Introduction to Goodwillie-Weiss manifold calculus.

Short summary: The historical root of Goodwillie-Weiss manifold calculus can be found in the famous "eversion of the sphere", in other words in the proof in the late 1950's by Steve Smale that a sphere in the usual 3-dimensional Euclidean space can be turned inside out through a continuous family of immersions. Indeed this proof leads to a description of the homotopy type of the space of immersions of the 2-dimensional sphere into the 3-dimensional euclidean space, $\mathrm{Imm}(S^2, \mathbf{R}^3)$, in terms of homotopical universal construction; the key fact here is that the homotopy functor $\mathrm{Imm}(\bullet, \mathbf{R}^3)$ defined on the category of open subsets of S^2 is a *linear functor*, that is it converts homotopy pushouts into homotopy pullbacks. This approach was generalized by Goodwillie, Klein, and Weiss in the 1990's to the study of spaces of smooth embedding of a smooth manifold M into another manifold W, $\mathrm{Emb}(M, W)$. In that case the functor $\mathrm{Emb}(\bullet, W)$ is not linear anymore but it can be arbitrarely well approximated by *polynomial* functor, leading to an approximation of functors akin to approximations of real functions by Taylor polynomials. The theory is also very much related to the little n-disks operad. In particular the rational formality of this operad, established by Kontsevich, has applications to the computation of the rational homology of the space of embeddings of M into $\mathbb{R}^n$.

Outline of the lectures

- Spaces of immersions $\mathrm{Imm}(M, W)$ and embeddings $\mathrm{Emb}(M, W)$ between smooth manifolds M and W.
- Quick review on homotopy limits and colimits.
- Linear, polynomial and analytic functor. A (contravariant) functor is called *linear* it it sends homotopy pushouts to homotopy pullbacks. A functor is *polynomial of degree* $\leq k$ if it has an analogous property where the square diagrams (of the homotopy pushout/pullback) are replaced by $k+1$-dimensional cubical diagrams. A functor is *analytic* if it is well approximated by polynomial functors.
- The Smale-Hirsh theorem: the functor

  $$\mathrm{Imm}(-, W) \colon \mathcal{O}(M) \to \mathrm{Top}\,,\; U \mapsto \mathrm{Imm}(U, W),$$

  where $\mathcal{O}(M)$ is the category of open subsets in M, is a linear functor.
- The Goodwillie-Klein-Weiss theorem: the functor $\mathrm{Emb}(-, W)$ is analytic. Description of its approximations as homotopy limits of configuration spaces.
- The operad E_n of little n-disks and its relation with Goodwillie-Weiss calculus applied to the study of $\mathrm{Emb}(M, \mathbb{R}^n)$.
- Kontsevich's formality of the little disks operad and its application to the computation of the rational homology of spaces of embeddings.

References

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