Joint spectral characteristics

A tale of three disciplines

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LIDS, MIT,
February 2013.

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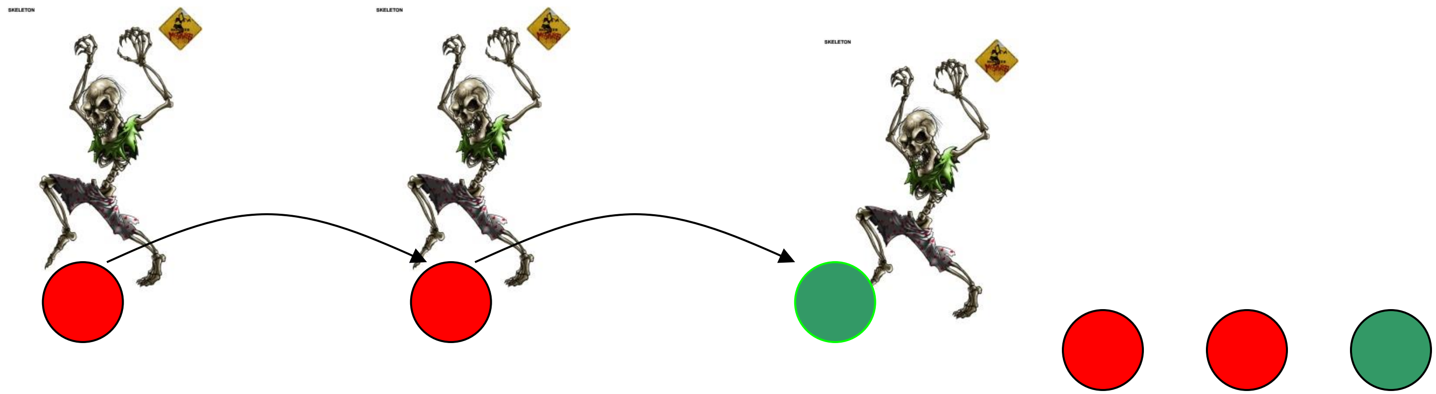
Trackable graphs



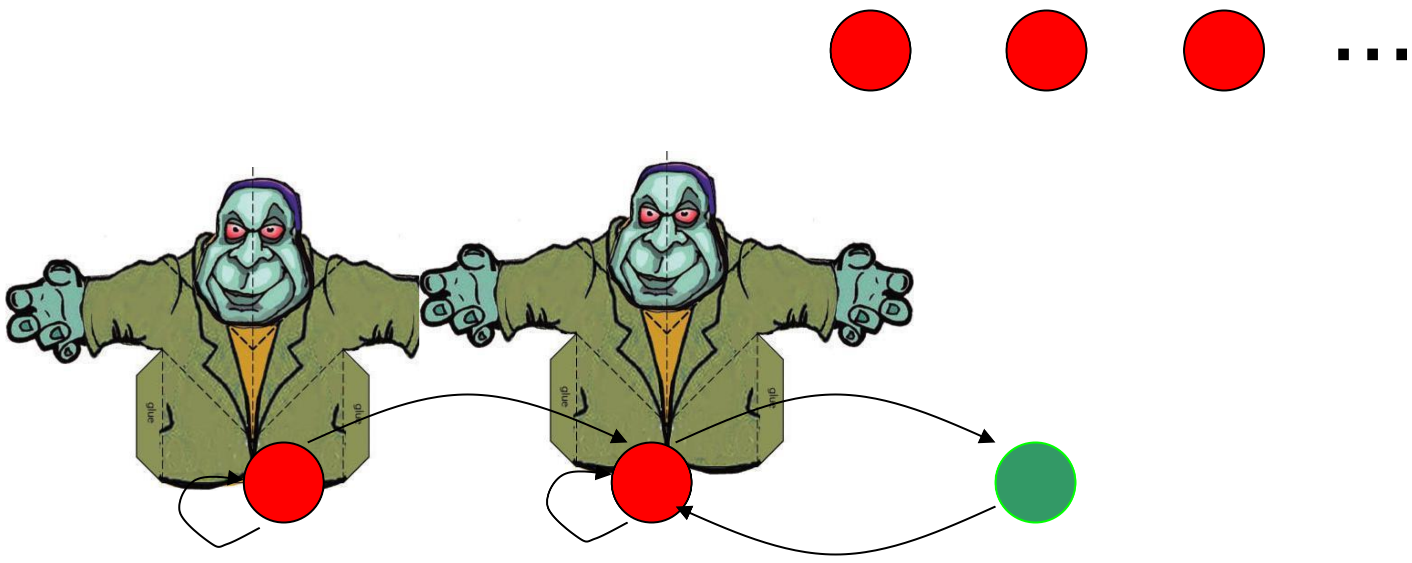
Let $N(t)$ be te worst possible number of trajectories compatible with an observation of length t
A network is trackable if $N(t)$ grows subexponentially

[Crespi et al. 05]

Here: number of possibilities asymptotically zero $\quad N(t) \approx 0$

➔ Trackable

Trackable graphs

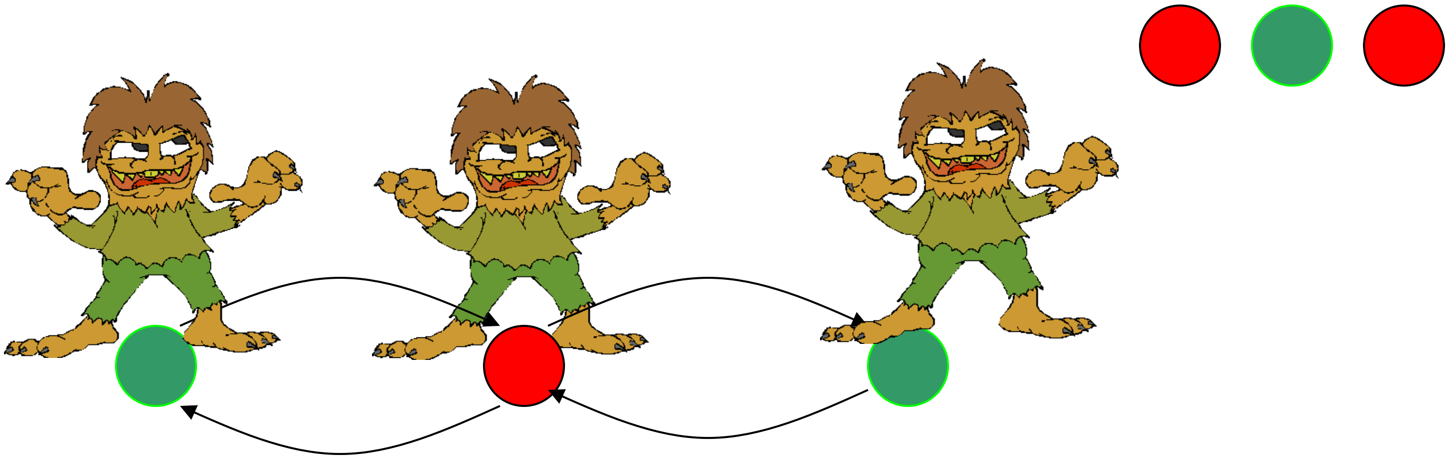


Worst case : RRRRRR… ➔ $N(t) \approx t$

Polynomial number of possibilities

➔ Trackable

Trackable graphs



Worst case : RGRGRG…➔

$$N(t) \approx 2^{t/2}$$

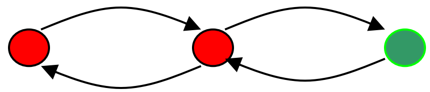
Exponential number of possibilities

➔ Not trackable

Trackability : the formal problem

We are given

- A graph G(V,E) :

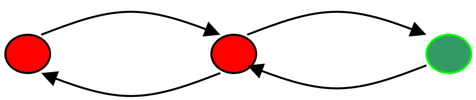


$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- A set of possible observations : defining a partition of the nodes

$$\begin{cases} R = \{1,2\} \\ G = \{3\} \end{cases}$$

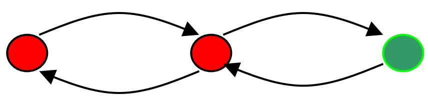
For each possible color, we define the corresponding matrix by erasing the incompatible columns from A :



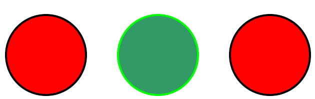
$$A_r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Trackability : the formal problem

To a given observation, associate the corresponding product:



$$A_r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



$$A_r A_g A_r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The number of possible trajectories is given by the sum of the entries of the matrix

Outline

- Matrix semigroups and some tools to analyze them
- LMI methods for switching systems stability
- Applications
- Conclusion and perspectives

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Switching systems

$$\mathbf{x_{t+1}} = \begin{matrix} \mathbf{A_0\, x_t} \\ \mathbf{A_1\, x_t} \end{matrix}$$

Point-to-point Given x_0 and x_*, is there a product (say, $A_0\ A_0\ A_1\ A_0 \ldots A_1$) for which $x_* = A_0\ A_0\ A_1\ A_0 \ldots A_1\ x_0$?

Mortality Is there a product that gives the zero matrix?

Boundedness Is the set of all products $\{A_0, A_1, A_0A_0, A_0A_1, \ldots\}$ bounded?

Switching systems

$$x_{t+1} = \begin{array}{l} A_0\, x_t \\ A_1\, x_t \end{array}$$

Global convergence to the origin Do all products of the type $A_0\, A_0\, A_1\, A_0 \ldots A_1$ converge to zero?

The spectral radius of a matrix A controls the growth or decay of powers of A

$$\rho(A) = \lim_{t \to \infty} ||A^t||^{1/t}$$

The powers of A converge to zero iff $\rho(A) < 1$

The joint spectral radius of $\Sigma = \{A_0, A_1\}$ is given by

$$\rho(\Sigma) = \lim_{t \to \infty} \max_{A_i \in \Sigma} ||A_1 A_2 \ldots A_t||^{1/t}$$

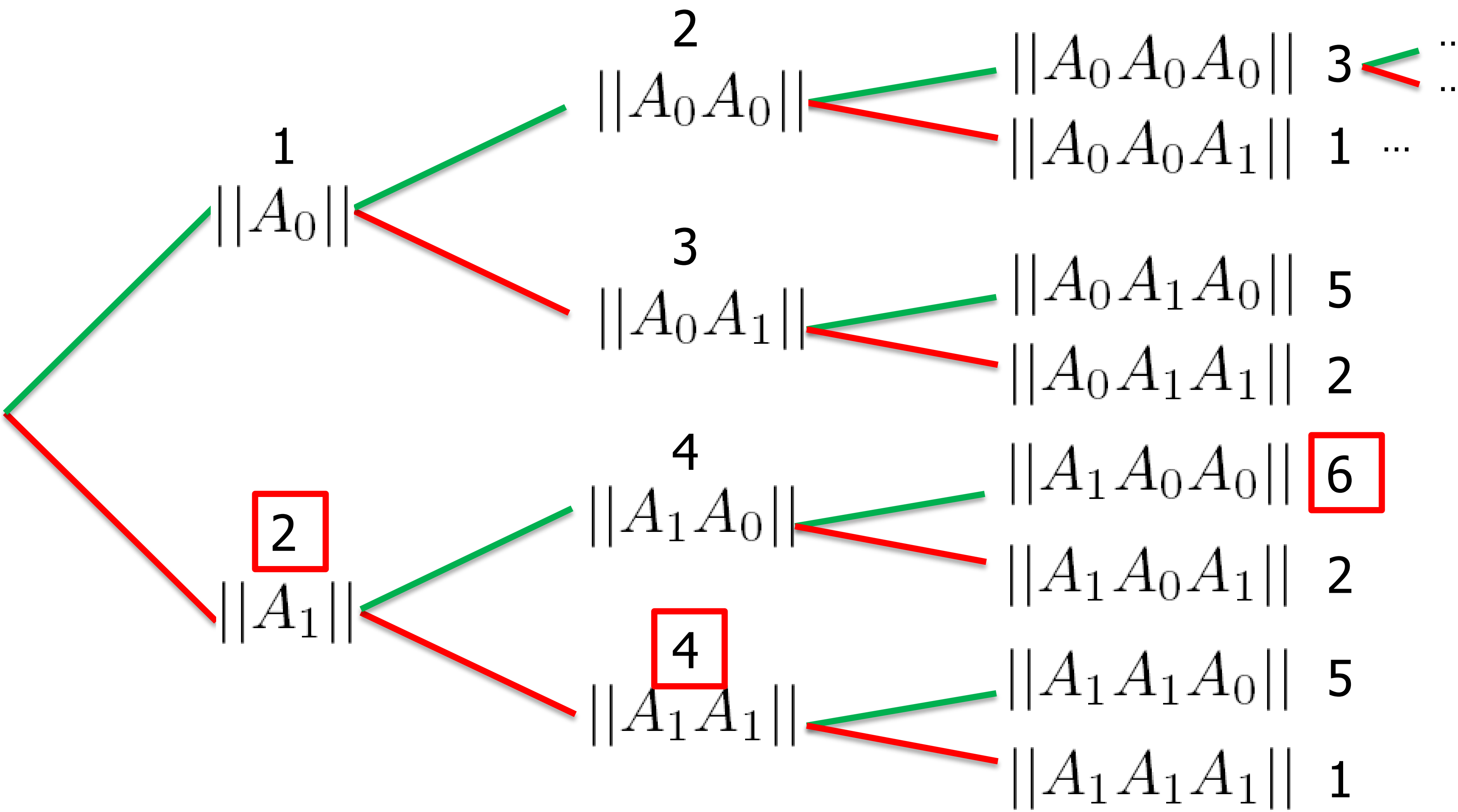
All products of A_0 and A_1 converge to zero iff $\rho(\Sigma) < 1$

[Rota, Strang, 1960]

The joint spectral characteristics

$$\boxed{\rho(\Sigma) = \lim_{t\to\infty} \left[\max_{A_i\in\Sigma} ||A_1 A_2 \ldots A_t||\right]^{1/t}}$$

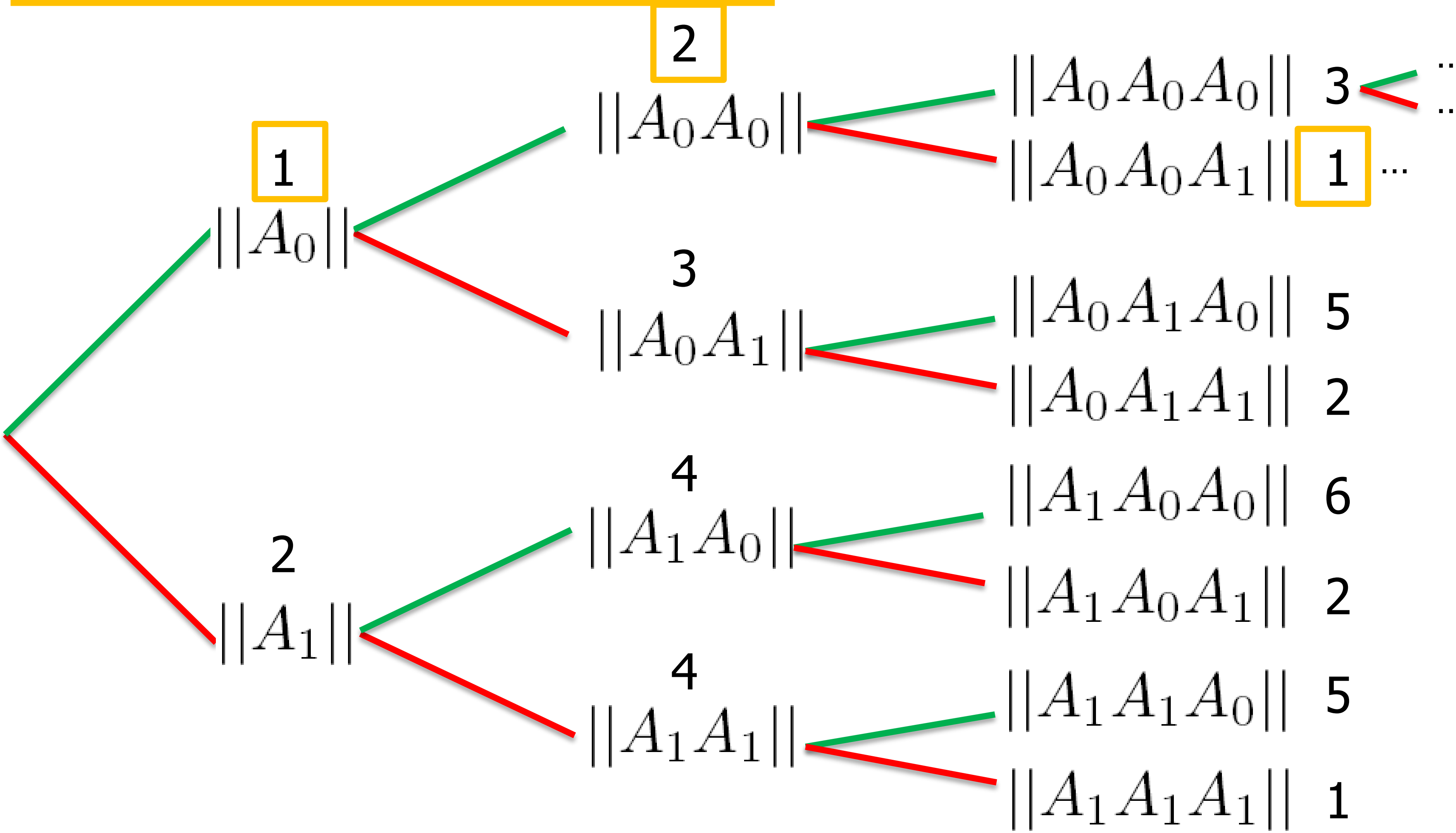
The joint spectral radius



The joint spectral characteristics

$$\check{\rho}(\Sigma) = \lim_{t\to\infty} \left[\min_{A_i \in \Sigma} ||A_1 A_2 \dots A_t|| \right]^{1/t}$$

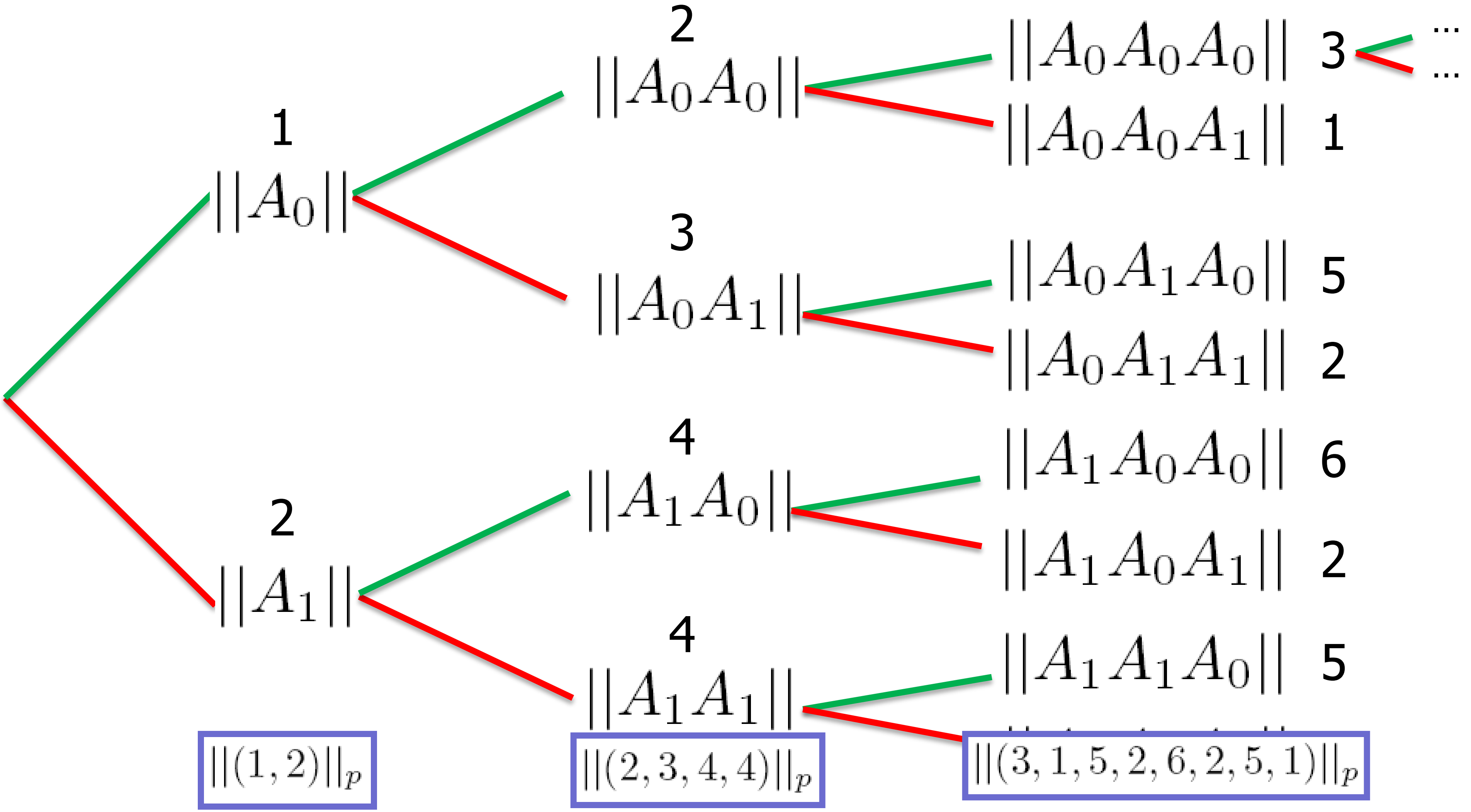
The joint spectral subradius



The joint spectral characteristics

$$\rho_p(\Sigma) = \lim_{t\to\infty} \left[\, m^{-t} \sum_{A_i \in \Sigma^t} \|A_1 A_2 \ldots A_t\|^p \, \right]^{1/(pt)}$$

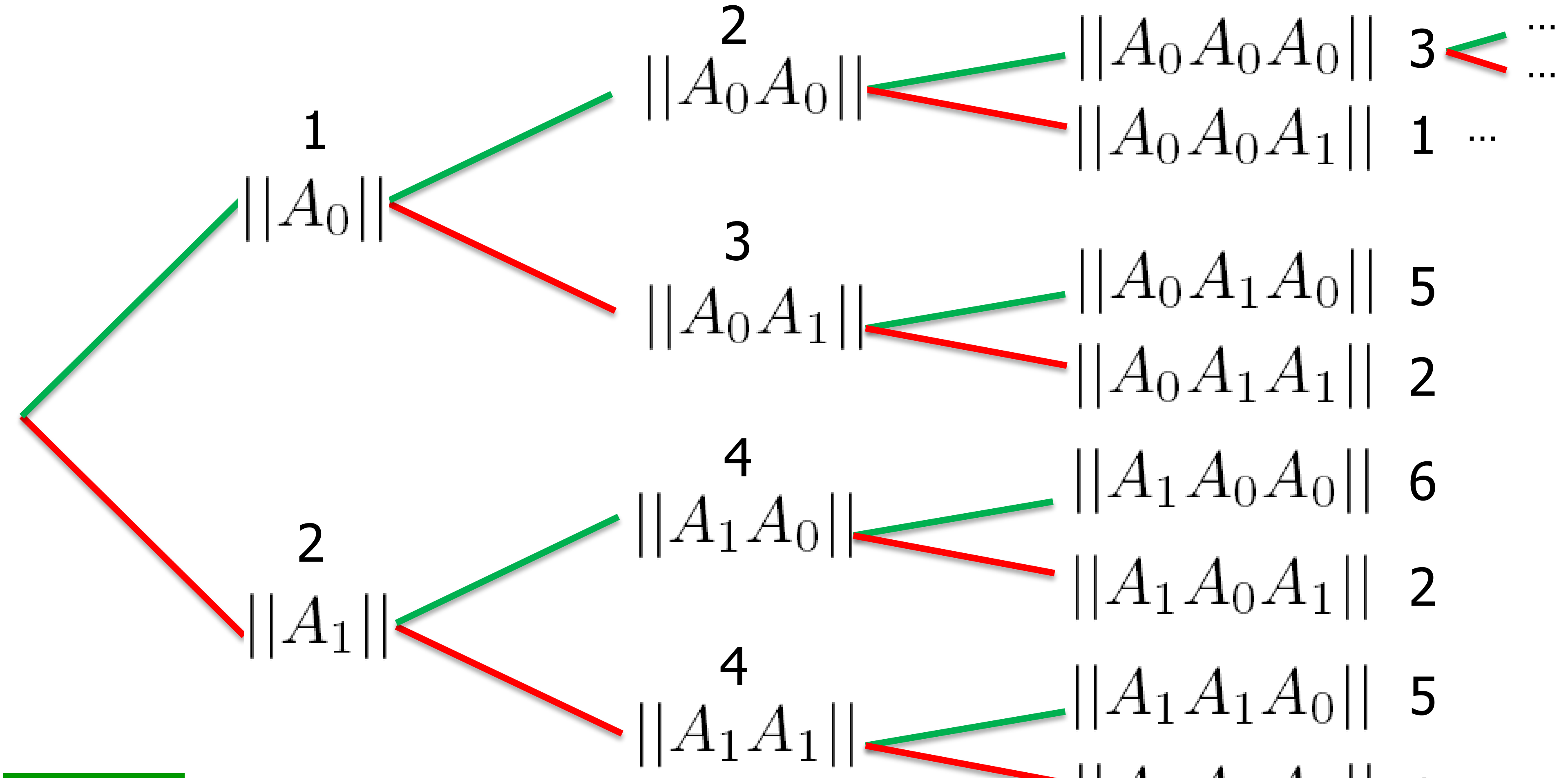
The p-radius



The joint spectral characteristics

$$\bar{\rho}(\Sigma) = \lim_{t\to\infty} \Big[\prod_{A_i \in \Sigma^t} \|A_1 A_2 \dots A_t\| \Big]^{1/(tm^t)}$$

The Lyapunov Exponent



$\boxed{p = 0}$ $\boxed{||(1,2)||_p}$ $\boxed{||(2,3,4,4)||_p}$ $\boxed{||(3,1,5,2,6,2,5,1)||_p}$

The joint spectral characteristics

$$\rho(\Sigma) = \lim_{t\to\infty} \left[\max_{A_i\in\Sigma} ||A_1 A_2 \dots A_t|| \right]^{1/t}$$

The joint spectral radius answers the **stability** problem

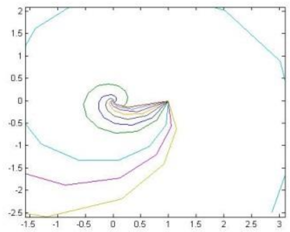
$$\check{\rho}(\Sigma) = \lim_{t\to\infty} \left[\min_{A_i\in\Sigma} ||A_1 A_2 \dots A_t|| \right]^{1/t}$$

The joint spectral subradius answers the **stabilizability** problem

$$\rho_p(\Sigma) = \lim_{t\to\infty} \left[m^{-t} \sum_{A_i\in\Sigma^t} \|A_1 A_2 \dots A_t\|^p \right]^{1/(pt)}$$

The p-radius addresses the… **p-weak stability**

[J. Protasov 10]



$$\bar{\rho}(\Sigma) = \lim_{t\to\infty} \left[\prod_{A_i\in\Sigma^t} \|A_1 A_2 \dots A_t\| \right]^{1/(tm^t)}$$

The Lyapunov exponent answers the **stability with probability one**

(Cfr. Oseledets Theorem)

The joint spectral characteristics: Mission Impossible?

Theorem Computing or approximating ρ is NP-hard

Theorem The problem $\rho \leq 1$ is algorithmically undecidable

Conjecture The problem $\rho < 1$ is algorithmically undecidable

Theorem Even the question « $|\check{\rho} - r| \leq a + b\check{\rho}$?» is algorithmically undecidable for all (nontrivial) a and b

Theorem The same is true for the Lyapunov exponent

Theorem The p-radius is NP-hard to approximate

See [Blondel Tsitsiklis 97, Blondel Tsitsiklis 00, J. Protasov 09]

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- LMI methods for switching systems stability
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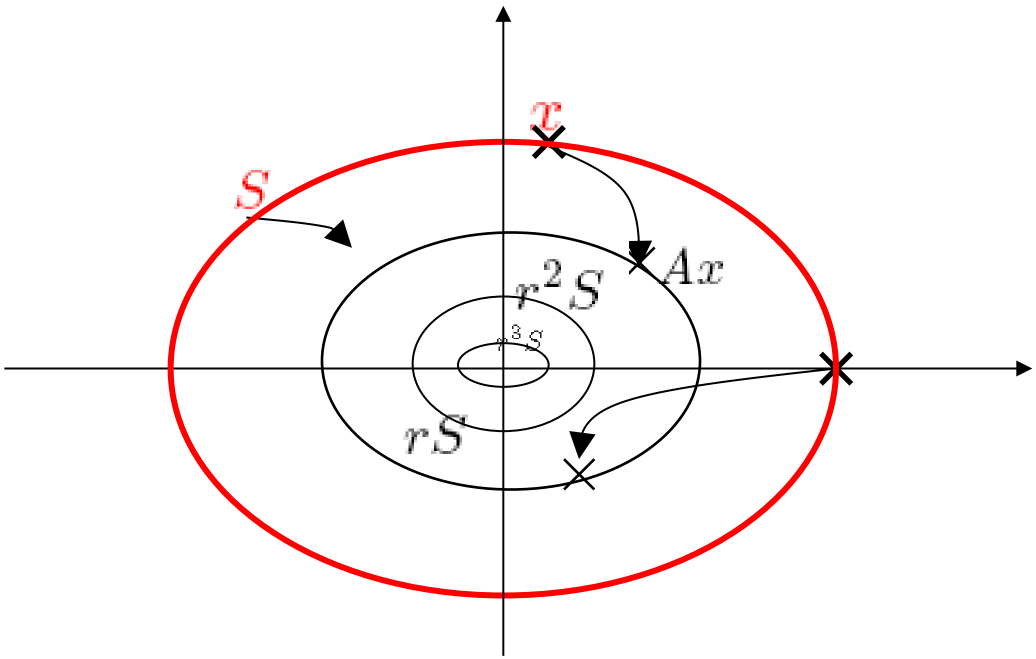
LMI methods

$$\boxed{\rho(\Sigma) = \lim_{t\to\infty} \left[\max_{A_i \in \Sigma} ||A_1 A_2 \dots A_t||\right]^{1/t}}$$

- Idea: look for a vector norm such that $\forall A \in \Sigma, \forall x : |x| \leq 1,\ |Ax| \leq r$



Boundedness (and $\rho \leq r$)

SDP methods

- Theorem For all $\epsilon > 0$ there exists a norm such that

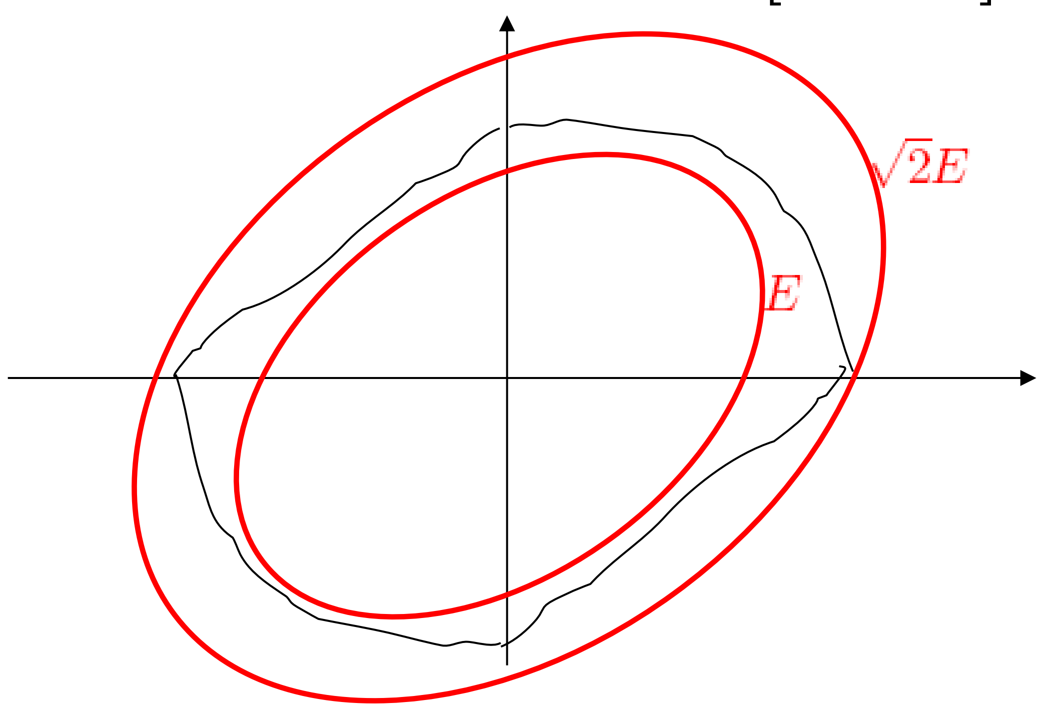
$$\forall A \in \Sigma, \forall x, |Ax| \leq (\rho + \epsilon)|x|$$

[Rota Strang, 60]

- John's ellipsoid Theorem: Let K be a compact convex set with nonempty interior symmetric about the origin. Then there is an ellipsoid E such that $E \subset K \subset \sqrt{n}E$

[John 1948]

- So we can approximate the unit ball of an extremal norm with an ellipsoid

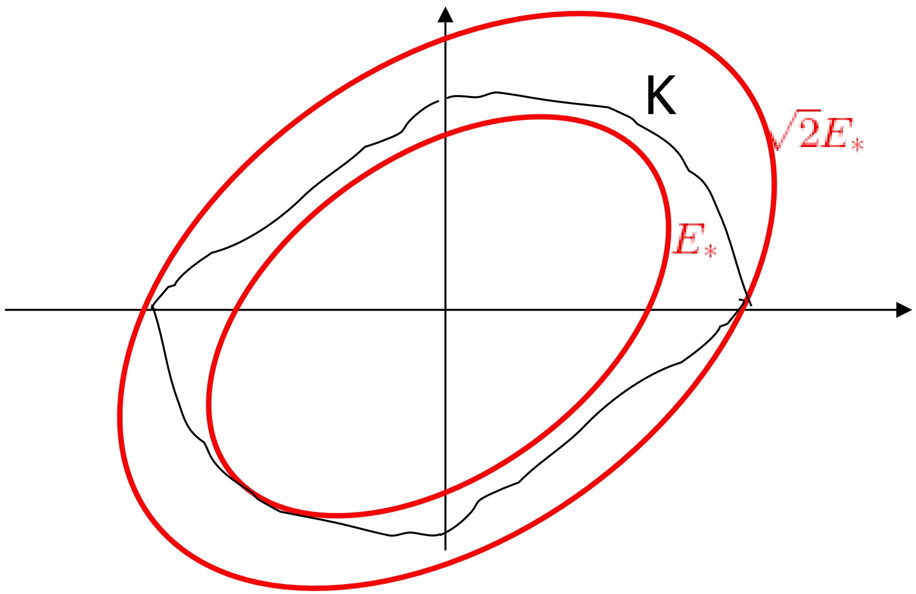


SDP methods

- Theorem The best ellipsoidal norm $||\cdot||_{E_*}$ approximates the joint spectral radius up to a factor $\sqrt{n}$ [Ando Shih 98]

$$\rho \leq \max ||A||_{E_*} \leq \sqrt{n}\rho$$

$$\frac{1}{\sqrt{n}}\rho* \leq \rho \leq \rho*$$



- Theorem The best ellipsoidal norm of a set of m (nonnegative) matrices also approximates the joint spectral radius up to a factor $\sqrt{m}$

$$\rho \leq \max ||A||_{E_*} \leq \sqrt{m}\rho$$

[Blondel Nesterov 05]

LMI methods

- Problem How to compute the best ellipsoidal norm?

$$\begin{array}{lcll} \inf_{r \in \mathbb{R}^+} & & r & \\ \text{s.t.} & & & \\ A^T P A & \preceq & r^2 P, & \forall A \in \Sigma \\ P & \succeq & 0. & \end{array}$$

$$\Leftrightarrow \frac{|Ax|_P}{|x|_P} \leq r$$

- This is « just » a sufficient condition for stability
- Computable in polynomial time (interior point methods)
- The « CQLF method » (see [Mason Shorten 04]...)

LMI methods

- Other methods using LMIs have been proposed
  - Based on symmetric algebras [Blondel Nesterov 05]
  - Based on sum of squares [Parrilo Jadbabaie 08]

- All these methods can be seen as trying to approximate the best norm with more and more complex curves

...Or, the symmetric algebras (the SOS) allow for lifting the matrices in a higher-dimensional (but more gentle) space

There exist **algorithms** that approximate the joint spectral radius of arbitrary sets of n by n matrices up to an arbitrary accuracy ϵ in $O(n^{k\frac{1}{\epsilon}})$ operations

Yet another LMI method

- A strange semidefinite program

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P & \succeq & 0.
\end{array}
$$

$$\rho \leq r$$

[Goebel, Hu, Teel 06]

- But also…

[Daafouz Bernussou 01]

[Bliman Ferrari-Trecate 03]

[Lee and Dullerud 06] …

Yet another LMI method

- An even stranger program:

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P A_1 & \preceq & r^2 P, \\
(A_2 A_1)^T P (A_2 A_1) & \preceq & r^4 P, \\
(A_2^2)^T P (A_2^2) & \preceq & r^4 P, \\
P & \succeq & 0.
\end{array}
$$

$$\rho \leq r$$

[Ahmadi, J., Parrilo, Roozbehani10]

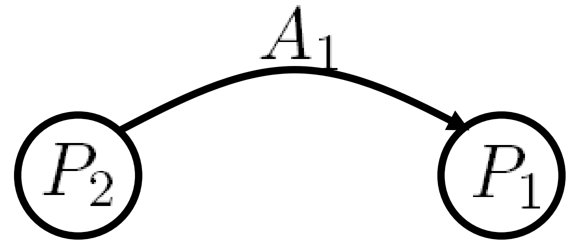
Yet another LMI method

- Questions:
  - Can we characterize all the LMIs that work, in a unified framework?
  - Which LMIs are better than others?
  - How to prove that an LMI works?
  - Can we provide converse Lyapunov theorems for more methods?

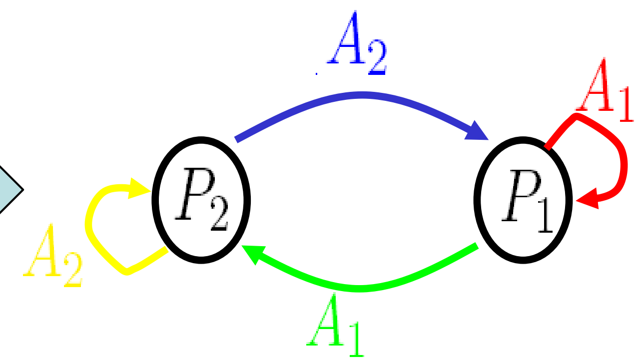
From an LMI to an automaton

- Automata representation Given a set of LMIs, construct an automaton like this:

$$A_1^T P_1 A_1 \preceq r^2 P_2$$



$$\begin{array}{lll} \min_{r \in \mathbb{R}^+} & & r \\ \text{s.t.} & & \\ A_1^T P_1 A_1 & \preceq & r^2 P_1, \\ A_2^T P_1 A_2 & \preceq & r^2 P_2, \\ A_1^T P_2 A_1 & \preceq & r^2 P_1, \\ A_2^T P_2 A_2 & \preceq & r^2 P_2, \\ P_i & \succeq & 0. \end{array}$$

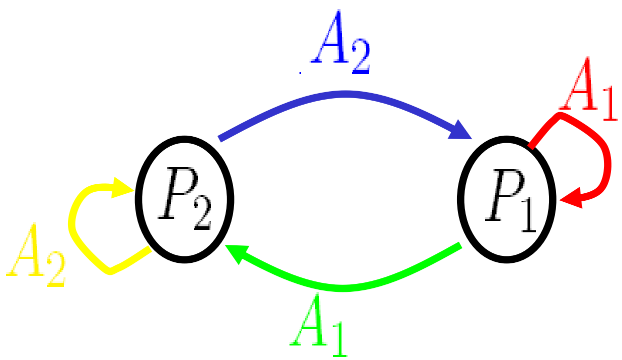


- Definition A labeled graph (with label set A) is path-complete if for any word on the alphabet A, there exists a path in the graph that generates the corresponding word.
- Theorem If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability.

[Ahmadi J. Parrilo Roozbehani 11]

An obvious question: are there other valid criteria?

- Theorem



$$
\begin{aligned}
&\min_{r\in\mathbb{R}+} && r \\
&\text{s.t.} && \\
&A_1^T P_1 A_1 &\preceq\ & r^2 P_1, \\
&A_2^T P_1 A_2 &\preceq\ & r^2 P_2, \\
&A_1^T P_2 A_1 &\preceq\ & r^2 P_1, \\
&A_2^T P_2 A_2 &\preceq\ & r^2 P_2, \\
&P_i &\succeq\ & 0.
\end{aligned}
$$

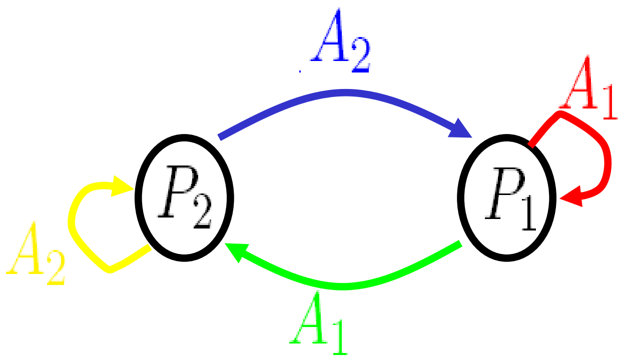
Path complete ??? Sufficient condition for stability

If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability.

- Are all valid sets of equations coming from path-complete graphs?
- …or are there even more valid LMI criteria?

Are there other valid criteria?

- Theorem Every non path-complete set of equations is not a sufficient condition for stability. [J. Ahmadi Parrilo Roozbehani 12]



$$
\begin{array}{lll}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}
$$

Path complete !!! Sufficient condition for stability

- Corollary:

  It is PSPACE complete to recognize sets of equations that are a sufficient condition for stability

- These results are not limited to LMIs, but apply to other families of conic inequalities

What about the other quantities?

	Arbitrary approximation	Arbitrary approximation in polynomial time	Arbitrary approximation for positive matrices	Decidability
Joint Spectral Radius	v	v	v	?
Joint Spectral Subradius	x	x	v	x
Lyapunov Exponent	x	x	v	x
p-radius	**Depends on p**	**Depends on p**	v	?

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  - MCNs and Delayed Systems
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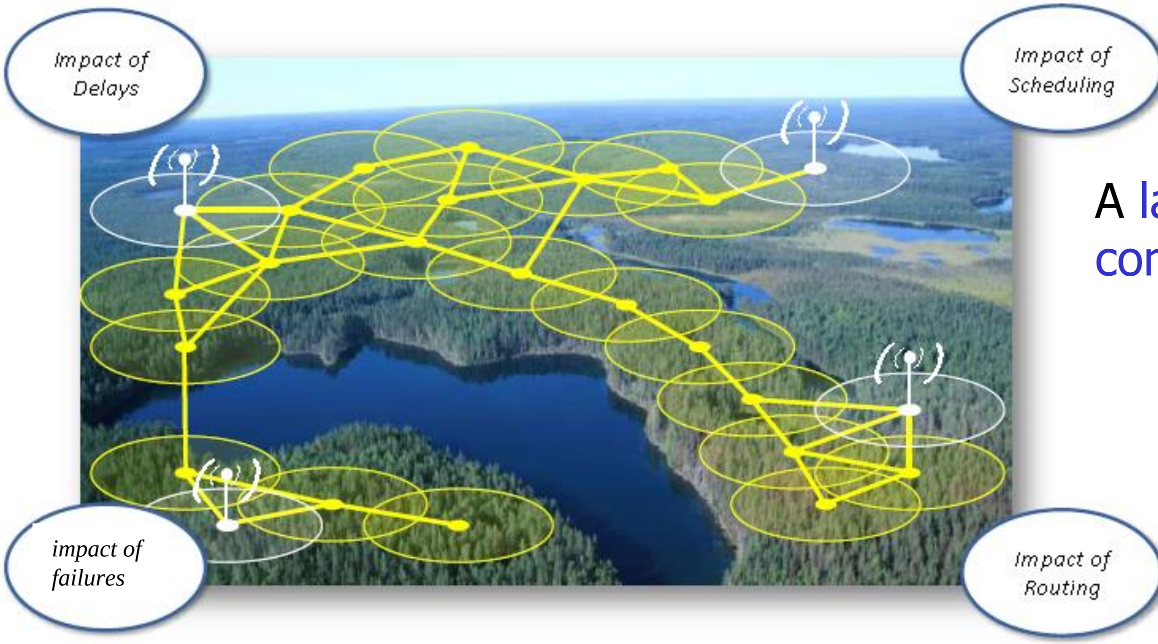
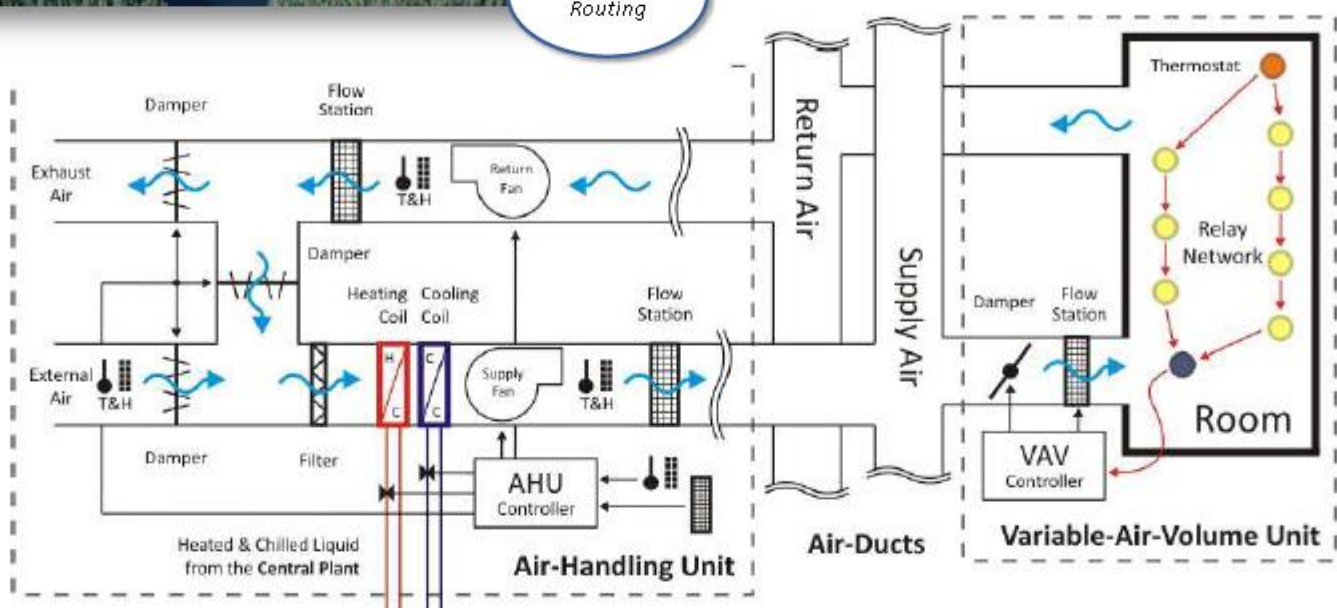
multihop control networks



A large scale decentralized control network

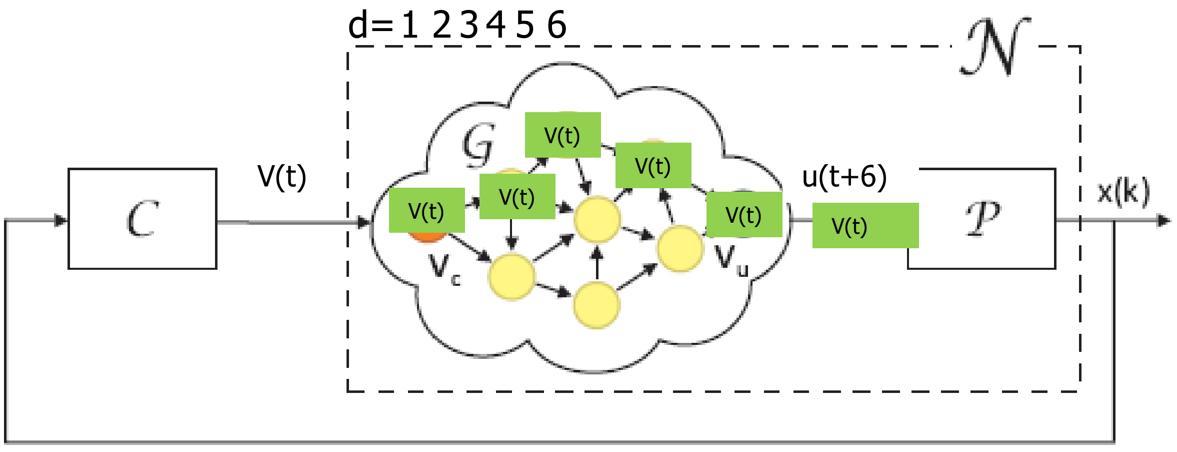
A green building

[Ramanathan Rosales-Hain 00]
[alur D'Innocenzo Johansson Pappas Weiss 10]
[Zhu Yuan Song Han Başar 12]



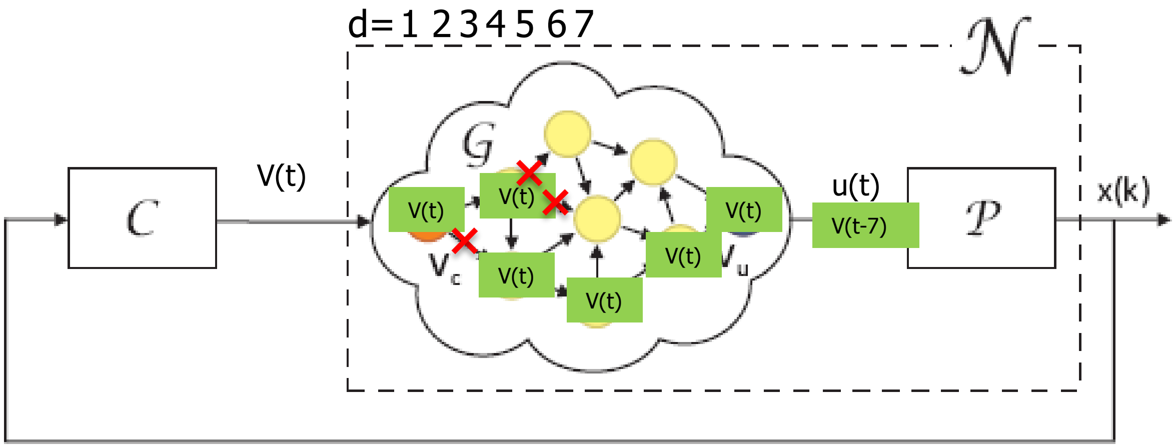
[J. D'Innocenzo Di Benedetto 12]

How to model failures?
LTIs with switched delays



$$x(t+1) = Ax + Bu(t-d)$$

How to model failures? LTIs with switched delays



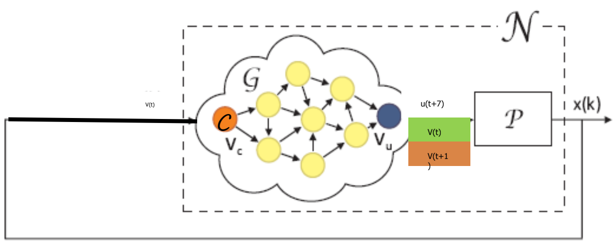
$$x(t+1) = Ax + Bu(t - d_2)$$

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$$

$$D = \{0, 1\}, \quad \sigma(t) = t \bmod 2$$

LTIs with switched delays
The linear controller

Delay dependent controller

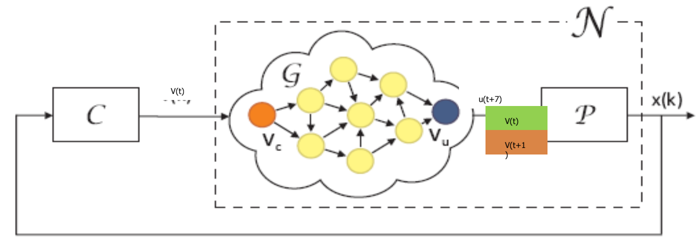


$$v(t) = K(d)\tilde{u}(t)$$

$$\tilde{u}(t) =$$

$$(x(t), u_1(t), u_2(t), \ldots, u_{d_{max}}(t))$$

Delay independent controller



$$v(t) = K\tilde{v}(t),$$

$$\tilde{v}(t) =$$

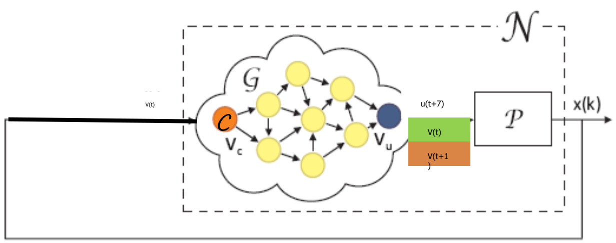
$$(x(t), v(t-d_{max}), \ldots, v(t-1)))$$

$$x(t+1) = Ax(t) + Bu(v(t-d_{max} : t), \sigma(t-d_{max} : t))$$

$$u_d(t) = \sum_{t' < t : t' + \sigma(t') = t + d} v(t')$$

LTIs with switched delays
The linear controller

Delay dependent controller



$$v(t) = K(d)\tilde{u}(t)$$

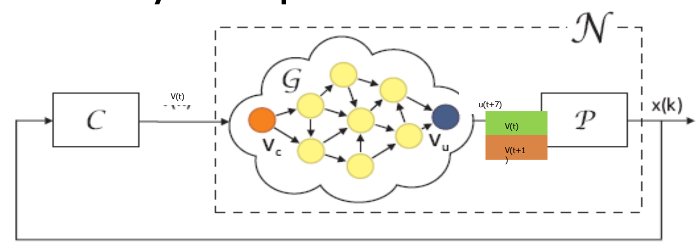
$$\tilde{u}(t) =$$

$$(x(t), u_1(t), u_2(t), \ldots, u_{d_{max}}(t))$$

$$\Sigma = \left\{ \begin{pmatrix} A & B & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ldots & I \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ K(d) \\ \vdots \\ 0 \end{pmatrix} \right\}$$

[Hetel Daafouz Iung 07]
[Weiss et al. 09]

Delay independent controller



$$v(t) = K\tilde{v}(t),$$

$$\tilde{v}(t) =$$

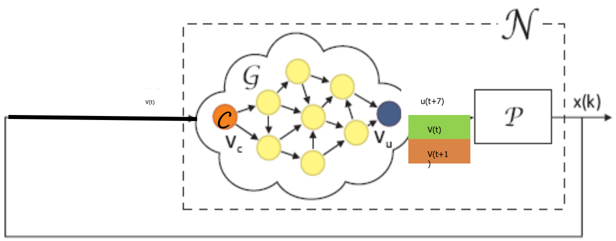
$$(x(t), v(t - d_{max}), \ldots, v(t-1)))$$

$$\Sigma = \left\{ \begin{pmatrix} A & 0 & \boxed{B} & \ldots\boxed{B} & 0 \\ 0 & 0 & I & \ldots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ K_0 & K_1 & K_2 & \ldots & K_{d_{max}} \end{pmatrix} \right\}$$

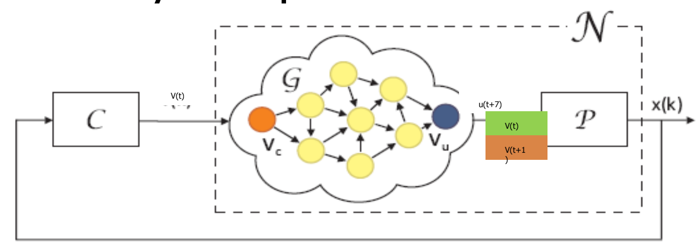
[J. D'Innocenzo Di Benedetto 12]

LTIs with switched delays

Delay dependent controller



Delay independent controller



- Corollary

  For both models there is a PTAS for the stability question:

  for any required accuracy, there is a polynomial-time algorithm for checking stability up to this accuracy

Previous sufficient conditions for stability in [Hetel Daafouz Iung 07, Zhang Shi Basin 08]

- However:

  Theorem the very stability problem is NP-hard

  [J. D'Innocenzo Di Benedetto 12]

- Open questions

  What about the stabilizability?

  What about the stability with probability one? (sufficient conditions in, e.g. [Weiss et 09])

Design of LTIs with switched delays

Delay dependent controller:

- Theorem for n=m=1, there is an explicit formula for a controller that achieves finite time stabilization.

  (based on a generalization of the Ackermann formula for delayed LTI)

- Example

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$$

$$D = \{0, 1\}, \quad \sigma(t) = t \bmod 2$$

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x_1 = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_2 = A^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Bv(1) + Bv(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v(1) + v(2) \end{pmatrix}$$

➔ uncontrollable

Design of LTIs with switched delays
the delay-independent case

- Example: n=m=1, D={0,1}: find k_1 , k_2 such that

$$
\begin{aligned}
x(t+1) = ax(t) + (1-\sigma(t)) \quad & (k_1 x(t) + k_2 x(t-1)) \\
+\sigma(t-1) \quad & (k_1 x(t-1) + k_2 x(t-2))
\end{aligned}
$$

is stable, whatever switching signal $\sigma(t)$ occurs?

  - a>3: impossible
  - a=2: possible, but one needs memory
  - a=1.1: possible with $k_2 = 0$
  - a<1: no need for a controller!

→ Question: how to algorithmically decide if stabilization is possible, even in this simplest case?

Outline

- Matrix semigroups and some tools to analyze them

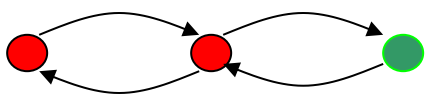
- LMI methods for switching systems stability

- Applications
  - Trackable graphs

- Conclusion and perspectives

Trackable graphs

To a given observation, associate the corresponding product:



$$A_r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_r A_g A_r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The number of possible trajectories is given by the sum of the entries of the matrix

Trackable graphs

The maximal total number of possibilities is

$$N(t) = \max\left\{ \left\|A\right\|_1 \ : A \in \Sigma^t \right\}$$

We are interested in the asymptotic worst case :

$$\lim_{t\to\infty} N(t)^{1/t} = \lim_{t\to\infty} \max\left\{ \left\|A\right\|_1^{1/t} : A \in \Sigma^t \right\}$$

This is a joint spectral radius!

Trackable graphs

The network is trackable iff

$$\rho \leq 1$$

[Crespi et al. 05]

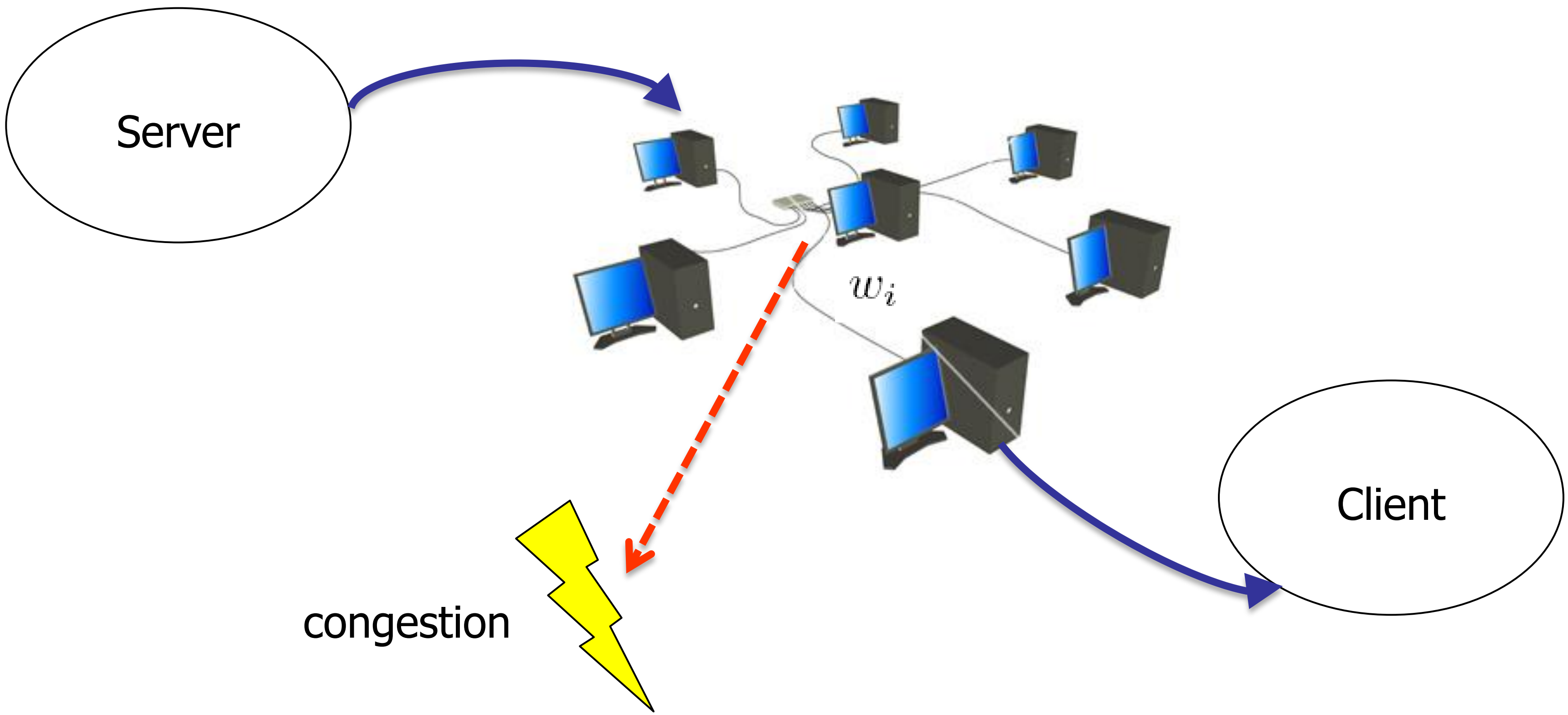
Theorem It is possible to check trackability in polynomial time

[J. Protasov Blondel 08]

Outline

- Matrix semigroups and some tools to analyze them
- LMI methods for switching systems stability
- Applications
  - TCP congestion control
- Conclusion and perspectives

TCP Congestion control

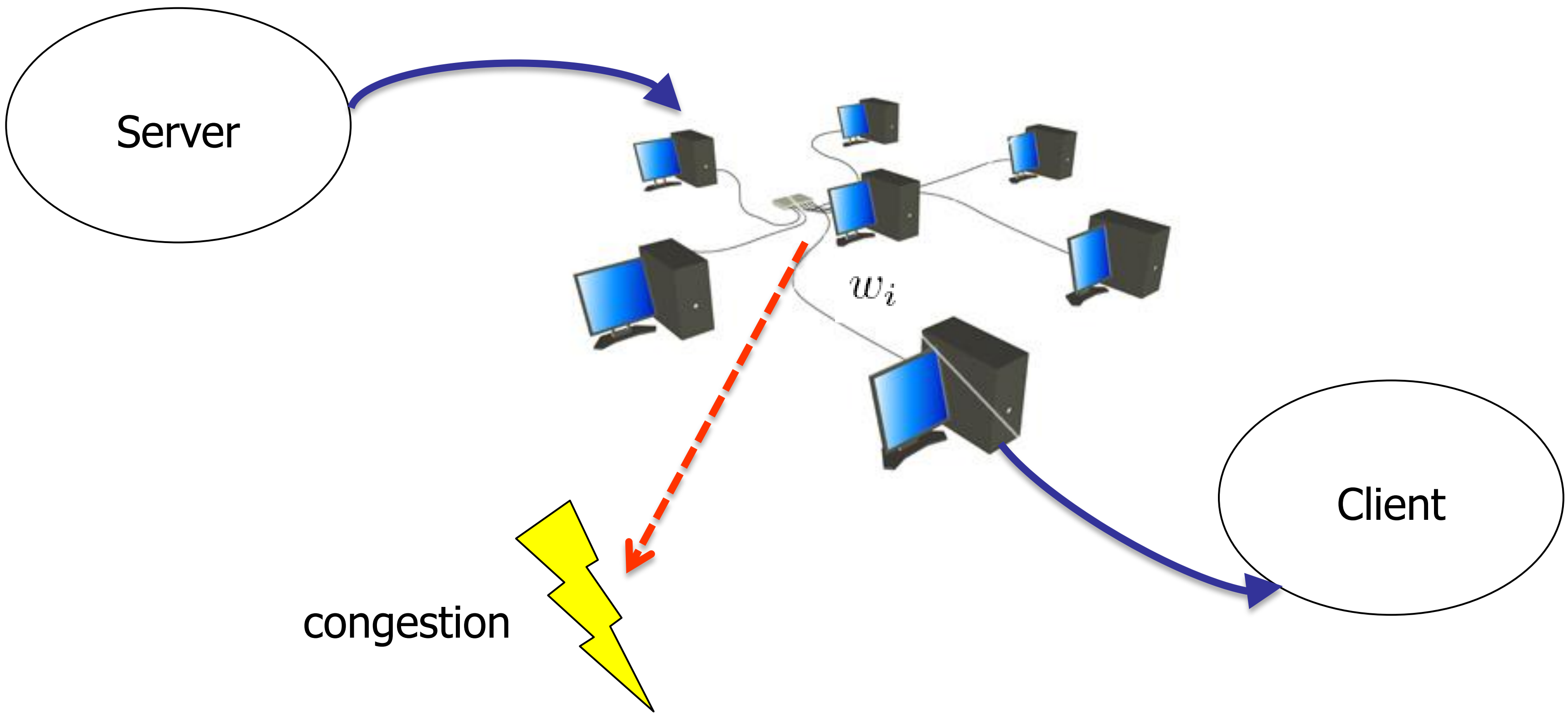


Congestion avoidance strategy: the **AIMD algorithm**
(Additive Increase Multiplicative Decrease)

If the node i does not see congestion $w_i(t+1) = w_i(t) + \alpha$ **AI**

If the node i sees congestion $w_i(t+1) = \beta w_i(t), \ \beta < 1$ **MD**

TCP Congestion control



Congestion avoidance strategy: the **AIMD algorithm**
(Additive Increase Multiplicative Decrease)

This is a switching system!

[Scholte, Berman, Shorten 12]

Outline

- Matrix semigroups and some tools to analyze them
- LMI methods for switching systems stability
- Applications
    - Other applications
- Conclusion and perspectives

Applications

[Protasov 04]

[Protasov J. Blondel 10]

[Hendrickx 08]

[Jadbabaie Lin Morse 03]

$$\mathrm{cap}(D) = \lim_{n\to\infty} \frac{\log_2 \delta_n(D)}{n}$$

$$\mathrm{cap}(D) = \log_2 \rho(\Sigma(D))$$

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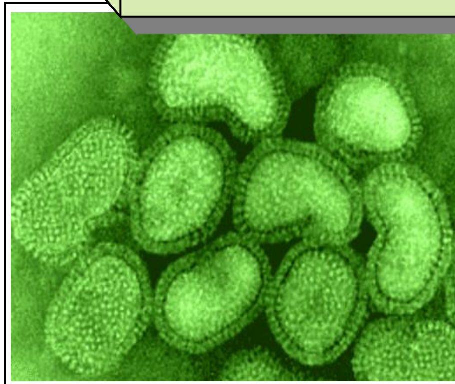
[Moision Orlitsky Siegel 01]

[Blondel J. Protasov 06]

[Dima Asarin 10]

[Beal crochemore Moision Siegel 12]



[Hernandez-Varga colaneri Middleton Blanchini 10]

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[Blondel Cassaigne J. 09]

[Rigo Berthe 10]

[Shorten et al. 07]
[Goebel, Sanfelice, Teel 09]
[J. 09]

Outline

- Matrix semigroups and some tools to analyze them
- LMI methods for switching systems stability
- Applications
- Conclusion and perspectives

Conclusion and perspectives

- Many open questions and Conjectures:
  - For computer scientists:

  Is 'ρ<1' decidable? [Blondel Megretski 05]

  Write protocols for optimal conrol of computer networks

  - For mathematicians:

  The finiteness conjecture

  - For control theorists: [Lagarias, Wang 95], [Bousch Mairesse 02] [J. Blondel 08]

  What are the best control schemes for LTIs with switched delays?

  How to design switching systems? [Lee Dullerud 06, Hetel Daafouz Iung 07, Zhang Shi Basin 08] [J. D'Innocenzo Di Benedetto 12]

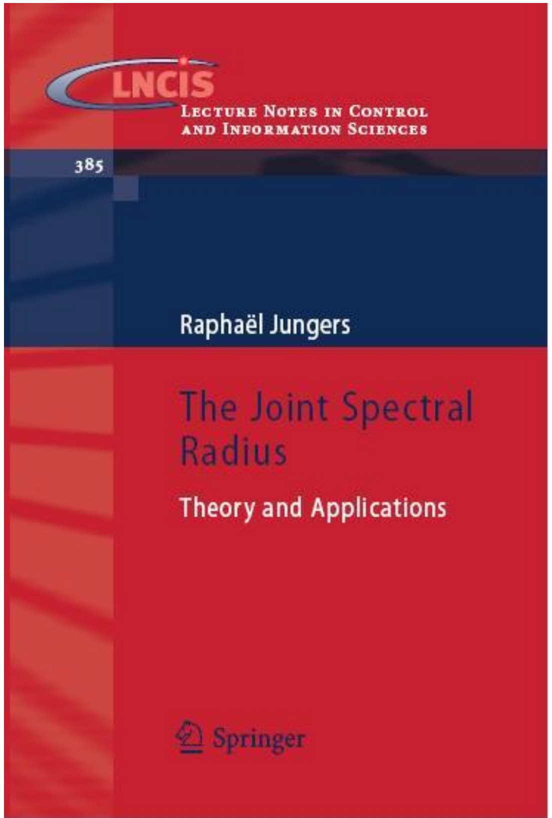
- Meta-conclusion: many hybrid systems boil down to the study of switching systems and matrix semigroups, for which nice mathematical tools exist

Thanks! Questions?

Ads

[The JSR Toolbox: http://www.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox](http://www.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox)

References: http://perso.uclouvain.be/raphael.jungers/



Joint work with

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