The existence of a bounded trajectory is decidable

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In this section we show that the following problem is decidable:

The `bounded trajectory problem`**:**
INSTANCE

- A finite set of nonnegative integer matrices $\mathcal{M} = \{A_1, \dots, A_m\} \subset \mathbb{N}^{n \times n}$.
- A finite set of nonnegative integer vectors $V = \{u_1, \dots, u_p\} \subset \mathbb{N}^n$.

PROBLEM
Determine whether there exists a sequence $(i_t)_{t=1}^{\infty}$, $i_t \in \{1, \dots, m\}$ and an *initial vector* $v_0 \in V$ such that the sequence of vectors determined by the recurrence

$$v_t = A_t v_{t-1},\ t = 1, 2, \dots \tag{1}$$

is bounded.

In the following, $\mathcal{M}^*, \mathcal{M}^t$ denote respectively the set of all products of matrices in $\mathcal{M}$, and the set of all products of length t of matrices in $\mathcal{M}$.

This problem is closely related to the so called *joint spectral subradius* of a set of matrices, which is the smallest asymptotic rate of growth of any long product of matrices in the set, when the length of the product increases. For a survey on the joint spectral subradius and similar quantities, see [2]. While the joint spectral subradius is notoriously Turing-uncomputable in general, we will see that in our precise situation, we are able to provide an algorithmic solution to the problem. We first show that one should not expect a polynomial time algorithm for the problem.

Proposition 1. *Unless $P = NP$, there is no polynomial time algorithm for solving the* `bounded trajectory problem`.

Proof. Our proof is by reduction from the `mortality problem` which is known to be NP-hard, even for nonnegative integer matrices [1, p. 286]. In this problem, one is given a set of matrices $\mathcal{M}$, and it is asked whether there exists a product of matrices in $\mathcal{M}^*$ which is equal to the zero matrix. We now construct an instance $\mathcal{M}', V$ of the `bounded trajectory problem` such that there is a bounded trajectory for this instance if and only if the set $\mathcal{M}$ is mortal: take $\mathcal{M}' = \{A' = 2A : A \in \mathcal{M}\}$ and $v_0 = e$ (the "all ones vector") as the unique vector in V.
Now, it is straightforward that there exists a sequence $(i_t)_{t=1}^{\infty} : i_t \in \{1, \ldots, m\}$ such that the sequence of vectors

$$A'_t \ldots A'_1 e = 2^t A_t \ldots A_1 e$$

is bounded if and only if the set $\mathcal{M}$ is mortal. Indeed, the matrices in $\mathcal{M}$ being nonnegative integer valued, if the vector $A_t \ldots A_1 e$ is different from zero, then its (say, Euclidean) norm is greater or equal to one. □

Also, if one relaxes the requirement that the matrices and the vectors are nonnegative, then the problem becomes undecidable, as shown in the next proposition.

Proposition 2. *The* `bounded trajectory problem` *is undecidable if the matrices and vectors in the instance can have negative entries.*

Proof. (sketch) It is known that the `mortality problem` with entries in $\mathbb{Z}$ is undecidable [2, Corollary 2.1]. We reduce this problem to the `bounded trajectory problem` in a way similar as in Proposition 1, except that we build much larger matrices: we make 2^n copies of each matrix in $\mathcal{M}$ and place them in a large block-diagonal matrix. That is, our matrices in $\mathcal{M}'$ are of the shape

$$\{\mathrm{diag}(2A, 2A, \ldots, 2A) : A \in \mathcal{M}\}.$$

Now we take $V = v_0$, where $v_0 \in \{-1, 1\}^{2^n n}$ is the concatenation of all the different n-dimensional $\{-1, 1\}$-vectors. This vector has a bounded trajectory if and only if there exists a zero product in $\mathcal{M}^*$. □

However, we show in the remainder of this section that the problem is decidable when restricted to nonnegative instances. The next lemma states that if there is a bounded trajectory, then it can be obtained with a periodic sequence of matrices.

Lemma 1. *Let $\mathcal{M}, V$ be an instance of the* `bounded trajectory problem`*. There exists a sequence (v_t) as given by Equation (1) which is bounded iff there exist matrices $A, B \in \mathcal{M}^*$, and a vector $v_0 \in V$ such that the sequence $(c_t) = A^t B v_0$ is bounded.*

Proof. The if-part is obvious.
In the other direction, if the set $\{v_t = A_t \dots A_1 v_0\}$ is bounded it must be finite. Thus, there actually exist A, B, v such that $Av = v, v = Bv_0$. $\square$

It appears that it is possible to check in polynomial time, given a matrix A and a vector v, whether the sequence $(c_t) = A^t v$ is bounded. In fact, as we show below, this does not really depend on the actual value of the entries of A and v, but only for each entry of A whether it is equal to zero, one, or larger than one, and for each entry of v whether it is equal to zero or larger than zero. For this reason we introduce two operators that get rid of the inessential information:

Definition 1. *Given any matrix (or vector) $M \in \mathbb{N}^{n_1 \times n_2}$, we denote by $\sigma(M)$ the matrix in $\{0,1,2\}^{n_1 \times n_2}$ in which all entries larger than two are set to two, while the other entries are equal to the corresponding ones in M.*
Similarly, we denote by $\tau(M)$ the matrix in $\{0,1\}^{n_1 \times n_2}$ in which all entries larger than one are set to one, while the other entries are equal to zero.

Theorem 1. *Given a matrix $A \in \mathbb{N}^{n \times n}$, and two indices $1 \leq i, j \leq n$, it is possible to check in polynomial time whether the sequence*

$$(A^t)_{i,j}$$

remains bounded when t grows.

Proof. Our algorithm adopts a different approach in the cases $i = j$ and $i \neq j$.

- **For the diagonal elements,** it is known (see [3, corollary 2] and the remarks before) that for any matrix $A \in \mathbb{N}^{n \times n}$, and any index i, its diagonal entry $(A^t)_{i,i}$ remains bounded if and only if $(A^t)_{i,i} \leq 1$ for $t = 1, \dots, n^2$.

- **Non-diagonal elements** $(1 \leq i, j \leq n, i \neq j)$. First, it is obvious that the condition

$$\exists t : 1 \leq t \leq n-1 : \quad A^t_{i,j} \geq 1 \tag{2}$$

  is necessary for having the (i,j)-entry unbounded.
  Indeed, for any power A^t such that $(A^t)_{i,j} > 0$, if $t \geq n$ one can find a $t' < t$ such that $(A^{t'})_{i,j} > 0$. To see this, simply remark that $(A^t)_{i,j} > 0$ implies the existence of a sequence

$$(i, i_1), (i_1, i_2), \dots, (i_{t-1}, j)$$

of nonzero entries in A. Now, if $t \geq n$, there is a particular index that appears twice in this sequence, and one can build a shorter sequence with the same property.

Thus, one can restrict its attention to such pairs (i, j) such that there exists a $t < n$ such that $A^t_{i,j} \geq 1$. We will prove that this entry is unbounded if and only if one of the following conditions must occur (and these conditions can be checked in polynomial time):

I. The (i, i)-entry or the (j, j)-entry is unbounded.

II. There exists t such that

$$A^t_{i,i}, A^t_{i,j}, A^t_{j,j} \geq 1. \tag{3}$$

Moreover, if this condition holds, there is such a t smaller than n^3 [3, Proposition 1].

III. There exists another index j' and an integer $t \leq n - 1$ such that either

- $A^t_{j',j} \geq 1$ and the (i, j')-entry satisfies the Condition II.,
- or conversely $A^t_{i,j'} \geq 1$, and (j', j) satisfies Condition II.

It is straightforward to check that any of these three conditions (together with the necessary condition in Equation (2)) implies that the (i, j)-entry is unbounded.
We now show that if the (i, j) entry is unbounded, but yet, I. and II. fail, then, III. should hold.
We claim that I. and II. being violated implies that either $\forall t, (A^t)_{i,i} = 0$, or $\forall t, (A^t)_{j,j} = 0$. Indeed, it is not difficult to see that if there exist t_1, t_2, t_3 such that $(A^{t_1})_{i,i} \geq 1$, $(A^{t_2})_{j,j} \geq 1$, $(A^{t_3})_{i,j} \geq 1$, then condition II. holds (see [3, proof of Proposition 1] for a proof). We thus suppose without loss of generality that

$$\forall t, (A^t)_{j,j} = 0. \tag{4}$$

(If it is not the case, then the proof is symmetrically the same replacing j with i.)

Now, since

$$(A^t)_{i,j} = \sum_k A^{t-1}_{i,k} A_{k,j},$$

it comes that there is an index $k_1 \neq i, j$, such that $(A^t)_{i,k_1}$ is unbounded and $A_{k_1,j} \geq 1$. Thus, if the pair (i, k_1) satisfies Condition II. the proof is done. If not, one can remove the row and column corresponding to j in the matrix A and reiterate the proof on the pair (i, k_1) since this entry is unbounded in the powers of this submatrix too. Reiterating the proof as long as necessary, one will find an index k such that the pair (i, k) satisfies condition II.

$\square$

We are now in position to present our algorithm:
Algorithm 1 for solving the `bounded trajectory problem`.

I. Construct a new instance of the `bounded trajectory problem`:

$$\mathcal{M}' = \{\sigma(A) : A \in \mathcal{M}\}, \tag{5}$$
$$V' = \{\tau(v) : v \in V\}. \tag{6}$$

II. REPEAT

- $V' \leftarrow V' \cup \{\tau(Av) : A \in \mathcal{M}', v \in V'\}$
- $\mathcal{M}' \leftarrow \mathcal{M}' \cup \{\sigma(AB) : A \in \mathcal{M}', B \in \mathcal{M}'\}$

UNTIL no new element is added to $V', \mathcal{M}'$.

III. For all pair $(A, v) \in \mathcal{M}' \times V'$,

(a) IF the sequence $(c_t) = A^t v$ is bounded,
   - RETURN YES
   - STOP

IV. RETURN NO.

Theorem 2. *Algorithm 1 is correct and stops in finite time.*

Proof. We first show how to implement Point III. (a) in the algorithm. For any column corresponding to a nonzero entry of v, one just has to check whether all the entries of this column remain bounded in the sequence of matrices A^t. Thanks to Theorem 1, it is possible to fulfill this requirement[1].

It remains to be seen that all the tests on the entries of A in the algorithm exposed in the proof of Theorem 1 do actually only depend on $\sigma(A)$ (indeed, the tests only ask whether the entry in some power of A is zero, one, or

[1]The algorithmic complexity of this particular line in the algorithm could be slightly improved thanks to techniques developped in [2], but it would not change the overall complexity.

larger than one, but this latter condition does not change if one considers A or $\sigma(A)$). Thus, one can restrict his attention to the finite set $\{\sigma(A) : A \in \mathcal{M}^*\}$, which is the set obtained after the loop at Line 2 in the algorithm.
Also, $A^t v$ is bounded if and only if $A^t \tau(v)$ is bounded, and one can restrict his attention to the finite set $\{\tau(v) : v = Av_0, A \in \mathcal{M}^*, v_0 \in V\}$, which is the set obtained after the loop at Line II. in the algorithm. $\square$

References

[1] V. D Blondel and J. N. Tsitsiklis. When is a pair of matrices mortal? *Information Processing Letters*, 63:283–286, 1997.

[2] R. M. Jungers. The joint spectral radius, theory and applications. In *Lecture Notes in Control and Information Sciences*, volume 385. Springer-Verlag, Berlin, 2009.

[3] R. M. Jungers, V. Protasov, and V. D. Blondel. Efficient algorithms for deciding the type of growth of products of integer matrices. *Linear Algebra and its Applications*, 428(10):2296–2311, 2008.