



Efficient algorithms for deciding the type of growth of products of integer matrices

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Received 11 April 2006; accepted 1 August 2007

Available online 25 September 2007

Submitted by F. Wirth

Abstract

For a given finite set Σ of matrices with nonnegative integer entries we study the growth with t of

$$\max\{\|A_1 \cdots A_t\| : A_i \in \Sigma\}.$$

We show how to determine in polynomial time whether this growth is bounded, polynomial, or exponential, and we characterize all possible behaviors.

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Keywords: Joint spectral radius; Integer matrices; Binary matrices; Semigroup of matrices; Bounded semigroup

1. Introduction

In the last decade the joint spectral radius of sets of matrices has been the subject of intense research due to its role for studying wavelets, switching systems, approximation algorithms, curve

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¹ Their research was partially supported by Communauté française de Belgique – Actions de Recherche Concertées, by the HYCON Network of Excellence (contract number FP6-IST-511368) and by the Belgian Programme on Interuniversity Attraction Poles initiated by the Belgian Federal Science Policy Office. The scientific responsibility rests with its authors. Raphaël Jungers is a FNRS fellow (Belgian Fund for Scientific Research).

² His research is supported by the grant RFBR 05-01-00066 and by the grant 5813.2006.1 for the Leading Scientific Schools.

design, etc. [6,17,29]. The particular case of matrices with integer or binary entries (rather than arbitrary real entries) is itself interesting due to the existence of many applications where such matrices arise. For instance, the rate of growth of the binary partition function in combinatorial number theory is expressed in terms of the joint spectral radius of binary matrices, that is, matrices whose entries are zeros and ones [25,28]. Moision et al. [21–23] have shown how to compute the capacity of codes under certain constraints (caused by the noise in a channel) by using the joint spectral radius of binary matrices. Recently the joint spectral radius of binary matrices has also been used to express trackability of mobiles in a sensor network [12].

For a given finite set Σ of matrices, the *joint spectral radius* of the set Σ , denoted $\rho(\Sigma)$, is defined by the limit

$$\rho(\Sigma) = \lim_{t \rightarrow \infty} \max\{\|A_1 \dots A_t\|^{1/t} : A_i \in \Sigma\}.$$

This limit exists for all finite sets of matrices and does not depend on the chosen norm. In the sequel we will mostly use the norm given by the sum of the absolute values of all matrix entries. Of course, for nonnegative matrices this norm is simply given by the sum of all entries.

The problem of computing the joint spectral radius is known to be algorithmically undecidable in the case of arbitrary matrices [3]. There are several known approximation algorithms [6,11,24,26], but all of them have exponential complexity either in the dimension of the matrices or in the accuracy of computation. Even in the case of binary matrices, computing the joint spectral radius is not easy: that problem has been shown to be NP-hard [7].

In this paper, we focus our attention on the case of *nonnegative integer* matrices and consider questions related to the growth with t of the quantity $F_t(\Sigma) = \max\{\|A_1 \dots A_t\| : A_i \in \Sigma\}$. The following cases can possibly occur:

- (1) $\rho(\Sigma) = 0$.
- (2) $\rho(\Sigma) = 1$ and the products of matrices in Σ are bounded, that is, there is a constant K such that $\|A_1 \dots A_t\| < K$ for all $A_i \in \Sigma$.
- (3) $\rho(\Sigma) = 1$ and the products of matrices in Σ are unbounded.
- (4) $\rho(\Sigma) > 1$.

Note that the situation $0 < \rho(\Sigma) < 1$ is not possible because the norm of a nonzero integer matrix is always larger or equal to one. When $\rho(\Sigma) = 0$, $F_t(\Sigma)$ takes the value 0 for all values of t larger than some t_0 and so all products of length at least t_0 are equal to zero. When $\rho(\Sigma) = 1$ and the products of matrices are unbounded, we will show that the growth of $F_t(\Sigma)$ is polynomial. Finally, when $\rho(\Sigma) > 1$ the growth of $F_t(\Sigma)$ is exponential.

The cases (1)–(4) already occur when there is only one matrix in the set Σ . Particular examples for each of these four cases are given by the matrices:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The problem of distinguishing between the four different cases has a long history. The polynomial-time decidability of the equality $\rho(\Sigma) = 0$ is shown in [16]. As mentioned by Blondel and Canterini [4], the decidability of the boundedness of products of nonnegative integer matrices follows from results proved in the 1970s. Indeed, the finiteness of a semigroup generated by a finite set of matrices has been proved to be decidable independently by Jacob [18] and by Mandel and Simon [20]. It is clear that for integer matrices, finiteness of the semigroup is equivalent to

its boundedness, and so boundedness is decidable for integer matrices. The decision algorithms proposed in [18,20] are based on the fact that if the semigroup is finite, then every matrix in the semigroup can be expressed as a product of length at most B of the generators, and the bound B only depends on the dimension of the matrices n and on the number of generators. The proposed algorithms consist of generating all products of length less than B ; and checking whether new products are obtained by considering products of length $B + 1$. The high value of the bound B does however lead to highly nonpolynomial algorithms and is therefore not practical. A sufficient condition for the unboundedness of $F_t(\Sigma)$ was also derived recently for the case of binary matrices by Crespi et al. [12]. We show here that the condition given there is also necessary. Moreover, we provide a polynomial algorithm that checks this condition, and thus we prove that boundedness of semigroups of integer matrices is decidable in polynomial time. Crespi et al. [12] also provide a criterion to verify the inequality $\rho(\Sigma) > 1$ for binary matrices and an algorithm based on that criterion. However, their algorithm is not polynomial.³

In this paper, we present a polynomial algorithm for checking $\rho(\Sigma) > 1$ for sets of nonnegative integer matrices. Let us note that the same problem for other joint spectral quantities (such as the lower spectral radius or the Lyapunov exponent) is proved to be NP-hard even for binary matrices [31]. The polynomial solvability of $\rho(\Sigma) > 1$ is therefore somewhat surprising.

Our results have direct implications for all the problems that can be formulated in terms of the joint spectral radius of nonnegative integer matrices. In particular, it follows from our results that the trackability problem for sensor networks as formulated in [12] can be decided in polynomial time. The trackability problem is as follows: we are given a directed graph with labelled nodes. Nodes may have identical labels and we consider successions of labels produced by directed paths in the graph. The function $N(t)$ gives the largest number of paths that are compatible with some label sequence of length t . When the growth of $N(t)$ is bounded or grows polynomially, the graph is said to be trackable. It has been shown in [12] that trackability can be decided by verifying that the joint spectral radius of a set of binary matrices constructed from the graph is less or equal to one. In this paper, this last property is shown to admit a polynomial time decision algorithm. Moreover, we provide an algorithm for computing the degree of the polynomial growth for trackable graphs.

Our main results can be summarized as follows. For any finite set of nonnegative integer $n \times n$ matrices Σ there is a polynomial algorithm that decides between the four cases $\rho = 0$, $\rho = 1$ and bounded growth, $\rho = 1$ and polynomial growth, $\rho > 1$ (see Theorems 1 and 2). Moreover, if $\rho(\Sigma) = 1$, then there exist constants C_1, C_2, k , such that $C_1 t^k \leq F_t(\Sigma) \leq C_2 t^k$ for all t ; the rate of growth k is an integer such that $0 \leq k \leq n - 1$, and there is a polynomial time algorithm for computing k (see Theorem 3). This sharpens previously known results on the asymptotic of the value $F_t(\Sigma)$ for nonnegative integer matrices. We discuss this aspect in Section 6. Thus, for nonnegative integer matrices, the only case for which we cannot decide the exact value of the joint spectral radius is $\rho > 1$. However, it is most likely that the joint spectral radius cannot be polynomially approximated in this case since it was proved that its computation is NP-hard, even for binary matrices [7,31].

The paper is organized as follows. Section 2 contains some notation and auxiliary facts from graph theory. In Section 3 we establish a criterion for separating the three main cases $\rho(\Sigma) <$

³ The comments made here on the results presented in the Technical Report [12] refer to the version of August 11, 2005 of that report. In a later version, and after a scientific exchange between RJ and VB with two of the authors of the report, the authors of [12] have improved some of their results and have incorporated some suggestions made by RJ and VB, as acknowledged in the updated version of the report dated December 19, 2005.

1, $\rho(\Sigma) = 1$ and $\rho(\Sigma) > 1$. Applying this criterion we derive a polynomial algorithm that decides each of these cases. In Section 4 we present a criterion for deciding product boundedness and provide a polynomial time implementation of this criterion. In Section 5 we find the asymptotic behavior of the value $F_t(\Sigma)$ as $t \rightarrow \infty$ for the case $\rho = 1$. We prove that this value is asymptotically equivalent to t^k for a certain integer k with $0 \leq k \leq n - 1$ and show how to find the rate of growth k in polynomial time. Finally, in Section 6 we formulate several open problems on possible generalizations of those results to arbitrary matrices.

2. Auxiliary facts and notation

For a given finite set of matrices Σ we denote by Σ^t the set of all products of length t of matrices from Σ . By Σ^* we denote the union of all Σ^t over all $t \geq 0$. For two nonnegative functions $f(t), g(t)$ we use the standard notation $f(t) = O(g(t))$, which means that there is a positive constant C such that $f(t) \leq Cg(t)$ for all t . The functions f and g are said to be asymptotically equivalent, which we denote $f(t) \asymp g(t)$ if $f(t) = O(g(t))$ and $g(t) = O(f(t))$.

We shall consider each nonnegative $n \times n$ matrix as the adjacency matrix of a directed weighted graph G . This graph has n nodes enumerated from 1 to n . There is an edge from node i to node j if the (i, j) entry of the matrix is positive and the weight of this edge is then equal to the corresponding entry. This graph may have loops, i.e., edges from a node to itself, which correspond to diagonal entries. If we are given a family Σ of square nonnegative integer matrices of identical dimensions, then we have several weighted graphs on the same set of nodes $\{1, \dots, n\}$. In addition we define the graph $G(\Sigma)$ associated to our family Σ as follows: There exists an edge in $G(\Sigma)$ from node i to node j if and only if there is a matrix $A \in \Sigma$ such that $A_{i,j} > 0$. The weight of this edge is equal to $\max_{A \in \Sigma} A_{i,j}$. We shall also use the graph G^2 , whose n^2 nodes represent the ordered pairs of our initial n nodes, and whose edges are defined as follows: there is an edge from a node (i, i') to (j, j') if and only if there is a matrix $A \in \Sigma$ such that both $A_{i,j}$ and $A_{i',j'}$ are positive for the same matrix. The edges of G^2 are not weighted.

Products of matrices from Σ can be represented by *cascade graphs*. In a cascade graph, a matrix $A \in \Sigma$ is represented by a bipartite graph with a left and a right set of nodes. The sets have identical size and there is an edge between the i th left node and the j th right node if $A_{i,j} > 0$. The weight of this edge is equal to the entry $A_{i,j}$. For instance, the non-weighted bipartite graph in Fig. 1 represents the matrix

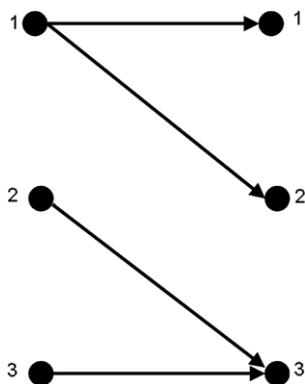


Fig. 1. A bipartite graph representing a binary matrix.

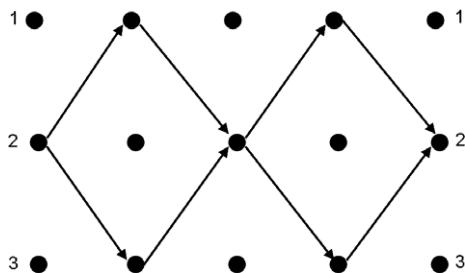


Fig. 2. A typical cascade graph.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, for a given product of matrices $A_{d_1} \cdots A_{d_t}$ we construct a cascade graph as follows: we concatenate the corresponding bipartite graphs in the order in which they appear in the product, with the right side of each bipartite graph directly connected to the left side of the following graph. For example, Fig. 2 shows a cascade graph representing the product $A_0 A_1 A_0 A_1$ of length four, with

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We say that the bipartite graph at the extreme left side begins at level $t = 0$ and the one at the extreme right side ends at the last level. We note (i, t) to refer to the node i at level t . We say that there is a path from node i to node j if one is able to construct a cascade graph with a path from some node (i, t) to some node (j, t') for some $t < t'$. A path is to be understood as a succession of edges from a level to the next level, i.e. always from left to right. One can check that the (i, j) entry of a matrix product of length t is equal to the number of directed paths from the node $(i, 0)$ to the node (j, t) in the corresponding cascade graph. We thus have a way of representing $F_t(\Sigma)$ as the maximal total number of paths from extreme left nodes to extreme right nodes in cascade graphs of length t .

Two nodes of a graph are said to be connected if they are connected by a path (not necessarily by an edge). A directed graph is strongly connected if for any pair of nodes (i, j) , i is connected to j . The following well known result states that we can partition the set of nodes of a directed graph in a unique way in strongly connected components, and that the links between those components form a tree [30].

Lemma 1. *For any directed graph G there is a partition of its nodes in nonempty disjoint sets V_1, \dots, V_l that are strongly connected and such that no two nodes belonging to different partitions are connected by directed paths in both directions. Such a maximal decomposition is unique up to renumbering. Moreover there exists a (non-necessarily unique) ordering of the subsets V_s such that any node $i \in V_k$ cannot be connected to any node $j \in V_l$, whenever $k > l$. There is an algorithm to obtain this partition in $O(n)$ operations (with n the number of nodes).*

In this lemma, we suppose by convention that a node that is not strongly connected to any other node is itself a strongly connected subset, even if it does not have a self-loop. In such a case we will say that the corresponding set is a *trivial strongly connected subset*. Consider the graph

$G(\Sigma)$ corresponding to a family of matrices Σ , as defined above. After possible renumbering, it can be assumed that the set of nodes is ordered, that is, for all nodes $i \in V_k$ and $j \in V_l$, if $k > l$ then $i > j$. In that case all the matrices of Σ have block upper-triangular form with I blocks corresponding to the sets V_1, \dots, V_I (I can be equal to one).

3. Deciding $\rho < 1$, $\rho = 1$, and $\rho > 1$

Let Σ be a finite set of nonnegative integer matrices and let $\rho = \rho(\Sigma)$ be their joint spectral radius. The goal of this section is to prove the following result.

Theorem 1. *For matrices with nonnegative integer entries there is a polynomial algorithm that decides the cases $\rho < 1$, $\rho = 1$ and $\rho > 1$.*

Proof. The proof will be split into several lemmas. The inequality $\rho < 1$ means that the maximum number of paths in a cascade graph of length t tends to zero as $t \rightarrow \infty$. Hence for sufficiently large t there are no paths of this length in the graph $G(\Sigma)$ corresponding to the whole family Σ , since this graph represents the set of all possible edges. This means that $G(\Sigma)$ has no cycles. So we get our first lemma:

Lemma 2. *For a finite set of nonnegative integer matrices Σ , we have $\rho(\Sigma) > 0$ if and only if the graph $G(\Sigma)$ has a cycle. In this case $\rho \geq 1$.*

This condition can be checked in $O(n)$ operations: one just has to find the strongly connected components of the graph $G(\Sigma)$ (a task that can be performed in $O(n)$ operations [30]); a cycle will be possible iff one of the subsets is nontrivial. The problem of deciding between $\rho = 1$ and $\rho > 1$ is more difficult. Let us start with the following lemma.

Lemma 3. *Let Σ be an arbitrary finite set of real matrices. If $\rho(\Sigma) > 1$, then there is a product $A \in \Sigma^*$, for which $A_{i,i} > 1$ for some i . If the matrices are nonnegative, then the converse is also true.*

Proof. *Sufficiency.* Since $A \in \Sigma^t$ has nonnegative elements, it follows that $\|A^k\| \geq A_{i,i}^k$, hence $\rho(A) \geq A_{i,i} > 1$. It is well-known that for all t , and for all $A \in \Sigma^t$, $\rho(\Sigma) \geq \rho(A)^{1/t}$; therefore $\rho(\Sigma) \geq [\rho(A)]^{1/t} > 1$.

Necessity. Since $\rho(\Sigma) > 1$ it follows that there is a product $B \in \Sigma^*$ such that $\rho(B) > 1$ [2]. Let λ_1 be one eigenvalue of B of largest magnitude, so $|\lambda_1| = \rho(B) > 1$ and let $\lambda_2, \dots, \lambda_n$ be the other eigenvalues. There exists a sufficiently large t such that $|\lambda_1|^t > 2n$ and $\arg(\lambda_k^t) \in (-\frac{\pi}{3}, \frac{\pi}{3})$ for all $k = 1, \dots, n$, where $\arg(z)$ is the argument of the complex number z [32]. Therefore $\operatorname{Re}(\lambda_k^t) \geq \frac{1}{2}|\lambda_k^t|$ for all k . We have $\sum_{k=1}^n (B^t)_{k,k} = \operatorname{tr} B^t = \sum_{k=1}^n \lambda_k^t = \sum_{k=1}^n \operatorname{Re} \lambda_k^t \geq \frac{1}{2}|\lambda_1^t| > n$. Since the sum of the n numbers $(B^t)_{k,k}$ exceeds n , one of them must exceed 1. \square

Corollary 1. *For any finite set of nonnegative integer matrices Σ , we have $\rho(\Sigma) > 1$ if and only if there is a product $A \in \Sigma^*$ such that $A_{i,i} \geq 2$ for some i .*

A different proof of this corollary can be found in Crespi et al. [12]. Thus, the problem is reduced to testing if there is a product $A \in \Sigma^*$ that has a diagonal element larger than or equal to 2. This is equivalent to the requirement that at least one of the following conditions is satisfied:

- (1) There is a cycle in the graph $G(\Sigma)$ containing at least one edge of weight greater or equal to 2.
- (2) There is a cycle in the graph G^2 containing at least one node (i, i) (with equal entries) and at least one node (p, q) with $p \neq q$.

Indeed, if $A_{i,i} \geq 2$ for some $A \in \Sigma^*$, then either there is a path on the graph $G(\Sigma)$ from i to i that goes through an edge of weight ≥ 2 (first condition), or there are two different paths from i to i in the cascade graph corresponding to the product A , this is equivalent to the second condition. The converse is obvious. To verify Condition 1 one needs to look over all edges of $G(\Sigma)$ of weight ≥ 2 and to check the existence of a cycle containing this edge. This requires at most $O(n^3)$ operations. To verify Condition 2 one needs to look over all $\frac{1}{2}n^2(n-1)$ triples (i, p, q) with $p > q$ and for each of them check the existence in the graph G^2 of paths from (i, i) to (p, q) and from (p, q) to (i, i) , which requires at most $O(n^2)$ operations. Thus, to test Condition 2 one needs to perform at most $O(n^5)$ operations. This completes the proof of Theorem 1. \square

Fig. 2 shows a cascade graph with the condition 2 of Corollary 1 satisfied: there are two paths from node 2 to node 2, and for every even t the number of paths is multiplied by two.

The shortest cycle in the graph G^2 with the required properties has at most n^2 edges. It therefore follows that whenever $\rho(\Sigma) > 1$, there is a product A of length less than n^2 such that $A_{i,i} \geq 2$ for some i . From this we deduce the following corollary.

Corollary 2. *Let Σ be a finite set of nonnegative integer matrices of dimension n . If $\rho(\Sigma) > 1$, then $\rho(\Sigma) \geq 2^{1/n^2}$.*

4. Deciding product boundedness

If $\rho = 1$, two different cases are possible: either the maximum norm of products of length t is bounded by a constant, or it grows with t . Deciding between these two cases is not trivial. In this section we present a simple criterion that allows us to decide whether the products are bounded. Our reasoning will be split into several lemmas. We begin with a simple but crucial observation.

Lemma 4. *Let Σ be a finite set of nonnegative integer matrices with $\rho(\Sigma) = 1$. If there is a product $A \in \Sigma^*$ that has an entry larger than 1, then the graph $G(\Sigma)$ is not strongly connected.*

Proof. Let $A_{i,j} \geq 2$, that is, counting with weights, there are two paths from i to j in the same cascade graph. If there is another cascade graph with a path from j to i , then, concatenating the two cascade graphs, we can find two different paths from i to itself, and by corollary 1 $\rho(\Sigma) > 1$, which is a contradiction. Hence $G(\Sigma)$ is not strongly connected. \square

Consider the partition of the nodes of $G(\Sigma)$ into strongly connected sets V_1, \dots, V_I (cf: Lemma 1). Applying Lemma 4 we get the following corollaries.

Corollary 3. *Let Σ be a finite set of nonnegative integer matrices. If $\rho(\Sigma) = 1$, but the products of these matrices are not uniformly bounded, then there exists a permutation matrix P such that for all matrices A in Σ , $P^T A P$ is block upper triangular with at least two blocks.*

Corollary 4. *Let Σ be a finite set of nonnegative integer matrices with joint spectral radius one. Then all products of those matrices restricted to any strongly connected set V_k are binary matrices.*

We are now able to prove the main result of this section. We first provide a result for the case of one matrix and then consider the case of several matrices.

Proposition 1. *Let A be a nonnegative integer matrix with $\rho(A) = 1$. The set $\{\|A^t\| : t \geq 1\}$ is unbounded if and only if there exists some $k \geq 1$, and a pair of indices (i, j) such that*

$$A_{i,i}^k, A_{i,j}^k, A_{j,j}^k \geq 1. \quad (1)$$

Proof. Sufficiency is easy: One can check that $(A^{kt})_{i,j} \geq t$ for any t , and hence $F_t(\Sigma)$ is unbounded. Let us prove the necessity: Consider the partition in strongly connected subsets V_1, \dots, V_I . By Corollary 3 we have $I \geq 2$.

We claim that there are two nontrivial sets V_a and V_b , $a < b$ that are connected by a path (there is a path from an element of V_a to an element of V_b). Otherwise any path in $G(\Sigma)$ intersects at most one nontrivial set, and we prove that their number must then be bounded: Let a path start from a set V_{a_1} , then go to V_{a_2} etc., until it terminates in V_{a_l} . We associate the sequence $a_1 < \dots < a_l$, $l \leq I$ to this path. As supposed, this sequence contains at most one nontrivial set, say V_{a_s} . There are at most K^l paths, counting with weights, corresponding to this sequence, where K is the largest number of edges between two given sets (still counting with weights). Indeed, each path of length $t > l$ begins with the only edge connecting V_{a_1} to V_{a_2} (since V_{a_1} is trivial), etc. until it arrives in V_{a_s} after $s - 1$ steps (for each of the previous steps we had at most K variants), and the reasoning is the same if one begins by the end of the path, while, given a starting node in V_{a_s} , and a last node in the same set, there is at most one path between these two nodes, by Corollary 4. Since there are finitely many sequences $\{a_j\}_{j=1}^l$, $l \leq I$, we see that the total number of paths of length t is bounded by a constant independent of t , which contradicts the assumption.

Hence there are two nontrivial sets V_a and V_b , $a < b$ connected by a path. Let this path go from a node $i_1 \in V_a$ to $j_1 \in V_b$ and have length l . Since both graphs V_a and V_b are strongly connected, it follows that there is a cycle $i_1 \rightarrow \dots \rightarrow i_p \rightarrow i_1$ in V_a and a path $j_1 \rightarrow \dots \rightarrow j_q \rightarrow j_1$ in V_b , $p, q \geq 1$. Take now a number $s \in \{1, \dots, p\}$ such that $l + s$ is divisible by p : $l + s = vp$, $v \in \mathbb{N}$. Take a nonnegative integer x such that $v + x$ is divisible by q : $v + x = uq$, $u \in \mathbb{N}$. Let us show that the matrix A^{upq} and the indices $i = i_{p-s+1}$, $j = j_1$ possess property 2. Indeed, a path of length upq along the first cycle, beginning at node i_{p-s+1} terminates in the same node, hence $A_{i_{p-s+1}, i_{p-s+1}}^{upq} \geq 1$. Similarly $(A^{upq})_{j_1, j_1} \geq 1$. On the other hand, the path going from $i_{p-s+1} \rightarrow \dots \rightarrow i_1$, then going x times around the first cycle from i_1 to itself, and then going from i_1 to j_1 , has a total length $s + xp + l = vp + xp = upq$, therefore $A_{i_{p-s+1}, j_1}^{upq} \geq 1$. \square

The fact that there must be two nontrivial sets connected by a path had already been proved by Mandel and Simon [20, Lemma 2.6]. We now provide a generalization of this result to the case of several matrices.

Proposition 2. *Let Σ be a finite set of integer nonnegative matrices with $\rho(\Sigma) = 1$. The set of products norms $\{\|A\| : A \in \Sigma^*\}$ is unbounded if and only if there exists a product $A \in \Sigma^*$, and indices i and j ($i \neq j$) such that*

$$A_{i,i}, A_{i,j}, A_{j,j} \geq 1. \quad (2)$$

Proof. The sufficiency is obvious by the previous proposition. Let us prove the necessity. We have a set Σ of nonnegative integer matrices, and their products in Σ^* are unbounded. Consider again the partition of the nodes in strongly connected sets V_1, \dots, V_I for Σ . Our proof proceeds

by induction on I . For $I = 1$ the products are bounded by Corollary 4, and there is nothing to prove. Let $I \geq 2$ and the theorem holds for any smaller number of sets in the partition. If the value $F_I(\Sigma, U)$ is unbounded on the set $U = \bigcup_{s=2}^I V_s$, then the theorem follows by induction. Suppose then that the products are bounded on this subset of nodes, by some constant M . Let us consider a product of t matrices, and count the paths from any leftmost node to any rightmost node. First, there are less than n^2 paths beginning in V_1 and ending in V_1 , since the corresponding adjacency matrix must have $\{0, 1\}$ entries (recall that n is the total number of nodes). Second, there are at most Mn^2 paths beginning and ending in U , since each entry is bounded by M . Let us count the paths beginning in V_1 and ending in U : Let $i_0 \rightarrow \dots \rightarrow i_t$ be one of these paths. The nodes i_0, \dots, i_{r-1} are in V_1 , the nodes i_r, \dots, i_t are in U and $i_{r-1}i_r$ is an edge connecting V_1 and U . The number r will be called a switching level. For any switching level there are at most KMn^2 different paths connecting V_1 with U , where K is the maximum number of edges jumping from V_1 to U at the same level, counting with weights. Indeed for one switching edge $i_{r-1}i_r$, the total number of paths from i_r to any node at the last level is bounded by M , and there are less than n nodes in U . By the same way of thinking, there is at most one path from each node in V_1 to i_{r-1} , and there are less than n nodes in V_1 . The number of switching levels is thus not bounded, because so would be the number of paths. To a given switching level r we associate a triple (A', A'', d) , where $A' = A_{d_1} \dots A_{d_{r-1}|V_1}$ and $A'' = A_{d_{r+1}} \dots A_{d_t|U}$ are matrices and $d = d_r$ is the index of the r th matrix. The notation $A|_{V_1}$ means the square submatrix of A corresponding to the nodes in V_1 . Since A' is a binary matrix (Corollary 4), A'' is an integer matrix with entries less than M , and d can take finitely many values, it follows that there exist finitely many, say N , different triples (A', A'', d) . Taking t large enough, it can be assumed that the number of switching levels $r \in \{2, \dots, t-1\}$ exceeds N , since for any switching level there are at most KMn^2 different paths. Thus, there are two switching levels r and $r+s$, $s \geq 1$ with the same triple. Define $d = d_r = d_{r+s}$ and

$$B = A_1 \dots A_{d_{r-1}}, \quad D = A_{d_{r+1}} \dots A_{d_{r+s-1}}, \quad E = A_{d_{r+s+1}} \dots A_{d_t} \quad (3)$$

(if $s = 1$, then D is the identity matrix). Thus, $A_{d_1} \dots A_{d_t} = B A_d D A_d E$. Since $A' = B|_{V_1} = B A_d D|_{V_1}$ it follows that $B|_{V_1} = B(A_d D)^k|_{V_1}$ for any k . Similarly $A'' = E|_U = D A_d E|_U$ implies that $E|_U = (D A_d)^k E|_U$. Therefore for any k the cascade graph corresponding to the product $B(A_d D)^k A_d E$ has at least $k+1$ paths of length $t + (k-1)s$ starting at i_0 . Those paths have switching levels $r, r+s, \dots, r+(k-1)s$ respectively. Indeed, for any $l \in \{0, \dots, k\}$ there is a path from i_0 to $i_{r-1+ls} = i_{r-1}$, because $B(A_d D)^l|_{V_1} = B|_{V_1}$; there is an edge from i_{r-1+ls} to $i_{r+ls} = i_r$, because $A_{d_{r+ls}} = A_{d_r} = A_d$; finally there is a path from $i_{r+ls} = i_r$ to $i_{t+(k-1)s} = i_t$, because $(D A_d)^{k-l} E|_U = E|_U$. Therefore, $\|B(A_d D)^k A_d E\| \geq k+1$ for any k , hence $\|B(A_d D)^k A_d E\| \rightarrow \infty$ as $k \rightarrow \infty$, and so $\|(A_d D)^k\| \rightarrow \infty$. Now we apply Proposition 1 for the matrix $A_d D$; since the powers of this matrix are unbounded it follows that some power $A = (A_d D)^k$, which is $(A_{d_r} \dots A_{d_{r+s-1}})^k$ possesses the property $A_{i,i}, A_{j,j}, A_{i,j} \geq 1$ for suitable i and j . \square

In the last proof, we find a matrix $A_d D \in \Sigma^*$ such that $\|(A_d D)^k\| \rightarrow \infty$. There is a different way to prove the existence of such a matrix that is based on the generic theorem of McNaughton and Zalcstein, that states that every torsion semigroup of matrices over a field is locally finite [9]. We have given here a self-contained proof that uses the combinatorics for nonnegative integer matrices.

A graph interpretation of the condition (2) in the proposition can be given with the example on Fig. 3. If one matrix in Σ has those three entries (and no other) equal to one, then we have

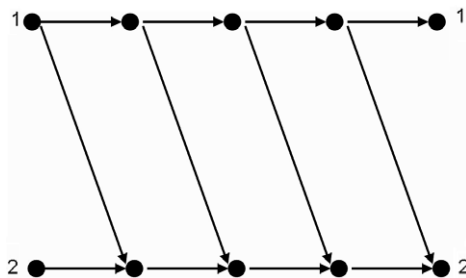


Fig. 3. A cascade graph with linear growth.

two infinite and separate paths: one is a circuit passing through the node i , the other is a circuit passing through the node j . Those cycles are linked in a unique direction, so that the first one is a source and the second one is a sink, that eventually collects all these paths.

We now prove that the criterion of Proposition 2 can be checked in polynomial time.

Theorem 2. *There is a polynomial time algorithm for verifying product boundedness of families of nonnegative integer matrices.*

Proof. Assume we are given a finite set of nonnegative integer matrices Σ . First, we decide between the cases $\rho = 0$, $\rho = 1$ and $\rho > 1$ with the algorithm provided in the previous section. In the first case $F_t(\Sigma)$ is bounded, in the latter it is not. The main problem is to check boundedness for the case $\rho = 1$. By Proposition 2 it suffices to check if there exists a product $A \in \Sigma^*$ possessing the property of Eq. (2) for some indices i, j . Consider the product graph G^3 with n^3 nodes defined as follows. The nodes of G^3 are ordered triples (i, j, k) , where $i, j, k \in \{1, \dots, n\}$. There is an edge from a vertex (i, j, k) to a vertex (i', j', k') if and only if there is a matrix $A \in \Sigma$, for which $(A)_{i,i'}, (A)_{j,j'}, (A)_{k,k'} \geq 1$. (The adjacency matrix of G^3 is obtained by taking the 3rd Kronecker power of each matrix in Σ , and by taking the maximum of these matrices componentwise.) The above condition means that there are indices $i \neq j$ such that there is a path in G^3 from the node (i, i, j) to the node (i, j, j) . The algorithm involves checking $n(n-1)$ pairs, and for each pair at most $O(n^3)$ operations to verify the existence of a path from (i, i, j) to (i, j, j) . In total one needs to perform $O(n^5)$ operations to check boundedness. \square

5. The rate of polynomial growth

We have provided in the previous section a polynomial time algorithm for checking product boundedness of sets of nonnegative integer matrices. In this section we consider sets of matrices that are not product bounded and we analyze the *rate of growth* of the value $F_t(\Sigma)$ when t grows. When the set Σ consists of only one matrix A with spectral radius equal to one, the norm of A^k increases polynomially with k and the degree of the polynomial is given by the size of the largest Jordan block of eigenvalue one. A generalization of this for several matrices is given in the following theorem.

Theorem 3. *For any finite set Σ of integer nonnegative matrices with $\rho(\Sigma) = 1$ there are positive constants C_1 and C_2 and an integer $k \geq 0$ (the rate of growth) such that*

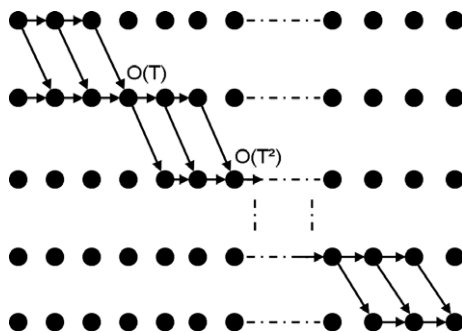


Fig. 4. A cascade graph with polynomial growth.

$$C_1 t^k \leq F_t(\Sigma) \leq C_2 t^k \quad (4)$$

for all t . The rate of growth k is the largest integer possessing the following property: there exist k different ordered pairs of indices $(i_1, j_1), \dots, (i_k, j_k)$ such that for every pair (i_s, j_s) there is a product $A \in \Sigma^*$, for which

$$A_{i_s, i_s}, A_{i_s, j_s}, A_{j_s, j_s} \geq 1, \quad (5)$$

and for each $1 \leq s \leq k-1$, there exists $B \in \Sigma^*$ such that $B_{j_s, i_{s+1}} \geq 1$.

The idea behind this theorem is the following: if we have a polynomial growth of degree k , we must have a combination of k linear growths that combine themselves successively to create a growth of degree k . This can be illustrated by the cascade graph in Fig. 4.

Before we give a proof of Theorem 3 let us observe one of its corollaries. Consider the ordered chain of maximal strongly connected subsets V_1, \dots, V_I for our set Σ . By Corollary 4 the elements i_s, j_s of each pair (i_s, j_s) belong to different sets, with, if $i_s \in V_{i_s}, j_s \in V_{j_s}, j_s > i_s$. This implies that there are fewer such couples than strongly connected subsets, and then:

Corollary 5. *The rate of growth k does not exceed $I-1$, where I is the number of strongly connected sets of the family Σ . In particular, $k \leq n-1$.*

We may now provide the proof of Theorem 3.

Proof. We shall say that a node i is $O(t^k)$ if there is a constant $C > 0$ such that $\max_{A \in \Sigma^*, 1 \leq j \leq n} A_{i,j} \leq C t^k$ for all t . Suppose that for some k we have k pairs $(i_1, j_1), \dots, (i_k, j_k)$ satisfying the assumption of the theorem. We construct a cascade graph similar to the one represented in Fig. 4: Let $A_s, s = 1, \dots, k$ and $B_s, s = 1, \dots, k-1$ be the corresponding products and m be their maximal length. Then for any s and any $p \in \mathbb{N}$ one has $(A_s^p)_{i_s, j_s} \geq p$, and therefore $(A_1^p B_1 A_2^p B_2 \cdots A_k^p)_{i_1, j_k} \geq p^k$ for any p . Denote this product by D_p and its length by l_p . Obviously $l_p \leq (pk + k - 1)m$. For an arbitrary $t > (2k-1)m$ take the largest p such that $l_p < t$. It follows that $l_p \geq t - km$, and therefore $p \geq \frac{l_p}{km} - 1 + \frac{1}{k} \geq \frac{t}{km} - 2 + \frac{1}{k}$. In order to complete the product, take for instance $A_k^{t-l_p}$. Then the product $D_p A_k^{t-l_p}$ has length t and its $(i_1 j_k)$ -entry is bigger than $p^k \geq \left(\frac{t}{km} - 2 + \frac{1}{k}\right)^k$, which is bigger than $C t^k$ for some positive constant C . This proves sufficiency.

It remains to establish the converse: if for some k there is a node that is not $O(t^{k-1})$, then there exist k required pairs of indices. We prove this by induction on the dimension n (number of nodes). For $n = 2$ and $k = 1$ it follows from Proposition 2. For $n = 2$ and $k > 1$ this is impossible, since one node (say, node 1) is an invariant by Corollary 3, then the edge $(1, 2)$ is forbidden, and there are at most $t + 2$ paths of length t (if all other edges occur at each level).

Suppose the theorem holds for all $n' \leq n - 1$. Let a node i_0 be not $O(t^{k-1})$. Assume first that there are two nodes i, j of the graph $G(\Sigma)$ that are not connected by any path. Therefore there are no paths containing these nodes. Hence one can remove one of these nodes (with all corresponding edges) so that i_0 is still not $O(t^{k-1})$. Now by induction the theorem follows. It remains to consider the case when any pair of nodes is (weakly) connected. Take the decomposition in strongly connected subsets V_1, \dots, V_l for Σ . The nodes are ordered so that all the matrices in Σ are in block upper triangular form. Let p be the smallest integer such that all nodes in $G_p = \bigcup_{s=p}^l V_s$ are $O(1)$, i.e., G_p is the biggest invariant on which the number of paths is bounded. By Corollary 4 such p does exist. On the other hand, by the assumption we have $p \geq 2$. Since the products in Σ^* restricted to the subspace corresponding to $G_{p-1} = G_p \cup V_{p-1}$ are unbounded, it follows from Proposition 2 that there is a pair $(i_k, j_k) \in G_{p-1}$ realizing equation 2. Observe that $i_k \in V_{p-1}$ and $j_k \in G_p$. Otherwise these nodes are either in V_{p-1} (hence the restriction of Σ^* to V_{p-1} is unbounded, which violates Corollary 4) or in G_p (contradicts the boundedness of Σ^* on G_p). Now consider the products restricted on the set $\bigcup_{s=1}^{p-1} V_s$. We claim that at least one node is not $O(t^{k-2})$ in this restriction: For any product in Σ^* of length t consider the corresponding cascade graph. Any path of length t starting at a node $i \in \bigcup_{s=1}^{p-1} V_s$ consists of 3 parts (some of them may be empty): a path $i \rightarrow v \in \bigcup_{s=1}^{p-1} V_s$ of some length l , an edge $v \rightarrow u \in G_p$, and a path from u inside G_p of length $t - l - 1$. Suppose that each entry in the restriction of the products to $\bigcup_{s=1}^{p-1} V_s$ is $O(t^{k-2})$, then for a given l there are at most $C l^{k-2}$ paths for the first part ($C > 0$ is a constant), for each of them the number of different edges $v \rightarrow u$ (counting with edges) is bounded by a constant K , and the number of paths from u to the end is bounded by C_0 by the assumption. Taking the sum over all l we obtain at most $\sum_{l=0}^t C K C_0 l^{k-2} = O(t^{k-1})$ paths, which contradicts our assumption.

Hence there is a node in $\bigcup_{s=1}^{p-1} V_s$ that is not $O(t^{k-2})$. Applying now the inductive assumption to this set of nodes we obtain $k - 1$ pairs (i_s, j_s) , $s = 1, \dots, k - 1$ with the required properties. Note that they are different from (i_k, j_k) , because $j_k \in G_p$. It remains to show that there is a path in $G(\Sigma)$ from j_{k-1} to i_k . Let us remember that $i_k \in V_{p-1}$. If $j_{k-1} \in V_{p-1}$ as well, then such a path exists, because V_{p-1} is strongly connected. Otherwise, if $j_{k-1} \in V_j$ for some $j < p - 1$, then there is no path from i_k to j_{k-1} , which yields that there is a path from j_{k-1} to i_k , since each pair of nodes is weakly connected. \square

Let us note that the products of maximal growth constructed in the proof of Theorem 3 are not periodic, that is, the optimal asymptotic product is not the power of one product. Indeed, we multiply the first matrix A_1 p times, and then the second one p times, etc. This leads to a family of products that are not the repetition of a period. In general, these aperiodic products can be optimal, as illustrated by the following simple example.

$$\Sigma = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Any finite product A of these matrices has spectral radius equal to one and has at most linear growth. Indeed, A has rank at most two, therefore the condition of Theorem 3 for $k = 2$ is

not satisfied for the product A . Nevertheless, the aperiodic sequence of products of the type $A_1^{t/2} A_2^{t/2}$ gives a quadratic growth in t . It is interesting to compare this phenomenon with the well-known finiteness property of linear operators [8,10,19]: for this set of matrices, the maximal behavior is a quadratic growth, which is possible only for aperiodic products. On the other hand, considering the boundedness of the products such phenomenon is impossible: by Proposition 2 if $F_t(\Sigma)$ is unbounded, this unbounded growth can always be obtained by a periodic sequence. This fact is true only for nonnegative integer matrices, since the following example gives a set of complex matrices for which the products are unbounded while all periodic products are bounded:

$$\Sigma = \left\{ \begin{pmatrix} e^{i\theta 2\pi} & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{i\theta 2\pi} & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

If $0 \leq \theta \leq 1$ is irrational, then the powers of any $A \in \Sigma^*$ are bounded, while $F_t(\Sigma)$ grows linearly in t .

Proposition 3. *The rate of growth of a set of nonnegative integer matrices can be found in polynomial time.*

Proof. For each pair (i, j) of vertices one can check in polynomial time whether there is a product A such that $A_{i,j} \geq 1$, $A_{i,i} = A_{j,j} = 1$. For each couple of those pairs $(i_1, j_1), (i_2, j_2)$, we can check in polynomial time whether there is a path from j_1 to i_2 , or from j_2 to i_1 . Finally we are left with a directed graph whose nodes are the couples (i, j) satisfying Eq. (2) and with an edge between the nodes $(i_1, j_1), (i_2, j_2)$ if there is a path from j_1 to i_2 . This graph is acyclic (because if there is also a path from j_2 to i_1 then there are two paths from i_1 to itself, and $\rho > 1$ by Lemma 3), and it is known that the problem of finding a longest path in a directed acyclic graph can be solved in linear time. \square

6. Polynomial growth for arbitrary matrices

Theorem 3 shows that for a finite family Σ of nonnegative integer matrices with joint spectral radius equal to one the value $F_t(\Sigma)$ is asymptotically equivalent to t^k , where k is an integer. Moreover, we have shown that the exponent k can be computed in polynomial time. A natural question arises: do these properties hold for sets of matrices with arbitrary real entries (without the constraint of having nonnegative integer entries)?

Problem 1. Is this true that for any family of matrices Σ (real or complex) with $\rho(\Sigma) = 1$ one has $F_t(\Sigma) \asymp t^k$ for some integer k ?

In other words, is the asymptotic behavior of the value $F_t(\Sigma)$ really polynomial with an integer rate of growth? This property can obviously be reformulated without the restriction $\rho(\Sigma) = 1$ as follows: is it true that for any family of matrices Σ we have

$$F_t(\Sigma) \asymp \rho^t t^k, \tag{6}$$

where $\rho = \rho(\Sigma)$ and k is an integer? A more general problem arises if we remove the strict requirements of asymptotic equivalence up to a positive constant:

Problem 2. Is this true that for any family of matrices Σ the following limit

$$\lim_{t \rightarrow \infty} \frac{\ln \rho^{-t} F_t(\Sigma)}{\ln t} \quad (7)$$

exists and is always an integer?

In particular, does property (6) or, more generally, property (7) hold for nonnegative integer matrices? If the answer is positive, can the rate of growth be computed? We have solved these problems only for the case $\rho = 1$. Thus, is it possible to obtain a sharper information on the asymptotic behavior of the value $F_t(\Sigma)$ as $t \rightarrow \infty$ than the well-known relation $\lim_{t \rightarrow \infty} \ln F_t(\Sigma)/t = \ln \rho(\Sigma)$? The question is reduced to the study of the value $r(t) = \rho^{-t} F_t(\Sigma)$. For some special families of matrices this question has appeared in the literature many times. Dubuc in 1986 studied it for a special pair of 2×2 matrices in connection with the rate of convergence of some approximation algorithm [15]. In 1991 Daubechies and Lagarias [13] estimated the value $r(t)$ for special pairs of $n \times n$ matrices to get a sharp information on the continuity of wavelets and refinable functions, and their technique was developed in many later works (see [17] for references). In 1990 Reznik [28] formulated several open problems on the asymptotic of binary partition functions (combinatorial number theory) that were actually reduced to computing the value $r(t)$ for special binary matrices [25]. This value also appeared in other works, in the study of various problems [11,14,29]. For general families of matrices very little is known about the asymptotic behavior of $r(t)$, although some estimations are available. First, if the matrices from Σ do not have a nontrivial common invariant subspace, then $r(t) \asymp 1$, i.e., $C_1 \leq \rho^{-t} F_t(\Sigma) \leq C_2$ for some positive constants C_1, C_2 [2,24,33]. So, in this case the answer to problem 1 is positive with $k = 0$. This assumption was relaxed for nonnegative matrices in [25]. It was shown that if a family of nonnegative matrices is irreducible (has no common invariant subspaces among the coordinate planes), then we still have $r(t) \asymp 1$. For all other cases, if the matrices are arbitrary and may have common invariant subspaces, we have only rough estimations. For the lower bound we always have $r(t) \geq C$ [24]. For the upper bound, as it was independently shown in [13] and [1], we have $r(t) \leq Ct^{n-1}$. This upper bound was sharpened in the following way [11]. Let l be the maximal integer such that there is a basis in \mathbb{R}^n , in which all the matrices from Σ get a block upper-triangular form with l blocks. Then $r(t) \leq Ct^{l-1}$. The next improvement was obtained in [27]. Let $\Sigma = \{A_1, \dots, A_N\}$ and each matrix $A_d \in \Sigma$ are in upper triangular form, with diagonal blocks B_d^1, \dots, B_d^l . Let s be the total number of indices $j \in \{1, \dots, l\}$ such that $\rho(B_1^j, \dots, B_N^j) = \rho(\Sigma)$. Then $r(t) \leq Ct^{s-1}$. Thus, for an arbitrary family of matrices we have $C_1 \leq \rho^{-t} F_t(\Sigma) \leq C_2 t^{s-1}$. To the best of our knowledge this is the sharpest information about the asymptotic behavior of $r(t)$ available thus far.

7. Conclusion and remarks

The results of this paper completely characterize finite sets of nonnegative integer matrices with bounded products and with polynomially growing products. Without any changes the results can be applied to general sets of nonnegative matrices that have no entries between zero and one. Unlike the proofs, which are quite technical, the results are easily implementable in algorithms. One question we are not addressing in this paper is that of the exact computation of the joint spectral radius when $\rho > 1$; but this problem is known to be NP-hard even for binary matrices. We also provide an example of two matrices whose joint spectral radius is one but for which the optimal asymptotic behavior is not periodic. This example may possibly help for the analysis

of the finiteness property that was conjectured in [5] to hold for binary matrices. Finally, in the last section we leave several open problems on possible generalizations of these results for more general sets of matrices.

Acknowledgments

This work was carried out while the second author was visiting the Department of Mathematical Engineering of the Université catholique de Louvain (Belgium). That author wishes to express his thanks to the University for its hospitality.

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