This looks like Fibonacci... But is it really? A combinatorial open problem

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A complexity problem

Linear Programming

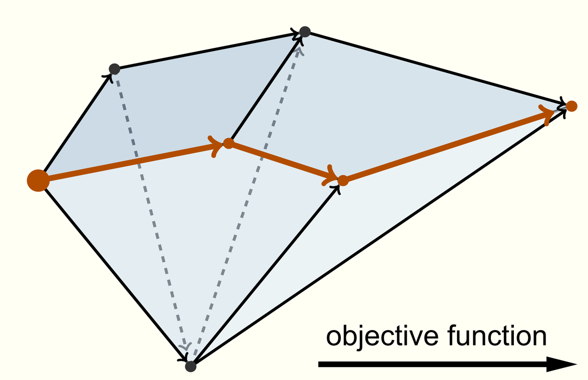
Markov Decision Processes

Resolution

Simplex algorithm

"Follow the "best" outgoing edge around you and iterate until you reach the sink"

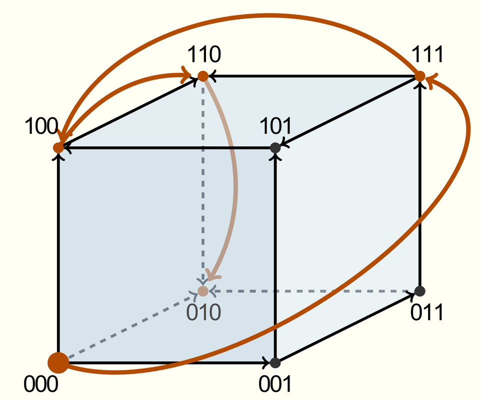
Worst case : exponential [Klee & Minty '72] [Friedmann '11]



Policy Iteration

"Follow all outgoing edges around you at once and iterate until you reach the sink"

Worst case : exponential [Fearnley '10]



How exponential can Policy Iteration be?

relaxation

Acyclic Unique Sink Orientation

Every sub-cube has a unique sink

There are no cycles

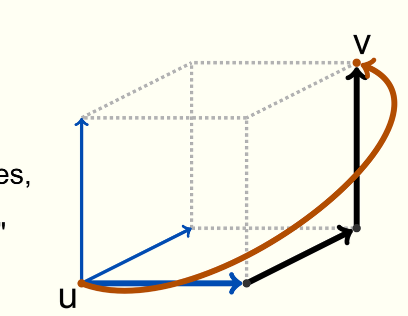
Goal : Bound the number of jumps of PI in any AUSO of dimension n

Non-inclusion property

"Let $T(w)$ be the set of dimensions towards which the outgoing edges of w point to and let u and v be two vertices of an AUSO such that there exists a directed path from u to v. Then $T(u) \not\subseteq T(v)$."

k outgoing edges → k jumps

"If at some step of PI, we are at a vertex u with k outgoing edges, then there exists a path of length $\geq k$ from u to the endpoint v"



relaxation

Order-Regular matrices

$\Pi \in \{0,1\}^{m \times n}$ is Order-Regular

iff $\forall\, 0 < i < j < m, \exists\, 0 < k \leq n$

s.t.: $\Pi_{i,k} \neq \Pi_{i+1,k} = \Pi_{j,k} = \Pi_{j+1,k}$

$$\begin{array}{cc} & \begin{array}{ccc} & k & \end{array} \\ \begin{array}{c} i \\ \\ \\ j \\ \\ \end{array} & \begin{array}{|ccc|} 0 & \boxed{0} & 0 \\ 1 & \boxed{1} & 1 \\ 1 & 0 & 0 \\ 1 & \boxed{1} & 0 \\ 0 & \boxed{1} & 0 \end{array} \end{array}$$

Goal : Bound the number of rows (iterations) of any OR matrix with n columns (dimensions)

n	1	2	3	4	5	6	7
Maximum m	2	3	5	8	13	21	≥34

Conjecture [Hansen & Zwick '12]

$$m \leq \underbrace{F_{n+2}}_{\text{Fibonacci numbers}} = O(\phi^n)$$

Fibonacci numbers

golden ratio

Towards new bounds

One combinatorial problem left to solve

First ingredient : the non-inclusion property

Let $\mathcal{T}$ be the set of all subsets of n elements: $\mathcal{T} = \{T \mid T \subseteq \{1,...,n\}\}$

We consider any ordering $T_1 \prec ... \prec T_{2^n}$ of the elements of $\mathcal{T}$ such that:

$$\forall\, i < j \;:\; T_i \not\subseteq T_j$$

Second ingredient : k outgoing edges → k jumps

Let $I \subseteq \{1,...,2^n\}$ be a subset of indices satisfying:

$$\forall\, i,j \in I, i < j \;:\; i + \mathrm{card}(T_i) \leq j$$

Our goal

Find the maximum cardinality of I under these constraints

or an upper bound on this cardinality

Two extreme cases for the ordering

Sets with larger cardinality first

T_1 T_2 T_3 T_4 T_5 T_6 T_7 T_8 T_9 T_{10} T_{11} T_{12} T_{13} T_{14} T_{15} T_{16}

1111 < 1110 < 1101 < 1011 < 0111 < 1100 < 1010 < 1001 < 0110 < 0101 < 0011 < 1000 < 0100 < 0010 < 0001 < 0000

4 jumps, 3 jumps, 2 jumps, 2 jumps, 1 jump, 1 jump, 1 jump, 1 jump

So $I = \{1,5,8,10,12,13,14,15,16\}$ and $\mathrm{card}(I) = 9$

Observation: the indices of I that maximize $\mathrm{card}(I)$ can be chosen greedily in this case

Asymptotically when n grows: $\mathrm{card}(I) \sim 2 \cdot \frac{2^n}{n}$ → **optimal ?**

Anti-lexicographic order

T_1 T_2 T_3 T_4 T_5 T_6 T_7 T_8 T_9 T_{10} T_{11} T_{12} T_{13} T_{14} T_{15} T_{16}

1111 < 1110 < 1101 < 1100 < 1011 < 1010 < 1001 < 1000 < 0111 < 0110 < 0101 < 0100 < 0011 < 0010 < 0001 < 0000

So $I = \{1,5,8,9,12,13,15,16\}$ and $\mathrm{card}(I) = 8$

The worst case should be somewhere between these two cases

Our hope:

Show that $\mathrm{card}(I) \leq 2 \cdot \frac{2^n}{n} + O\left(p(n)\right)$

the constant is important

a polynomial

n	1	2	3	4	5	6	7
Maximum m	2	3	5	8	13	21	≥34
$2 \cdot \frac{2^n}{n}$	4	4	5.3	8	12.8	21.3	36.6