Algebraic Techniques for Switching Systems

And applications to Wireless Control Neworks

Raphaël Jungers (UCL, Belgium)

UCLA
May 2014

LA LIBERTÉ DE CHERCHER

Outline

- Joint spectral characteristics
- Application: WCNs and switching delays
- LMI methods for switching systems stability
- Conclusion and perspectives

Outline

- Joint spectral characteristics
- Application: WCNs and switching delays
- LMI methods for switching systems stability
- Conclusion and perspectives

Switching systems

$$\mathbf{x}_{t+1} = \begin{matrix} \mathbf{A}_0\ \mathbf{x}_t \\ \mathbf{A}_1\ \mathbf{x}_t \end{matrix}$$

Point-to-point Given x_0 and x_*, is there a product (say, $A_0\ A_0\ A_1\ A_0\ \ldots\ A_1$) for which $x_* = A_0\ A_0\ A_1\ A_0\ \ldots\ A_1\ x_0$?

Mortality Is there a product that gives the zero matrix?

Boundedness Is the set of all products $\{A_0, A_1, A_0A_0, A_0A_1, \ldots\}$ bounded?

Switching systems

$$x_{t+1} = \begin{matrix} A_0\, x_t \\ A_1\, x_t \end{matrix}$$

Global convergence to the origin Do all products of the type $A_0\, A_0\, A_1\, A_0 \ldots A_1$ converge to zero?

The spectral radius of a matrix A controls the growth or decay of powers of A

$$\rho(A) = \lim_{t \to \infty} ||A^t||^{1/t}$$

The powers of A converge to zero iff $\rho(A) < 1$

The joint spectral radius of a set of matrices Σ is given by

$$\rho(\Sigma) = \lim_{t \to \infty} \max_{A_i \in \Sigma} ||A_1 A_2 \ldots A_t||^{1/t}$$

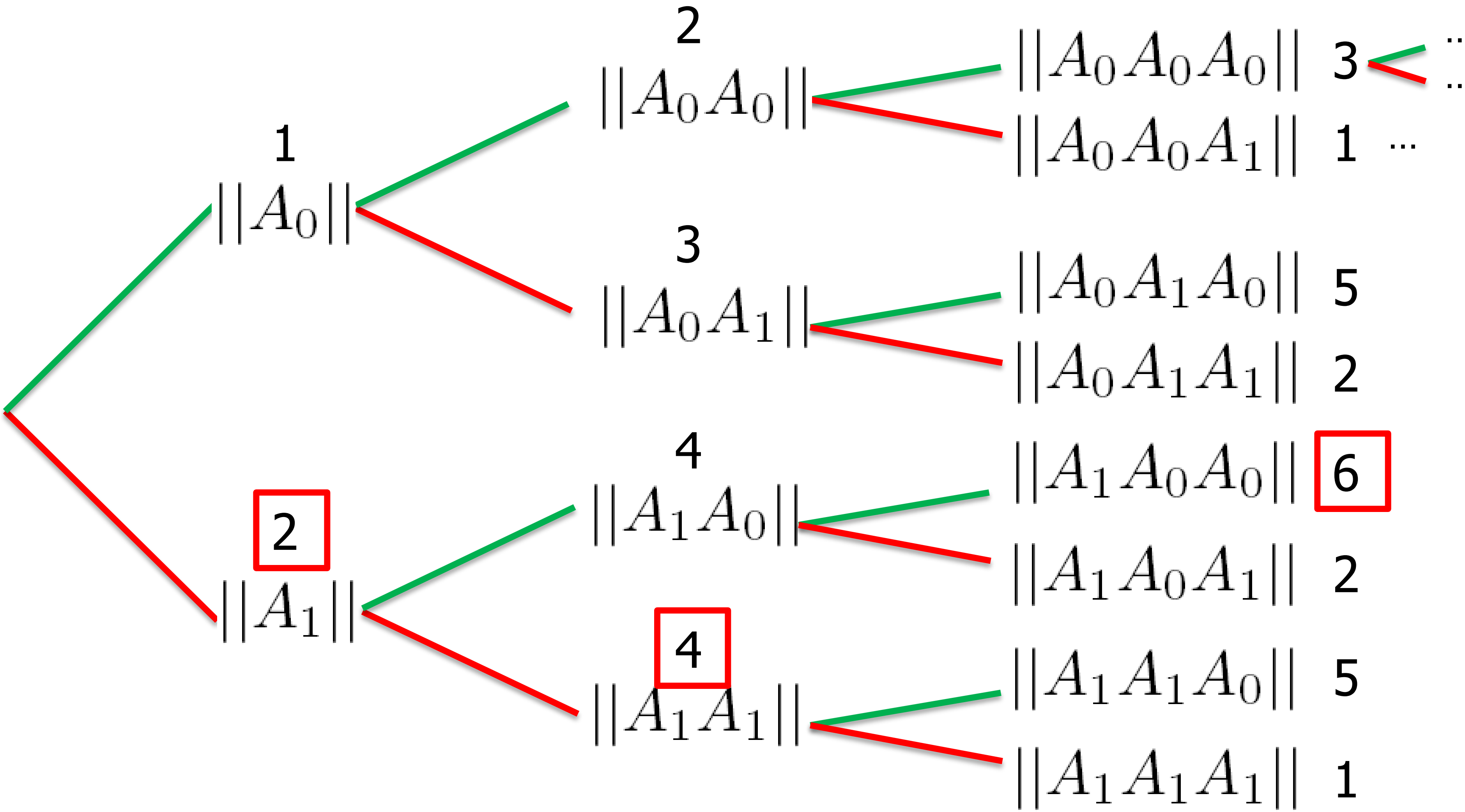
All products of matrices in Σ converge to zero iff $\rho(\Sigma) < 1$

[Rota, Strang, 1960]

The joint spectral characteristics

$$\boxed{\rho(\Sigma) = \lim_{t\to\infty} \left[\max_{A_i \in \Sigma} ||A_1 A_2 \dots A_t|| \right]^{1/t}}$$

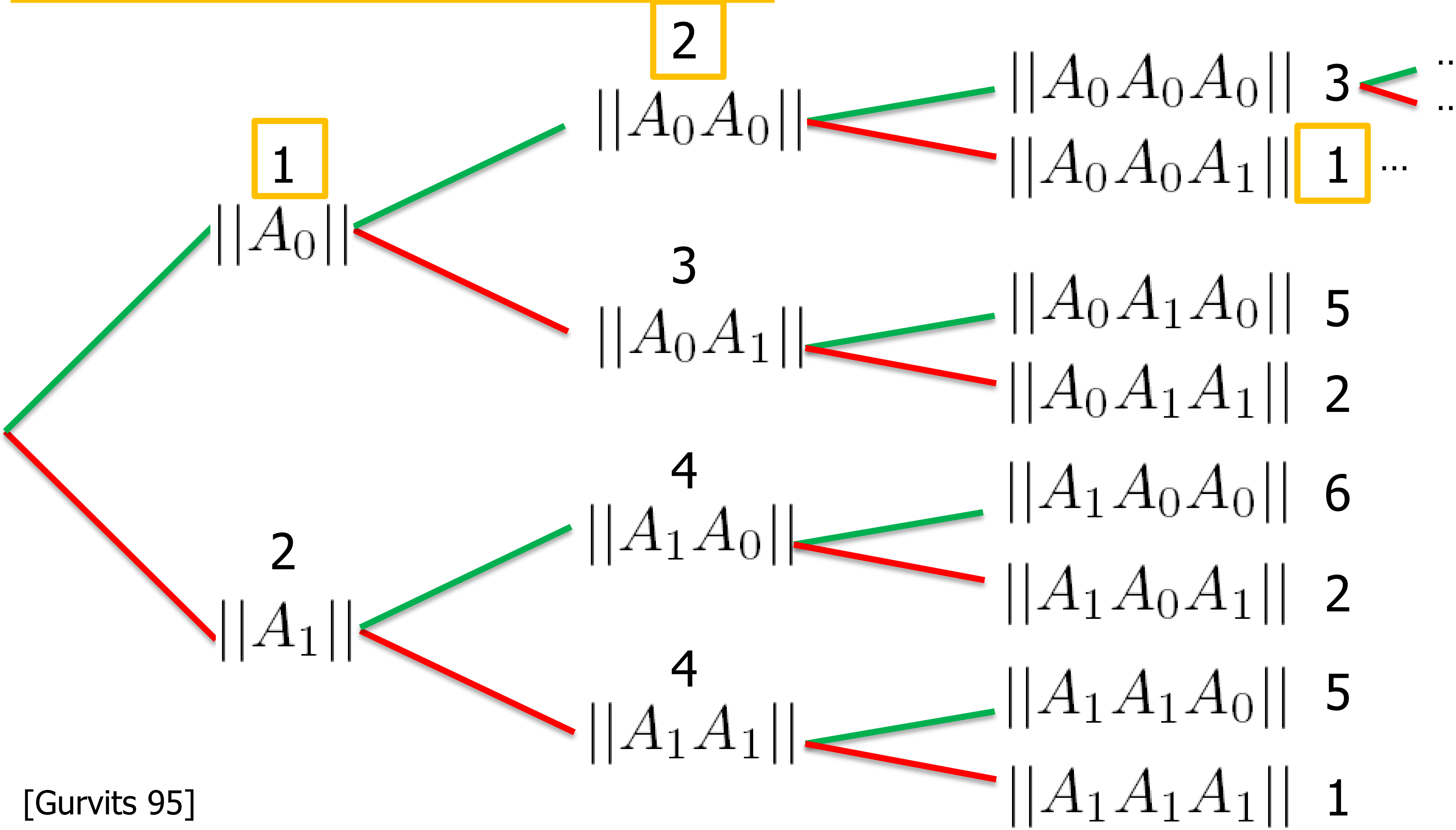
The joint spectral radius



The joint spectral characteristics

$$\check{\rho}(\Sigma) = \lim_{t\to\infty} \left[\min_{A_i \in \Sigma} ||A_1 A_2 \dots A_t|| \right]^{1/t}$$

The joint spectral subradius



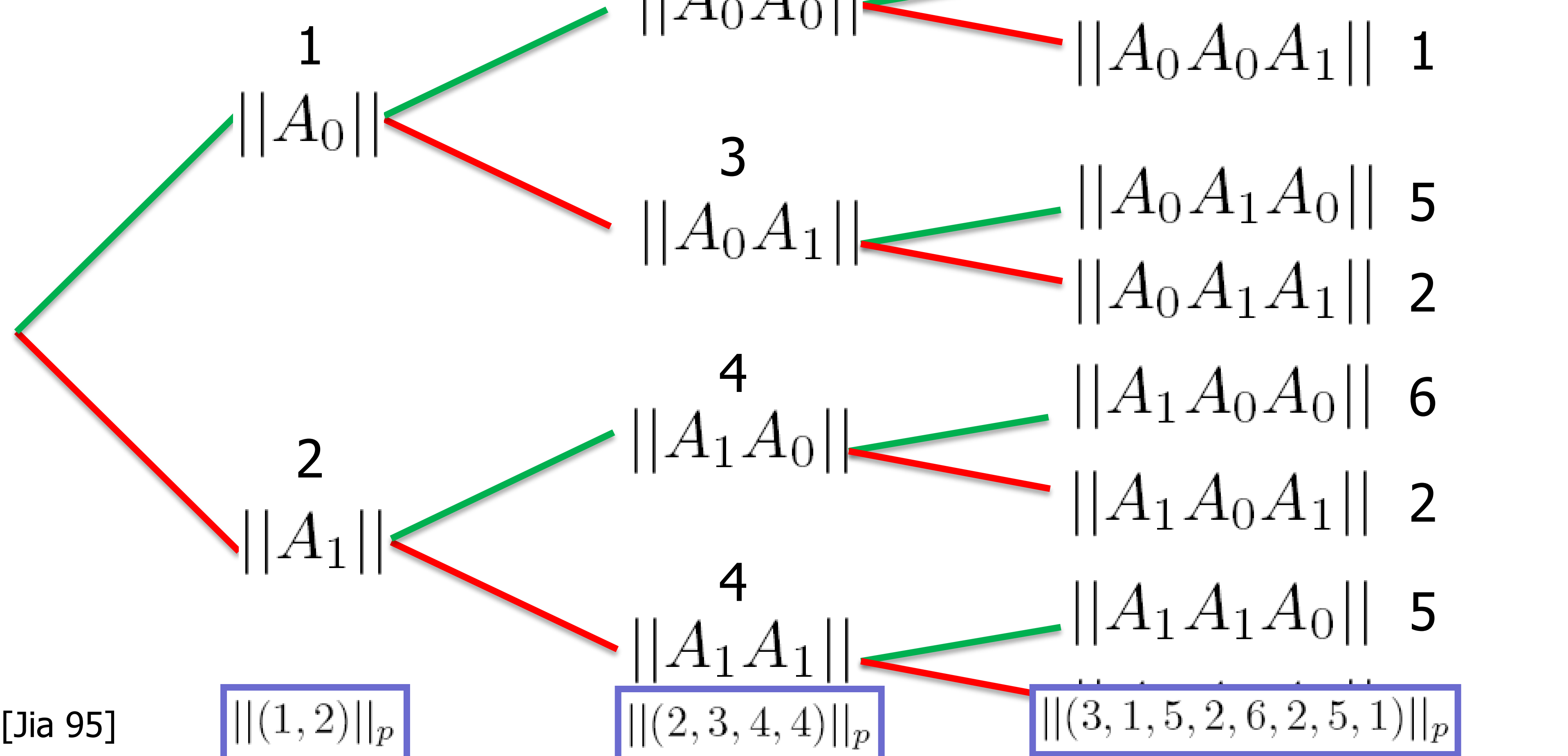
[Gurvits 95]

The joint spectral characteristics

$$\rho_p(\Sigma) = \lim_{t\to\infty} \left[m^{-t} \sum_{A_i \in \Sigma^t} \|A_1 A_2 \dots A_t\|^p \right]^{1/(pt)}$$

The p-radius

(m is the number of matrices in Σ)



[Jia 95]

The joint spectral characteristics

$$\boxed{\bar{\rho}(\Sigma) = \lim_{t\to\infty}\left[\prod_{A_i\in\Sigma^t}\|A_1A_2\ldots A_t\|\right]^{1/(tm^t)}}$$

The Lyapunov Exponent

(m is the number of matrices in Σ)

1 $||A_0||$

2 $||A_0A_0||$

$||A_0A_0A_0||$ 3 …

…

$||A_0A_0A_1||$ 1 …

3 $||A_0A_1||$

$||A_0A_1A_0||$ 5

$||A_0A_1A_1||$ 2

4 $||A_1A_0||$

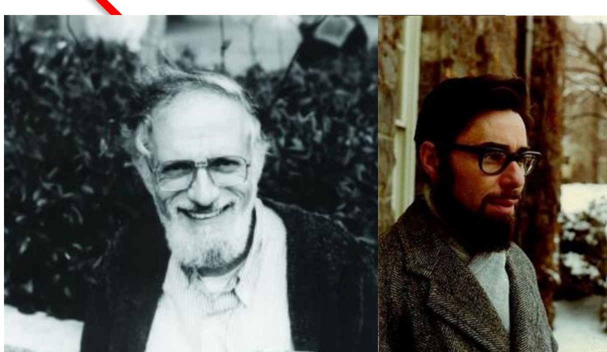
$||A_1A_0A_0||$ 6

$||A_1A_0A_1||$ 2

4 $||A_1A_1||$

$||A_1A_1A_0||$ 5

$||A_1A_1A_1||$ 1



[Furstenberg Kesten, 1960]

The joint spectral characteristics

$$\rho(\Sigma) = \lim_{t\to\infty} \left[\max_{A_i \in \Sigma} ||A_1 A_2 \dots A_t|| \right]^{1/t}$$

The joint spectral radius addresses the **stability** problem

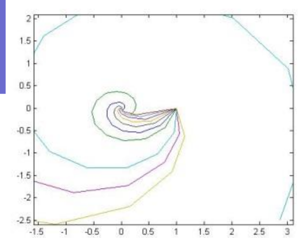
$$\check{\rho}(\Sigma) = \lim_{t\to\infty} \left[\min_{A_i \in \Sigma} ||A_1 A_2 \dots A_t|| \right]^{1/t}$$

The joint spectral subradius addresses the **stabilizability** problem

$$\rho_p(\Sigma) = \lim_{t\to\infty} \left[m^{-t} \sum_{A_i \in \Sigma^t} \|A_1 A_2 \dots A_t\|^p \right]^{1/(pt)}$$

The p-radius addresses the… **p-weak stability**

[J. Protasov 10]



$$\bar{\rho}(\Sigma) = \lim_{t\to\infty} \left[\prod_{A_i \in \Sigma^t} \|A_1 A_2 \dots A_t\| \right]^{1/(tm^t)}$$

The Lyapunov exponent addresses the **stability with probability one**

(Cfr. Oseledets Theorem)

The joint spectral characteristics: Mission Impossible?

Theorem Computing or approximating ρ is NP-hard

Theorem The problem $\rho\cdot 1$ is algorithmically undecidable

Conjecture The problem $\rho<1$ is algorithmically undecidable

Theorem Even the question « $|\check{\rho} - r| \leq a + b\check{\rho}$?» is algorithmically undecidable for all (nontrivial) a and b

Theorem The same is true for the Lyapunov exponent

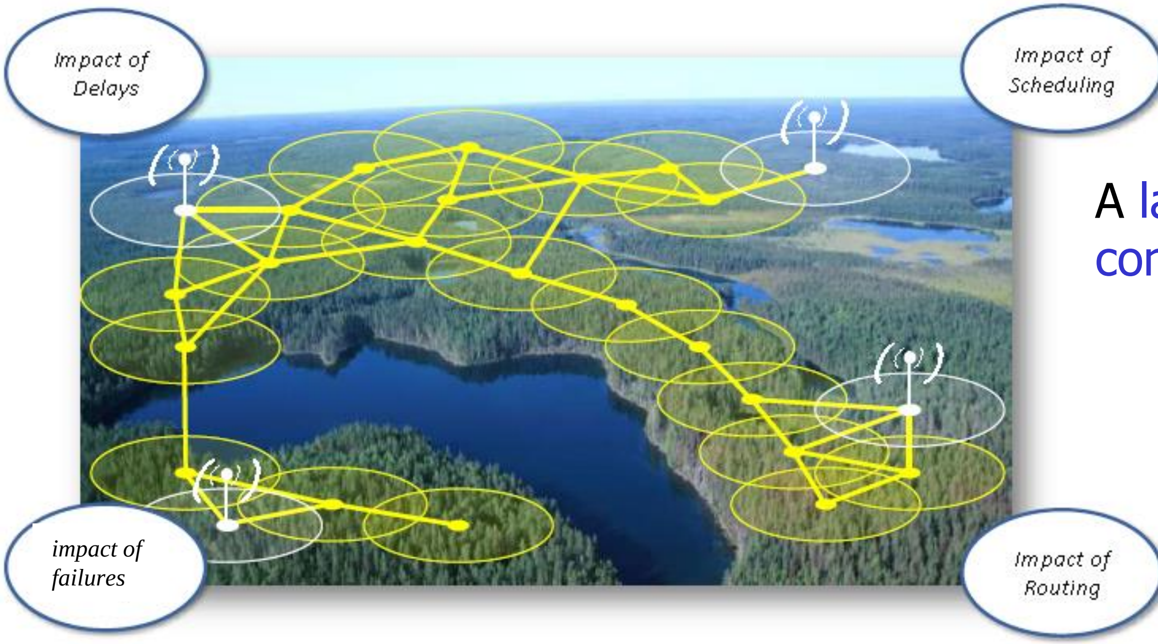
Theorem The p-radius is NP-hard to approximate

See [Blondel Tsitsiklis 97, Blondel Tsitsiklis 00, J. Protasov 09]

Outline

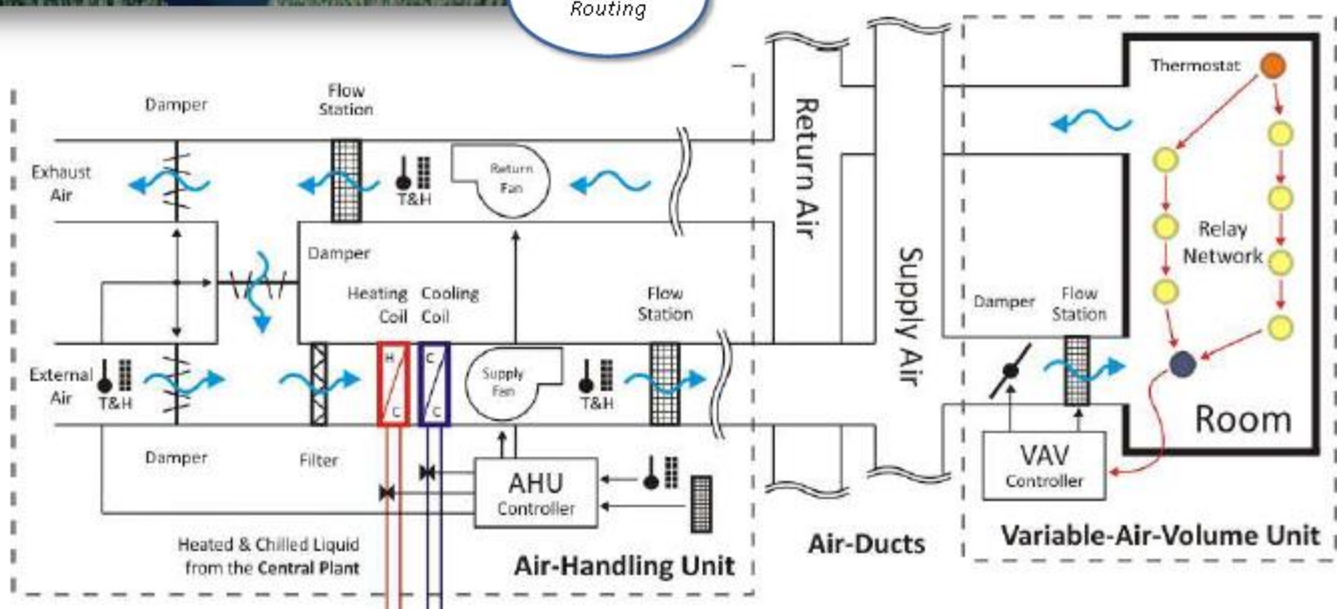
- Joint spectral characteristics
- Application: WCNs and switching delays
    - Modeling
    - Stability analysis
    - Controllability with infinite look-ahead
    - Approximate controllability (with small look-ahead)
- LMI methods for switching systems stability
- Conclusion and perspectives

Wireless control networks



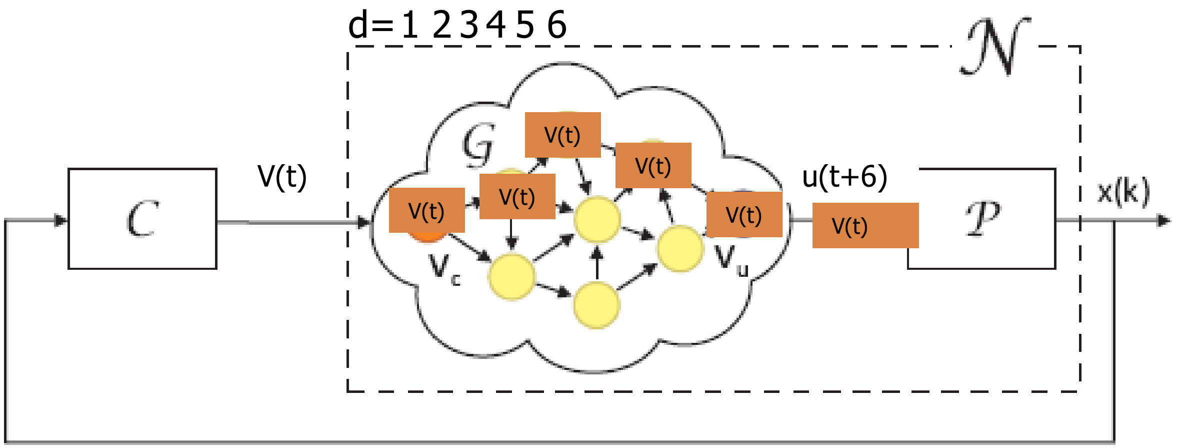
A large scale decentralized control network

A green building



[Ramanathan Rosales-Hain 00]
[alur D'Innocenzo Johansson Pappas Weiss 10]
[Mazo Tabuada 10]
[Zhu Yuan Song Han Başar 12]

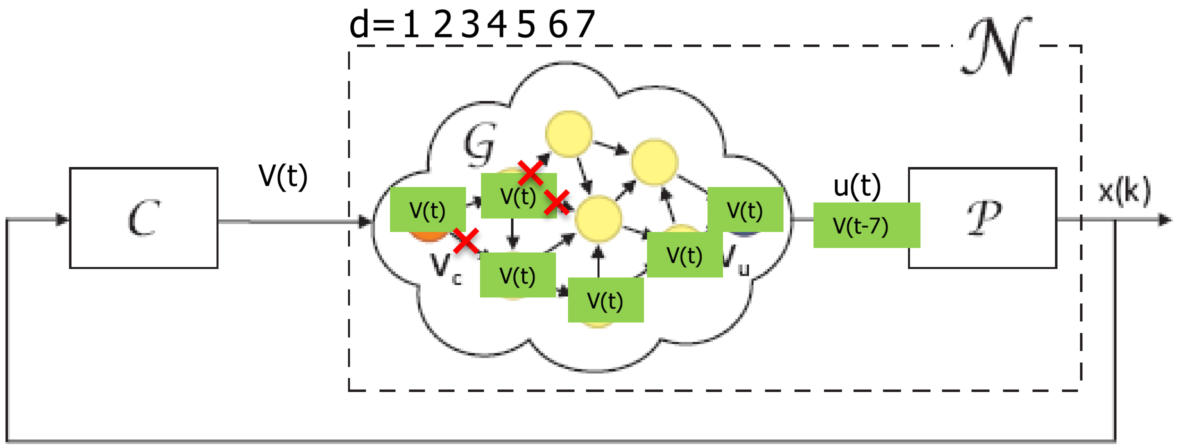
How to model failures?



WCNs are delay systems:

$$x(t+1) = Ax + B\,\mathrm{v}(t-d)$$

How to model failures?
LTIs with switched delays



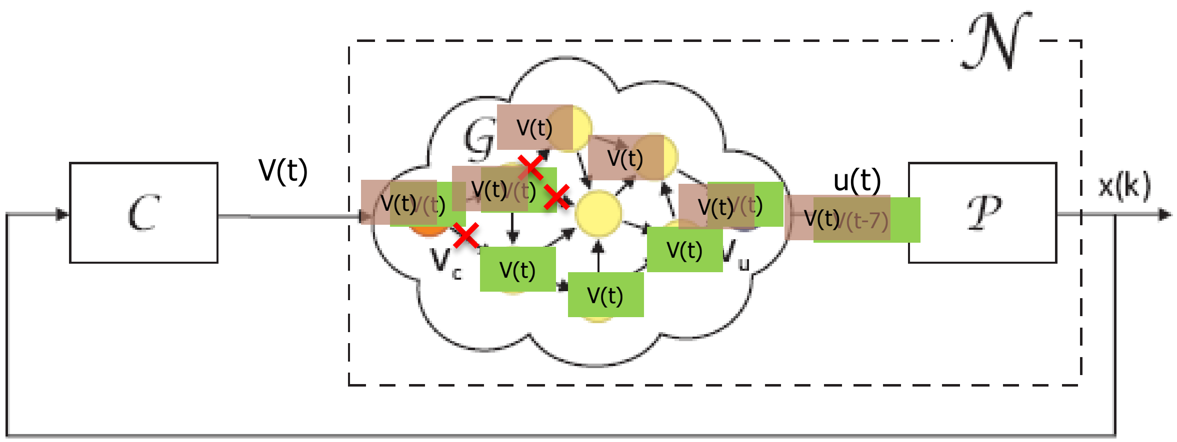
$$x(t+1) = Ax + B\,\mathsf{v}(t - d_2)$$

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$$

$$D = \{0, 1\}$$

LTIs with switched delays
Devil's in details…

1) Two control signals can arrive at the same time



What to do with these two signals?

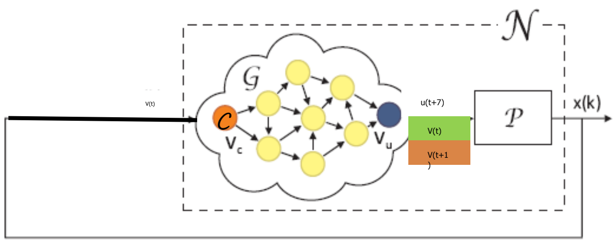
$$x(t+1) = Ax(t) + Bu(v(t-d_{max}:t), \sigma(t-d_{max}:t))$$

$$u_d(t) = \sum_{t'<t : t'+\sigma(t')=t+d} v(t')$$

LTIs with switched delays
Devil's in details…

2) Does the controller C know about the future delay?

Delay dependent controller

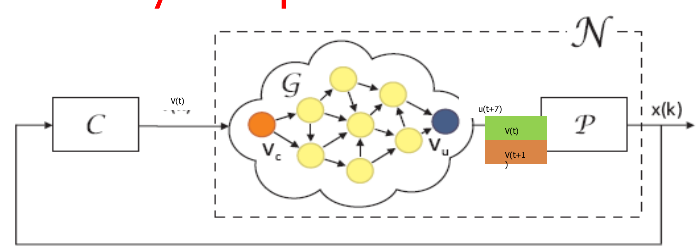


$$v(t) = K(d)\tilde{u}(t)$$

$$\tilde{u}(t) =$$

$$(x(t), u_1(t), u_2(t), \ldots, u_{d_{max}}(t))$$

Delay independent controller



$$v(t) = K\tilde{v}(t),$$

$$\tilde{v}(t) =$$

$$(x(t), v(t-d_{max}), \ldots, v(t-1))$$

More generally, let us define the look-ahead N such that C knows d(t), d(t+1), … , d(t+N-1)

LTIs with switched delays
Devil's in details…

3) When is the delay determined?

Previous models suppose that the delay is determined 'at the plant':

$$\dot{x}(t) = Ax(t) + Bu(t - \tau(t))$$

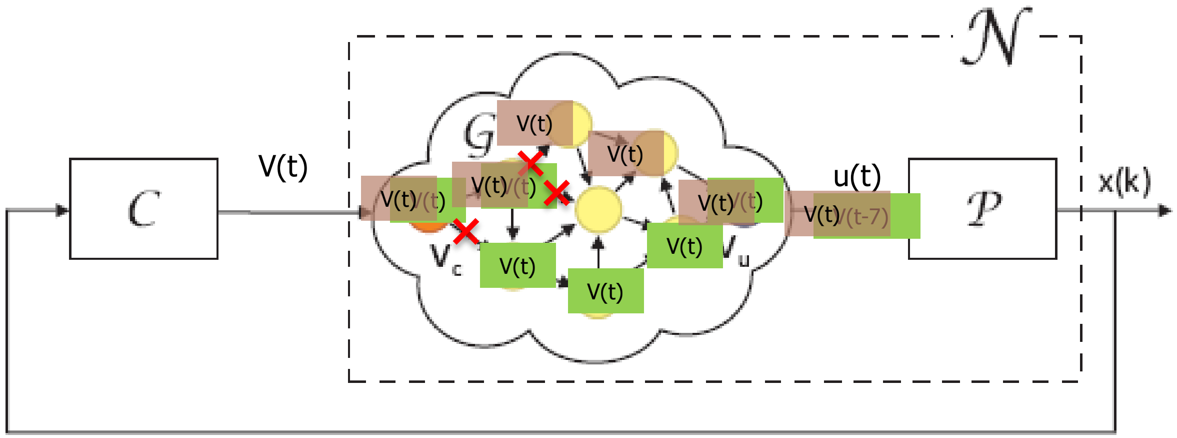
Our delays are determined when the control signal is issued

$$u_d(t) = \sum_{t' < t : t' + \sigma(t') = t + d} v(t')$$

There can be times with an empty sum

LTIs with switched delays
Devil's in details…

4) How much memory should we give to the controller?



Normal (fixed) delay systems need d

→ We (arbitrarily) chose the maximum delay

$$x(t+1) = Ax(t) + Bu(v(t-d_{max}:t), \sigma(t-d_{max}:t))$$

$$u_d(t) = \sum_{t'<t:\, t'+\sigma(t')=t+d} v(t')$$

LTIs with switched delays
An example to wrap up…

- Example the delay independent case

n=m=1, D={0,1}:

$$x(t+1) = ax(t) \quad \boxed{+k_1x(t) + k_2x(t-1) \quad \text{if } \sigma(t) = 0}$$
$$\boxed{+k_1x(t-1) + k_2x(t-2) \quad \text{if } \sigma(t-1) = 1}$$

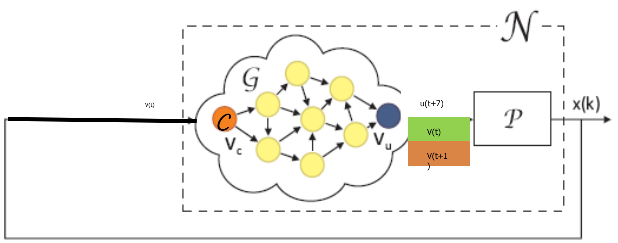
➔ Question: how to algorithmically decide if stabilization is possible, even in this simplest case?

Outline

- Joint spectral characteristics
- Application: WCNs and switching delays
  - Modeling
  - Stability analysis
  - Controllability with infinite look-ahead
  - Approximate controllability (with small look-ahead)
- LMI methods for switching systems stability
- Conclusion and perspectives

LTIs with switched delays
The linear controller

Delay dependent controller

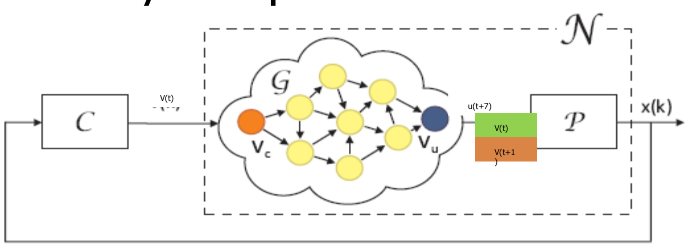


$$v(t) = K(d)\tilde{u}(t)$$

$$\tilde{u}(t) =$$

$$(x(t), u_1(t), u_2(t), \dots, u_{d_{max}}(t))$$

Delay independent controller



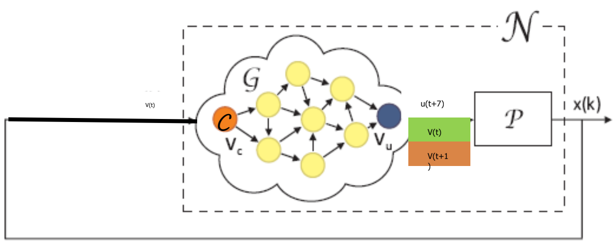
$$v(t) = K\tilde{v}(t),$$

$$\tilde{v}(t) =$$

$$(x(t), v(t - d_{max}), \dots, v(t-1)))$$

LTIs with switched delays
The linear controller

Delay dependent controller



$$v(t) = K(d)\tilde{u}(t)$$

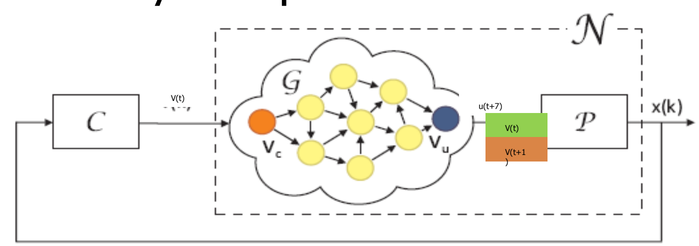
$$\tilde{u}(t) =$$

$$(x(t), u_1(t), u_2(t), \ldots, u_{d_{max}}(t))$$

$$\Sigma = \left\{ \begin{pmatrix} A & B & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ldots & I \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ K(d) \\ \vdots \\ 0 \end{pmatrix} \right\}$$

[Hetel Daafouz Iung 07]
[Weiss et al. 09]

Delay independent controller



$$v(t) = K\tilde{v}(t),$$

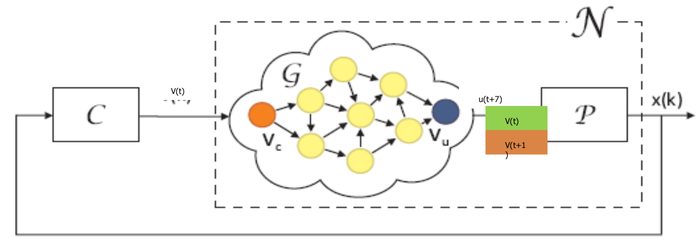
$$\tilde{v}(t) =$$

$$(x(t), v(t-d_{max}), \ldots, v(t-1)))$$

$$\Sigma = \left\{ \begin{pmatrix} A & 0 & \boxed{B} & \ldots \boxed{B} & 0 \\ 0 & 0 & I & \ldots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ K_0 & K_1 & K_2 & \ldots & K_{d_{max}} \end{pmatrix} \right\}$$

LTIs with switched delays
The linear controller

Delay independent controller



Correspond to all the delayed controls that arrive at time t

$$v(t) = K\tilde{v}(t),$$

- There is an exponential number of matrices
- The occurrences of the matrices are correlated

$$\Sigma = \left\{ \begin{pmatrix} A & 0 & \boxed{B} & \ldots \boxed{B} & 0 \\ 0 & 0 & I & \ldots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ K_0 & K_1 & K_2 & \ldots & K_{d_{max}} \end{pmatrix} \right\}$$

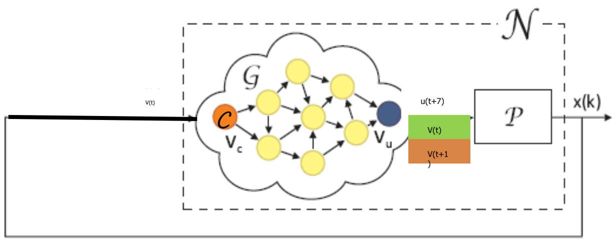
We propose an alternative representation as an unconstrained switching system, with a polynomial sized set of matrices!

$(n+2d_{max}m)$-dimensional with only d matrices. Idea: double the state space:

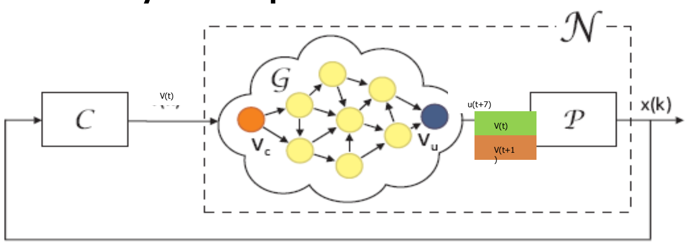
$$\tilde{w}(t) = (x(t), u_1(t), \ldots, u_{d_{max}}(t), v(t-d_{max}), \ldots, v(t-1))$$

LTIs with switched delays

Delay dependent controller



Delay independent controller



- Corollary

  For both models there is a PTAS for the stability question:

  for any required accuracy, there is a polynomial-time algorithm for checking stability up to this accuracy

Previous sufficient conditions for stability in [Hetel Daafouz Iung 07, Zhang Shi Basin 08]

- However:

  Theorem the very stability problem is NP-hard

  Theorem the boundedness problem is even Turing-undecidable!

[J. D'Innocenzo Di Benedetto 12]

Outline

- Joint spectral characteristics

- Application: WCNs and switching delays
  - Modeling
  - Stability analysis
  - Controllability with infinite look-ahead
  - Approximate controllability (with small look-ahead)

- LMI methods for switching systems stability

- Conclusion and perspectives

Design of LTIs with switched delays

What are the intrinsically uncontrollable systems?

- Given a good old linear system

$$x(t+1) = Ax(t) + Bu(t)$$

Is it controllable if the control signal is subject to delays varying in a set?

$$\tilde{u}(t) = (x(t), u_1(t), u_2(t), \ldots, u_{d_{max}}(t)) \qquad \sigma(t) \in D$$

$$\Sigma = \left\{ \begin{pmatrix} A & B & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ldots & I \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ K(d) \\ \vdots \\ 0 \end{pmatrix} \right\}$$

- Definition A system is controllable (with look-ahead N) if for any delay sequence and any pair (x,y), there is a control sequence that maps x on y

- Observation If the pair (A,b) is uncontrollable, then the delayed system is uncontrollable.

- It seems hard to do… Let us first suppose we know all the future delays (infinite look-ahead)

Design of LTIs with switched delays

The infinite look-ahead case

- Theorem for n=m=1, there is an explicit formula for a linear controller that achieves deadbeat stabilization, even if N=1

  (based on a generalization of the Ackermann formula for delayed LTI)

$$K^*(d) = (-a^{d+1}/b, -a^d, -a^{d-1}, \ldots, -a)$$

- So, does a controllable system always remain controllable with delays?

- No! when n>1, nastier things can happen…

  Example:

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$$

$$x_1 = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad D = \{0, 1\}, \quad \sigma(t) = t \bmod 2$$

$$x_2 = A^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Bv(1) + Bv(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v(1) + v(2) \end{pmatrix}$$

➔ The system is not stabilizable, even with infinite lookahead

Design of LTIs with switched delays
The infinite look-ahead case

- A sufficient condition for uncontrollability (informal): if A,B can be put in the following form (under similarity transformation):

$$A = \left(\begin{array}{c|c|c} 0 & X & 0 \\ \hline 0 & 0 & X \\ \hline X & 0 & 0 \end{array}\right) \qquad B = \left(\begin{array}{c} X \\ \hline 0 \\ \hline 0 \end{array}\right)$$

An adversary strategy can make this system uncontrollable:

$$\forall t, t + d(t) \neq 1 \mod 3$$

Is it also necessary?

Would be nice, because we can prove …

- Theorem There is a polynomial time algorithm that decides whether such an adversary strategy is possible

Design of LTIs with switched delays

The infinite look-ahead case

- Answer: No! There are more intricate examples

$$A = \begin{pmatrix} \sin\theta_1 & -\cos\theta_1 & 0 & 0 \\ \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ 0 & 0 & \sin\theta_2 & -\cos\theta_2 \\ 0 & 0 & \cos\theta_2 & \sin\theta_2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$D = \{0, 1, \ldots, 121\} \qquad \theta_1 = \tfrac{\pi}{120} \qquad \theta_2 = \tfrac{\pi}{60}$$

$$\sigma(t) = \begin{cases} 0 & if \;\; 0 \le t \le 2 \\ 121 - t \; mod(121) & if \;\; t \ge 3 \end{cases}$$

Design of LTIs with switched delays

The infinite look-ahead case

- Theorem Controllability is decidable (in exponential time)

  Proof Split the problem into a nihilpotent matrix and a regular matrix

$$TAT^{-1} = \begin{pmatrix} J_{0,k} & 0 \\ 0 & A' \end{pmatrix}, \quad Tb = \begin{pmatrix} b_0 \\ b' \end{pmatrix}$$

  - Lemma: The nihilpotent case is completely combinatorial
  - Lemma: The regular case can be decided thanks to a finite dimension argument

Algo: try every delay sequence of length smaller than some bound L and look for a 'loop'

L= $\binom{n+2|D|}{2|D|}$

- Corollary controllability with infinite look-ahead = controllability with arbitrarily large but finite look-ahead

Outline

- Joint spectral characteristics

- Application: WCNs and switching delays
  - Modeling
  - Stability analysis
  - Controllability with infinite look-ahead
  - Approximate controllability (with small look-ahead)

- LMI methods for switching systems stability

- Conclusion and perspectives

Design of LTIs with switched delays

The small look-ahead case

- Example the delay independent case

n=m=1, D={0,1}: find k_1 , k_2 such that

$$x(t+1) = ax(t) \quad \begin{array}{l} +k_1 x(t) + k_2 x(t-1) \quad \text{if } \sigma(t) = 0 \\ +k_1 x(t-1) + k_2 x(t-2) \quad \text{if } \sigma(t-1) = 1 \end{array}$$

is stable, whatever switching signal $\sigma(t)$ occurs?

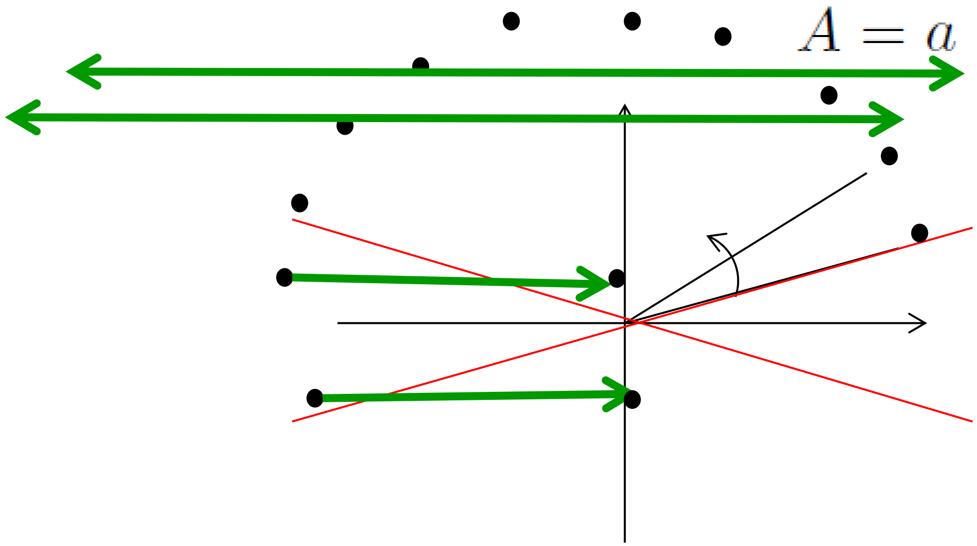
  - a>3: impossible
  - a=2: possible, but you'll need memory
  - a=1.1: possible with $k_2 = 0$
  - a<1: no need for a controller!

➔ Question: how to algorithmically decide if stabilization is possible, even in this simplest case?

Design of LTIs with switched delays the delay-independent case

A linear controller is not always sufficient:

Example: a 2D system with two possible delays



$$A = a\begin{pmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(alpha) \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$D = \{0, 1\}$$

- Theorem For the above system, there exist values of the parameters such that no linear controller can stabilize the system, but a nonlinear bang-bang controller does the job.

LTIs with switched delays
Open questions

- Theorem: controllability is decidable (in exponential time)

Proof: split the problem into a nihilpotent matrix and a regular matrix

$$TAT^{-1} = \begin{pmatrix} J_{0,k} & 0 \\ 0 & A' \end{pmatrix}, \quad Tb = \begin{pmatrix} b_0 \\ b' \end{pmatrix}$$

- Lemma : The nihilpotent case is completely combinatorial
- Lemma : The regular case can be decided thanks to a dimensionality argument

Algo: try every delay sequence of length smaller than L and look for a 'loop'

L= $\begin{pmatrix} n+2|D| \\ 2|D| \end{pmatrix}$

What if there are several inputs?

LTIs with switched delays
Open questions

- Answer: No! There are more intricate examples

$$A = \begin{pmatrix} \sin\theta_1 & -\cos\theta_1 & 0 & 0 \\ \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ 0 & 0 & \sin\theta_2 & -\cos\theta_2 \\ 0 & 0 & \cos\theta_2 & \sin\theta_2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$D = \{0, 1, \ldots, 121\} \qquad \theta_1 = \frac{\pi}{120} \qquad \theta_2 = \frac{\pi}{60}$$

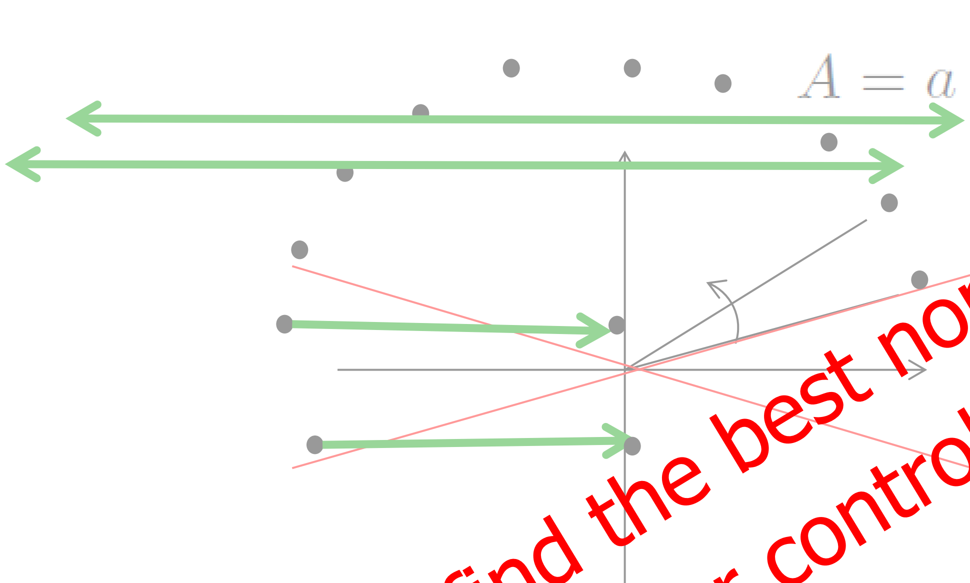
$$\sigma(t) = \begin{cases} 0 & if \;\; 0 \leq t \leq 2 \\ 121 - t \; mod(121) & if \;\; t \geq 3 \end{cases} \qquad L = \binom{n + 2|D|}{2|D|}$$

Are there examples with exponentially large periods?

LTIs with switched delays
Open questions

A linear controller is not always sufficient:

Exemple: a 2D system with two possible delays



$$A = a\begin{pmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(alpha) \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$D = \{0, 1\}$$

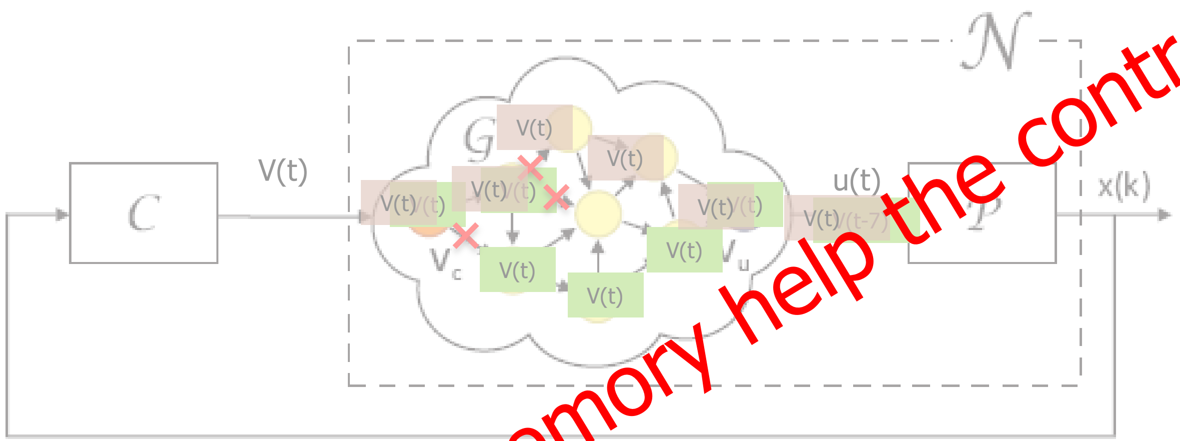
- Theorem: For the above system, there exist values of the parameters such that no linear controller can stabilize the system, but a nonlinear bang-bang controller does the job.

How to find the best nonlinear controller?

The best linear controller?

LTIs with switched delays
Open questions

4) How much memory should we give to the controller?



Normal (fixed) delay systems need d

→ We (arbitrarily) chose the maximum delay

$$x(t+1) = Ax(t) + Bu(v(t-d_{max} : t), \sigma(t-d_{max} : t))$$

$$u_d(t) = \sum_{t' < t : t' + \sigma(t') = t+d} v(t')$$

Can additional memory help the controller?

Outline

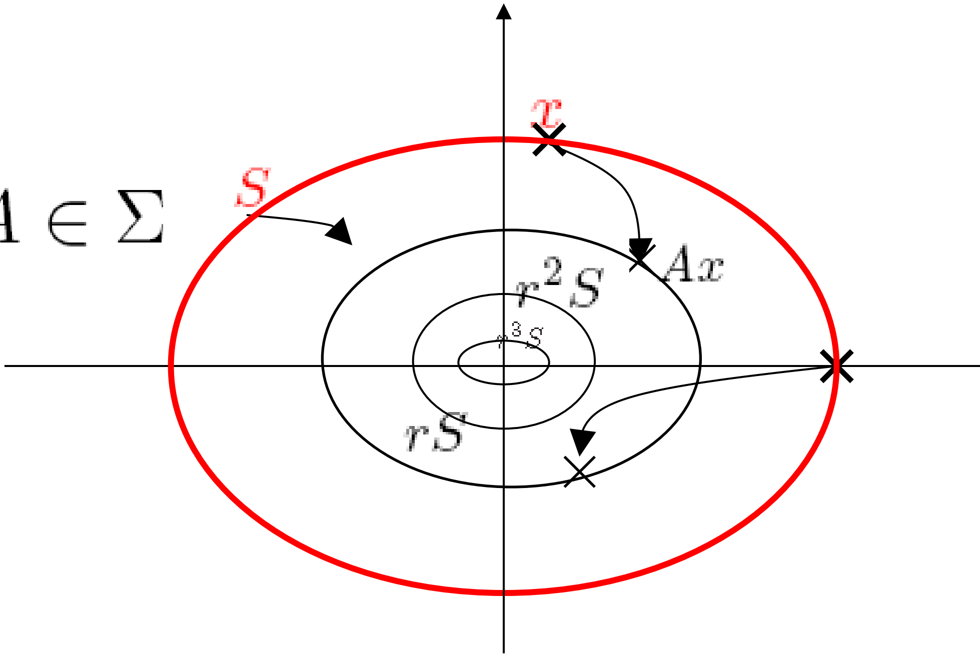
- Joint spectral characteristics
- Application: WCNs and switching delays
- LMI methods for switching systems stability
- Conclusion and perspectives

LMI methods

- The CQLF method

$$\begin{array}{lll} \inf_{r\in\mathbb{R}^+} & r & \\ \text{s.t.} & & \\ A^TPA & \preceq & r^2P, \quad \forall A\in\Sigma \\ P & \succeq & 0. \end{array}$$

$$\Leftrightarrow \frac{|Ax|_P}{|x|_P} \leq r$$



- Theorem The best ellipsoidal norm $||\cdot||_{E_*}$ approximates the joint spectral radius up to a factor $\sqrt{n}$ [Ando Shih 98]

LMI methods

- Other methods using LMIs have been proposed
  - Based on symmetric algebras [Blondel Nesterov 05]
  - Based on sum of squares [Parrilo Jadbabaie 08]

- All these methods can be seen as trying to approximate the best norm with more and more complex curves

…Or, the symmetric algebras (the SOS) allow for lifting the matrices in a higher-dimensional (but more gentle) space

There exists a PTAS for the jsr computation
Algorithm that approximate the joint spectral radius of arbitrary sets of n by n matrices up to an arbitrary accuracy ϵ in $\mathcal{O}(n^{m^{\frac{1}{\epsilon}}})$ operations

Yet another LMI method

- A strange semidefinite program

$$
\begin{array}{lcl}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P & \succeq & 0.
\end{array}
$$

$$\Rightarrow \qquad \rho \leq r$$

[Goebel, Hu, Teel 06]

- But also… [Daafouz Bernussou 01]

[Bliman Ferrari-Trecate 03]

[Lee and Dullerud 06] …

Yet another LMI method

- An even stranger program:

$$\begin{array}{lll} \min_{r\in\mathbb{R}^+} & & r \\ \text{s.t.} & & \\ A_1^T P A_1 & \preceq & r^2 P, \\ (A_2A_1)^T P (A_2A_1) & \preceq & r^4 P, \\ (A_2^2)^T P (A_2^2) & \preceq & r^4 P, \\ P & \succeq & 0. \end{array}$$

$$\rho \leq r$$

[Ahmadi, J., Parrilo, Roozbehani10]

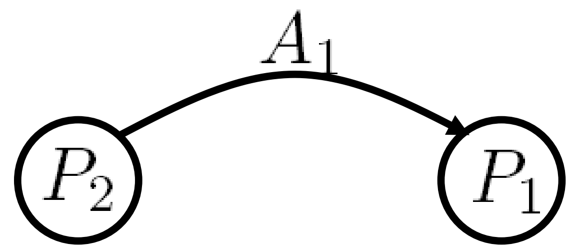
Yet another LMI method

- Questions:
  - Can we characterize all the LMIs that work, in a unified framework?
  - Which LMIs are better than others?
  - How to prove that an LMI works?
  - Can we provide converse Lyapunov theorems for more methods?

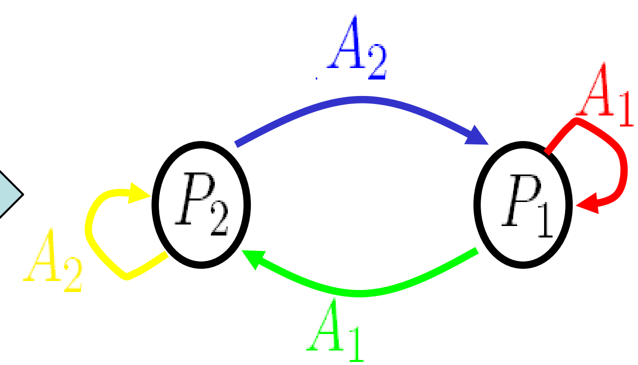
From an LMI to an automaton

- Automata representation Given a set of LMIs, construct an automaton like this:

$$A_1^T P_1 A_1 \preceq r^2 P_2 \quad \Longleftrightarrow$$



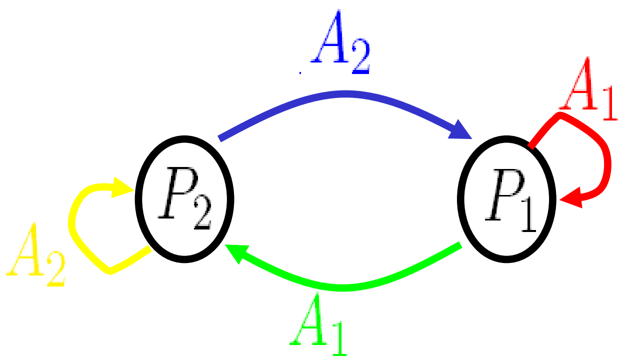
$$\begin{array}{lcl} \min_{r \in \mathbb{R}^+} & & r \\ \text{s.t.} & & \\ A_1^T P_1 A_1 & \preceq & r^2 P_1, \\ A_2^T P_1 A_2 & \preceq & r^2 P_2, \\ A_1^T P_2 A_1 & \preceq & r^2 P_1, \\ A_2^T P_2 A_2 & \preceq & r^2 P_2, \\ P_i & \succeq & 0. \end{array} \quad \Longleftrightarrow$$



- Definition A labeled graph (with label set A) is path-complete if for any word on the alphabet A, there exists a path in the graph that generates the corresponding word.
- Theorem If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability. [Ahmadi J. Parrilo Roozbehani 11]

An obvious question: are there other valid criteria?

- Theorem



$$
\begin{array}{lcl}
\min_{r\in\mathbb{R}+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}
$$

Path complete

???

Sufficient condition for stability

If G is path-complete, the corresponding semidefinite program is a sufficient condition for stability.

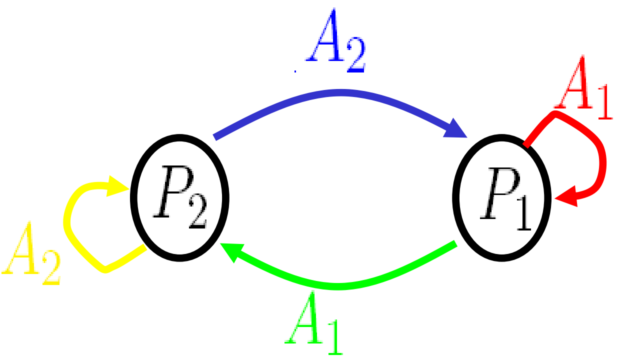
- Are all valid sets of equations coming from path-complete graphs?

- …or are there even more valid LMI criteria?

Are there other valid criteria?

- Theorem Non path-complete sets of LMIs are not sufficient for stability.

[J. Ahmadi Parrilo Roozbehani 12]



$$
\begin{array}{lll}
\min_{r \in \mathbb{R}^+} & & r \\
\text{s.t.} & & \\
A_1^T P_1 A_1 & \preceq & r^2 P_1, \\
A_2^T P_1 A_2 & \preceq & r^2 P_2, \\
A_1^T P_2 A_1 & \preceq & r^2 P_1, \\
A_2^T P_2 A_2 & \preceq & r^2 P_2, \\
P_i & \succeq & 0.
\end{array}
$$

Path complete ??? Sufficient condition for stability

- Corollary

  It is PSPACE complete to recognize sets of equations that are a sufficient condition for stability

- These results are not limited to LMIs, but apply to other families of conic inequalities

What about the other quantities?

	Arbitrary approximation	Arbitrary approximation in polynomial time	Arbitrary approximation for positive matrices	Decidability
Joint Spectral Radius	v	v	v	?
Joint Spectral Subradius	x	x	v	x
Lyapunov Exponent	x	x	v	x
p-radius	**Depends on p**	**Depends on p**	v	?

Outline

- Joint spectral characteristics
- Application: WCNs and switching delays
- LMI methods for switching systems stability
- Conclusion and perspectives

Conclusion

[Furstenberg Kesten, 1960]

[Gurvits, 1995]

[Rota, Strang, 1960]

[Blondel Tsitsiklis, 98+]

[Daafouz Bernussou 03]

[Johansson Rantzer 98]

[Lee Dullerud 06]

60s 70s

Mathematical properties

90s

TCS inspired
Negative
Complexity results

2000s

Lyapunov/LMI
Techniques
(S-procedure)

now

CPS applic.
Ad hoc
techniques

Applications

[Tsitsiklis 84]

[Jadbabaie Lin Morse 03]

[Chevalier Hendrickx J. 13]

$$\mathrm{cap}(D) = \lim_{n\to\infty} \frac{\log_2 \delta_n(D)}{n}$$

$$\mathrm{cap}(D) = \log_2 \rho(\Sigma(D))$$

IOOOIIIOIOI

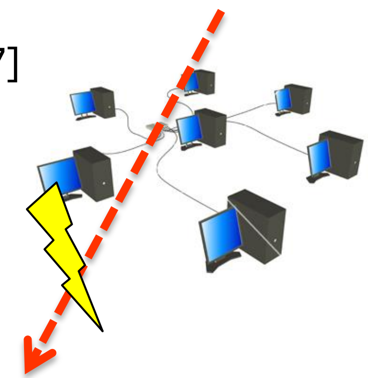
[Moision Orlitsky Siegel 01]

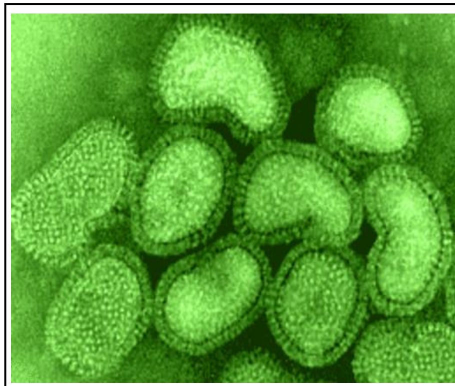
[Blondel J. Protasov 06]

[Dima Asarin 10]

[Beal crochemore Moision Siegel 12]

[Shorten et al. 07]





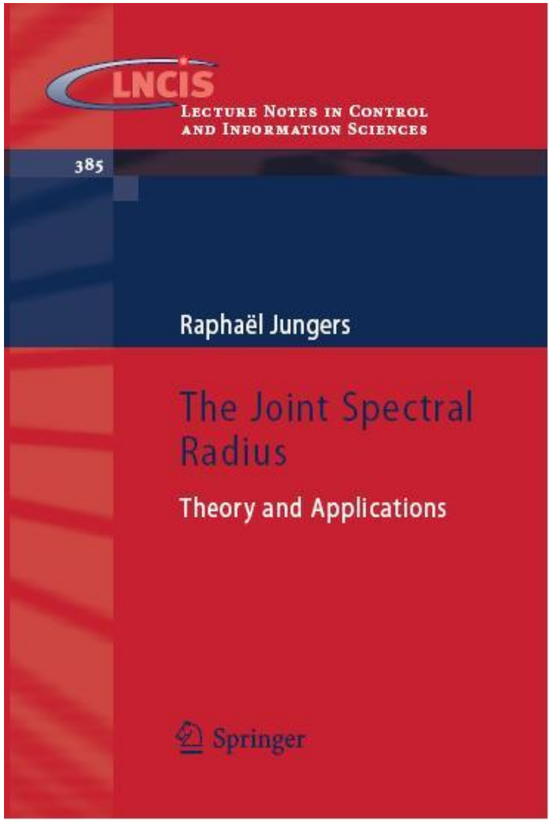
[Hernandez-Varga colaneri Middleton Blanchini 10]

Thanks!

Questions?

Ads

The JSR Toolbox:
http://www.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox
[Van Keerberghen, Hendrickx, J. HSCC 2014]



Several open positions:
raphael.jungers@uclouvain.be

References:
http://perso.uclouvain.be/raphael.jungers/

Joint work with
A.A. Ahmadi (Princeton), M-D di Benedetto (DEWS), A. D'innocenzo (DEWS), P. Parrilo (MIT), M. Roozbehani (MIT)