Geometric structures generation for switching systems stability

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CIMPA 2012, Dakar

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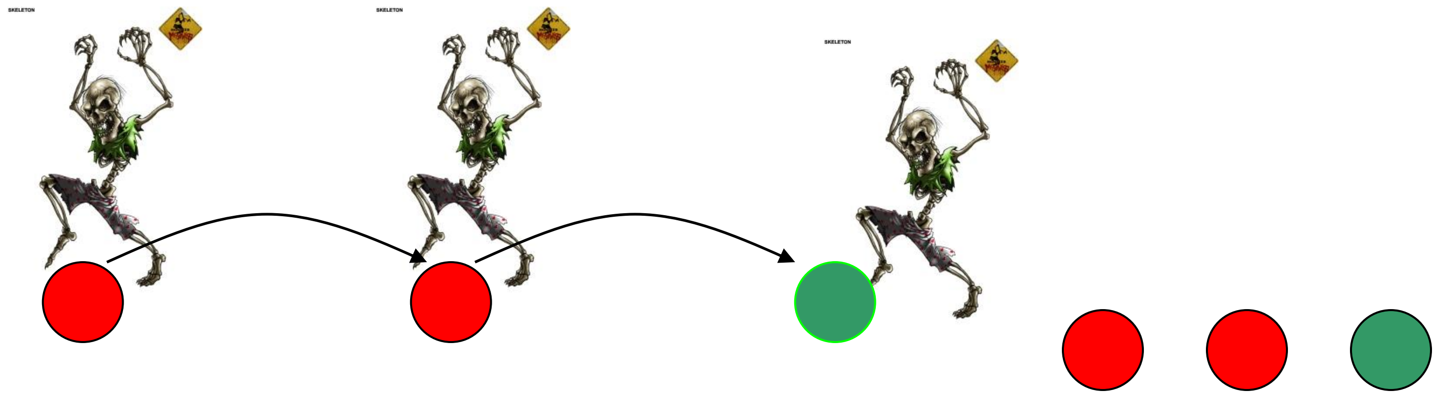
Trackability : example (1)

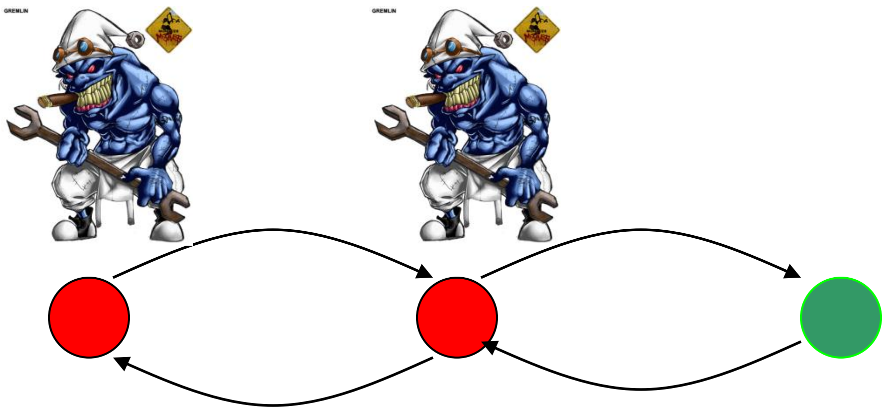


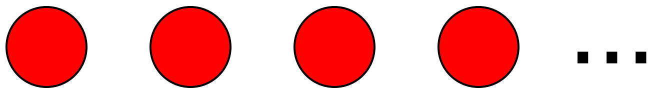
Worst case : RRG ➔

Number of possibilities asymptotically zero

$$N(t) \approx 0$$

Trackability : example (2)



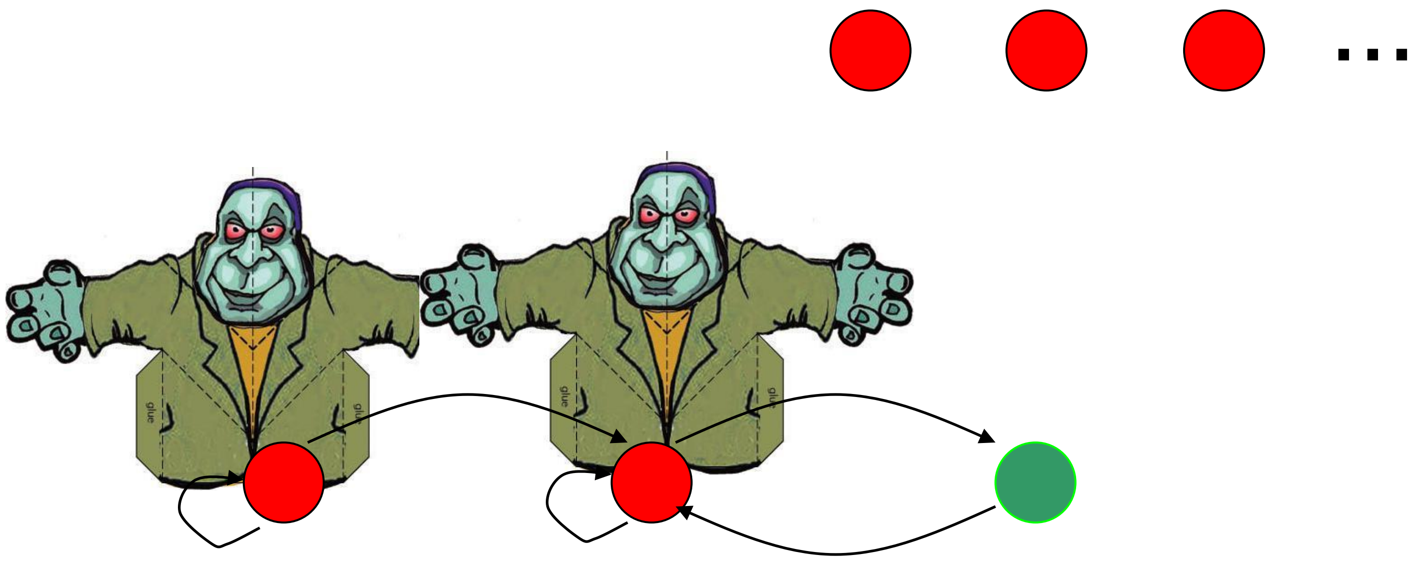


Worst case : RRRRR… ➔

Bounded number of possibilities

$$N(t) \approx 2$$

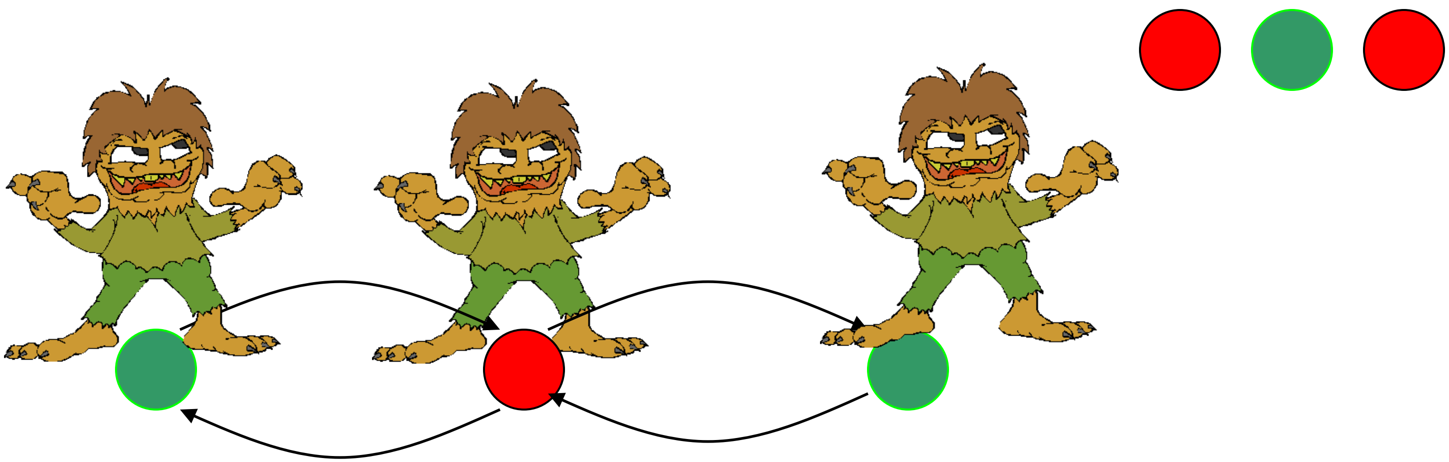
Trackability : example (3)



Worst case : RRRRRR… ➔ $N(t) \approx t$

Polynomial number of possibilities

Trackability : example (4)



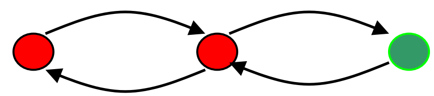
Worst case : RGRGRG…➔ $N(t) \approx 2^{t/2}$

Exponential number of possibilities

Trackability : the formal problem

We are given

- A graph G(V,E) :

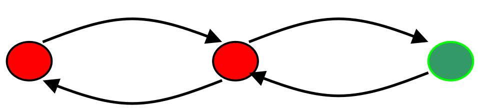


$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- A set of possible observations : defining a partition of the nodes

$$\begin{cases} R = \{1, 2\} \\ G = \{3\} \end{cases}$$

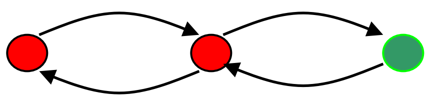
For each possible color, we define the corresponding submatrix of A by erasing the impossible columns :



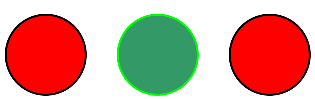
$$A_r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Trackability : the formal problem

To a given observation, associate the corresponding product:



$$A_r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



$$A_r A_g A_r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The number of possible trajectories is given by the sum of the entries of the matrix

Trackability : the formal problem

The maximal total number of possibilities is

$$N(t) = \max \left\{ \|A\|_1 \ : A \in \Sigma^t \right\}$$

We are interested in the asymptotic worst case :

$$\lim_{t \to \infty} N(t)^{1/t} = \lim_{t \to \infty} \max \left\{ \|A\|_1^{1/t} : A \in \Sigma^t \right\}$$

This is a joint spectral radius!

Trackability : the formal problem

The network is trackable iff

$$\rho \leq 1$$

[Crespi et al. 05]

Outline

- Matrix semigroups and some tools to analyze them
- LMI methods for hybrid systems stability
- An automata theoretic setting
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  - A characterization of valid LMI criteria
- Conclusion and perspectives

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Switching systems

$$x_{t+1} = \begin{cases} A_0\, x_t \\ A_1\, x_t \end{cases}$$

Point-to-point Given x_0 and x_*, is there a product (say, $A_0\ A_0\ A_1\ A_0\ \ldots\ A_1$) for which $x_* = A_0\ A_0\ A_1\ A_0\ \ldots\ A_1\ x_0$?

Mortality Is there a product that gives the zero matrix?

Boundedness Is the set of all products $\{A_0, A_1, A_0A_0, A_0A_1, \ldots\}$ bounded?

Switching systems

$$\mathbf{x}_{t+1} = \begin{matrix} \mathbf{A}_0\, \mathbf{x}_t \\ \mathbf{A}_1\, \mathbf{x}_t \end{matrix}$$

Global convergence to the origin Do all products of the type $A_0\, A_0\, A_1\, A_0 \ldots A_1$ converge to zero?

The spectral radius of a matrix A controls the growth or decay of powers of A

$$\rho(A) = \lim_{t \to \infty} ||A^t||^{1/t}$$

The powers of A converge to zero iff $\rho(A) < 1$

The joint spectral radius of $\Sigma = \{A_0, A_1\}$ is given by

$$\rho(\Sigma) = \lim_{t \to \infty} \max_{A_i \in \Sigma} ||A_1 A_2 \ldots A_t||^{1/t}$$

All products of A_0 and A_1 converge to zero iff $\rho(\Sigma) < 1$

The joint spectral characteristics

$$\rho(\Sigma) = \lim_{t\to\infty} \max_{A_i\in\Sigma} ||A_1 A_2 \ldots A_t||^{1/t}$$

$$\check{\rho}(\Sigma) = \lim_{t\to\infty} \min_{A_i\in\Sigma} ||A_1 A_2 \ldots A_t||^{1/t}$$

The joint spectral radius is the analysis question, while

The joint spectral subradius would be the design question

Theorem Computing or approximating ρ is NP-hard

Theorem The problem $\rho\leq 1$ is algorithmically undecidable

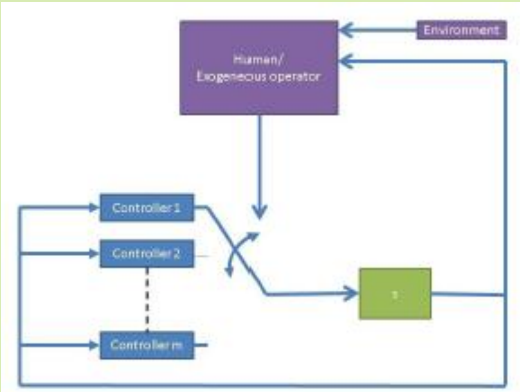
But also,… the p-radius, the Lyapunov exponent… [Oseledets 68, J. Protasov 11,]

…and many other negative results! [Alur Dill 94, Henzinger Raskin 00, Tsitsiklis Blondel 97]

Conjecture The problem $\rho<1$ is algorithmically undecidable

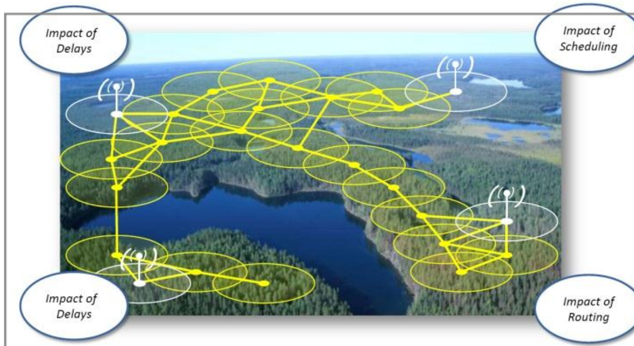
Theorem Even the question « $|\check{\rho} - r| \leq a + b\check{\rho}$?» is algorithmically undecidable for all a and b

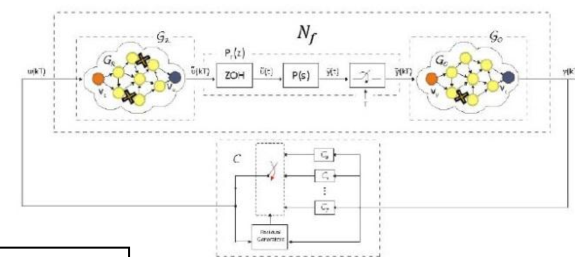
Other applications



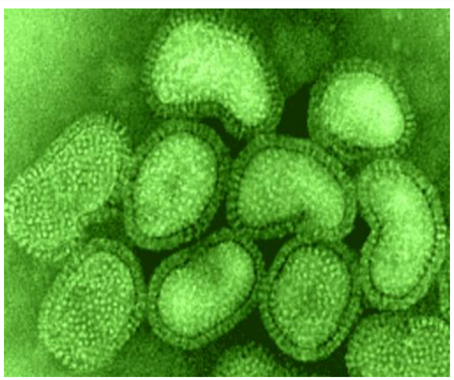
[Liberzon 03]
[J. 09]
[Shorten Wirth Mason Wulff King 07]

[Hendrickx 08]





[Alur Pappas D'Innocenzo Johannson Weiss 10]



[Hernandez-Varga colaneri Middleton Blanchini 10]

$$\frac{d\,[mRNA_X]}{dt} = \frac{\beta}{1+\left(\frac{[X](t-\Delta_1)}{K}\right)^n} - (\delta_{dil}+\delta_{deg1})\cdot[mRNA_X](t)$$

$$\frac{d\,[X]}{dt} = \gamma\cdot[mRNA_X](t-\Delta_2) - (\delta_{dil}+\delta_{deg2})\cdot[X](t)$$

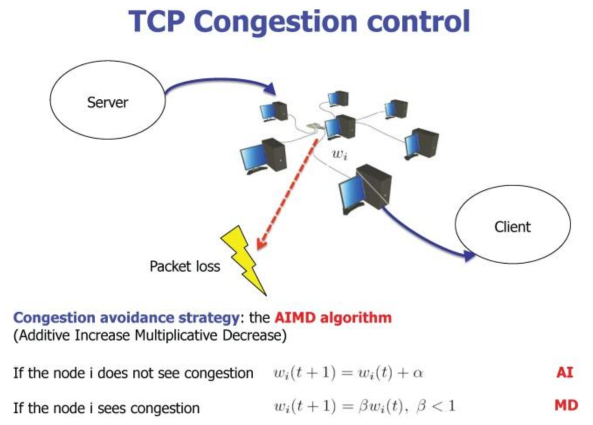
$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + bx(\lambda t)$$

[Guglielmi Zennaro 03]

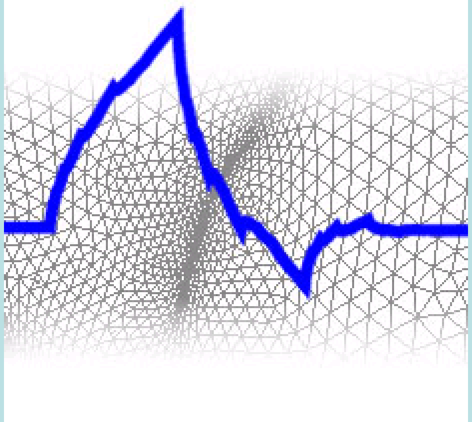
Other applications

[Protasov 04]

[Protasov J. Blondel 10]



[Scholte, Berman, Shorten 12]

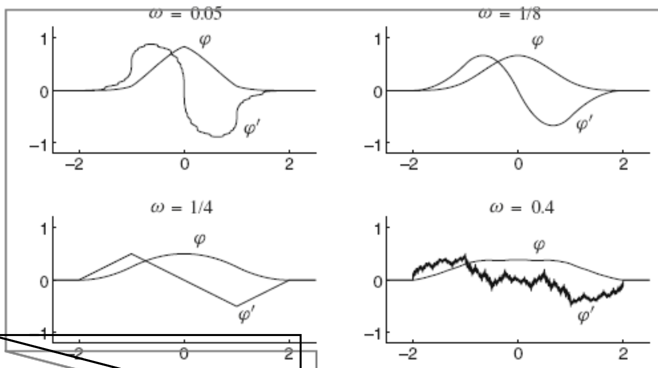


[Daubechies Lagarias 91]

1001011001101001…

[Blondel Cassaigne J. 09]

[Rigo Berthe 10]



[Dyn, Levin, Gregory 87]

[Dubuc, Merrien 09]

[Guglielmi, Manni, Vitale 10]

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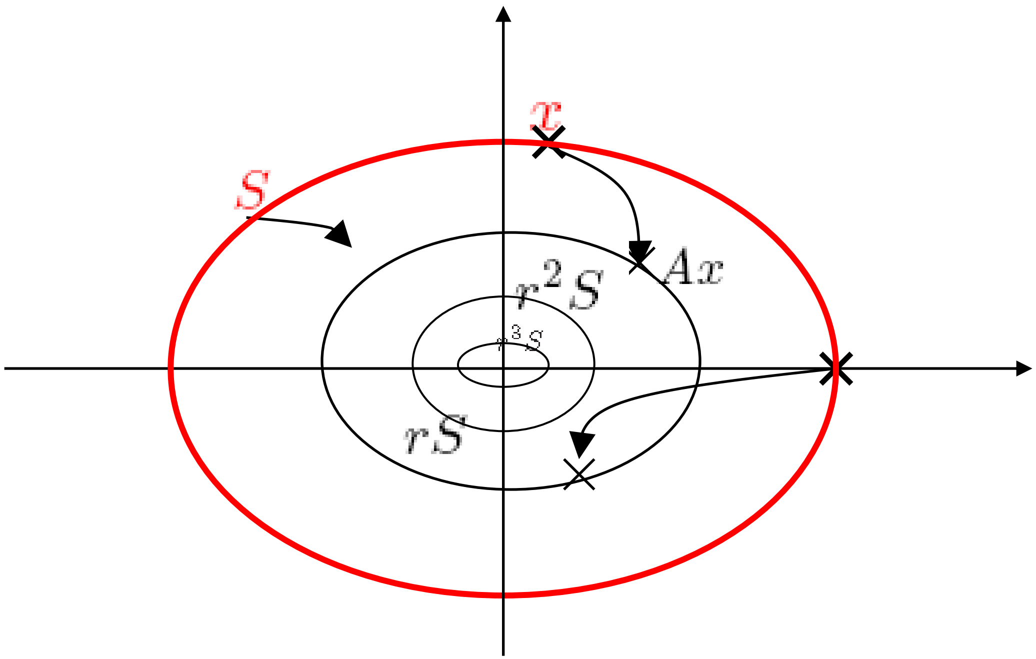
LMI methods

$$\rho(\Sigma) = \lim_{t \to \infty} \max_{A_1, \dots, A_t \in \Sigma} ||A_1 \dots A_t||^{1/t}$$

- Is there a vector norm such that $\forall A \in \Sigma, \ \forall x, |x| \leq 1, |Ax| \leq r$



Boundedness

SDP methods

- Theorem For all $\epsilon > 0$ there exists a norm such that

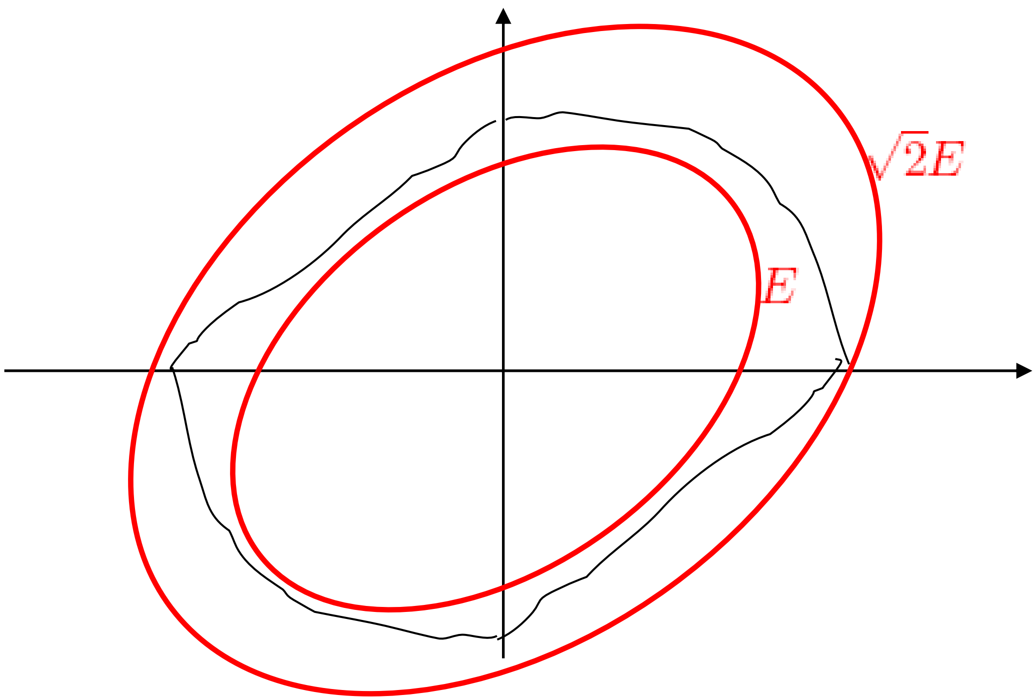
$$\forall A \in \Sigma, \forall x, |Ax| \leq (\rho + \epsilon)|x|$$

[Rota Strang, 60]

- John's ellipsoid Theorem: Let K be a compact convex set with nonempty interior symmetric about the origin. Then there is an ellipsoid E such that $E \subset K \subset \sqrt{n}E$

[John 1948]

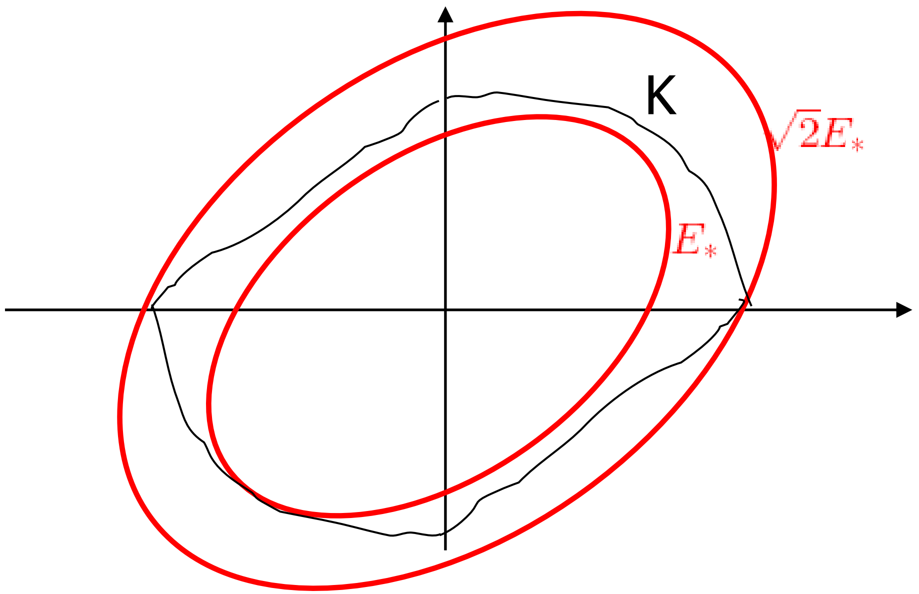
- So we can approximate the unit ball of an extremal norm with an ellipsoid



SDP methods

- Theorem The best ellipsoidal norm $||\cdot||_{E_*}$ approximates the joint spectral radius up to a factor $\sqrt{n}$ [Ando Shih 98]

$$\rho \leq \max ||A||_{E_*} \leq \sqrt{n}\rho$$
$$\frac{1}{\sqrt{n}}\rho * \leq \rho \leq \rho *$$



- Theorem The best ellipsoidal norm of a set of m (nonnegative) matrices also approximates the joint spectral radius up to a factor $\sqrt{m}$

$$\rho \leq \max ||A||_{E_*} \leq \sqrt{m}\rho$$

[Blondel Nesterov 05]

LMI methods

- Problem How to compute the best ellipsoidal norm?

$$\begin{array}{lll} \inf_{r\in\mathbb{R}^+} & & r \\ \text{s.t.} & & \\ A^TPA & \preceq & r^2P, \quad \forall A\in\Sigma \\ P & \succeq & 0. \end{array}$$

$$\Leftrightarrow \frac{|Ax|_P}{|x|_P} \leq r$$

- This is « just » a sufficient condition for stability
- Computable in polynomial time (interior point methods)
- The « CQLF method » (see [Mason Shorten 04])

Yet another LMI method

- A strange semidefinite program

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P & \succeq & 0.
\end{array}
$$

$$\Rightarrow \quad \rho \leq r$$

[Goebel, Hu, Teel 06]

Yet another LMI method

- An even stranger program:

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P A_1 & \preceq & r^2 P, \\
(A_2 A_1)^T P (A_2 A_1) & \preceq & r^4 P, \\
(A_2^2)^T P (A_2^2) & \preceq & r^4 P, \\
P & \succeq & 0.
\end{array}
$$

$$\rho \leq r$$

[Ahmadi, J., Parrilo, Roozbehani10]

Yet another LMI method

- Questions:

  - Can we characterize all the LMIs that work, in a unified framework?

  - Which LMIs are better than others?

  - How to prove that an LMI works?

  - Can we provide converse Lyapunov theorems for more methods?

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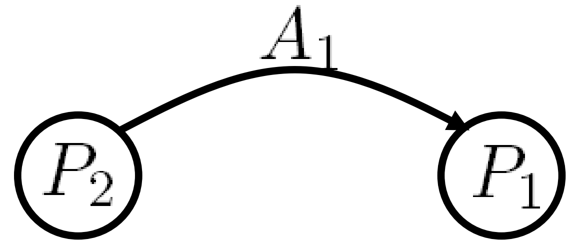
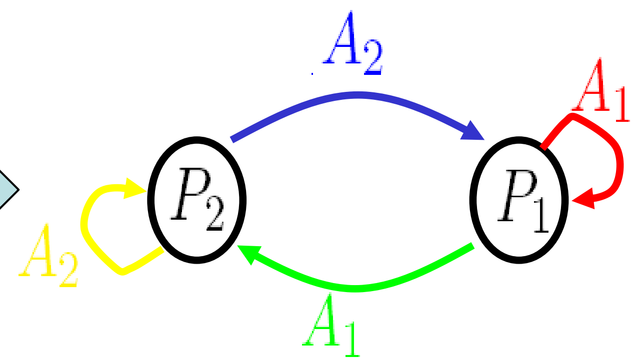
From an LMI to an automaton

- Automata representation: Given a set of LMIs, construct an automaton like this:

$$A_1^T P_1 A_1 \preceq r^2 P_2$$



$$\begin{array}{lcl} \min_{r \in \mathbb{R}^+} & & r \\ \text{s.t.} & & \\ A_1^T P_1 A_1 & \preceq & r^2 P_1, \\ A_2^T P_1 A_2 & \preceq & r^2 P_2, \\ A_1^T P_2 A_1 & \preceq & r^2 P_1, \\ A_2^T P_2 A_2 & \preceq & r^2 P_2, \\ P_i & \succeq & 0. \end{array}$$

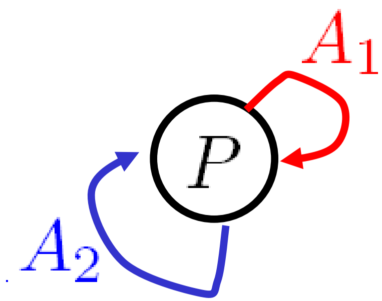


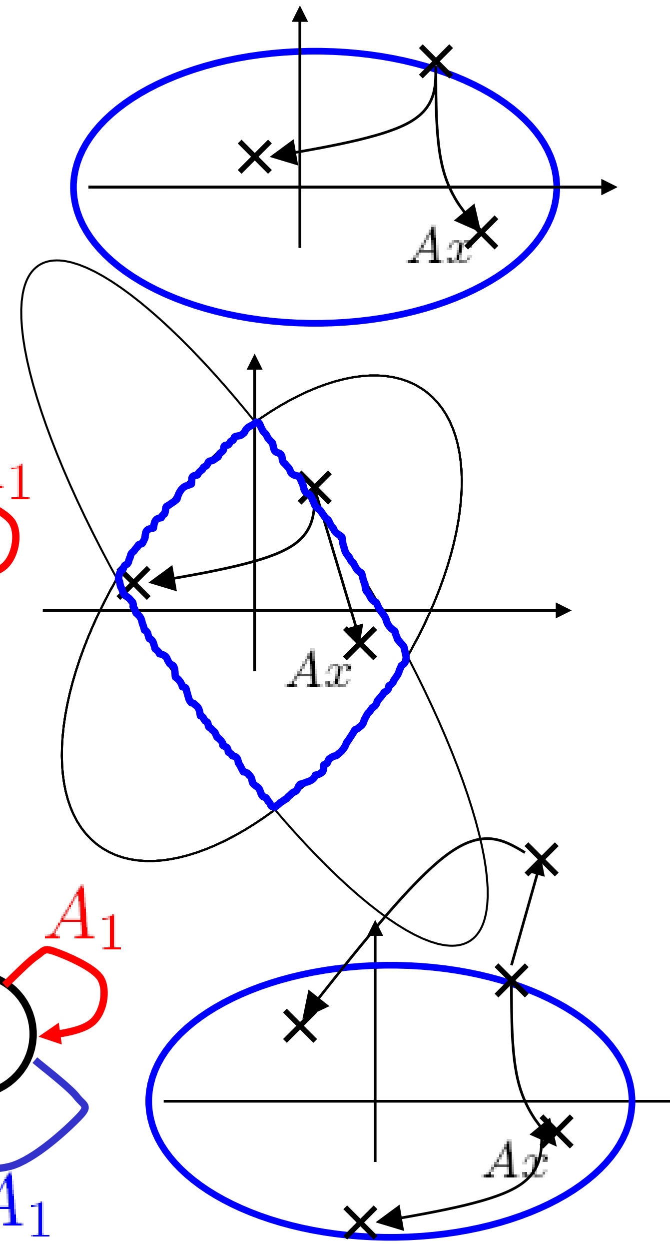
- Definition: A labeled graph (with label set A) is path-complete if for any word on the alphabet A, there exists a path in the graph that generates the corresponding word.
- Theorem: If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability.

[Ahmadi J. Parrilo Roozbehani 11]

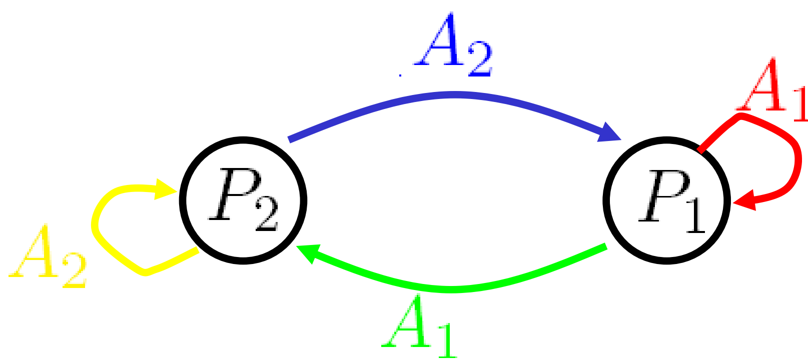
Some examples

- Examples:
  - CQLF





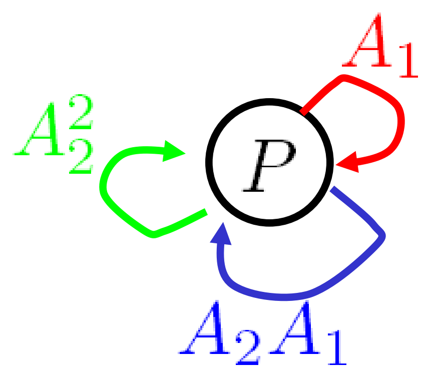
  - Example 1



    This type of graph gives a max-of-quadratics Lyapunov function (i.e. intersection of ellipsoids)

  - Example 2

    This type of graph gives a common Lyapunov function for a generating set of words



- Converse Lyapunov theorems

Example

- Consider $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ [Ando Shih 98]

- The CQLF behaves « as badly as possible »:

$$\rho(\Sigma) = 1 \qquad \text{but} \qquad r^* = \sqrt{n}$$

- Even worse: $r_k^* = \sqrt[2k]{n}$

- Example 2 makes it in one step $r_k^*(G_2) = 1$

- In general, we observed that this particular graph seems to behave very well!

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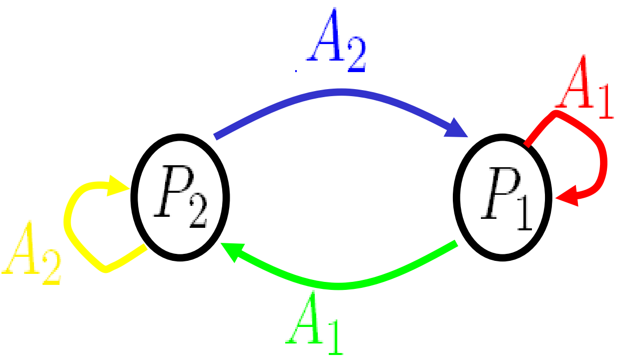
An obvious question: are there other valid criteria?

- Theorem:



$$
\begin{array}{lll}
\min_{r\in\mathbb{R}+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}
$$

Path complete ??? Sufficient condition for stability

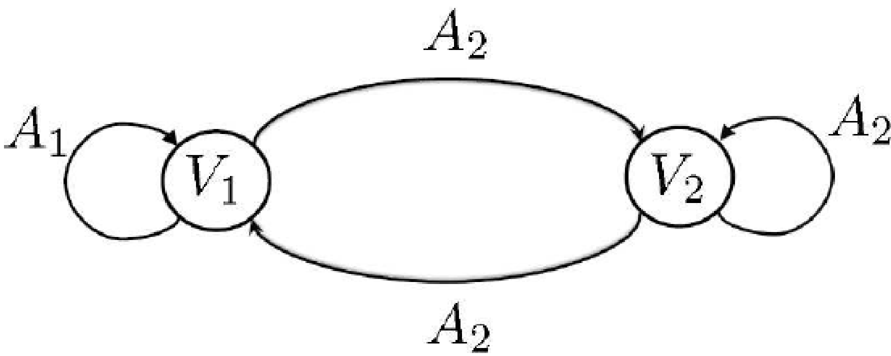
If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability.

- Are all valid sets of equations coming from path-complete graphs?

- …or are there even more valid LMI criteria?

Are there other valid criteria?

- Example:



$$
\begin{aligned}
A_1^T P_1 A_1 &\prec P_1 \\
A_2^T P_1 A_2 &\prec P_2 \\
A_2^T P_2 A_2 &\prec P_1 \\
A_2^T P_2 A_2 &\prec P_2 \\
P_1, P_2 &\succ 0.
\end{aligned}
$$

is not a valid set because the following matrices satisfy it, but, yet, are not stable

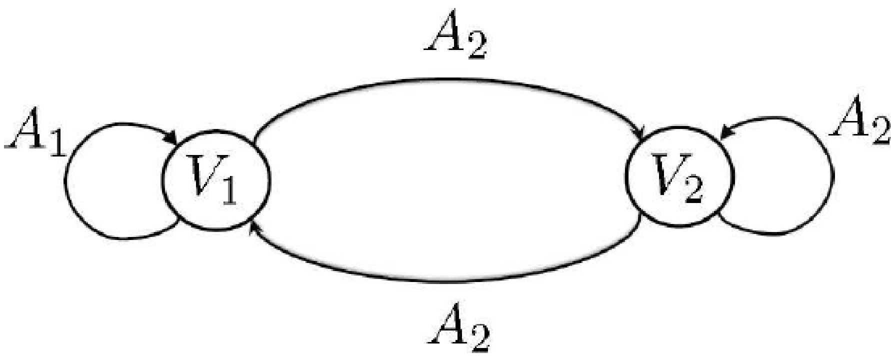
$$
\Sigma = \left\{ \begin{pmatrix} -0.7 & 0.3 & 0.4 \\ 0.4 & 0 & 0.8 \\ -0.7 & 0.5 & 0.7 \end{pmatrix}, \begin{pmatrix} -0.3 & -0.95 & 0 \\ 0.4 & 0.5 & 0.8 \\ -0.6 & 0 & 0.2 \end{pmatrix} \right\}
$$

$$
\rho(\Sigma) \geq \rho(A_1 A_2 A_1)^{1/3} = 1.01\ldots
$$

- Intuition: if a graph is not path complete, it must « miss » some products, that might well be unstable for some sets of matrices

Are there other valid criteria?

- Example:



$$
\begin{aligned}
A_1^T P_1 A_1 &\prec P_1 \\
A_2^T P_1 A_2 &\prec P_2 \\
A_2^T P_2 A_2 &\prec P_1 \\
A_2^T P_2 A_2 &\prec P_2 \\
P_1, P_2 &\succ 0.
\end{aligned}
$$

- It is actually a difficult question!

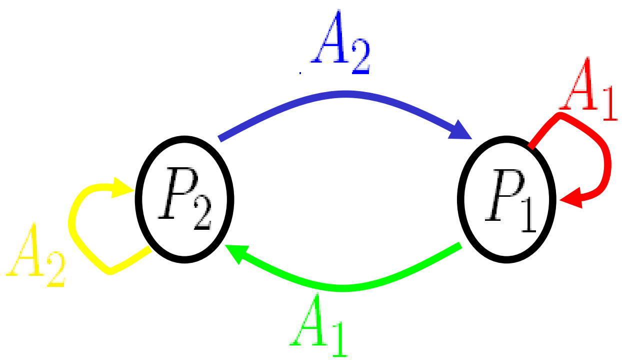
  - For any « non path complete » criteria, we have to find the right counterexample (an unstable set, which satisfies the LMIs)

  - For this counterexample (implicitely defined) prove that it satisfies the LMIs (i.e. construct the solutions P_i)

Are there other valid criteria?

- Theorem: Every non path-complete set of equations is not a sufficient condition for stability.

[J. Ahmadi Parrilo Roozbehani 12]



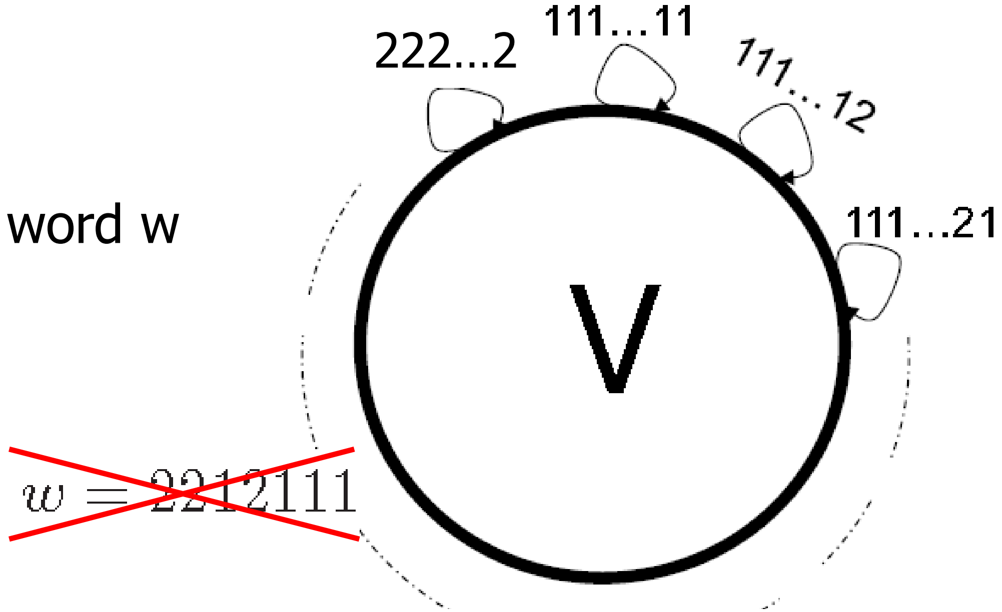
$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}
$$

Path complete !!! Sufficient condition for stability

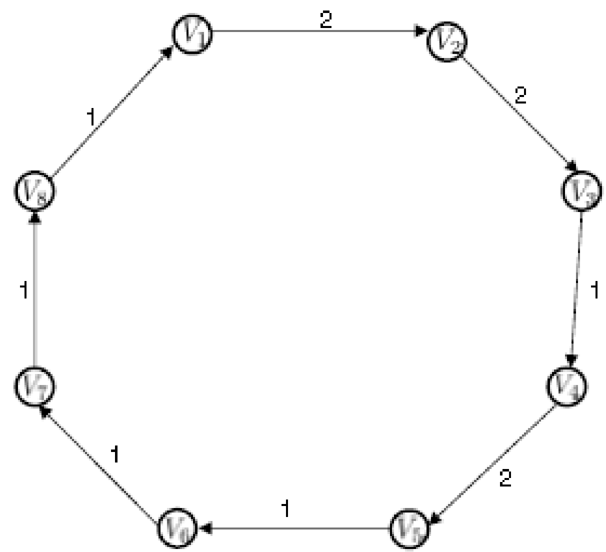
- Proof: constructive (but implicit!)
  - If the set of LMIs is not path complete, there exists a missing word w
  - Build a set of matrices whose « unique unstable word is w »
  - Show that this set of matrices satisfies the LMIs (compute the solution P)

Are there other valid criteria?

- Proof: constructive (but implicit!)
  - If the set of LMIs is not path complete, there exists a missing word w



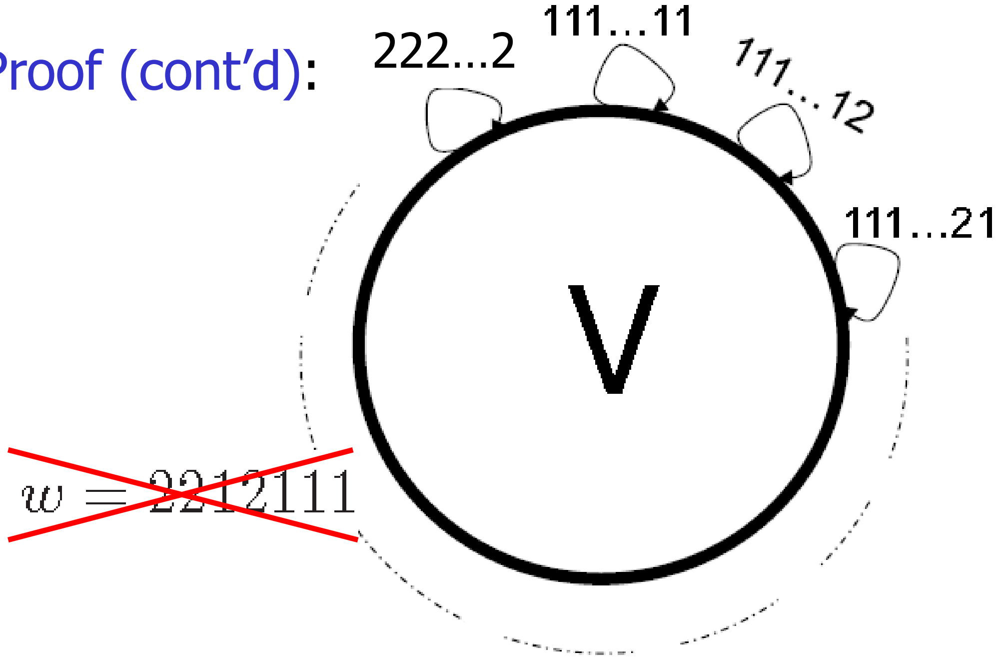
  - Build a set of matrices whose « unique unstable product is A_{w1} »

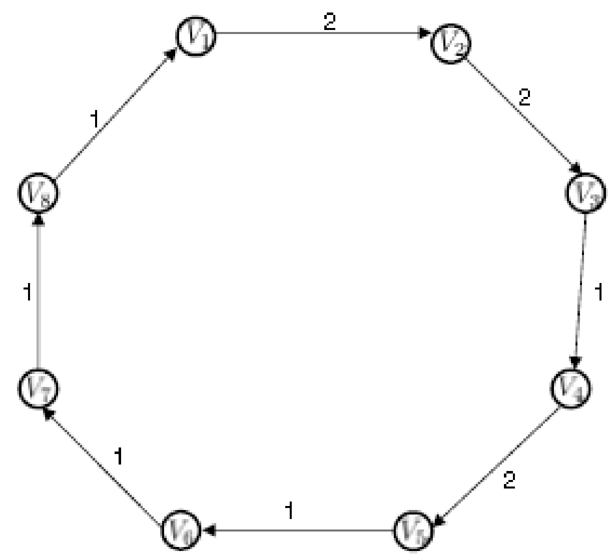


$A_2A_2A_1A_2A_1A_1A_1A_1$ is the « only unstable » product

Are there other valid criteria?

- Proof (cont'd):





- Now, we have to find a valid Lyapunov function V such that these equations are satisfied

- Theorem: if $\rho < \frac{1}{\sqrt{n}}$, then, there exists a CQLF

  [Ando Shih 98]

- One can show that indeed, if we miss the product A_w , all the other products taken together have zero jsr

- But what about more complicated graphs?

Are there other valid criteria?

- Corollary:

  It is PSPACE complete to recognize sets of equations that are a sufficient condition for stability

- Proof: recognizing them amounts to recognize automata that accept the full language

- These results are not limited to LMIs, but apply to other families of conic inequalities:

  - Theorem: Entrywise inequalities are valid stability criteria (for nonnegative matrices) iff the corresponding graph is path-complete

  - Theorem: Sum-Of-Squares inequalities are valid stability criteria iff the corresponding graph is path-complete

[J. Ahmadi Parrilo Roozbehani 12]

Outline

- Matrix semigroups and some tools to analyze them
- LMI methods for hybrid systems stability
- An automata theoretic setting
  - Model reduction for new LMI methods
  - A characterization of valid LMI criteria
- Conclusion and perspectives

LMI methods: to be continued

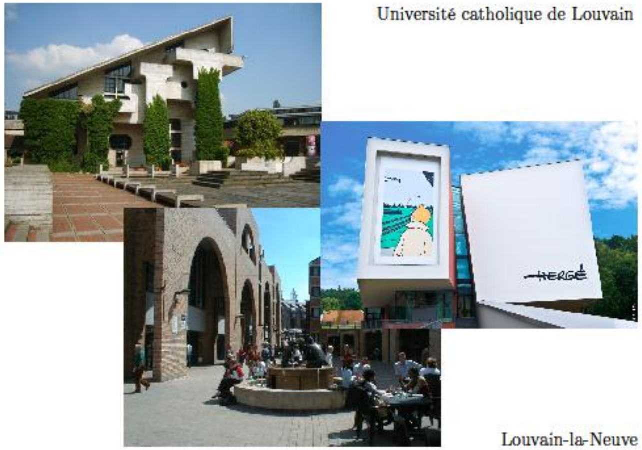
- Conjecture: For any family of conic inequalities, a criterion is valid iff it is path-complete

- One could apply the same ideas on restricted dynamics (subshifts)

- Apply these methods
  - for other joint spectral characteristics
  - For the design of switched systems

- If a classical application makes use of matrices, there is a big chance that semigroups of matrices are worth studying

14th Mons Days of Theoretical Computer Science
Université catholique de Louvain, Belgium
September 11th to 14th, 2012



Conference themes :
- combinatorics and algorithmics on words,
- automata theory and formal languages theory,
- discrete dynamical systems and symbolic dynamics,

and their links with other fields (number theory, calculability, logical aspects, model checking, theory of semigroups, game theory, discrete geometry, decentralized algorithms, biocomputing, ...).

Submissions will be open in **April**.

A few **student grants** should be available.

http://sites.uclouvain.be/JM2012/

icteam

http://sites.uclouvain.be/JM2012/

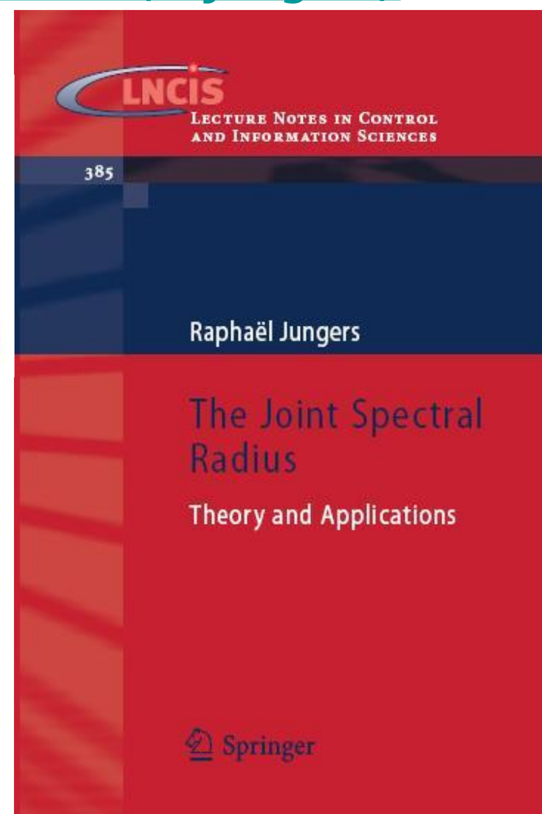
Jëréjëf!

Questions?

Ads

The JSR Toolbox: http://www.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox

References:
http://www.inma.ucl.ac.be/~jungers/



Joint work with
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