The commutator condition for higher central extensions

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joint work with Diana Rodelo

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- aim of the talk: explaining some (co)homological consequences of the *Smith is Huq* property
- main result based on unpublished work of Tomas Everaert's
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- context: semi-abelian categories

Overview

1. Two problems, one solution
   - Homology
   - The commutator condition (CC)
   - Cohomology
2. The commutator condition (CC)
   - Two commutators
   - Degree 1
   - Degree 2
   - Main result
3. A counterexample
4. Conclusion

First problem: homology

Theorem [Everaert, Gran & VdL, 2008]

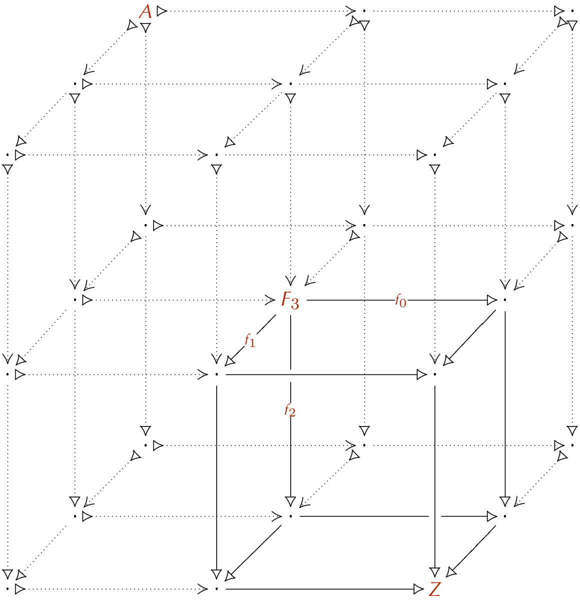
In a semi-abelian monadic category $\mathcal{A}$, for any n-presentation F of Z,

$$\mathrm{H}_{n+1}(Z, \mathbf{Ab}(\mathcal{A})) \cong \frac{[F_n, F_n] \wedge \bigwedge_{i \in n} \mathrm{Ker}(f_i)}{L_n[F]}.$$

Here F_n is the initial object of F and the f_i are the initial arrows. $\square$

$A = \bigwedge_{i \in n} \mathrm{Ker}(f_i)$ is called the **direction** of F.

A three-fold (central) extension of Z by A



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Problem

The denominator is not explicit!

$L_n[F]$ is the smallest normal subobject of F_n which, when divided out, makes F central.

In the examples it is a join of commutators.

Solution

Characterise higher central extensions in terms of commutators.

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The commutator condition (CC)

Definition

An n-fold extension F is **H-central** when

$$\Big[\bigwedge_{i\in I} \mathrm{Ker}(f_i), \bigwedge_{i\in n\setminus I} \mathrm{Ker}(f_i)\Big] = 0$$

for all $I \subseteq n$.

Huq or Higgins commutators

Definition

A semi-abelian category satisfies the **commutator condition (CC)** when H-centrality is equivalent to centrality.

Degree-wise:
a semi-abelian category satisfies **(CCn)** when
an n-fold extension is H-central iff it is central.

- This means $L_n[F] = \bigvee_{I\subseteq n} \big[\bigwedge_{i\in I} \mathrm{Ker}(f_i), \bigwedge_{i\in n\setminus I} \mathrm{Ker}(f_i)\big]$.

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Second problem: cohomology

Proposition [Rodelo & VdL, 2011]

Let F be an n-fold extension. Then $1 \Rightarrow 2 \Rightarrow 3$:

1. F is central; $\leftarrow$ **Galois theory**
2. F is an n-torsor; $\leftarrow$ **Duskin–Glenn cohomology**
3. F is H-central. $\square$

(CC) says $3 \Rightarrow 1$, so:

Theorem [Rodelo & VdL, 2011]

Let Z be an object, A an abelian object in a semi-abelian category $\mathcal{A}$. When $\mathcal{A}$ has (CC), there is an isomorphism

$$\mathrm{H}^{n+1}(Z, A) \cong \mathrm{Centr}^{n}(Z, A)$$

for all $n \geqslant 1$. $\square$

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Two commutators

Huq

For K, $L \lhd X$, the **Huq commutator** $[K, L]$ is the kernel of q:

$$
\begin{tikzcd}
& K \arrow[dl, "\langle 1_K, 0\rangle"'] \arrow[d, dotted] \arrow[dr, "k"] & \\
K \times L \arrow[r, dotted] & Q & X \arrow[l, dotted, "q"'] \\
& L \arrow[ul, "\langle 0, 1_L\rangle"] \arrow[u, dotted] \arrow[ur, "l"'] &
\end{tikzcd}
$$

Smith

For equivalence relations R, S on X

$$
\begin{tikzcd}
R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "\Delta_R"'] \arrow[r, "\Delta_S"] & S, \arrow[l, shift right=2, "s_2"'] \arrow[l, shift left=2, "s_1"]
\end{tikzcd}
$$

the **Smith commutator** $[R, S]^{\mathrm{S}}$ is the kernel pair of t:

$$
\begin{tikzcd}
& R \arrow[dl, "\langle 1_R, \Delta_S \circ r_1\rangle"'] \arrow[d, dotted] \arrow[dr, "r_2"] & \\
R \times_X S \arrow[r, dotted] & T & X \arrow[l, dotted, "t"'] \\
& S \arrow[ul, "\langle \Delta_R \circ s_2, 1_S\rangle"] \arrow[u, dotted] \arrow[ur, "s_1"'] &
\end{tikzcd}
$$

Two commutators

$$K \rhd\!\xrightarrow{r_2 \circ \ker(r_1)} X \xleftarrow{s_2 \circ \ker(s_1)}\!\lhd L \text{ normalisations of } R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \underset{s_1}{\overset{s_2}{\leftleftarrows}} S$$

- $[R, S]^{\mathrm{S}} = \Delta_X$ implies always $[K, L] = 0$ [Bourn & Gran, 2002].
- The converse is the **Smith is Huq** condition **(SH)**.

The situation in degree 1

$$X \xrightarrow{f} Z \quad \text{extension,} \quad K = \mathrm{Ker}(f) \quad \text{and} \quad R = \mathrm{R}[f]$$

is H-central when $[K, X] = 0$

is central when $[R, \nabla_X]^{\mathrm{S}} = \Delta_X$ [Gran, 2004]

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$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d, "g"] \\ D \arrow[r, "f"'] & Z \end{tikzcd}$$ double extension, $\begin{cases} K = \mathrm{Ker}(c) \\ L = \mathrm{Ker}(d) \end{cases}$ and $\begin{cases} R = \mathrm{R}[c] \\ S = \mathrm{R}[d] \end{cases}$

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Main result

Question

Is (CC) a higher-dimensional version of (SH)?

Answer

No!

Theorem

If a semi-abelian category has (CC2) then it has (CC).
In particular, (SH) $\Rightarrow$ (CC). $\square$

- So (CC) stays within bounds:
  under (SH) both homology and cohomology are well-behaved!
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Overview

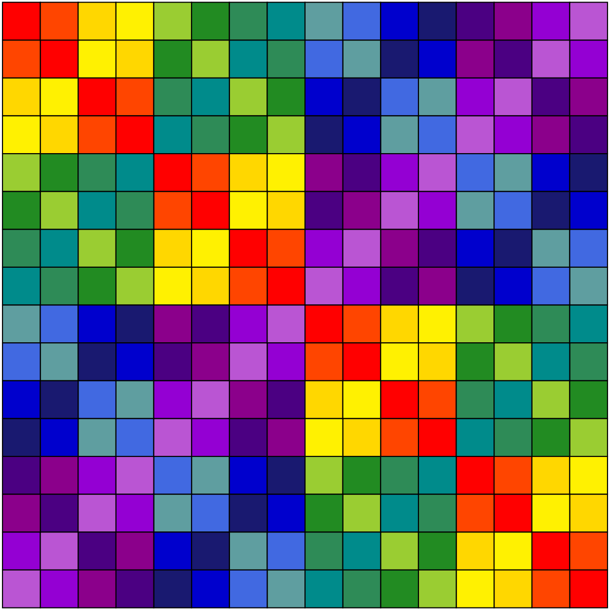
The category **Loop** of loops and loop homomorphisms

- A **loop** is a quasigroup with a neutral element: an algebraic structure $(X, \cdot, \backslash, /, 1)$ that satisfies $x \cdot 1 = x = 1 \cdot x$ and

$$y = x \cdot (x \backslash y) \qquad y = x \backslash (x \cdot y)$$
$$x = (x / y) \cdot y \qquad x = (x \cdot y) / y$$

- semi-abelian variety: $n = 1, \quad t(x, y) = x \cdot y, \quad t_1(x, y) = x / y$
- associative loop = group
- multiplication table of a loop = Latin square with unit
- associator: $[\![x, y, z]\!] = ((x \cdot y) \cdot z) / (x \cdot (y \cdot z))$ for $x, y, z \in X$

A counterexample



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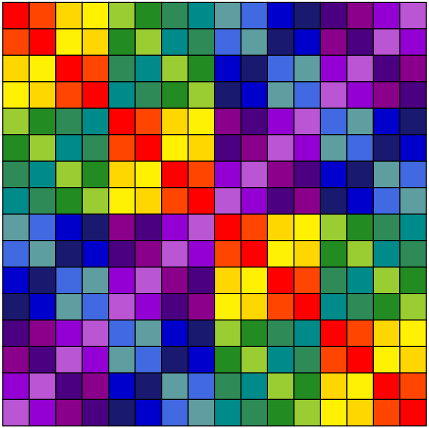
X is the (commutative) loop given by the Latin square



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- $[K, L] = 0 = [K \wedge L, X]$ but $[R, S]^{\mathrm{S}}$ **is non-trivial**
- indeed $1 \neq [\![k, l, x]\!]$ while $([\![k, l, x]\!], 1) \in [R, S]^{\mathrm{S}}$

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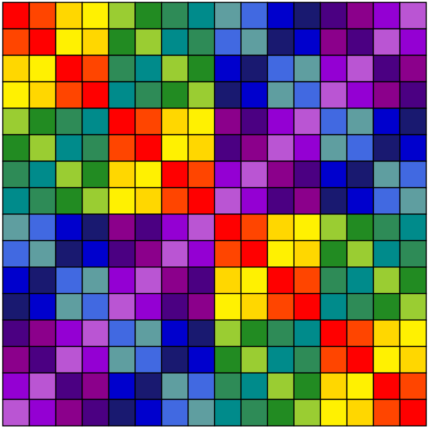

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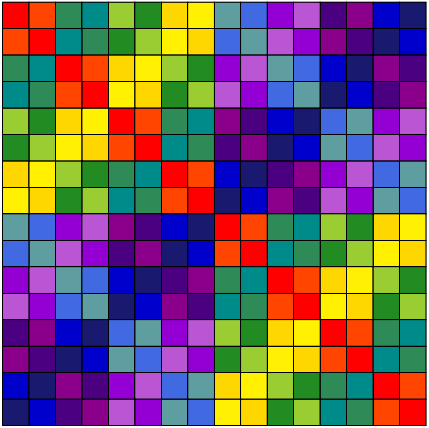

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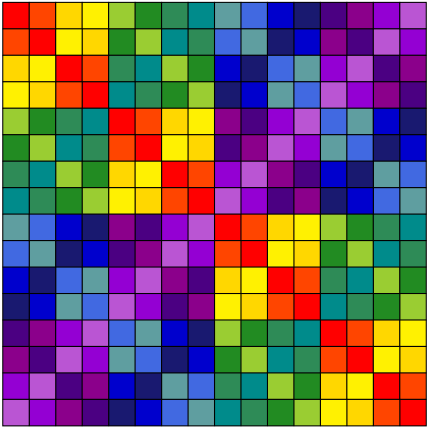

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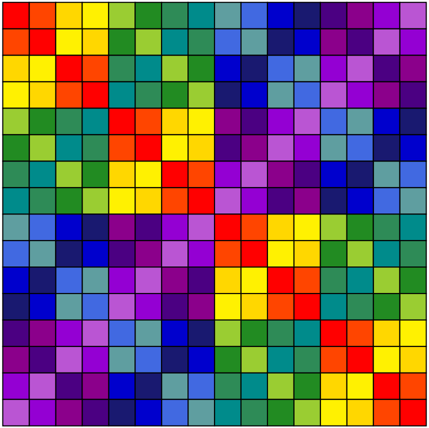

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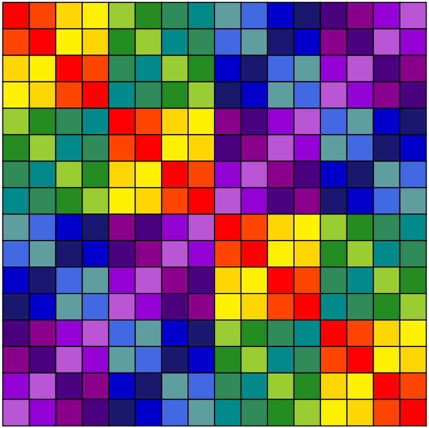

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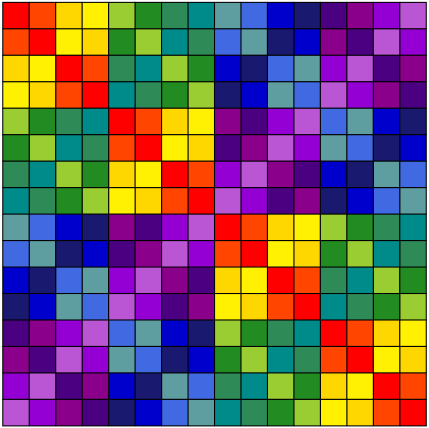

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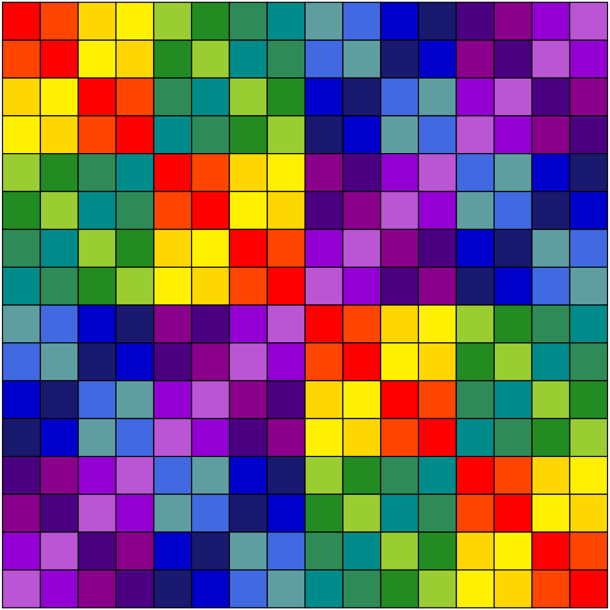

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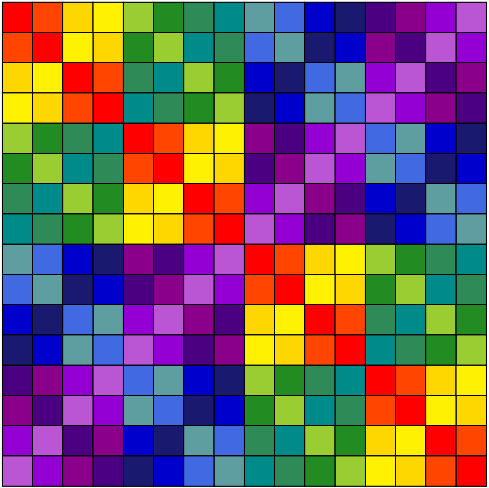

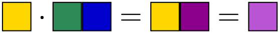

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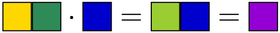

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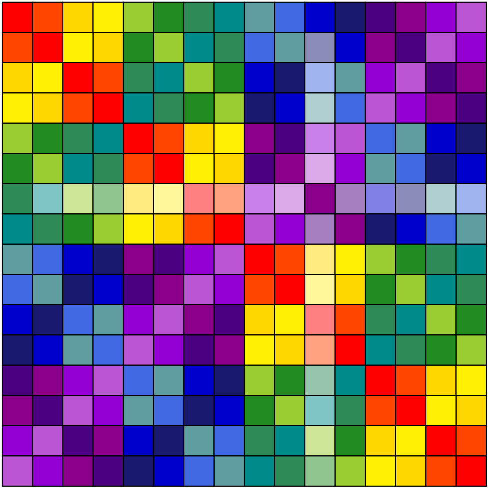
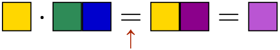
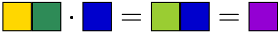
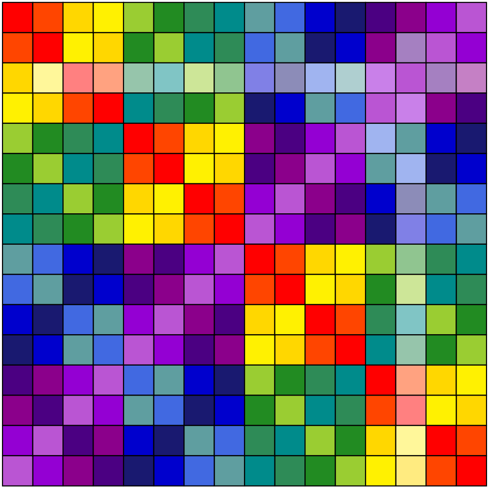
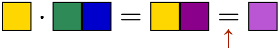
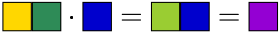


 while 

A counterexample



 while 

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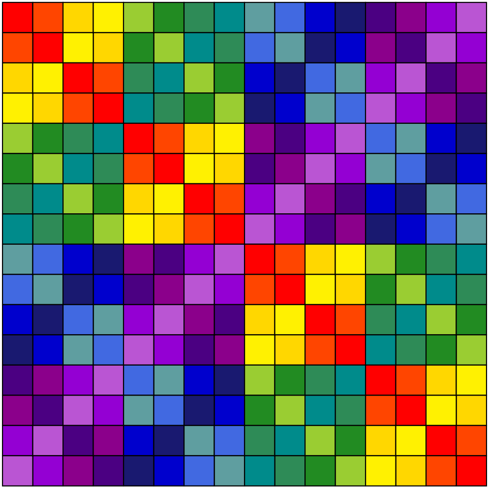

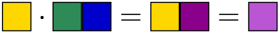

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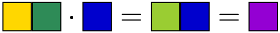

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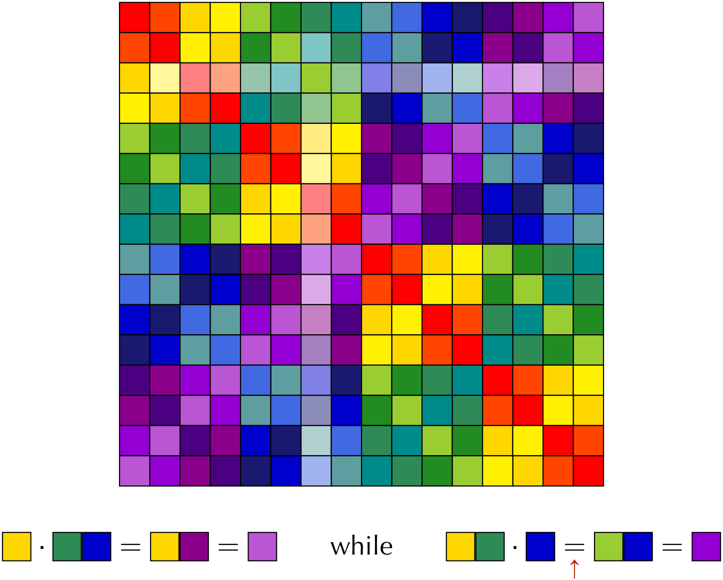

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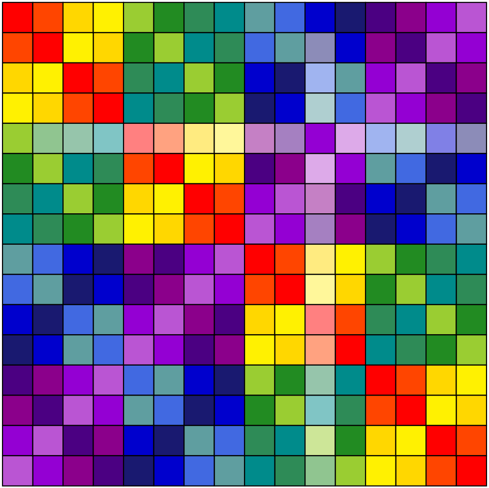

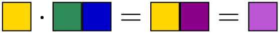

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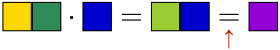

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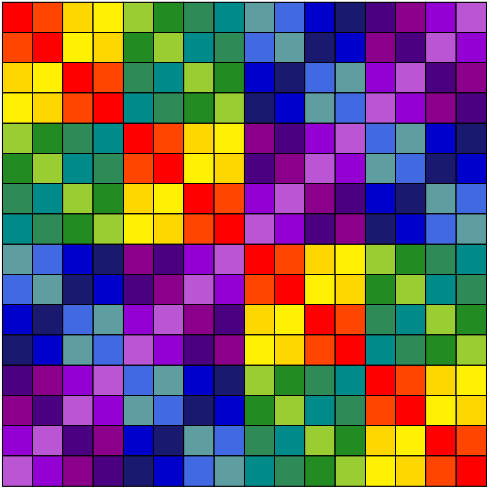

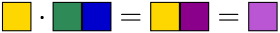

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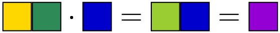
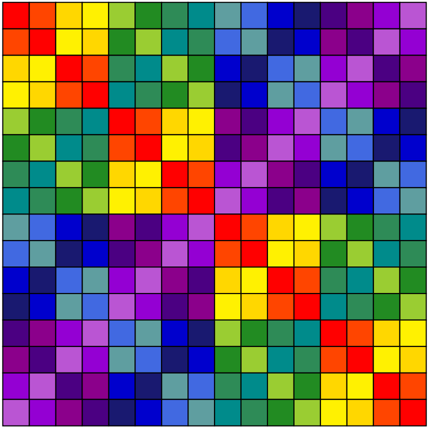


A counterexample

X is the (commutative) loop given by the Latin square



- $K =$  and $L =$  are normal in X
- $[K, L] = 0 = [K \wedge L, X]$ but $[R, S]^{\mathrm{S}}$ **is non-trivial**
- indeed $1 \neq [\![k, l, x]\!]$ while $([\![k, l, x]\!], 1) \in [R, S]^{\mathrm{S}}$

  $1 =$  and $k =$  $l =$  $x =$ 

Two commutators

Huq

For K, $L \lhd X$, the **Huq commutator** $[K, L]$ is the kernel of q:

$$
\begin{tikzcd}
 & K \arrow[dl, "\langle 1_K, 0\rangle"'] \arrow[d, dotted] \arrow[dr, "k"] & \\
K \times L \arrow[r, dotted] & Q & X \arrow[l, dotted, "q"'] \\
 & L \arrow[ul, "\langle 0, 1_L\rangle"] \arrow[u, dotted] \arrow[ur, "l"'] &
\end{tikzcd}
$$

Smith

For equivalence relations R, S on X

$$
\begin{tikzcd}
R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "\Delta_R" description] \arrow[r, "\Delta_S" description] & S, \arrow[l, shift right=2, "s_2"'] \arrow[l, shift left=2, "s_1"]
\end{tikzcd}
$$

the **Smith commutator** $[R, S]^{\mathrm{S}}$ is the kernel pair of t:

$$
\begin{tikzcd}
 & R \arrow[dl, "\langle 1_R, \Delta_S \circ r_1\rangle"'] \arrow[d, dotted] \arrow[dr, "r_2"] & \\
R \times_X S \arrow[r, dotted] & T & X \arrow[l, dotted, "t"'] \\
 & S \arrow[ul, "\langle \Delta_R \circ s_2, 1_S\rangle"] \arrow[u, dotted] \arrow[ur, "s_1"'] &
\end{tikzcd}
$$

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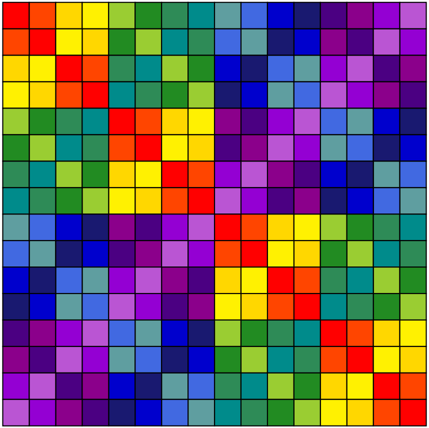

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