AN INTRODUCTION TO KAC–MOODY GROUPS OVER FIELDS: CORRECTIONS (28/11/2023)

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1. Chapter 1

• **Exercise 1.4 should read as follows.**

Exercise 1.4. Recall that if a subgroup G of $\mathrm{GL}_n(\mathbb{R})$ is of the form $G = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid f(A) = 0\}$ for some smooth function $f\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^m$ such that $T_x f$ is surjective for all $x \in G$, then G inherits from $\mathrm{GL}_n(\mathbb{R})$ a Lie group structure and $T_1 G = \{B \in \mathfrak{gl}_n(\mathbb{R}) \mid T_1 f(B) = 0\}$.

(1) Deduce that the group $\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = 1\}$ is a Lie group, with Lie algebra $\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathrm{Mat}_n(\mathbb{R}) \mid \operatorname{tr} A = 0\}$ (and same bracket and exponential map as $\mathrm{GL}_n(\mathbb{R})$).

(2) Compute the Lie algebra of the group $\{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^T \Omega A = \Omega\}$, where Ω is a given non-degenerate $n \times n$ matrix such that $\Omega^T \in \{\Omega, -\Omega\}$.

2. Chapter 3

• **The following text should replace the text in Section 3.7 from the sentence before (3.15) on page 52 until the end of the second paragraph on page 53 (i.e. until line 11):**

In order to have the simplest possible formulas for $(\cdot,\cdot)$ over $\mathfrak{h}$, we would now like to find a complement $\mathfrak{h}''$ of $\mathfrak{h}' = \bigoplus_{i\in I} \mathbb{C}\alpha_i^\vee$ in $\mathfrak{h}$ (i.e. $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$) such that

$$(h, h') = 0 \quad \text{for all } h, h' \in \mathfrak{h}''. \tag{3.15}$$

For this, we will need to assume that the restriction of $(\cdot,\cdot)$ to $\mathfrak{h}$ is symmetric (as we have seen, the restriction of $(\cdot,\cdot)$ to $\mathfrak{h}'$ is automatically symmetric; in particular, this extra assumption is automatically satisfied if A is invertible). With this additional assumption, we may find such a complement $\mathfrak{h}''$ of $\mathfrak{h}'$ as follows.

Set $\ell = \mathrm{rank} A$. For a choice of basis $\{h_1, \dots, h_{2n-\ell}\}$ of $\mathfrak{h}$, let $C = (c_{ij})_{1\leq i,j\leq 2n-\ell}$ with $c_{ij} := (h_i, h_j)$ be the matrix of the bilinear form $(\cdot,\cdot)|_{\mathfrak{h}}$ in this basis. Write $C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$, where C_{11} is an $\ell \times \ell$ matrix and C_{22}, C_{33} are $(n-\ell)\times(n-\ell)$ matrices.

Since $(\cdot,\cdot)|_{\mathfrak{h}'}$ is a symmetric bilinear form, it is diagonal in some basis of $\mathfrak{h}'$. In other words, we can choose $h_1, \dots, h_n$ forming a basis of $\mathfrak{h}'$ such that C_{11} is an invertible diagonal matrix and C_{12}, C_{21}, C_{22} are zero matrices. Thus we can assume that C has the form

$$C = \begin{pmatrix} C_{11} & 0 & C_{13} \\ 0 & 0 & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

and it remains to find suitable modifications of $h_{n+1}, \dots, h_{2n-\ell}$ so that also $C_{33} = 0$.

Note that the matrix $\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{pmatrix} = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \\ C_{31} & C_{32} \end{pmatrix}$ has rank n (since the restriction of $(\cdot,\cdot)$ to $\mathfrak{h} \times \mathfrak{h}'$ has rank n by (3.14)), and hence C_{32} is invertible. By Gaussian elimination, we can then find, for each $i = 1, \dots, n-\ell$, some $\lambda_j^{(i)} \in \mathbb{C}$ such that

$$(h_{n+k}, h_{n+i} - \sum_{j=\ell+1}^{n} \lambda_j^{(i)} h_j) = 0 \quad \text{for all } i,k \in \{1,\dots,n-\ell\}.$$

The desired modifications of $h_{n+1}, \dots, h_{2n-\ell}$ is then obtained by setting

$$h'_{n+i} := h_{n+i} - \tfrac{1}{2} \sum_{j=\ell+1}^{n} \lambda_j^{(i)} h_j \quad \text{for } i = 1,\dots,n-\ell,$$

since we then have (using the symmetry of $(\cdot,\cdot)|_{\mathfrak{h}}$)

$$(h'_{n+k}, h'_{n+i}) = (h_{n+k}, h_{n+i}) - \tfrac{1}{2}(h_{n+k}, \sum_{j=\ell+1}^{n} \lambda_j^{(i)} h_j) - \tfrac{1}{2}(h_{n+i}, \sum_{j=\ell+1}^{n} \lambda_j^{(k)} h_j) = 0.$$

We can therefore assume that the matrix C has the form $C = \begin{pmatrix} C_{11} & 0 & C_{13} \\ 0 & 0 & C_{23} \\ C_{13} & C_{23} & 0 \end{pmatrix}$ with C_{11}, C_{23} invertible, and hence C is invertible (its $2n-\ell$ column vectors are linearly independent). In other words, the restriction of $(\cdot,\cdot)$ to $\mathfrak{h}$ is nondegenerate. In particular, the radical $\mathcal{R}$ of $(\cdot,\cdot)$ is an ideal of $\mathfrak{g}(A) = \mathfrak{g}(A)_{\mathrm{Kac}}$ intersecting $\mathfrak{h}$ trivially, and hence $\mathcal{R} = \{0\}$, i.e. $(\cdot,\cdot)$ is nondegenerate.

- **The first sentence of the statement of Proposition 3.29 has to be modified accordingly:**

Proposition 3.29. *Let A be a GCM, and assume that $\mathfrak{g}(A)$ admits a nontrivial invariant bilinear form $(\cdot,\cdot)$ whose restriction to $\mathfrak{h}$ is symmetric. Set (...)*

3. Chapter 4

- **Section 4.2, page 69, Hint after (4.7): the hint is not very efficient and can be replaced by the following:**

[Hint: For the second equality, use Proposition 4.18(1) and the Serre relations.]

4. Chapter 7

- **Section 7.3.3, page 123, Exercise 7.26(1): replace "bijective" by "surjective".**
- **Section 7.4.5, page 147, Exercise 7.67: the exercise should read as follows.**

Exercise 7.67. Assume that $\mathcal{D}$ is free, cofree and cotorsion-free. Let $\varphi\colon \mathcal{D}^A_{\mathrm{Kac}} \to \mathcal{D}$ be the morphism provided by Exercise 7.20. Show that $\mathfrak{G}_{\mathcal{D}}(k)$ is a semi-direct product of $\mathfrak{G}_{\mathcal{D}^A_{\mathrm{Kac}}}(k)$ by a direct factor subtorus of $\mathfrak{T}_\Lambda(k)$. [Hint: Proceed as in the proof of Proposition 7.66.]

5. Appendix A

- **At the beginning of Example A.17, include the following assumption.**

Throughout this example, $\mathcal{A}$ is assumed to be free as a k-module.

6. Appendix B

• **Exercise B.16 should read as follows.**

Exercise B.16. Let (W, S) be a Coxeter system. The **Bruhat order** on W is the partial order $\leq$ defined as follows: for any $v, w \in W$, we have $v \leq w$ if and only if a reduced decomposition of v can be obtained from a reduced decomposition of w by deleting some (possibly none) fundamental reflections. Show that $v \leq w$ if the chamber $vC_0 \in \mathrm{Ch}(\Sigma)$ is on a minimal gallery from C_0 to wC_0.