t} = \lim_{t\to\infty} \max\left\{ \|A\|_1^{1/t} : A \in \Sigma^t \right\}$$

This is a joint spectral radius!

Trackable graphs

The network is trackable iff

$$\rho \leq 1$$

[Crespi et al. 05]

Theorem It is possible to check trackability in polynomial time

[J. Protasov Blondel 08]

Outline

- Joint spectral characteristics
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Applications of Wireless Control Networks

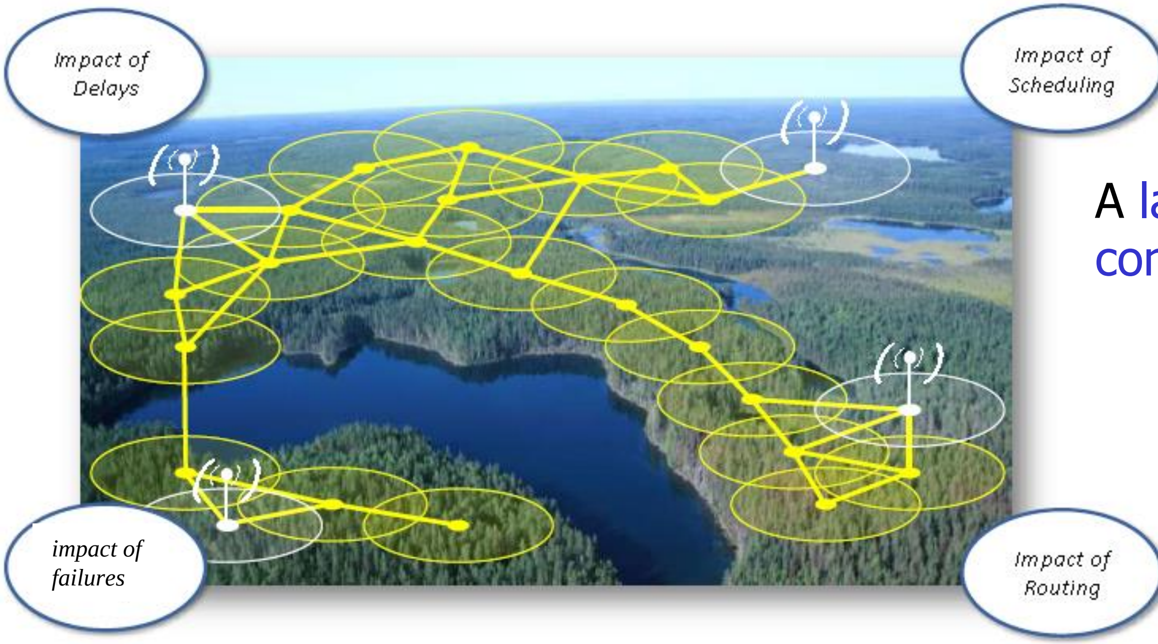
Industrial automation

Physical Security and Control

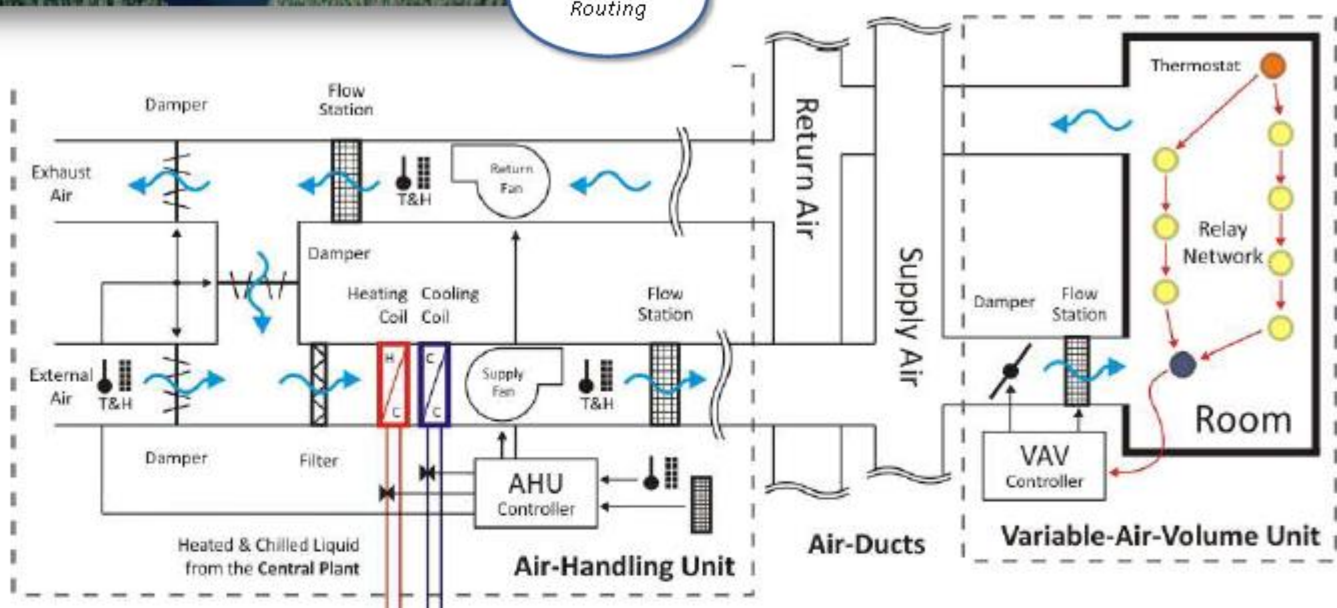
Supply Chain and Asset Management

Environmental Monitoring, Disaster Recovery and Preventive Conservation

Wireless control networks



A large scale decentralized control network



A green building

[Ramanathan Rosales-Hain 00]
[alur D'Innocenzo Johansson Pappas Weiss 10]
[Mazo Tabuada 10]
[Zhu Yuan Song Han Başar 12]

Motivation

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 55, NO. 8, AUGUST 2010 1781

Networked Control Systems With Communication Constraints: Tradeoffs Between Transmission Intervals, Delays and Performance

W. P. Maurice H. Heemels, *Member, IEEE*, Andrew R. Teel, *Fellow, IEEE*, Nathan van de Wouw, *Member, IEEE*, and Dragan Nešić, *Fellow, IEEE*

Roughly speaking, the network-induced imperfections and constraints can be categorized in five types:

(i) Quantization errors in the signals transmitted over the network due to the finite word length of the packets;
(ii) Packet dropouts caused by the unreliability of the network;
(iii) Variable sampling/transmission intervals;
(iv) Variable communication delays;
(v) Communication constraints caused by the sharing of the network by multiple nodes and the fact that only one node is allowed to transmit its packet per transmission.

Previous work

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[Jungers D'Innocenzo Di Benedetto, TAC 2015]

Today

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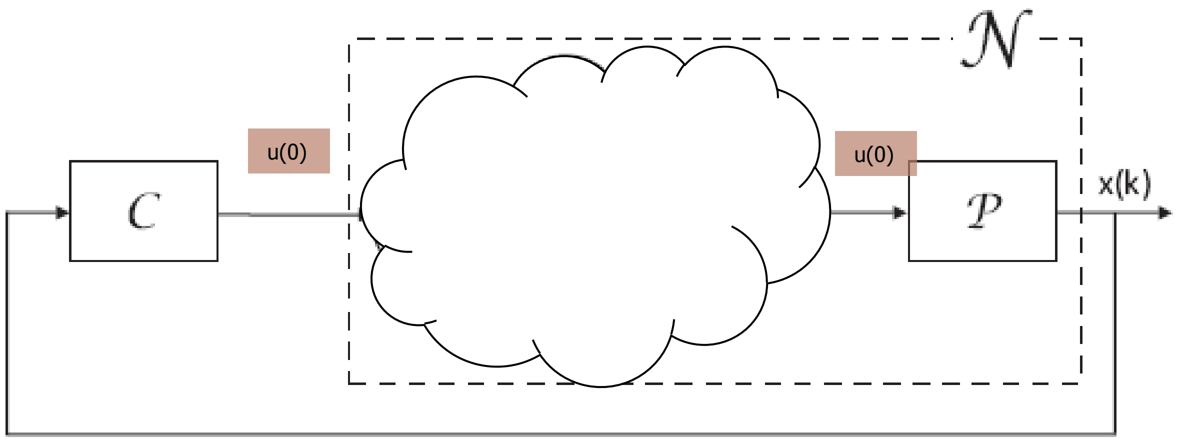
Controllability with packet dropouts

The delay is constant, but some packets are dropped

$$x(1) = Ax(0) + Bu(0)$$

$\boxed{\sigma = 1001\ldots}$

$\sigma(0) = 1$



A data loss signal determines the packet dropouts $\sigma(t) =$ 1 or 0

...this is a switching system! $$x(t+1) = \begin{cases} Ax(t) + bu(t), & \text{if } \sigma(t) = 1, \\ Ax(t), & \text{if } \sigma(t) = 0 \end{cases}$$

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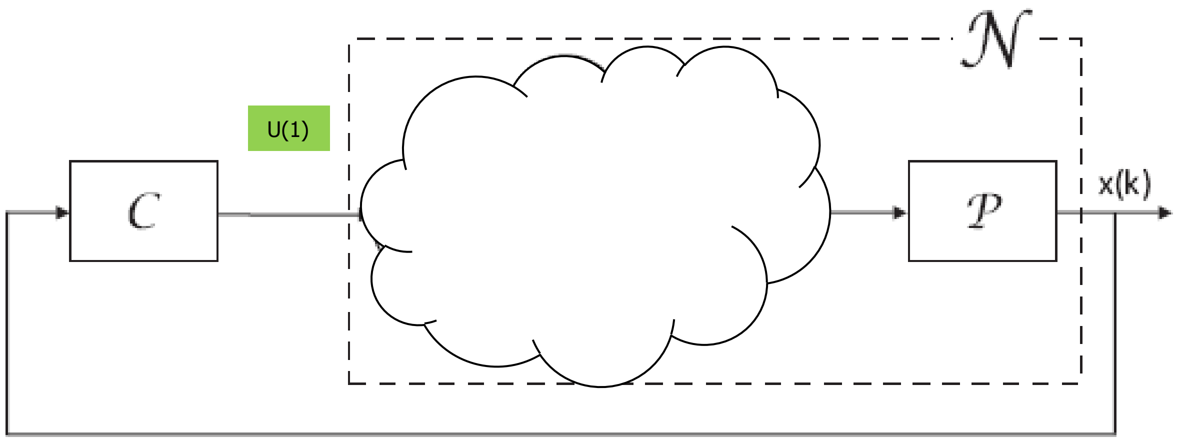

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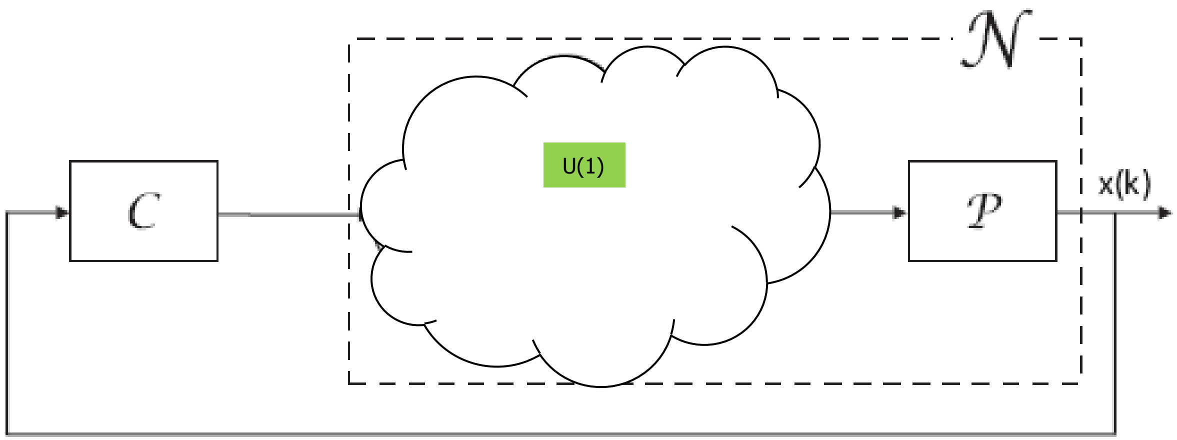
$$x(2) = A^2x(0) + ABu(0)$$



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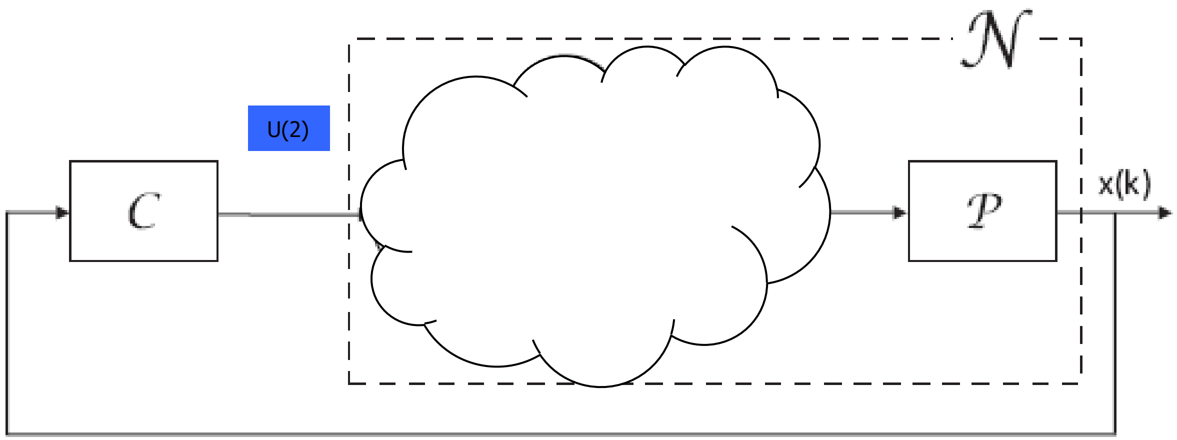

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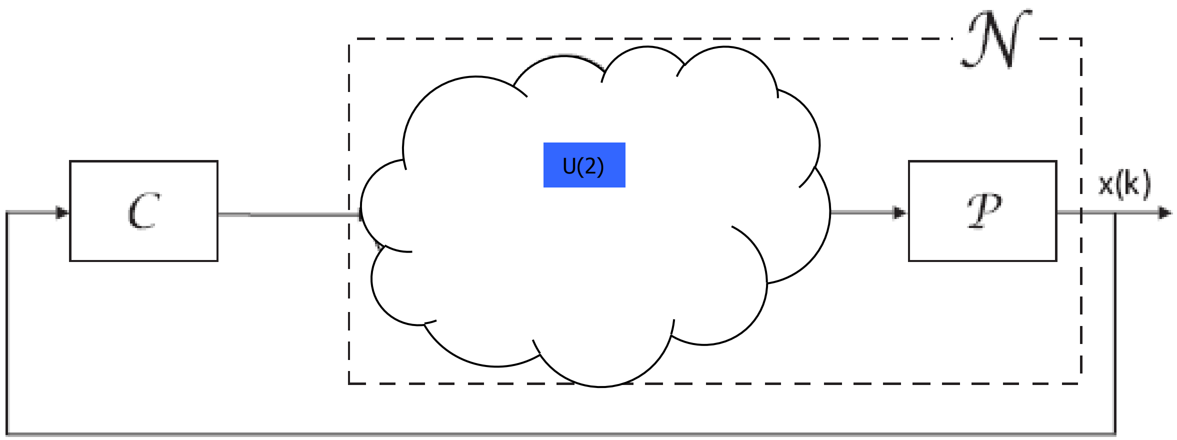
$$\boxed{\sigma = 1001\ldots}$$

$\sigma(0) = 1$
$\sigma(1) = 0$
$\sigma(2) = 0$

$$x(1) = Ax(0) + Bu(0)$$
$$x(2) = A^2x(0) + ABu(0)$$
$$x(3) = A^3x(0) + A^2Bu(0)$$



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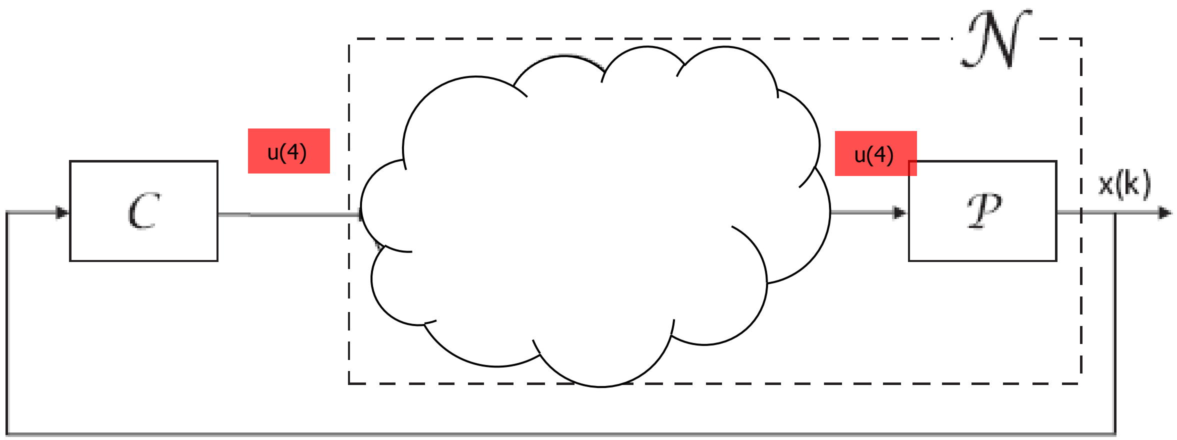
$$\boxed{\sigma = 1001\ldots}$$

$\sigma(0) = 1$
$\sigma(1) = 0$
$\sigma(2) = 0$

$$x(1) = Ax(0) + Bu(0)$$
$$x(2) = A^2x(0) + ABu(0)$$
$$x(3) = A^3x(0) + A^2Bu(0)$$
$$x(4) = A^4x(0) + A^3Bu(0) + Bu(3)$$



A data loss signal determines the packet dropouts $\sigma(t) =$ 1 or 0

…this is a switching system!

$$x(t+1) = \begin{cases} Ax(t) + bu(t), & \text{if } \sigma(t) = 1, \\ Ax(t), & \text{if } \sigma(t) = 0 \end{cases}$$

The switching signal

We are interested in the controllability of such a system

$$\boxed{\sigma = 1001\ldots}$$

$$\sigma(0) = 1 \qquad \sigma(1) = 0 \qquad \sigma(2) = 0$$

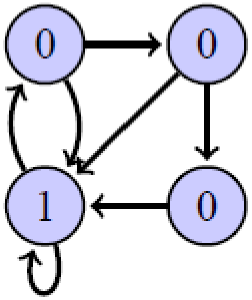
$$\begin{aligned} x(1) &= Ax(0) + Bu(0) \\ x(2) &= A^2x(0) + ABu(0) \\ x(3) &= A^3x(0) + A^2Bu(0) \\ x(4) &= A^4x(0) + A^3Bu(0) + Bu(3) \end{aligned}$$

Of course we need an assumption on the switching signal

The switching signal is constrained by an automaton

Example:

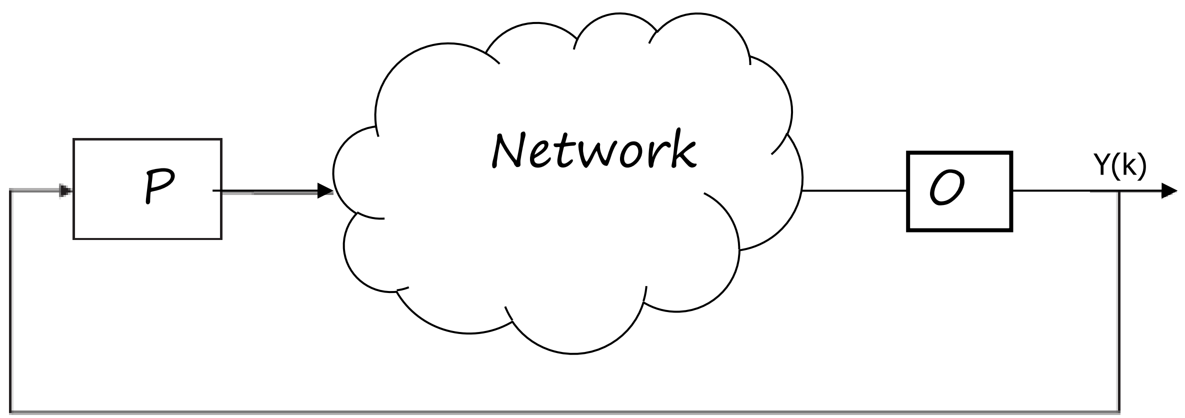
Bounded number of consecutive dropouts (here, 3)



The controllability problem: For any starting point x(0), and any target x*, does there exist, for any switching signal, a control signal u(.) and a time T such that x(T)=x* ?

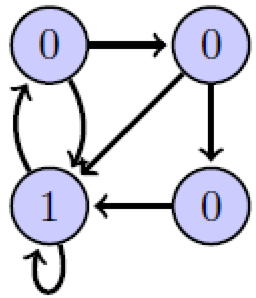
The dual observability problem

Observability under intermittent outputs is algebraically equivalent (and perhaps more meaningful)



$$x(t+1) = Ax(t),$$
$$y(t) = \sigma(t)Cx(t)$$

Controllability with Packet Dropouts



We are given a pair (A,b) and an automaton

$$\sigma(0) = 1$$
$$\sigma(1) = 0$$
$$\sigma(2) = 0$$

$$x(1) = Ax(0) + Bu(0)$$
$$x(2) = A^2x(0) + ABu(0)$$
$$x(3) = A^3x(0) + A^2Bu(0)$$
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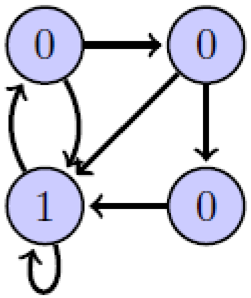
$$\boxed{\sigma = 1001\ldots}$$

The controllability problem: for any starting point x(0), and any target x*, does there exist, for any switching signal, a control signal u(.) and a time T such that x(T)=x* ?

Theorem: Deciding controllability of switching systems is undecidable in general (consequence of [Blondel Tsitsiklis, 97])

Controllability with Packet Dropouts



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$\boxed{\sigma = 1001\ldots}$

The controllability problem: for any starting point x(0), and any target x*, does there exist, for any switching signal, a control signal u(.) and a time T such that x(T)=x* ?

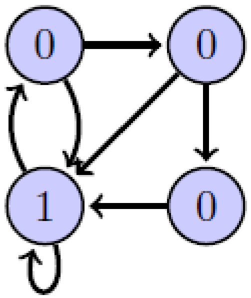
Baabali & Egerstedt's framework (2005)

$X(t+1)=Ax + B_i\, u(t)$ → Here, the switching is on the input matrix B_i

Theorem [Baabali Egerstedt 2005]: There exists some l such that : If for all l<L, the pairs (A^l,B_i) are controllable, then the system is controllable

- Only a sufficient condition
- The set of pairs to check can be huge (more than exponential)

Controllability with Packet Dropouts



We are given a pair (A,b) and an automaton

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$$x(1) = Ax(0) + Bu(0)$$
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$$x(4) = A^4x(0) + A^3Bu(0) + Bu(3)$$

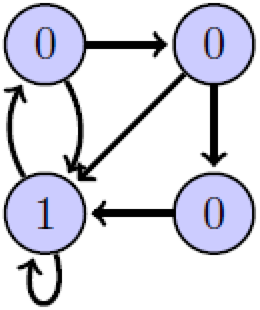
$$\boxed{\sigma = 1001\ldots}$$

The controllability problem: for any starting point x(0), and any target x*, does there exist, for any switching signal, a control signal u(.) and a time T such that x(T)=x* ?

Proposition: The system is controllable iff the generalized controllability matrix

$$\boxed{C_\sigma(t) = [A^{(t-1)}b\sigma(0)|A^{(t-2)}b\sigma(1)|\ldots|Ab\sigma(t-2)|b\sigma(t-1)]}$$

is bound to become full rank at some time t



Our algorithm

Thus, we have a purely algebraic problem: is it possible to find a path in the automaton such that the controllability matrix is never full rank?

$$C_\sigma(t) = [A^{(t-1)}b\sigma(0) | A^{(t-2)}b\sigma(1) | \ldots | Ab\sigma(t-2) | b\sigma(t-1)]$$

➔ Theorem: Given a matrix A and two vectors b,c, the set of paths such that

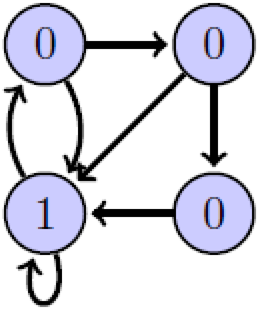
$$C_\sigma(t)$$

is never full rank is either empty, or contains a cycle in the automaton.

From this, we obtain an algorithm to decide controllability:

Semi-algorithm 1: For every cycle of the automaton, check if it leads to an infinite uncontrollable signal

Semi-algorithm 2: For every finite path, check whether it leads to a controllable signal (i.e. a full rank controllability matrix).



Proof of our theorem

Theorem ([Skolem 34]): Given a matrix A and two vectors b,c, the set of values n such that

$$c^\top A^n b = 0$$

is eventually periodic.

We managed to rewrite our controllability conditions in terms of a linear iteration

→ Theorem: Given a matrix A and two vectors b,c, the set of paths such that

$$C_\sigma(t)$$

is never full rank is either empty, or contains a cycle in the automaton.

Now, how to optimally chose the control signal, if one does not know the switching signal in advance?

Outline

- Joint spectral characteristics
- Path-complete methods for switching systems stability
- Applications:
  - Trackability
  - WCNs and packet dropouts
  - Switching delays
- Conclusion and perspectives

Previous work

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 55, NO. 8, AUGUST 2010 1781

Networked Control Systems With Communication Constraints: Tradeoffs Between Transmission Intervals, Delays and Performance

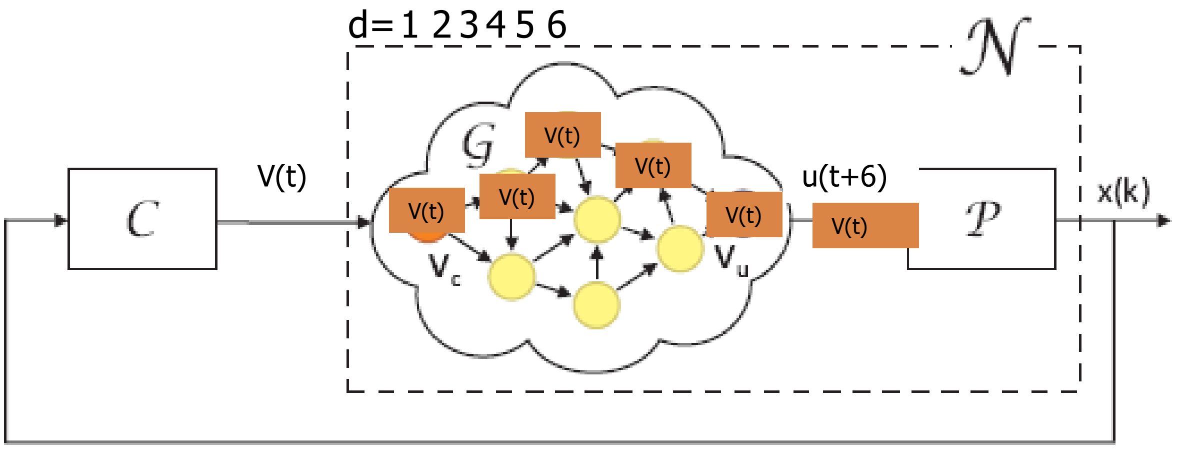
W. P. Maurice H. Heemels, *Member, IEEE*, Andrew R. Teel, *Fellow, IEEE*, Nathan van de Wouw, *Member, IEEE*, and Dragan Nešić, *Fellow, IEEE*

Roughly speaking, the network-induced imperfections and constraints can be categorized in five types:

(i) Quantization errors in the signals transmitted over the network due to the finite word length of the packets;
(ii) Packet dropouts caused by the unreliability of the network;
(iii) Variable sampling/transmission intervals;
(iv) Variable communication delays;
(v) Communication constraints caused by the sharing of the network by multiple nodes and the fact that only one node is allowed to transmit its packet per transmission.

[Jungers D'Innocenzo Di Benedetto, TAC 2015]

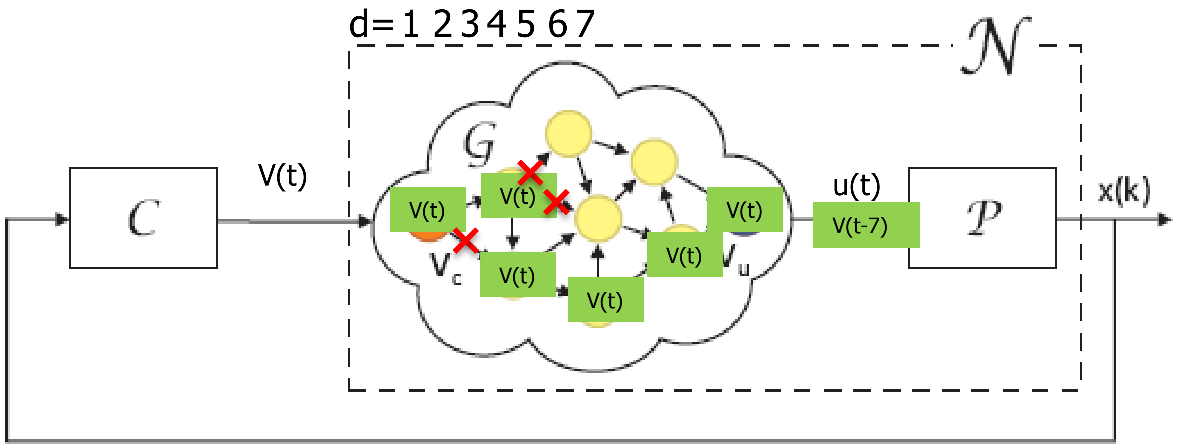
Previous work



WCNs are delay systems:

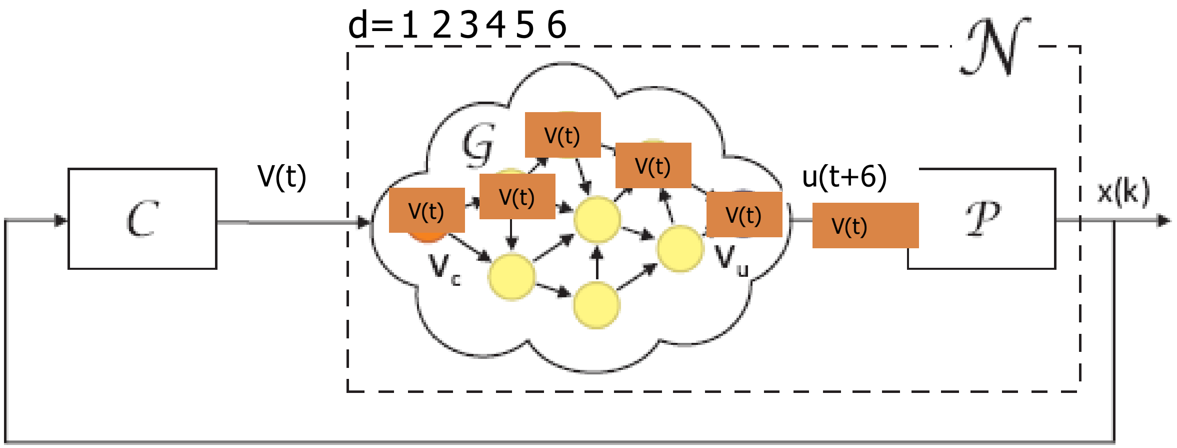
$$x(t+1) = Ax + B\,\mathrm{v}(t-d)$$

Previous work
LTIs with switched delays



$$x(t+1) = Ax + B\,\mathrm{v}(t - d_2)$$

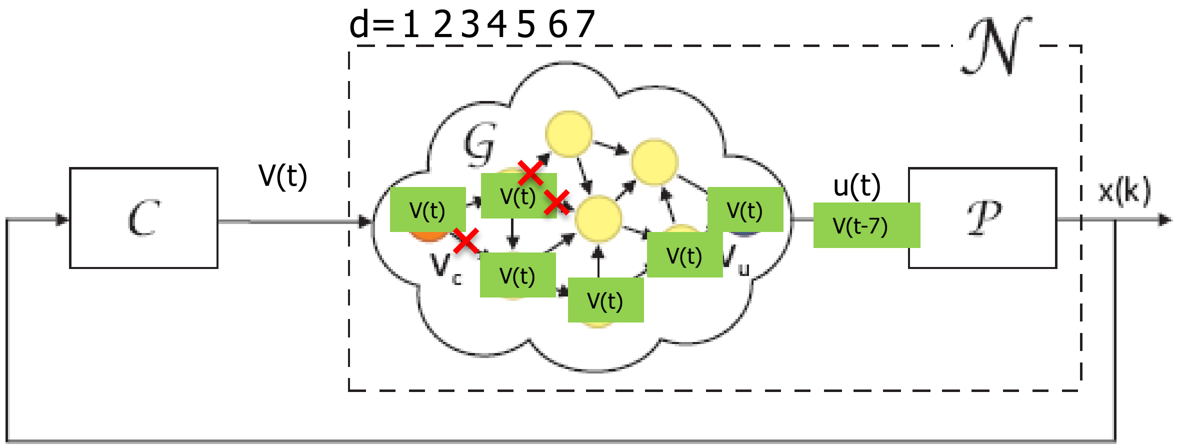
How to model failures?



WCNs are delay systems:

$$x(t+1) = Ax + B\,\mathrm{v}(t-d)$$

How to model failures?



WCNs are time-varying delay systems:

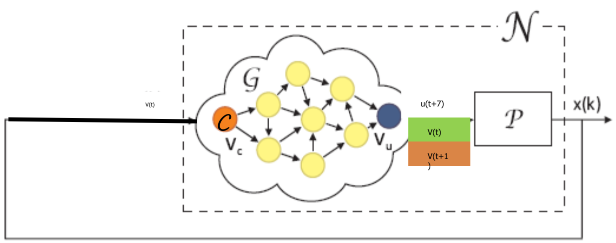
$$x(t+1) = Ax + B\,\mathrm{v}(t - d_2)$$

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$$

$$D = \{0, 1\}$$

LTIs with switched delays stability analysis

Delay dependent controller



$$v(t) = K(d)\tilde{u}(t)$$

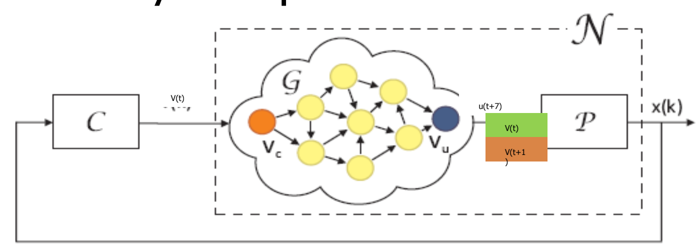
$$\tilde{u}(t) =$$

$$(x(t), u_1(t), u_2(t), \dots, u_{d_{max}}(t))$$

$$\Sigma = \left\{ \begin{pmatrix} A & B & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & I \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ K(d) \\ \vdots \\ 0 \end{pmatrix} \right\}$$

[Hetel Daafouz Iung 07]
[Weiss et al. 09]

Delay independent controller



$$v(t) = K\tilde{v}(t),$$

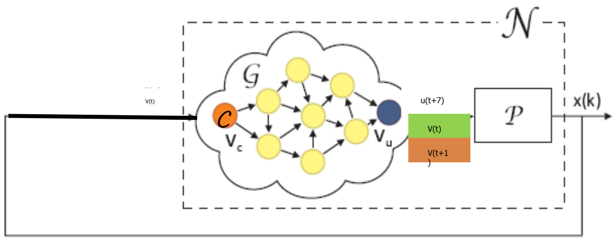
$$\tilde{v}(t) =$$

$$(x(t), v(t - d_{max}), \dots, v(t-1)))$$

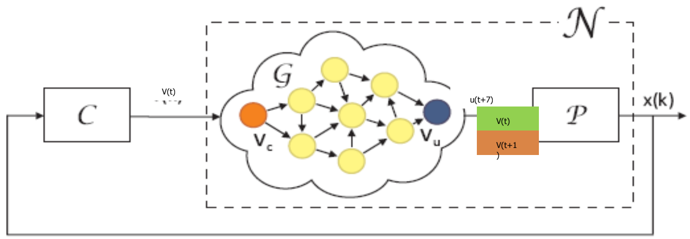
$$\Sigma = \left\{ \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ K_0 & K_1 & K_2 & \dots & K_{d_{max}} \end{pmatrix} \right\}$$

LTIs with switched delays stability analysis

Delay dependent controller



Delay independent controller



- Corollary

  For both models there is a PTAS for the stability question:

  for any required accuracy, there is a polynomial-time algorithm for checking stability up to this accuracy

Previous sufficient conditions for stability in [Hetel Daafouz Iung 07, Zhang Shi Basin 08]

- However:

  Theorem the very stability problem is NP-hard

  Theorem the boundedness problem is even Turing-undecidable!

[J. D'Innocenzo Di Benedetto 12]

Design of LTIs with switched delays

The infinite look-ahead case

- Theorem for n=m=1, there is an explicit formula for a linear controller that achieves deadbeat stabilization, even if N=1

  (based on a generalization of the Ackermann formula for delayed LTI)

$$K^*(d) = (-a^{d+1}/b, -a^d, -a^{d-1}, \dots, -a)$$

- So, does a controllable system always remain controllable with delays?

- No! when n>1, nastier things can happen…

  Example:

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$$

$$x_1 = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad D = \{0,1\}, \quad \sigma(t) = t \bmod 2$$

$$x_2 = A^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Bv(1) + Bv(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v(1) + v(2) \end{pmatrix}$$

➔ The system is not stabilizable, even with infinite lookahead

Design of LTIs with switched delays
The infinite look-ahead case

- A sufficient condition for uncontrollability (informal): if A,B can be put in the following form (under similarity transformation):

$$A = \left(\begin{array}{c|c|c} 0 & X & 0 \\ \hline 0 & 0 & X \\ \hline X & 0 & 0 \end{array}\right) \qquad B = \left(\begin{array}{c} X \\ \hline 0 \\ \hline 0 \end{array}\right)$$

An adversary strategy can make this system uncontrollable:

$$\forall t, t + d(t) \neq 1 \mod 3$$

Is it also necessary?

Would be nice, because we can prove …

- Theorem There is a polynomial time algorithm that decides whether such an adversary strategy is possible

Design of LTIs with switched delays

The infinite look-ahead case

- Answer: No! There are more intricate examples

$$A = \begin{pmatrix} \sin\theta_1 & -\cos\theta_1 & 0 & 0 \\ \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ 0 & 0 & \sin\theta_2 & -\cos\theta_2 \\ 0 & 0 & \cos\theta_2 & \sin\theta_2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$D = \{0, 1, \ldots, 121\} \qquad \theta_1 = \tfrac{\pi}{120} \qquad \theta_2 = \tfrac{\pi}{60}$$

$$\sigma(t) = \begin{cases} 0 & if \;\; 0 \leq t \leq 2 \\ 121 - t \; mod(121) & if \;\; t \geq 3 \end{cases}$$

Design of LTIs with switched delays

The infinite look-ahead case

- **Theorem:** Controllability is decidable (in exponential time)

Proof Split the problem into a nilpotent matrix and a regular matrix

$$TAT^{-1} = \begin{pmatrix} J_{0,k} & 0 \\ 0 & A' \end{pmatrix}, \quad Tb = \begin{pmatrix} b_0 \\ b' \end{pmatrix}$$

- Lemma: The nilpotent case is completely combinatorial
- Lemma: The regular case can be decided thanks to a finite dimension argument

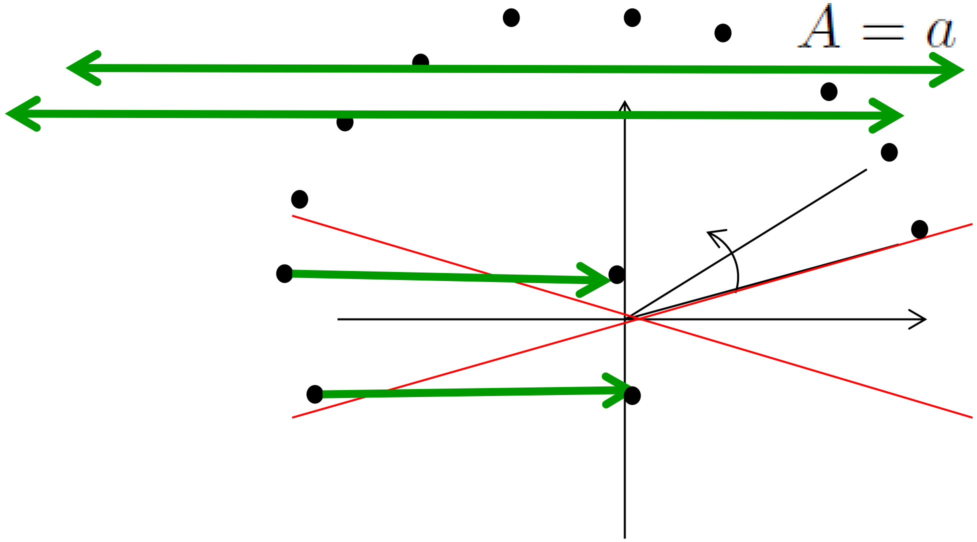
Algo: try every delay sequence of length smaller than some bound L and look for a 'loop'

L= $\binom{n+2|D|}{2|D|}$

- **Corollary:** controllability with infinite look-ahead = controllability with arbitrarily large but finite look-ahead = stabilizability!

LTIs with switched delays
Example

The controller design problem: a 2D system with two possible delays



$$A = a\begin{pmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(alpha) \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$D = \{0, 1\}$$

- **Theorem:** For the above system, there exist values of the parameters such that no linear controller can stabilize the system, but a nonlinear bang-bang controller does the job. [J. D'Innocenzo Di Benedetto 2014]

In the delay independent case, a linear controller is not always sufficient

Outline

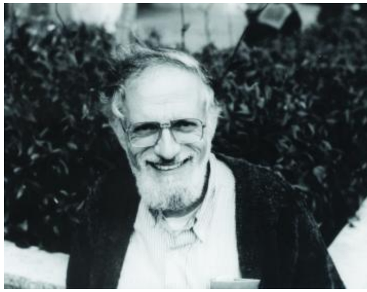
- Joint spectral characteristics
- Path-complete methods for switching systems stability
- Applications:
  - WCNs and packet dropouts
  - Switching delays
- Conclusion and perspectives

A few words about…

- Bisimulation
  - Link with Oliver's talk
  - Could Koopman eigenfunctions help to mesh?
  - Could path-complete methods help?

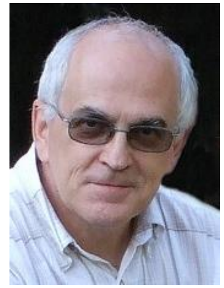
- Continuous time switching systems

Conclusion: a perspective on switching systems




[Furstenberg Kesten, 1960]

[Gurvits, 1995]



[Kozyakin, 1990]




[Rota, Strang, 1960]


[Blondel Tsitsiklis, 98+]


(sensor) networks

Wireless control

Bisimulation design

consensus problems

Social/big data control

…

60s 70s

Mathematical properties

90s

TCS inspired
Negative
Complexity results

2000s

Lyapunov/LMI
Techniques
(S-procedure)

now

CPS applic.
Ad hoc
techniques

Thanks!

Questions?

Ads

The JSR Toolbox:
http://www.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox
[Van Keerberghen, Hendrickx, J. HSCC 2014]
The CSS toolbox, 2015

Several open positions:
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References:
http://perso.uclouvain.be/raphael.jungers/

EECI Course,
L'Aquila, April 4-8

Joint work with

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Thanks! Questions?

References:

- R. M. Jungers, A. D'Innocenzo and M. D. Di Benedetto. **Modeling, analysis, and design of linear systems with switching delays.** IEEE TAC, 2015.
- R. M. Jungers, A. D'Innocenzo and M. D. Di Benedetto. **Further results on controllability of linear systems with switching delays.** *Proc. of IFAC WC 2014.*
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- R. M. Jungers, A. D'Innocenzo and M. D. Di Benedetto. **Feedback stabilization of dynamical systems with switched delays.** *Proceedings of the IEEE Conference on Decision and Control 2012, Hawai, 2012.*
- R. M. Jungers, M. Heemels. **Controllability of linear systems suject to packet losses**. *Proc. Of ADHS, Atlanta, 2015*.
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… these and more on
http://perso.uclouvain.be/raphael.jungers/