Algebraic Techniques for Switching Systems

And applications

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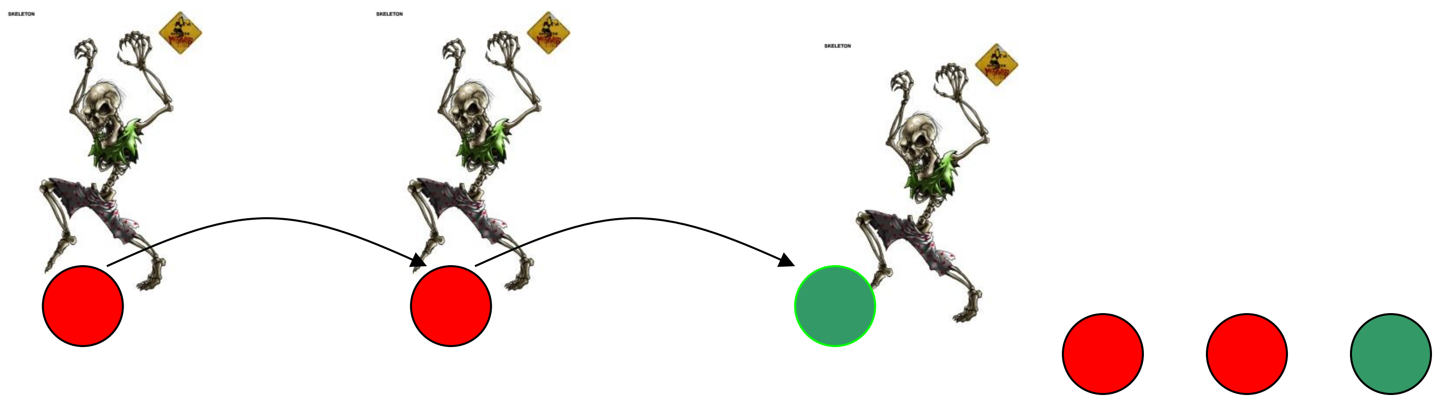
Trackable graphs



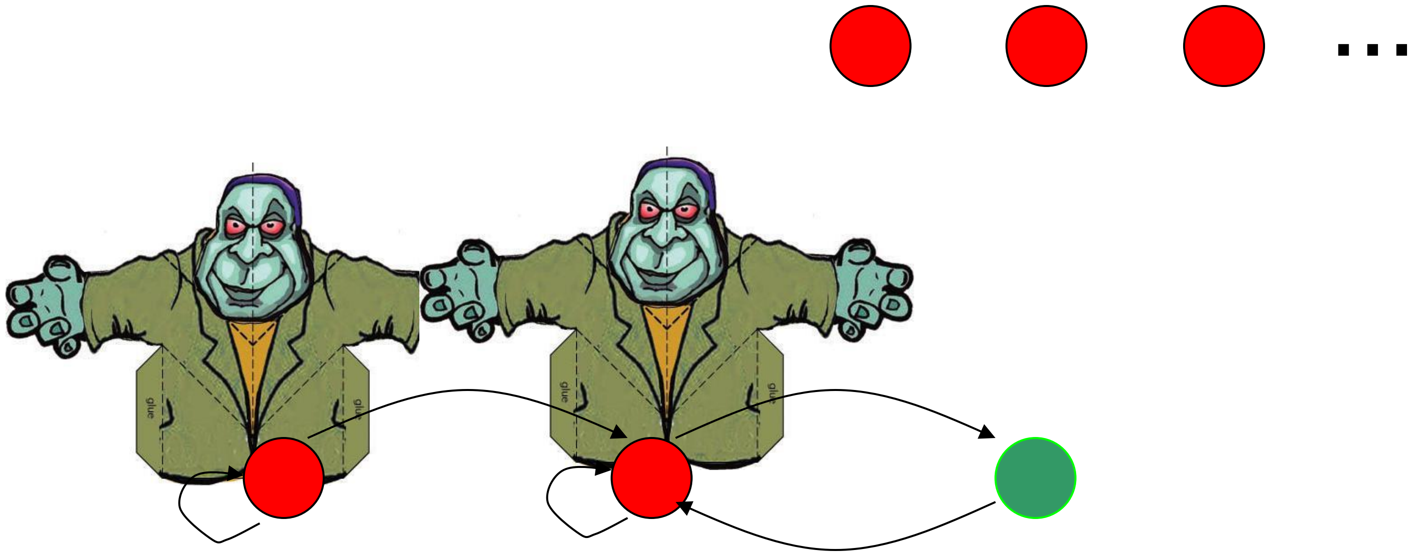
Let $N(t)$ be te worst possible number of trajectories compatible with an observation of length t
A network is trackable if $N(t)$ grows subexponentially

[Crespi et al. 05]

Here: number of possibilities asymptotically zero $\quad N(t) \approx 0$

➔ Trackable

Trackable graphs

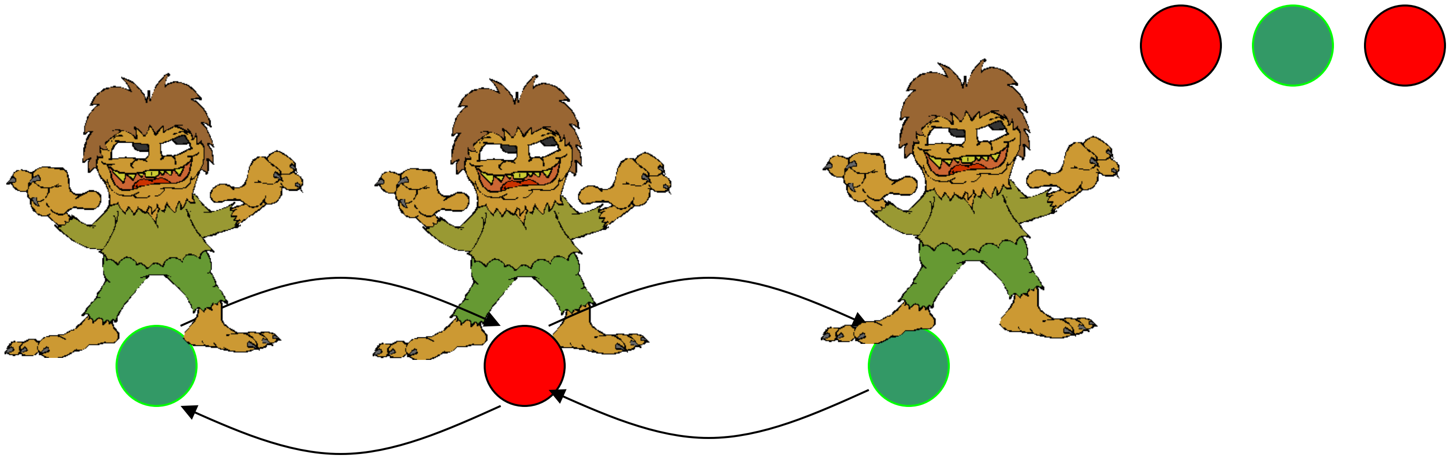


Worst case : RRRRRR… ➔ $N(t) \approx t$

Polynomial number of possibilities

➔ Trackable

Trackable graphs



Worst case : RGRGRG…➔ $$N(t) \approx 2^{t/2}$$

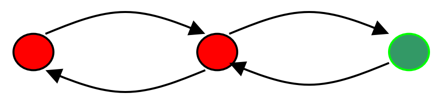
Exponential number of possibilities

➔ Not trackable

Trackability : the formal problem

We are given

– A graph G(V,E) :

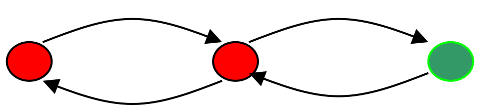


$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

– A set of possible observations : defining a partition of the nodes

$$\begin{cases} R = \{1,2\} \\ G = \{3\} \end{cases}$$

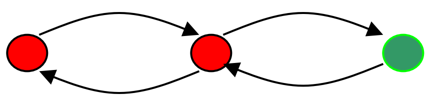
For each possible color, we define the corresponding matrix by erasing the incompatible columns from A :



$$A_r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Trackability : the formal problem

To a given observation, associate the corresponding product:



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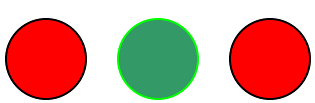

$$A_r A_g A_r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The number of possible trajectories is given by the sum of the entries of the matrix

Outline

- Joint spectral characteristics
- Automatic methods for switching systems stability
- Applications:
  - Trackable graphs
  - WCNs and switching delays
  - Consensus problems
- Conclusion and perspectives

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Switching systems

$$x_{t+1} = \begin{matrix} A_0\, x_t \\ A_1\, x_t \end{matrix}$$

Point-to-point Given x_0 and x_*, is there a product (say, $A_0\ A_0\ A_1\ A_0 \ldots A_1$) for which $x_* = A_0\ A_0\ A_1\ A_0 \ldots A_1\ x_0$?

Mortality Is there a product that gives the zero matrix?

Boundedness Is the set of all products $\{A_0, A_1, A_0A_0, A_0A_1, \ldots\}$ bounded?

Switching systems

$$\mathbf{x}_{t+1} = \begin{matrix} \mathbf{A}_0\, \mathbf{x}_t \\ \mathbf{A}_1\, \mathbf{x}_t \end{matrix}$$

Global convergence to the origin Do all products of the type $A_0\ A_0\ A_1\ A_0\ \ldots\ A_1$ converge to zero?

The spectral radius of a matrix A controls the growth or decay of powers of A

$$\rho(A) = \lim_{t\to\infty} ||A^t||^{1/t}$$

The powers of A converge to zero iff $\rho(A) < 1$

The joint spectral radius of a set of matrices Σ is given by

$$\rho(\Sigma) = \lim_{t\to\infty} \max_{A_i\in\Sigma} ||A_1 A_2 \ldots A_t||^{1/t}$$

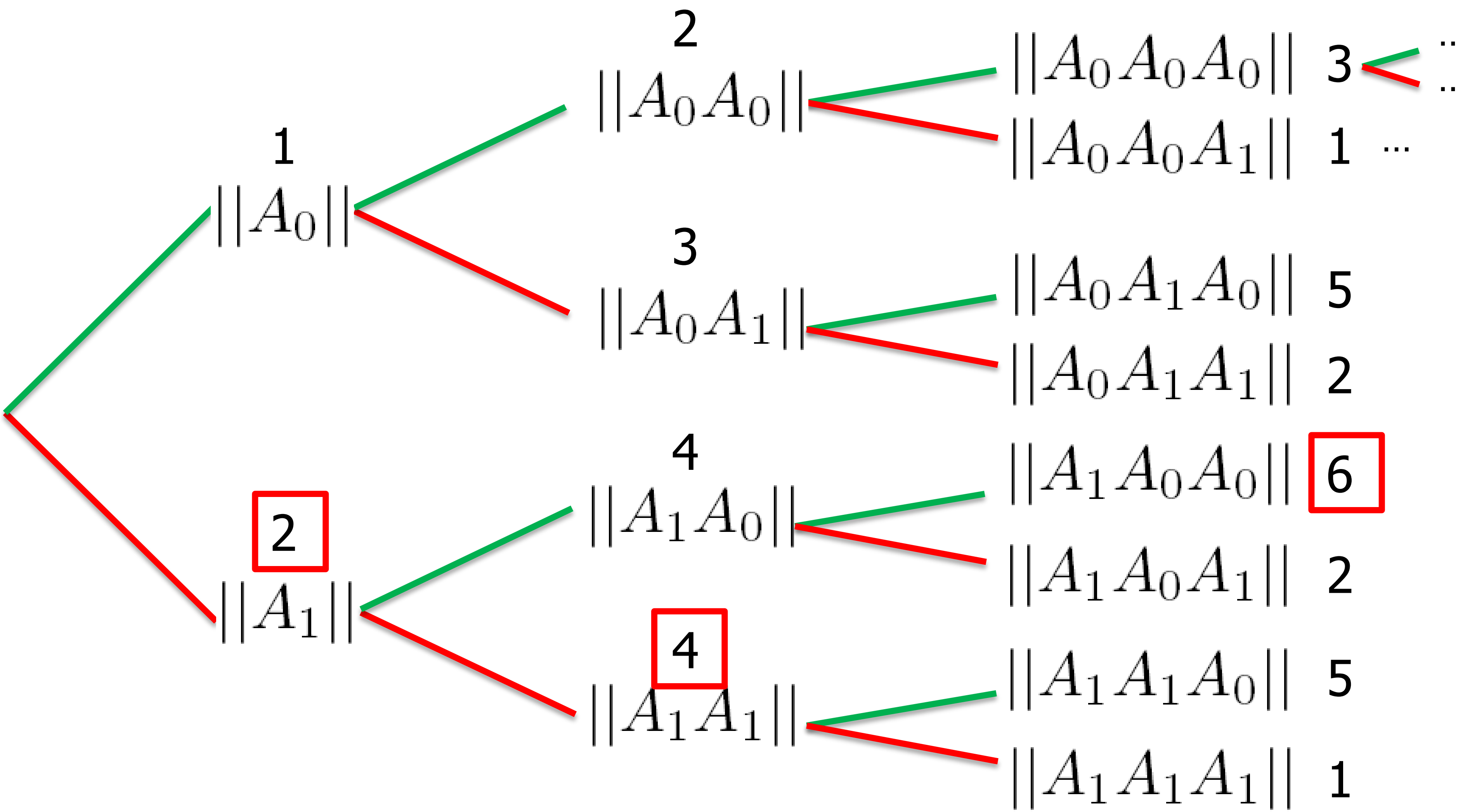
All products of matrices in Σ converge to zero iff $\rho(\Sigma) < 1$

[Rota, Strang, 1960]

The joint spectral characteristics

$$\boxed{\rho(\Sigma) = \lim_{t\to\infty} \left[\max_{A_i \in \Sigma} ||A_1 A_2 \dots A_t||\right]^{1/t}}$$

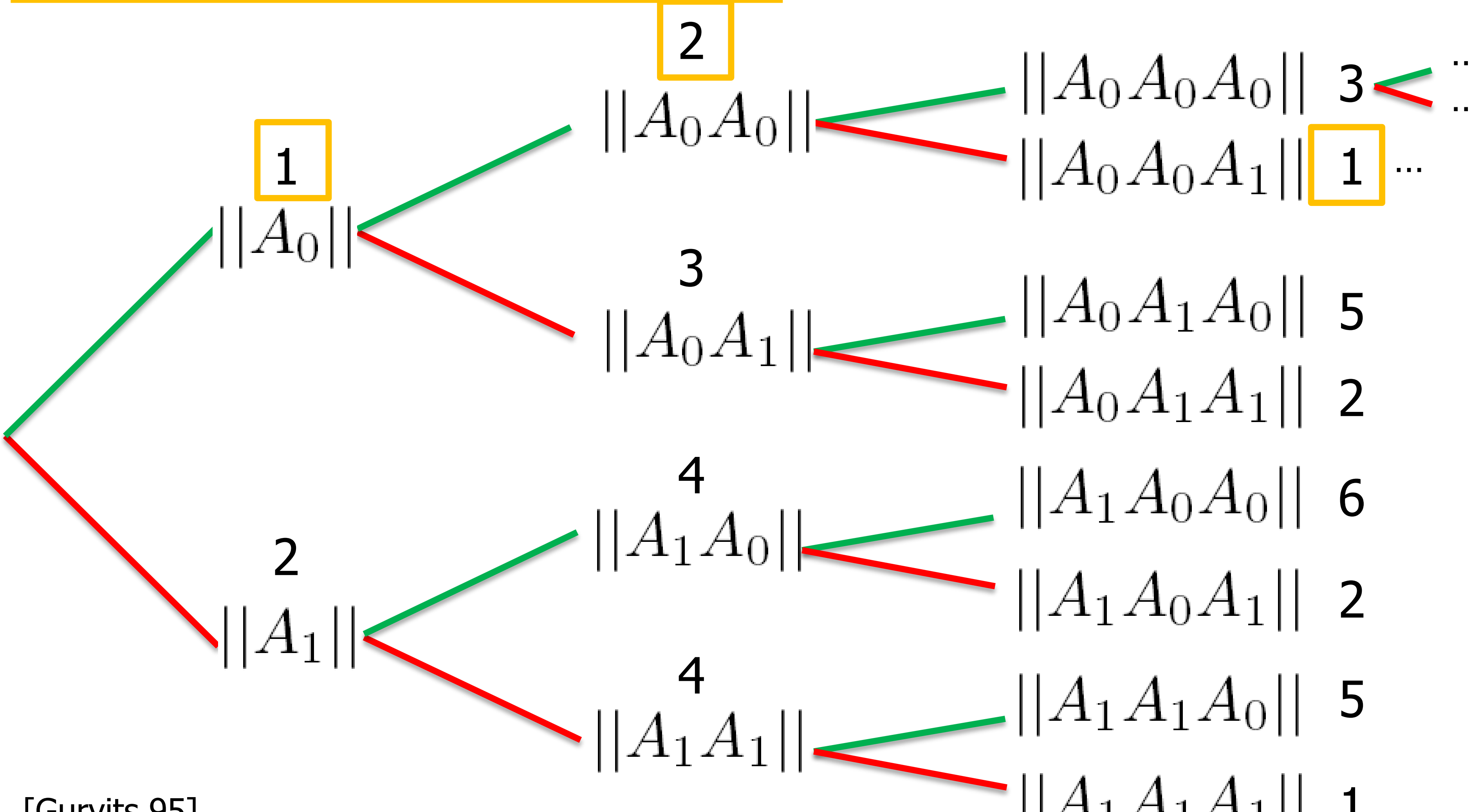
The joint spectral radius



The joint spectral characteristics

$$\boxed{\check{\rho}(\Sigma) = \lim_{t \to \infty} \left[\min_{A_i \in \Sigma} ||A_1 A_2 \dots A_t|| \right]^{1/t}}$$

The joint spectral subradius



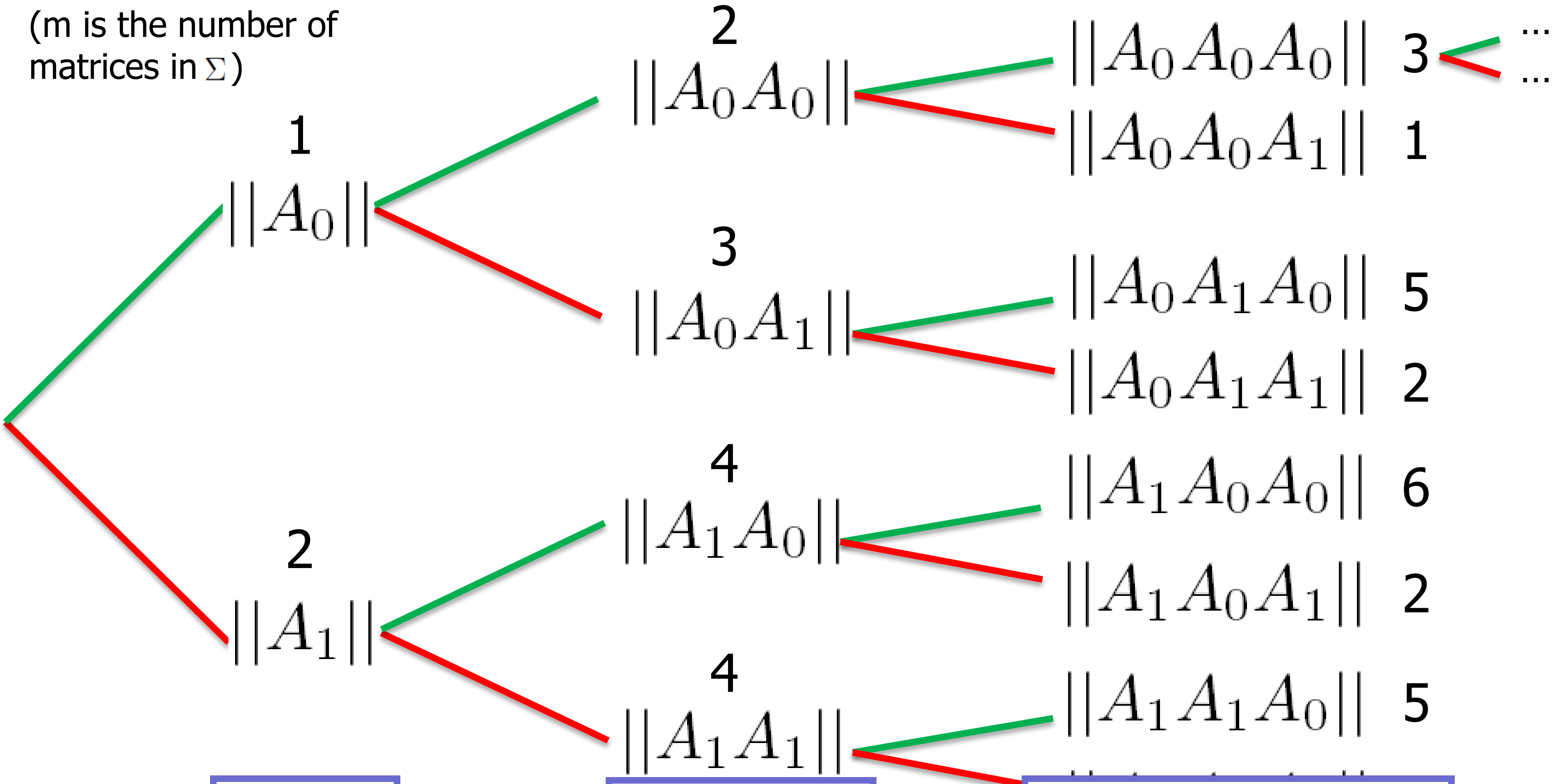
[Gurvits 95]

The joint spectral characteristics

$$\rho_p(\Sigma) = \lim_{t\to\infty} \left[m^{-t} \sum_{A_i \in \Sigma^t} \|A_1 A_2 \dots A_t\|^p \right]^{1/(pt)}$$

The p-radius

(m is the number of matrices in Σ)



[Jia 95]

The joint spectral characteristics

$$\bar{\rho}(\Sigma) = \lim_{t\to\infty} \left[\prod_{A_i \in \Sigma^t} \|A_1 A_2 \dots A_t\| \right]^{1/(tm^t)}$$

The Lyapunov Exponent

(m is the number of matrices in Σ)

1 $||A_0||$

2 $||A_0A_0||$

$||A_0A_0A_0||$ 3 …

…

$||A_0A_0A_1||$ 1 …

3 $||A_0A_1||$

$||A_0A_1A_0||$ 5

$||A_0A_1A_1||$ 2

4 $||A_1A_0||$

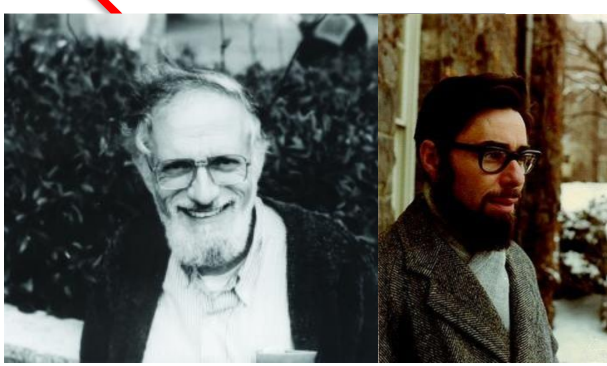
$||A_1A_0A_0||$ 6

$||A_1A_0A_1||$ 2

4 $||A_1A_1||$

$||A_1A_1A_0||$ 5

$||A_1A_1A_1||$ 1



[Furstenberg Kesten, 1960]

The joint spectral characteristics

$$\rho(\Sigma) = \lim_{t \to \infty} \left[\max_{A_i \in \Sigma} ||A_1 A_2 \dots A_t|| \right]^{1/t}$$

The joint spectral radius addresses the **stability** problem

$$\check{\rho}(\Sigma) = \lim_{t \to \infty} \left[\min_{A_i \in \Sigma} ||A_1 A_2 \dots A_t|| \right]^{1/t}$$

The joint spectral subradius addresses the **stabilizability** problem

$$\rho_p(\Sigma) = \lim_{t \to \infty} \left[m^{-t} \sum_{A_i \in \Sigma^t} \|A_1 A_2 \dots A_t\|^p \right]^{1/(pt)}$$

The p-radius addresses the **quadratic stability** (p=2), and more generally the **p-weak stability** [J. Protasov 10] [Ogura J. 14]

$$\bar{\rho}(\Sigma) = \lim_{t \to \infty} \left[\prod_{A_i \in \Sigma^t} \|A_1 A_2 \dots A_t\| \right]^{1/(tm^t)}$$

The Lyapunov exponent addresses the **stability with probability one** (Cfr. Oseledets Theorem)

The joint spectral characteristics: Mission Impossible?

Theorem Computing or approximating ρ is NP-hard

Theorem The problem $\rho \cdot 1$ is algorithmically undecidable

Conjecture The problem $\rho<1$ is algorithmically undecidable

Theorem Even the question « $|\check{\rho} - r| \leq a + b\check{\rho}$?» is algorithmically undecidable for all (nontrivial) a and b

Theorem The same is true for the Lyapunov exponent

Theorem The p-radius is NP-hard to approximate

See [Blondel Tsitsiklis 97, Blondel Tsitsiklis 00, J. Protasov 09]

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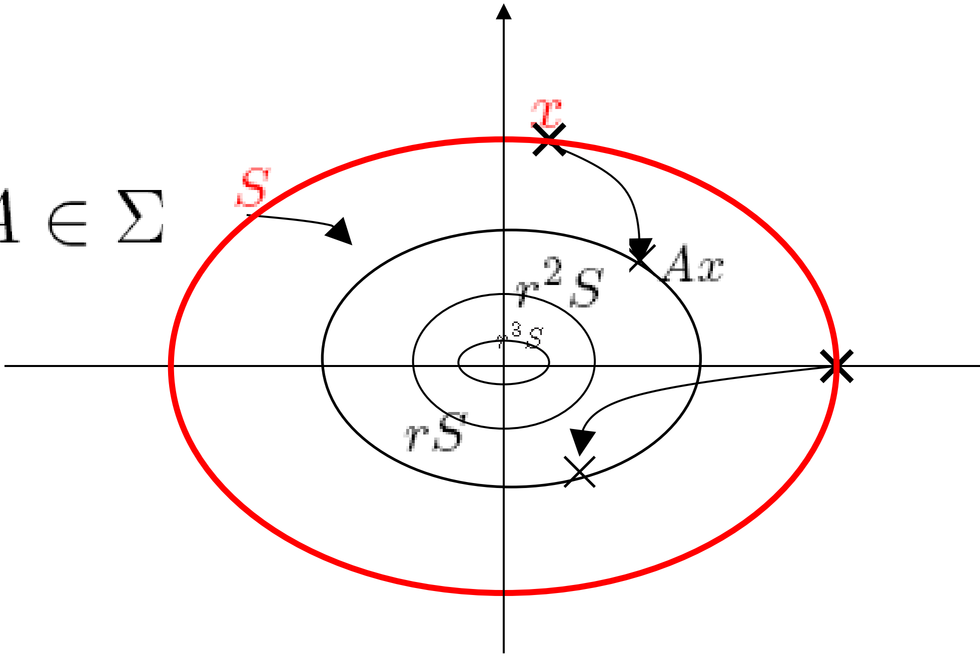
LMI methods

- The CQLF method

$$
\begin{array}{lcll}
\inf_{r\in\mathbb{R}^+} & & r & \\
\text{s.t.} & & & \\
A^T P A & \preceq & r^2 P, & \forall A \in \Sigma \\
P & \succeq & 0. &
\end{array}
$$

$$\Leftrightarrow \frac{|Ax|_P}{|x|_P} \leq r$$



SDP methods

- Theorem For all $\epsilon > 0$ there exists a norm such that

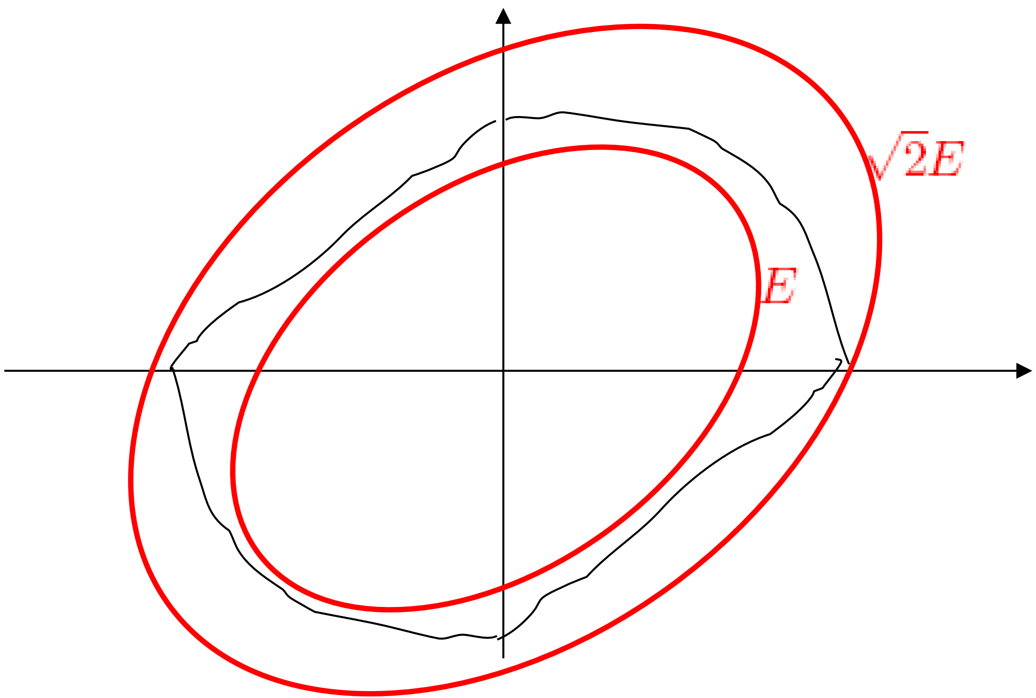
$$\forall A \in \Sigma, \forall x, |Ax| \leq (\rho + \epsilon)|x|$$

[Rota Strang, 60]

- John's ellipsoid Theorem: Let K be a compact convex set with nonempty interior symmetric about the origin. Then there is an ellipsoid E such that $E \subset K \subset \sqrt{n}E$

[John 1948]

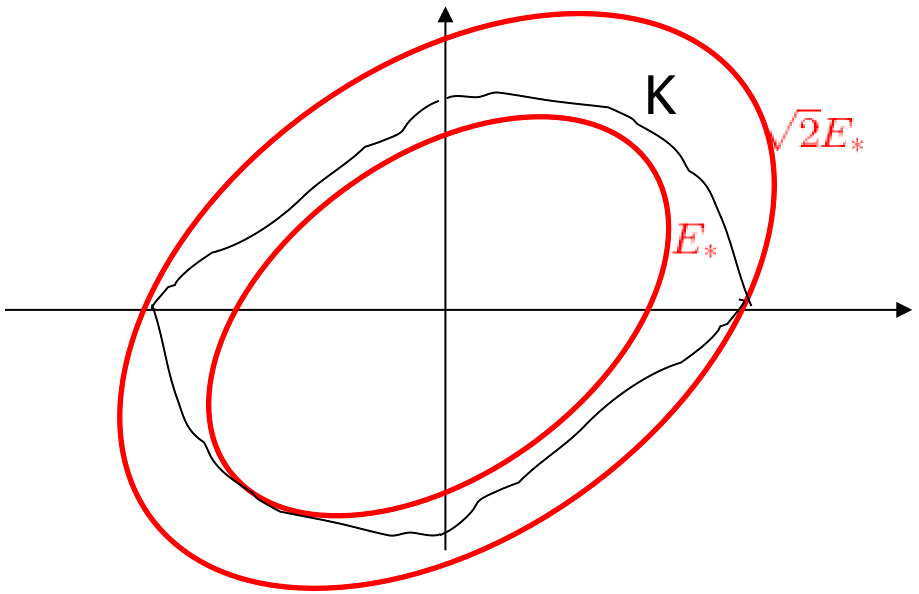
- So we can approximate the unit ball of an extremal norm with an ellipsoid



SDP methods

- Theorem The best ellipsoidal norm $||\cdot||_{E_*}$ approximates the joint spectral radius up to a factor $\sqrt{n}$ [Ando Shih 98]

$$\rho \leq \max ||A||_{E_*} \leq \sqrt{n}\rho$$
$$\frac{1}{\sqrt{n}}\rho* \leq \rho \leq \rho*$$



One can improve this method by lifting the state space
There exists a PTAS for the jsr computation :
Algorithm that approximates the joint spectral radius of arbitrary sets of m (nXn)-matrices up to an arbitrary accuracy ϵ in $\mathcal{O}(n^{m\frac{1}{\epsilon}})$ operations

Yet another LMI method

- A strange semidefinite program

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P & \succeq & 0.
\end{array}
$$

$$\rho \leq r$$

[Goebel, Hu, Teel 06]

- But also…

[Daafouz Bernussou 01]

[Bliman Ferrari-Trecate 03]

[Lee and Dullerud 06] …

Yet another LMI method

- An even stranger program:

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P A_1 & \preceq & r^2 P, \\
(A_2 A_1)^T P (A_2 A_1) & \preceq & r^4 P, \\
(A_2^2)^T P (A_2^2) & \preceq & r^4 P, \\
P & \succeq & 0.
\end{array}
$$

$$\rho \le r$$

[Ahmadi, J., Parrilo, Roozbehani10]

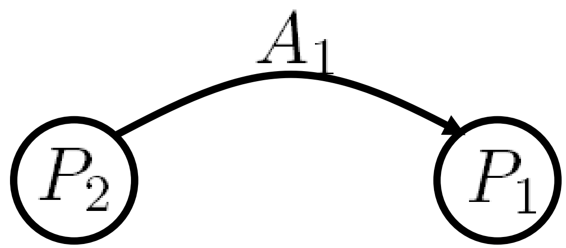
Yet another LMI method

- Questions:
  - Can we characterize all the LMIs that work, in a unified framework?
  - Which LMIs are better than others?
  - How to prove that an LMI works?
  - Can we provide converse Lyapunov theorems for more methods?

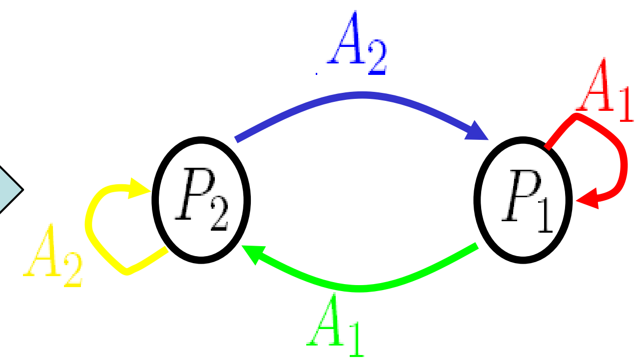
From an LMI to an automaton

- Automata representation Given a set of LMIs, construct an automaton like this:

$$A_1^T P_1 A_1 \preceq r^2 P_2$$



$$\begin{array}{lcl} \min_{r\in\mathbb{R}^+} & & r \\ \text{s.t.} & & \\ A_1^T P_1 A_1 & \preceq & r^2 P_1, \\ A_2^T P_1 A_2 & \preceq & r^2 P_2, \\ A_1^T P_2 A_1 & \preceq & r^2 P_1, \\ A_2^T P_2 A_2 & \preceq & r^2 P_2, \\ P_i & \succeq & 0. \end{array}$$

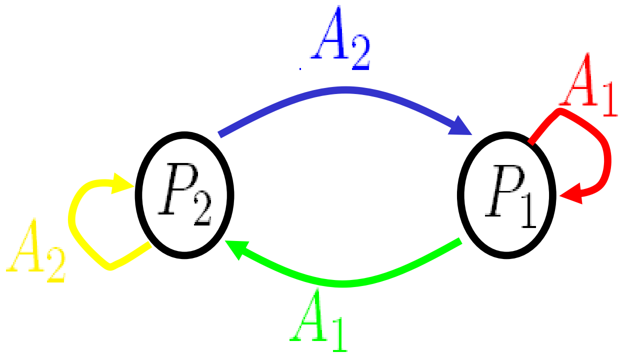


- Definition A labeled graph (with label set A) is path-complete if for any word on the alphabet A, there exists a path in the graph that generates the corresponding word.
- Theorem If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability.

[Ahmadi J. Parrilo Roozbehani 11]

An obvious question: are there other valid criteria?

- Theorem



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$$

Path complete

???

Sufficient condition for stability

If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability.

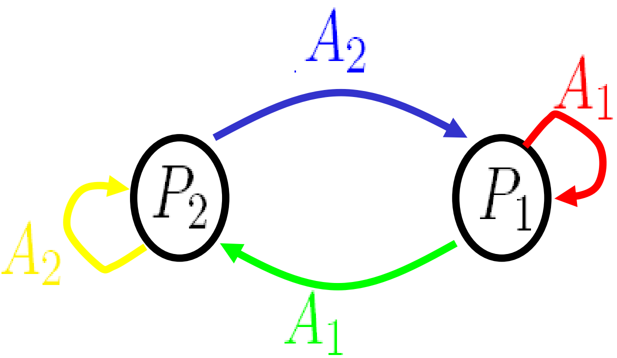
- Are all valid sets of equations coming from path-complete graphs?

- …or are there even more valid LMI criteria?

Are there other valid criteria?

- Theorem Non path-complete sets of LMIs are not sufficient for stability.

[J. Ahmadi Parrilo Roozbehani 12]



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Path complete

???

Sufficient condition for stability

- Corollary

  It is PSPACE complete to recognize sets of equations that are a sufficient condition for stability

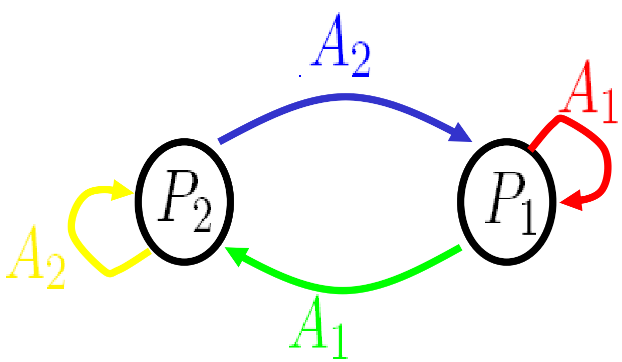
- These results are not limited to LMIs, but apply to other families of conic inequalities

What about the other quantities?

	Arbitrary approximation	Arbitrary approximation in polynomial time	Arbitrary approximation for positive matrices	Decidability	
Joint Spectral Radius	v	v	v	?	
Joint Spectral Subradius	x	x	v	x	
Lyapunov Exponent	x	x	v	x	
p-radius	**Depends on p**	**Depends on p**	v	?	

So what now?

- After all, what are all these results useful for?



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- ➔ this framework is generalizable to harder problems
  - Constrained switching systems
  - Controller design for switching systems
  - Automatically optimized abstraction of cyber-physical systems

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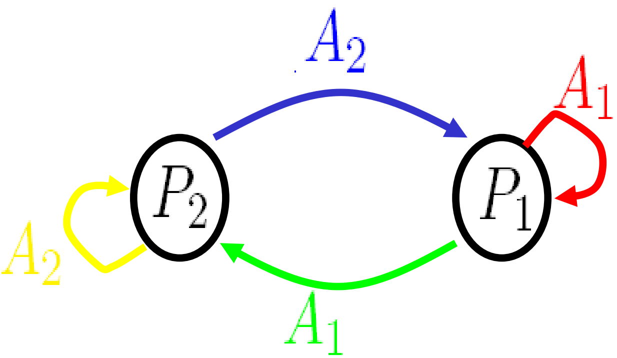

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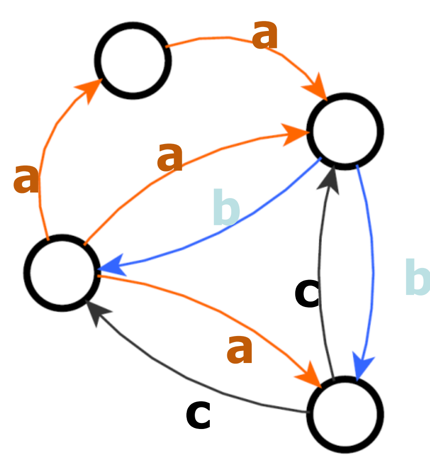
Constrained switching sequences

Switching sequences on regular languages

$G(V,E)$ Directed & Labeled $\quad e=(v_i,v_j,k)\in E \quad k\in\{1,\cdots,N\}$

$\sigma(1),\sigma(2),\cdots$ **admissible** if $\exists p=\{(v_i,v_j,\sigma(1)),\,(v_j,v_\ell,\sigma(2)),\,\cdots\}$



Constrained switching sequences

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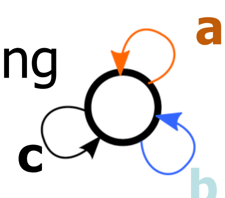
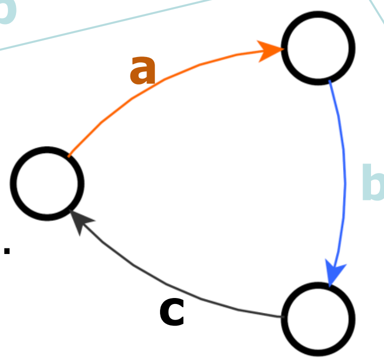
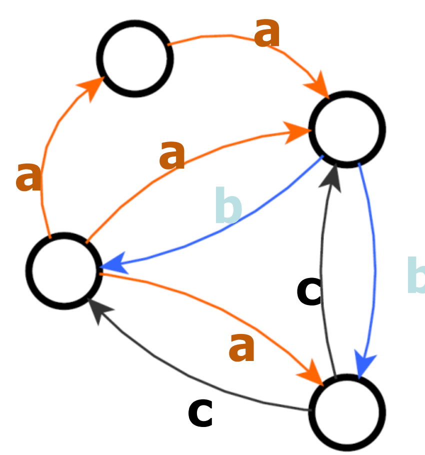
everything





... abcabcabc ...

~~... ac ...~~



~~... bb ...~~

~~... cc ...~~

... aab

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Constrained switching sequences

Switching sequences on regular languages

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$\sigma(1),\sigma(2),\cdots$ **admissible** if $\exists p=\{(v_i,v_j,\sigma(1)),\ (v_j,v_\ell,\sigma(2)),\ \cdots\}$

Stability

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} A_{\sigma(t-1)}\cdot\ldots\cdot A_{\sigma(0)}x_0 = 0$$
$$\forall x_0\in\mathbb{R}^n,\ \forall\,\sigma(0),\sigma(1),\cdots\in G$$

Theorem:

$\rho(G(V,E),\ M) < 1/\sqrt{n} \Rightarrow$ The system admits a Quadratic Lyapunov Multinorm

[Philippe J. 2014]

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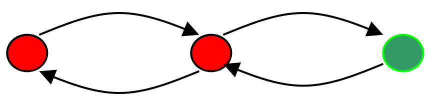
Trackable graphs

To a given observation, associate the corresponding product:



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$$A_r A_g A_r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The number of possible trajectories is given by the sum of the entries of the matrix

Trackable graphs

The maximal total number of possibilities is

$$N(t) = \max\left\{ \left\| A \right\|_1 \ : A \in \Sigma^t \right\}$$

We are interested in the asymptotic worst case :

$$\lim_{t \to \infty} N(t)^{1/t} = \lim_{t \to \infty} \max\left\{ \left\| A \right\|_1^{1/t} : A \in \Sigma^t \right\}$$

This is a joint spectral radius!

Trackable graphs

The network is trackable iff

$$\rho \leq 1$$

[Crespi et al. 05]

Theorem It is possible to check trackability in polynomial time

[J. Protasov Blondel 08]

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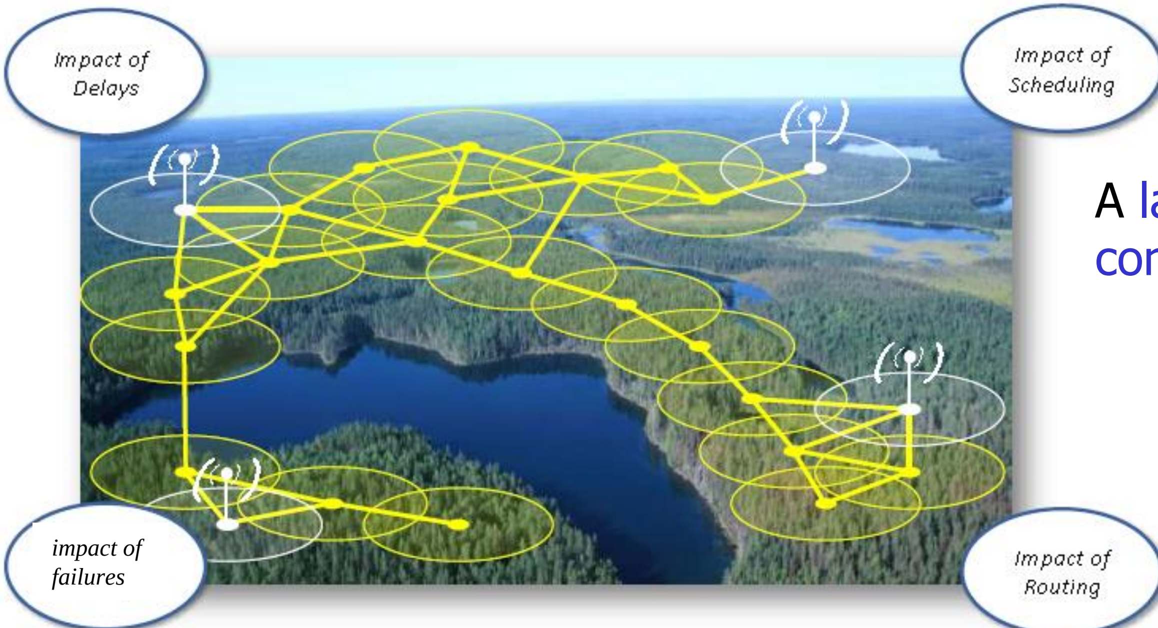
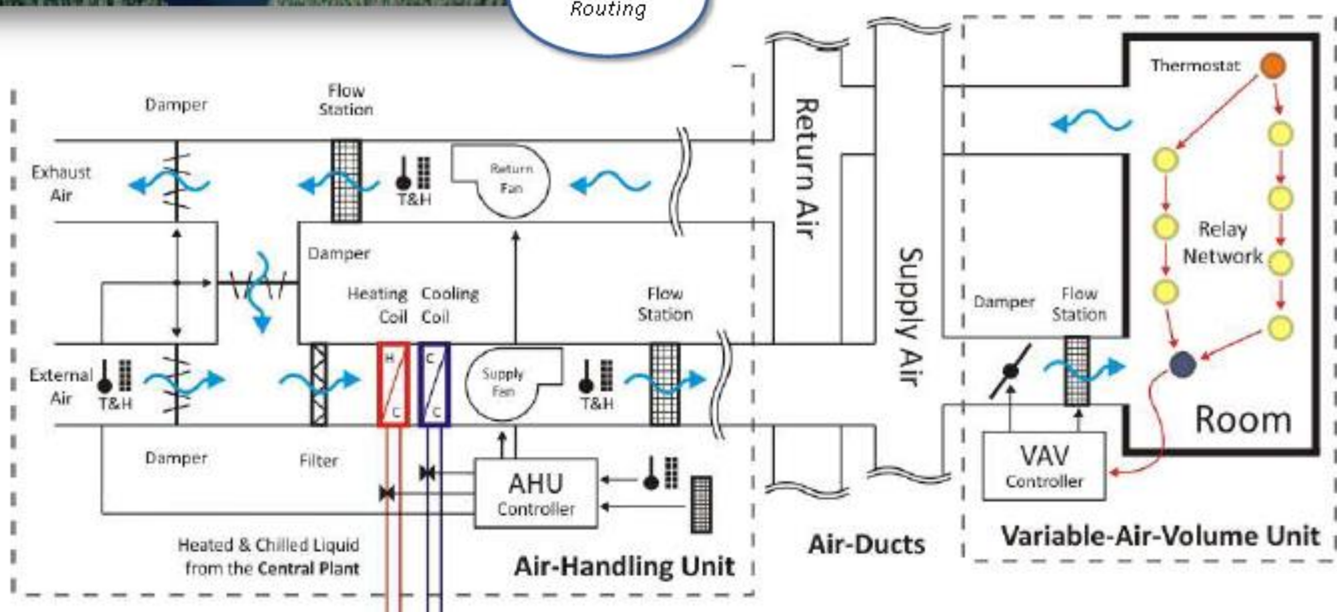
Wireless control networks



A large scale decentralized control network

A green building



[Ramanathan Rosales-Hain 00]
[alur D'Innocenzo Johansson Pappas Weiss 10]
[Heemels Teel Van de Wouw Nesic 10]
[Mazo Tabuada 10]
[Zhu Yuan Song Han Başar 12]
[Gommans Heemels Bauer Van de Wouw 13]…

Applications of Wireless Control Networks

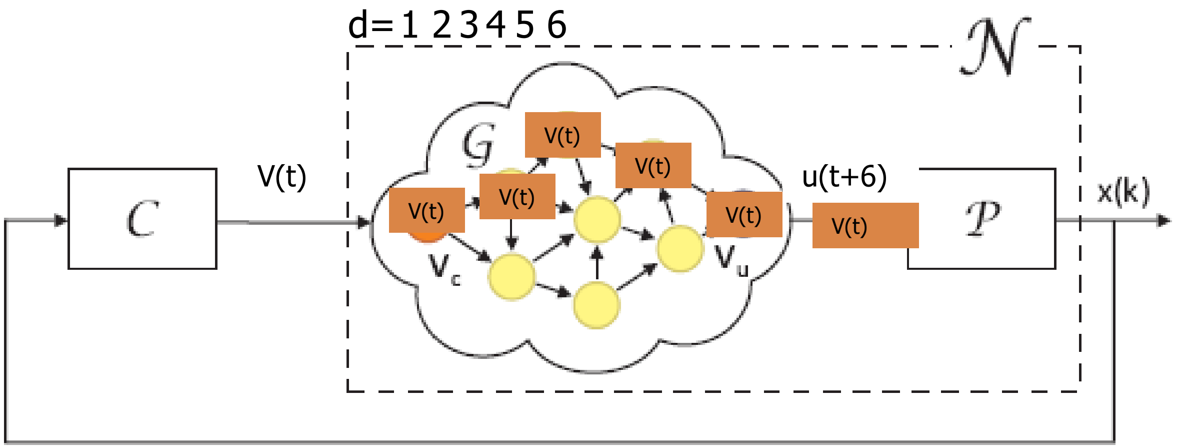
Industrial automation

Physical Security and Control

Supply Chain and Asset Management

Environmental Monitoring, Disaster Recovery and Preventive Conservation

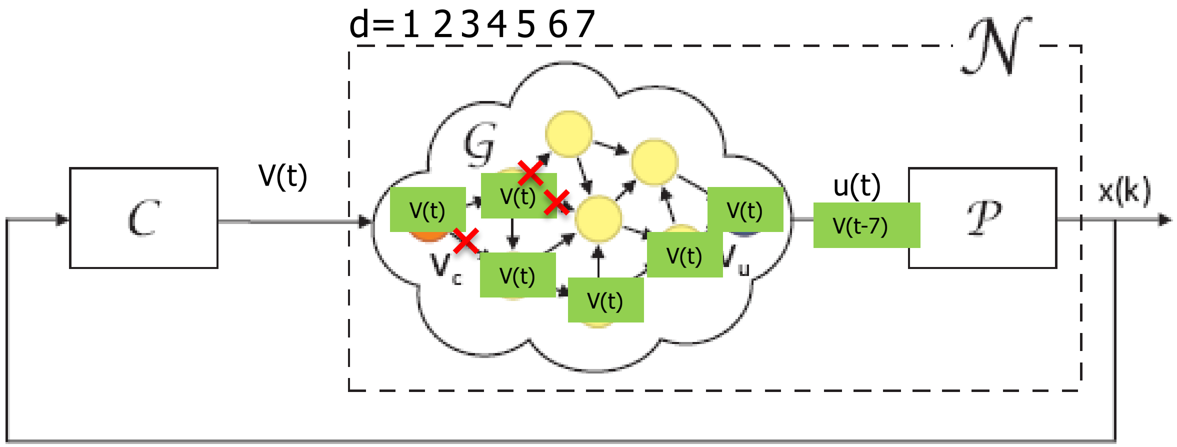
How to model failures?



WCNs are delay systems:

$$x(t+1) = Ax + B\,\mathrm{v}(t-d)$$

How to model failures?



WCNs are systems with switching delays :

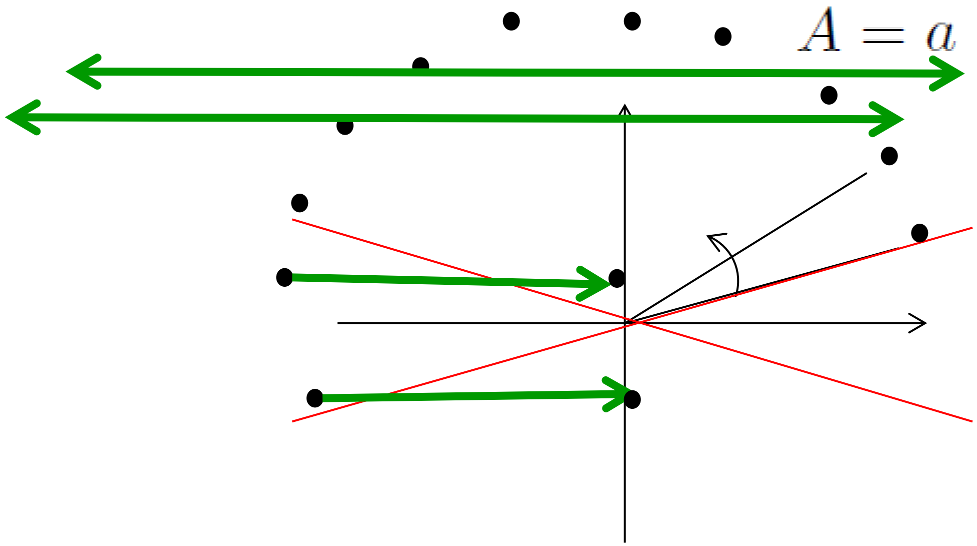
$$x(t+1) = Ax + B\,\mathrm{v}(t - d_2)$$

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$$

$$D = \{0, 1\}$$

LTIs with switched delays
Example

A 2D system with two possible delays



$$A = a \begin{pmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(alpha) \end{pmatrix}$$

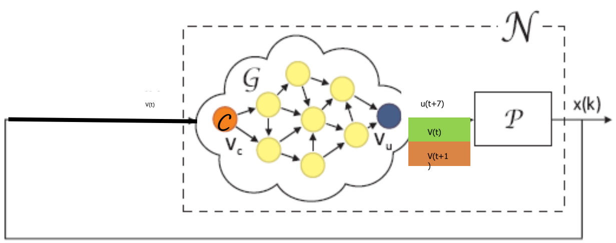
$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$D = \{0, 1\}$$

- **Theorem:** For the above system, there exist values of the parameters such that no linear controller can stabilize the system, but a nonlinear bang-bang controller does the job. [J. D'Innocenzo Di Benedetto 2014]

LTIs with switched delays stability analysis

Delay dependent controller



$$v(t) = K(d)\tilde{u}(t)$$

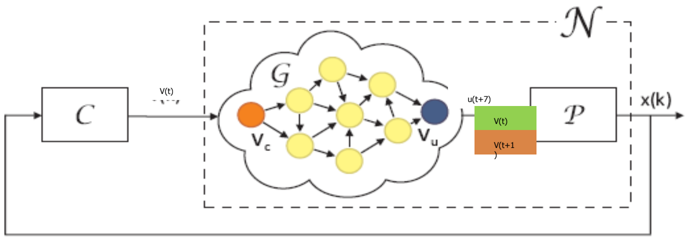
$$\tilde{u}(t) =$$

$$(x(t), u_1(t), u_2(t), \ldots, u_{d_{max}}(t))$$

$$\Sigma = \left\{ \begin{pmatrix} A & B & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ldots & I \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ K(d) \\ \vdots \\ 0 \end{pmatrix} \right\}$$

[Hetel Daafouz Iung 07]
[Weiss et al. 09]

Delay independent controller



$$v(t) = K\tilde{v}(t),$$

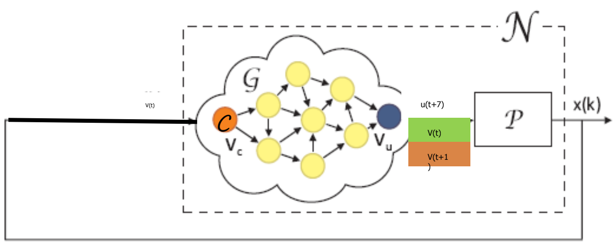
$$\tilde{v}(t) =$$

$$(x(t), v(t - d_{max}), \ldots, v(t-1)))$$

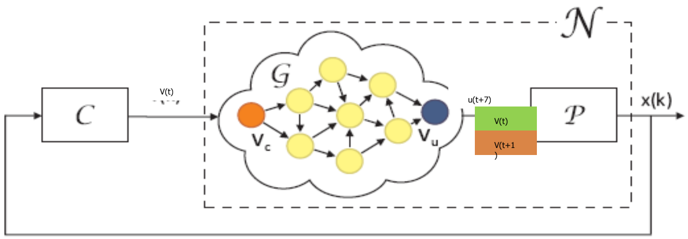
$$\Sigma = \left\{ \begin{pmatrix} A & 0 & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ K_0 & K_1 & K_2 & \ldots & K_{d_{max}} \end{pmatrix} \right\}$$

LTIs with switched delays stability analysis

Delay dependent controller



Delay independent controller



- Corollary

  For both models there is a PTAS for the stability question:

  for any required accuracy, there is a polynomial-time algorithm for checking stability up to this accuracy

Previous sufficient conditions for stability in [Hetel Daafouz Iung 07, Zhang Shi Basin 08]

- However:

  Theorem the very stability problem is NP-hard

  Theorem the boundedness problem is even Turing-undecidable!

[J. D'Innocenzo Di Benedetto 12]

Outline

- Joint spectral characteristics
- Automatic methods for switching systems stability
- Applications:
  - Trackable graphs
  - WCNs and switching delays
  - Consensus problems
- Conclusion and perspectives

Consensus systems

- Agents trying to agree on a common value

- **Applications:** control of vehicles formations, distributed computing etc.

- **Update** as weighted average:

$$x_i(t+1) = \sum_j a_{ij}(t) x_j(t)$$

with $\sum_j a_{ij}(t) = 1$ and $a_{ij}(t) \geq 0$

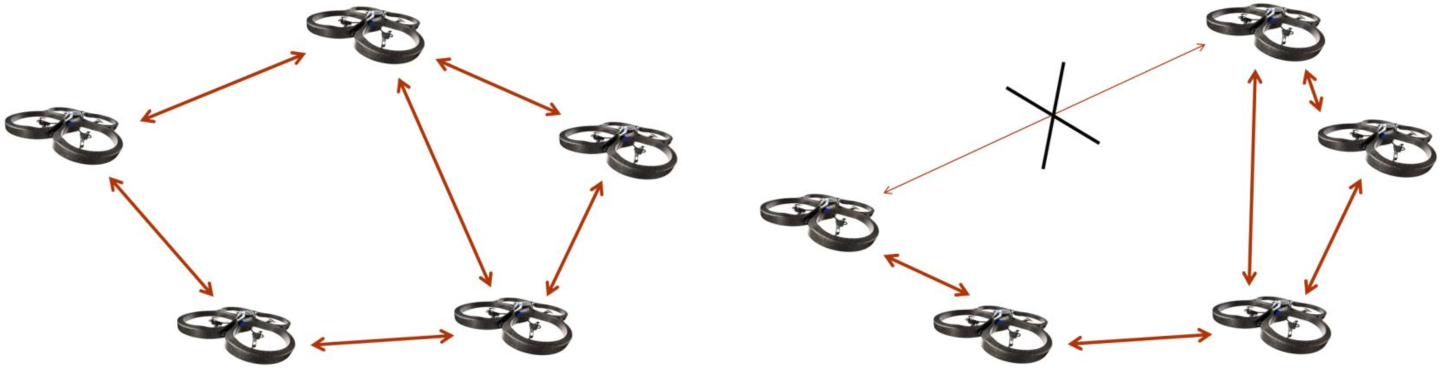
- **Question:** Convergence to consensus (multiple of $\mathbf{1} = (1 \quad \cdots \quad 1)^T$)?

Consensus systems

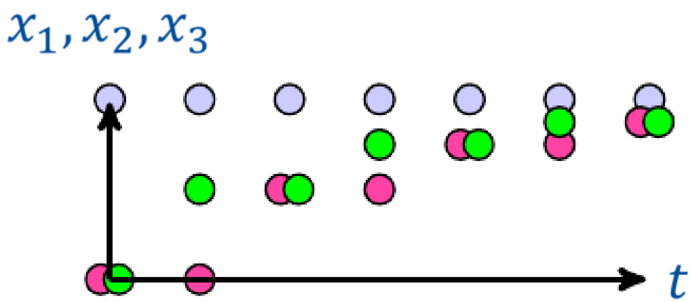
- $A(t)$ changes with time
- **Assumption:** set of possible transition matrices is known
  $S = \{A_1, \dots, A_m\}$

$$x(t+1) = A_{\sigma(t)} x(t), \qquad x(0) = x_0$$

- A_i stochastic
- σ sequence of transition matrices

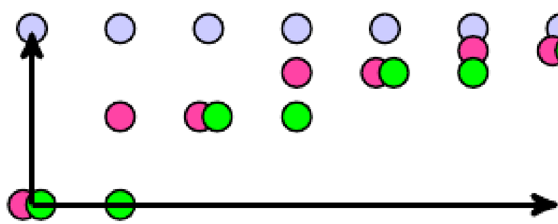


Switching can harm convergence

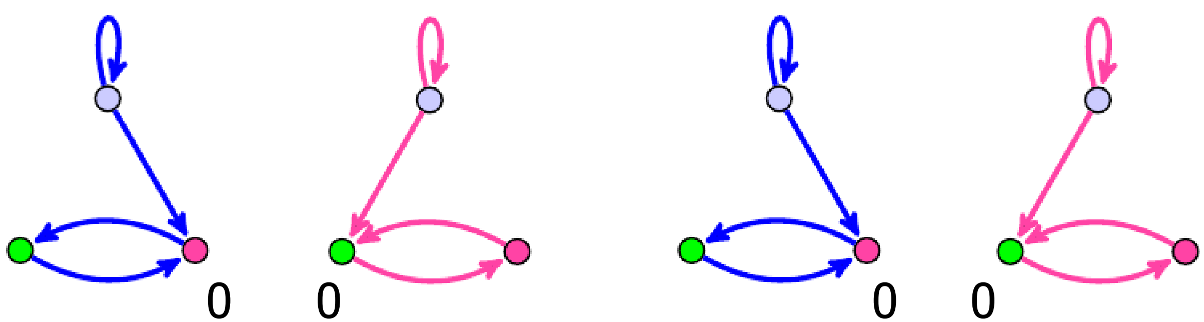


$\sigma = 1, 1, 1, \ldots$

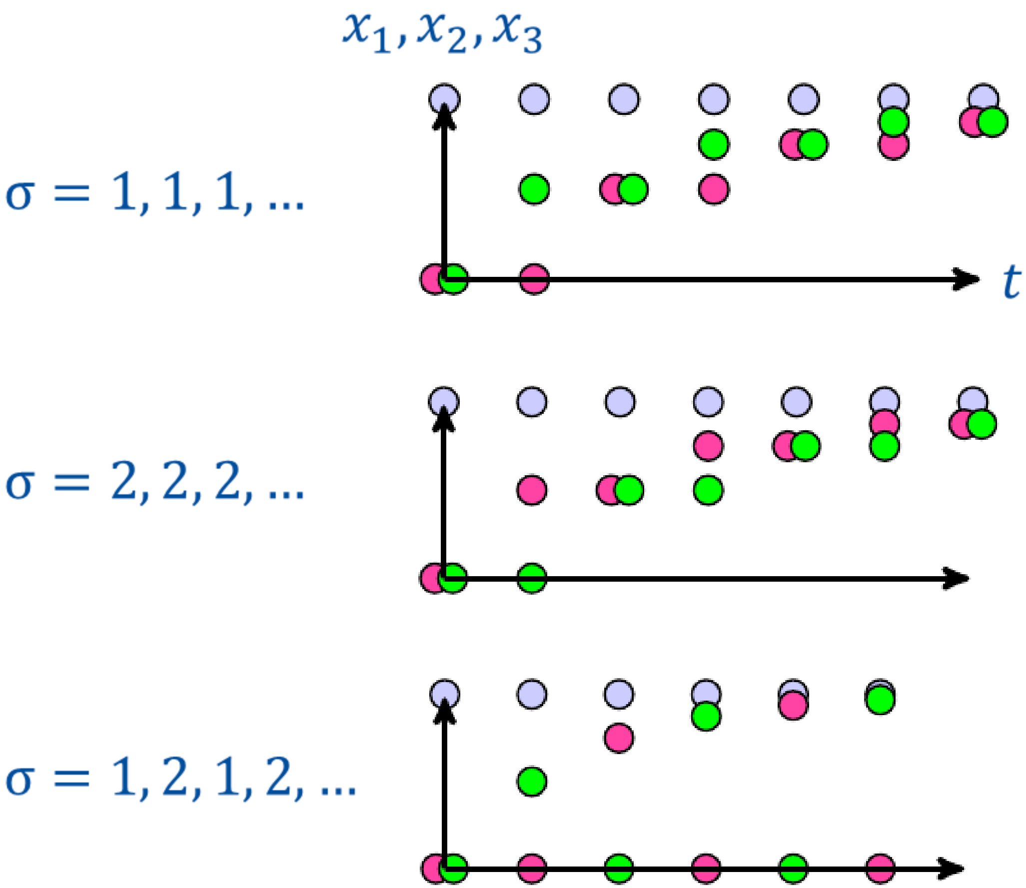
$\sigma = 2, 2, 2, \ldots$



$\sigma = 1, 2, 1, 2, \ldots$



Switching can harm convergence



Two decision problems

- **Problem 1** (stability): Given set S, does system converge to consensus for any σ, x_0?

Goal:

Algorithm:

Input: S

Output: "Yes" if system converges for any x_0, σ

"No" otherwise

- **Problem 2** (controllability): Given a set S, does there exist, for any x_0, a sequence σ such that the system converges to consensus?

Two decision problems

Previous results (only on Problem 1):

- Decidable: there exists an algorithm (doubly exponential complexity:O(m^{3^n}))
- NP-Hard

V. D. Blondel, A. Olshevsky, *How to decide consensus? A combinatorial necessary and sufficient condition and a proof that consensus is decidable but NP-hard*, to appear in SICON.

- Problem 1 reduces to a joint spectral radius computation! [Jadbabaie Lin Morse 2003]

Our results:

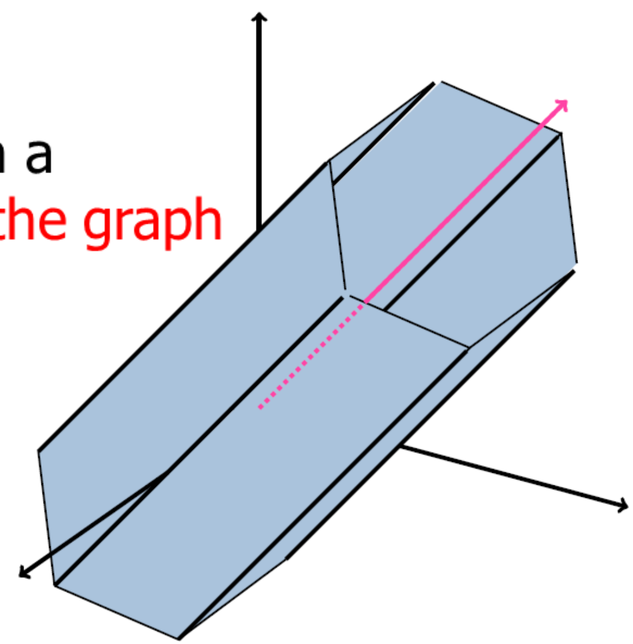
- The first singly exponential algorithm for problem 1
- First algorithm for Problem 2

Joint spectral characteristics of stochastic matrices

Property: Any consensus state is an equilibrium, and $P = \left\{ x \mid \max_i x_i - \min_i x_i \leq 2 \right\}$ is an invariant polyhedron

Theorem: [Lagarias Wang 95] If jsr=1 and there is an invariant polyhedron, every open face is mapped in an open face

Corollary: We can represent the (non)-convergence to consensus on a purely combinatorial, finite object: the graph of faces



Algorithms for problems 1 and 2

Theorem 0: The graph of faces is constructible in O$(|E| + |V|)$

Theorem 1: Problem 1 (stability) $\equiv$ Is graph of faces acyclic (other that the int(P) self-loop)?

Theorem 2: Problem 2 (controllability) ≡ Is there a path in the graph of faces from any node to int(P)

These problems are easy: O$(|E| + |V|)$

Construction of the graph dominates complexity

[Chevalier Hendrickx J. 2014]

Outline

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- Applications:
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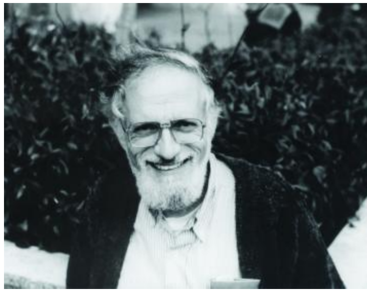
Conclusion




[Furstenberg Kesten, 1960]

[Gurvits, 1995]



[Rota, Strang, 1960]


[Blondel Tsitsiklis, 98+]

[Daafouz Bernussou 03]

[Johansson Rantzer 98]

[Lee Dullerud 06]

60s 70s

Mathematical properties

90s

TCS inspired
Negative
Complexity results

2000s

Lyapunov/LMI
Techniques
(S-procedure)

now

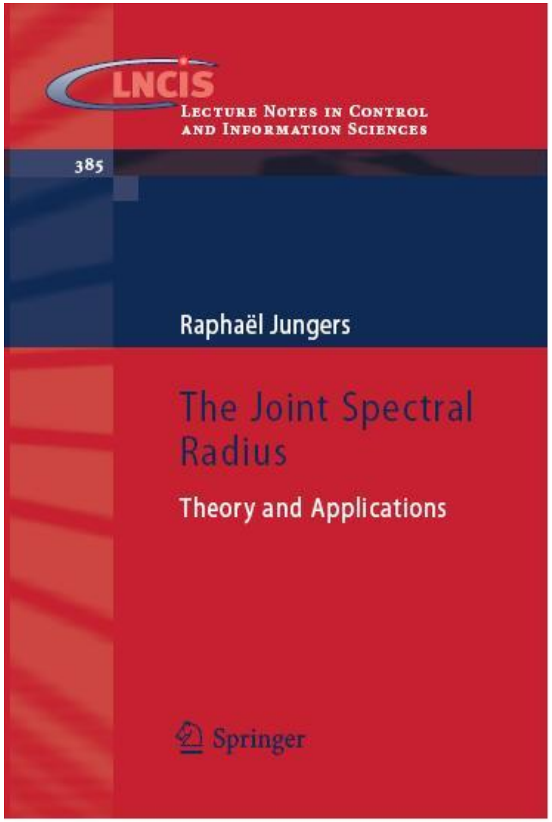
CPS applic.
Ad hoc
techniques

Thanks!

Questions?

Ads

[The JSR Toolbox: http://www.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox](http://www.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox)
[Van Keerberghen, Hendrickx, J. HSCC 2014]



Several open positions:
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References:
http://perso.uclouvain.be/raphael.jungers/

Joint work with

A.A. Ahmadi (Princeton), M-D di Benedetto (l'Aquila), V. Blondel (UCLouvain), P-Y Chevalier (UCLouvain), J. Hendrickx (UCLouvain) A. D'innocenzo (l'Aquila), M. Ogura (UPenn), P. Parrilo (MIT), M. Philippe (UCLouvain), V. Protasov (Moscow), M. Roozbehani (MIT)

Design of LTIs with switched delays

The infinite look-ahead case

- Theorem for n=m=1, there is an explicit formula for a linear controller that achieves deadbeat stabilization, even if N=1

  (based on a generalization of the Ackermann formula for delayed LTI)

$$K^*(d) = (-a^{d+1}/b, -a^d, -a^{d-1}, \ldots, -a)$$

- So, does a controllable system always remain controllable with delays?

- No! when n>1, nastier things can happen…

  Example:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$$

$$D = \{0,1\}, \quad \sigma(t) = t \bmod 2$$

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x_1 = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_2 = A^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Bv(1) + Bv(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v(1) + v(2) \end{pmatrix}$$

➔ The system is not stabilizable, even with infinite lookahead

Design of LTIs with switched delays
The infinite look-ahead case

- A sufficient condition for uncontrollability (informal): if A,B can be put in the following form (under similarity transformation):

$$A = \left(\begin{array}{c|c|c} 0 & X & 0 \\ \hline 0 & 0 & X \\ \hline X & 0 & 0 \end{array}\right) \qquad B = \left(\begin{array}{c} X \\ \hline 0 \\ \hline 0 \end{array}\right)$$

An adversary strategy can make this system uncontrollable:

$$\forall t, t + d(t) \neq 1 \mod 3$$

Is it also necessary?

Would be nice, because we can prove …

- Theorem There is a polynomial time algorithm that decides whether such an adversary strategy is possible

Design of LTIs with switched delays
The infinite look-ahead case

- Answer: No! There are more intricate examples

$$A = \begin{pmatrix} \sin\theta_1 & -\cos\theta_1 & 0 & 0 \\ \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ 0 & 0 & \sin\theta_2 & -\cos\theta_2 \\ 0 & 0 & \cos\theta_2 & \sin\theta_2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$D = \{0, 1, \ldots, 121\} \qquad \theta_1 = \tfrac{\pi}{120} \qquad \theta_2 = \tfrac{\pi}{60}$$

$$\sigma(t) = \begin{cases} 0 & if \;\; 0 \le t \le 2 \\ 121 - t \; mod(121) & if \;\; t \ge 3 \end{cases}$$

Conclusion and perspectives

- Many open questions and Conjectures:
  - For computer scientists:

  Is '$\rho<1$' decidable? [Blondel Megretski 05]

  Write protocols for optimal control of computer networks

  - For mathematicians:

  The finiteness conjecture

  [Lagarias, Wang 95], [Bousch Mairesse 02] [J. Blondel 08]

  - For control theorists:

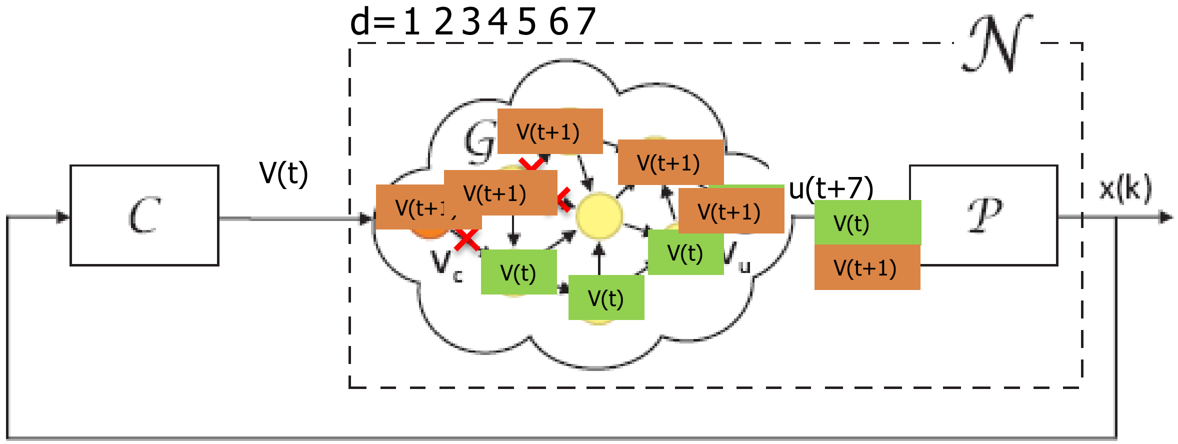
  What are the best path-complete graphs and why?

  Can we apply these path-complete methods to more general hybrid systems? (à la [Johansson Rantzer 98])

  How to design and control switching systems?

- Meta-conclusion: Is switching systems theory useful for modern CPS engineering?

How to model failures?
LTIs with switched delays



$$x(t+1) = Ax(t) + Bu(v(t-d_{max} : t), \sigma(t-d_{max} : t))$$

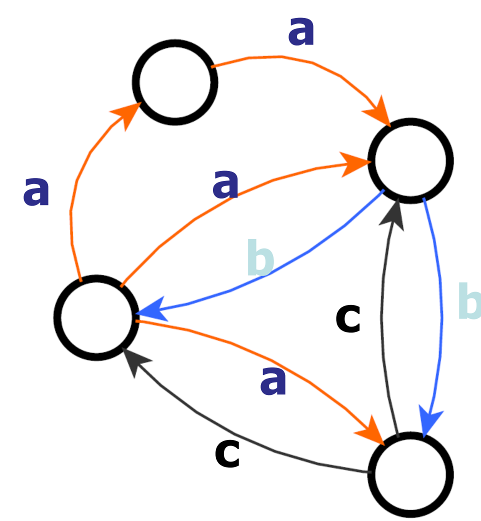
$D = \{d_1, \ldots, d_{|D|}\}$ Is the set of possible delays

$$x(t+1) = Ax + Bu(t - d_2)$$

d_{max} Is the maximal delay

Constrained switching sequences

$$x_{t+1} = A_{\sigma(t)} x_t$$
$$x_0 \in \mathbb{R}^n$$
$$\sigma(0), \sigma(1), \cdots \in G$$
$$A_{\sigma(t)} \in M \subset \mathbb{R}^{n \times n}$$



- Constrained Joint Spectral Radius [X. Dai 2012]

$$\rho(G, M) = \lim_{t\to\infty} \sup_{\sigma(\cdot)\in G} \{ \|A_{\sigma(t-1)} \cdot \ldots \cdot A_{\sigma(0)}\|^{1/t} \}$$

Stability and CJSR [X. Dai 2012 – Corr. 2.8]

$$\rho(G, M) < 1 \Leftrightarrow \begin{cases} \lim_{t\to\infty} x_t = \lim_{t\to\infty} A_{\sigma(t-1)} \cdot \ldots \cdot A_{\sigma(0)} x_0 = 0 \\ \forall [\sigma(0), \sigma(1), \sigma(2), \cdots] \in G \end{cases}$$

$$|x_t| \leq C \rho(G, M)^t$$

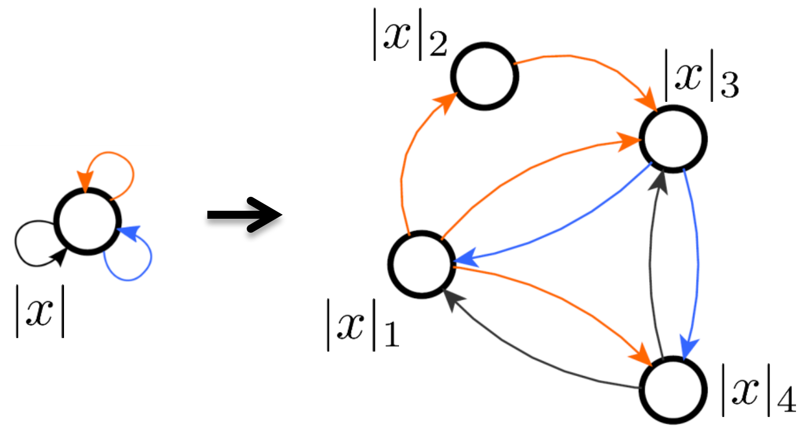
Constrained switching and multinorms

- CJSR as an infimum over sets of norms

$$\rho(G(V,E),\, M) = \inf_{{\color{red}|\cdot|_1, \cdots, |\cdot|_{|V|}}} \gamma$$

$$|A_k x|_j \leq \gamma |x|_i,$$

$$\forall (v_i, v_j, k) \in E,\ \forall x \in \mathbb{R}^n$$



Theorem: Stability iff there exists a multiple Lyapunov function

$$\rho(G(V,E),\, M) < 1 \Leftrightarrow \begin{cases} \exists |\cdot|_i,\ 1 \leq i \leq |V| \\ \qquad |A_k x|_j < |x|_i, \\ \qquad \forall (v_i, v_j, k) \in E,\ \forall x \in \mathbb{R}^n \end{cases}$$

[Philippe J. 2014]

Generalizes Path Complete Lyap Func.

Quadratic multinorms

- How to decide when a system is stable?

$$\exists\,?\,|\cdot|_i,\ 1 \le i \le |V|$$
$$|A_k x|_j < |x|_i,$$
$$\forall (v_i, v_j, k) \in E,\ \forall x \in \mathbb{R}^n$$

+ Necessary and Sufficient ☺

− Infinite dimensional program …

Quadratic multinorms

- How to decide when a system is stable?

$$\exists ? \, |\cdot|_i, \; 1 \leq i \leq |V|$$
$$|A_k x|_j < |x|_i,$$
$$\forall (v_i, v_j, k) \in E, \; \forall x \in \mathbb{R}^n$$

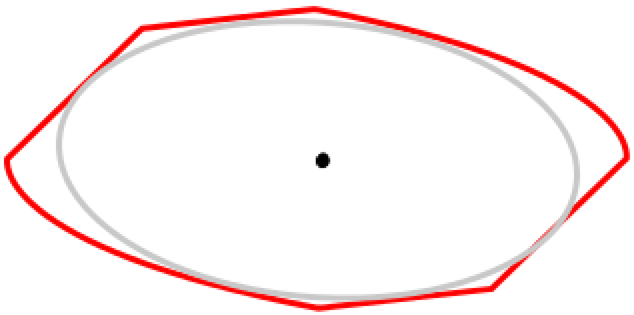
+ Necessary and Sufficient ☺

− Infinite dimensional program …

- Quadratic Approximation of Lyapunov functions

+ Quadratic is “easy” (LMIs)

$$|x|_i \simeq (x' Q_i x)^{1/2}$$



− Conservative,
Sufficient condition.

[Daafouz & Bernussou, 2001] – Param. Var. systs.
[Daafouz, Riedinger, Iung, 2002] – Switched Lyapunov
[Lee & Dullerud, 2006] – Path Dependant Lyap Functions

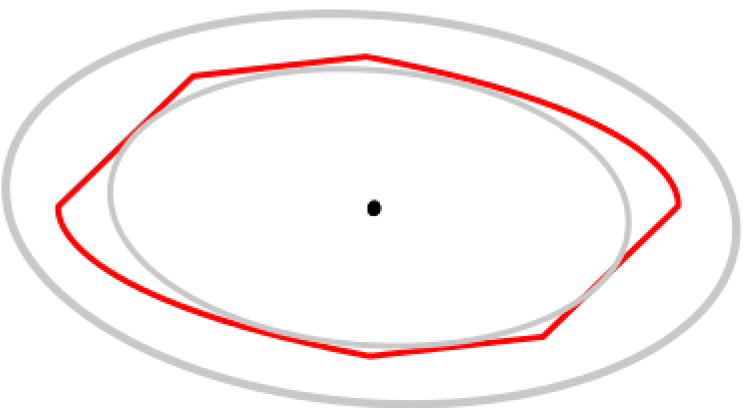
Converse Lyapunov Theorem

Theorem:

$$\rho(G(V,E),\, M) < 1/\sqrt{n} \Rightarrow \left\{ \begin{array}{l} \exists Q_i,\ 1 \leq i \leq |V| \\ \qquad A_k' Q_j A_k - Q_i \prec 0, \\ \qquad \forall (v_i, v_j, k) \in E \end{array} \right.$$

[Philippe J. 2014]

Generalizes the result of [Ando & Shih , 1998] for arbitrary switching systems



John's Ellipsoid Theorem

$$(x'Q_i x)^{1/2} \leq |x|_i \leq \sqrt{n}(x'Q_i x)^{1/2}$$

Requirement becomes heavy as n grows!

Hierarchy of converse Lyapunov Theorems

- Application of previous results on augmented systems

$$\rho(G_T, M^T) < 1/\sqrt{n} \Rightarrow S(G_T, M^T)$$ admits a Quadratic Lyapunov Multinorm

Theorem:

$$\rho(G,\, M) < n^{-1/2T} \Rightarrow S(G_T, M^T)$$ admits a Quadratic Lyapunov Multinorm

[Philippe J. 2014]

Stable system : there exist *T* ...