KAC–MOODY GEOMETRY IN KIEL
PREREQUISITES MINICOURSE ON KAC–MOODY GROUPS

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For this minicourse, I will assume the audience is familiar with the content of Sections 1–3 below. The appendices contain some extra information, mainly in the form of examples illustrating some concepts that will appear at some point in the minicourse; a previous familiarity with those concepts would be helpful, but is not absolutely necessary to get a global understanding of the minicourse.

1. Preliminaries

1.1. **Universal enveloping algebra of a Lie algebra.** Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$. Its **universal enveloping algebra** is the (unital, associative) $\mathbb{K}$-algebra $\mathcal{U}(\mathfrak{g})$, defined as the quotient of the tensor algebra $T(\mathfrak{g}) = \mathbb{K} \oplus \bigoplus_{n\geq 1} \mathfrak{g}^{\otimes n}$ by the two-sided ideal generated by the relations $x \otimes y - y \otimes x = [x, y]$ for all $x, y \in \mathfrak{g}$.

The canonical map $\iota\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is an injective Lie algebra morphism (when considering $\mathcal{U}(\mathfrak{g})$ as a Lie algebra with respect to the commutator bracket). The algebra $\mathcal{U}(\mathfrak{g})$ satisfies the following universal property: if $\mathcal{A}$ is a unital associative algebra with a Lie algebra morphism $\varphi\colon \mathfrak{g} \to \mathcal{A}$, there is a unique algebra morphism $\widetilde{\varphi}\colon \mathcal{U}(\mathfrak{g}) \to \mathcal{A}$ such that $\widetilde{\varphi} \circ \iota = \varphi$.

1.2. **Gradations.** Let M be an abelian group (e.g., $M \cong \mathbb{Z}^\ell$). A Lie algebra $\mathfrak{g}$ is M**-graded** if it admits a vector space decomposition $\mathfrak{g} = \bigoplus_{\alpha\in M} \mathfrak{g}_\alpha$, where the $\mathfrak{g}_\alpha$ are vector subspaces such that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in M$.

An associative algebra $\mathcal{A}$ is M**-graded** if it admits a vector space decomposition $\mathcal{A} = \bigoplus_{\alpha\in M} \mathcal{A}_\alpha$, where the $\mathcal{A}_\alpha$ are vector subspaces such that $\mathcal{A}_\alpha \cdot \mathcal{A}_\beta \subseteq \mathcal{A}_{\alpha+\beta}$ for all $\alpha, \beta \in M$.

Example 1.1. Let $\mathfrak{g} = \bigoplus_{\alpha\in M} \mathfrak{g}_\alpha$ be an M-graded Lie algebra. Then its universal enveloping algebra $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ is also M-graded: $\mathcal{U} = \bigoplus_{\alpha\in M} \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha$ is spanned by all products $x_1 \dots x_n$ ($x_i \in \mathfrak{g}$) with $\sum_{i=1}^n \deg(x_i) = \alpha$.

An element $x \in \mathfrak{g}_\alpha$ (resp. $x \in \mathcal{A}_\alpha$) is called **homogeneous**, of **degree** $\deg(x) := \alpha$.

2. Finite-dimensional simple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, such as $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ (the traceless $n \times n$ matrices, with Lie bracket $[A, B] := AB - BA$). Thus $\mathfrak{g}$ is a complex vector space with a Lie bracket $[\cdot, \cdot]$, which is encoded in the **adjoint representation**

$$\mathrm{ad}\colon\ \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad \mathrm{ad}(x)y := [x, y] \quad \text{for all } x, y \in \mathfrak{g}$$

of $\mathfrak{g}$ on itself.

The first step in trying to understand the structure of $\mathfrak{g}$ is to prove the existence of a **Cartan subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$, namely, of a nontrivial subalgebra $\mathfrak{h}$ all whose elements h are ad-diagonalisable (i.e. $\mathrm{ad}(h) \in \mathrm{End}(\mathfrak{g})$ is diagonal in some suitable basis of $\mathfrak{g}$) and that is maximal for this property. The elements of $\mathfrak{h}$ are then simultaneously ad-diagonalisable: in other words, $\mathfrak{g}$ admits a **root space decomposition**

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \tag{1}$$

where

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$$

is the α-eigenspace of $\mathrm{ad}(\mathfrak{h})$. The nonzero elements $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq \{0\}$ are called **roots**, and their set is denoted Δ. One shows that $\mathfrak{g}_0 = \mathfrak{h}$, so that (1) may be rewritten as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha. \tag{2}$$

Example 2.1. Let $\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbb{C})$, and write E_{ij} for the $(\ell + 1) \times (\ell + 1)$ matrix with an entry "1" in position (i, j) and "0" elsewhere. The subalgebra

$$\mathfrak{h} := \mathrm{span}_{\mathbb{C}}\langle \alpha_i^\vee := E_{ii} - E_{i+1,i+1} \mid 1 \leq i \leq \ell \rangle$$

of all diagonal matrices in $\mathfrak{sl}_{\ell+1}(\mathbb{C})$ is a Cartan subalgebra: the ad-diagonalisability of $\mathfrak{h}$ follows from the computation

$$[\alpha_i^\vee, E_{jk}] = (\delta_{ij} - \delta_{ik} - \delta_{i+1,j} + \delta_{i+1,k})E_{jk} = (\varepsilon_j - \varepsilon_k)(\alpha_i^\vee)E_{jk} \quad \text{for all } i, j, k,$$

where $\varepsilon_j(E_{ii}) := \delta_{ij}$. The corresponding set of roots and root spaces are then given by

$$\Delta = \{\alpha_{jk} := \varepsilon_j - \varepsilon_k \mid 1 \leq j \neq k \leq \ell + 1\} \quad \text{and} \quad \mathfrak{g}_{\alpha_{jk}} = \mathbb{C}E_{jk},$$

yielding the root space decomposition $\mathfrak{sl}_{\ell+1}(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{j \neq k} \mathbb{C}E_{jk}$.

The second step is to establish some properties of the $\mathfrak{g}_\alpha$'s. Here are some important ones:

(1) $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$.
(2) For any nonzero $x_\alpha \in \mathfrak{g}_\alpha$ ($\alpha \in \Delta$), there is some $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that the assignment

$$x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad x_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \alpha^\vee := [x_{-\alpha}, x_\alpha] \in \mathfrak{h} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

defines an isomorphism $\mathbb{C}x_{-\alpha} \oplus \mathbb{C}\alpha^\vee \oplus \mathbb{C}x_\alpha \to \mathfrak{sl}_2(\mathbb{C})$ of Lie algebras. The element $\alpha^\vee \in \mathfrak{h}$ depends only on α and is called the **coroot** of α.

(3) $\alpha(\beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$.

Example 2.2. In the notations of Example 2.1: for each $j, k \in \{1, \dots, \ell+1\}$ with $j \neq k$, we get an embedded copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{sl}_{\ell+1}(\mathbb{C})$ by considering submatrices indexed by $\{j, k\}$. One can then take

$$x_{\alpha_{jk}} := E_{jk} \in \mathfrak{g}_{\alpha_{jk}}, \quad x_{-\alpha_{jk}} := -E_{kj} \in \mathfrak{g}_{-\alpha_{jk}} \quad \text{and} \quad \alpha_{jk}^\vee := E_{jj} - E_{kk}.$$

We set $\alpha_i := \alpha_{i,i+1}$ for each $i \in \{1, \dots, \ell\}$, so that $\alpha_i^\vee = E_{ii} - E_{i+1,i+1}$ is consistent with our previous notations.

The third step is to study the root system Δ and to show that, together with the integers $\alpha(\beta^\vee)$ $(\alpha, \beta \in \Delta)$, it completely determines $\mathfrak{g}$. Here are key properties of Δ:

(4) Δ admits a **root basis** $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$: every $\alpha \in \Delta$ can be uniquely expressed as a linear combination of the **simple roots** $\alpha_1, \dots, \alpha_\ell$: $\alpha = \varepsilon_\alpha \sum_{i=1}^\ell n_i \alpha_i$ for some $n_i \in \mathbb{N}$ and $\varepsilon_\alpha \in \{\pm 1\}$. Roots α with $\varepsilon_\alpha = +$ (resp. $\varepsilon_\alpha = -$) are called **positive** (resp. **negative**) and their set is denoted Δ^+ (resp. Δ^-). We then have $\Delta^- = -\Delta^+$.

(5) The subgroup

$$W := \langle s_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^* : \beta \mapsto \beta - \beta(\alpha^\vee)\alpha \mid \alpha \in \Delta \rangle$$

of $\mathrm{GL}(\mathfrak{h}^*)$, called the **Weyl group** of $\mathfrak{g}$, is generated by the **simple reflections** $s_i := s_{\alpha_i}$ $(1 \leq i \leq \ell)$. It stabilises $\Delta \subseteq \mathfrak{h}^*$: in fact, $\Delta = W(\Pi)$. The pair $(W, S := \{s_i \mid 1 \leq i \leq \ell\})$ is a (finite) Coxeter system.

The Lie algebra $\mathfrak{g}$ is then uniquely determined, up to isomorphism, by its **Cartan matrix**

$$A = (a_{ij})_{1 \leq i,j \leq \ell} := (\alpha_j(\alpha_i^\vee))_{1 \leq i,j \leq \ell}.$$

More precisely, choosing elements $e_i = x_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$ and $f_i = x_{-\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ as above, $\mathfrak{g}$ is generated by the ℓ copies $\mathbb{C}f_i \oplus \mathbb{C}\alpha_i^\vee \oplus \mathbb{C}e_i$ of $\mathfrak{sl}_2(\mathbb{C})$ $(1 \leq i \leq \ell)$, and can even be reconstructed as the complex Lie algebra $\mathfrak{g}_A$ on the 3ℓ generators $e_i, f_i, \alpha_i^\vee$ and with the following defining relations $(1 \leq i, j \leq \ell)$:

$$[\alpha_i^\vee, \alpha_j^\vee] = 0,\ [\alpha_i^\vee, e_j] = a_{ij}e_j,\ [\alpha_i^\vee, f_j] = -a_{ij}f_j,\ [f_i, e_j] = \delta_{ij}\alpha_i^\vee, \tag{3}$$

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0,\ (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0 \qquad \text{for } i \neq j. \tag{4}$$

Note that the relations (4), called the **Serre relations**, make sense, as the $a_{ij} \in \mathbb{Z}$ in fact satisfy $a_{ij} \leq 0$ whenever $i \neq j$.

Example 2.3. In the notations of Examples 2.1 and 2.2: $\Pi = \{\alpha_i \mid 1 \leq i \leq \ell\}$ is a root basis of Δ, and $\Delta^+ = \{\alpha_{jk} \mid j < k\}$. The Weyl group W of $\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbb{C})$ is isomorphic to the Coxeter group $\mathrm{Sym}(\ell+1)$, with the simple reflection s_i acting on $\{1, \dots, \ell+1\}$ as the transposition $(i, i+1)$. The Lie algebra $\mathfrak{sl}_{\ell+1}(\mathbb{C})$ is generated, as a Lie algebra, by the elements $e_i := E_{i,i+1}$ and $f_i := -E_{i+1,i}$ $(1 \leq i \leq \ell)$. The Cartan matrix $A = (\alpha_j(\alpha_i^\vee))_{1 \leq i,j \leq \ell}$ has 2's on the main diagonal, -1's on the diagonals $(i, i+1)$ and $(i+1, i)$, and 0's elsewhere.

3. Kac–Moody algebras

To define infinite-dimensional generalisations of the simple Lie algebras, we follow the opposite path to the one leading to the classification of simple Lie algebras: we start from "generalised" Cartan matrices A, then define a Lie algebra associated to A.

More precisely, the presentation of the Lie algebra $\mathfrak{g}_A$ introduced in the previous section still makes sense if $A = (a_{ij})_{1\leq i,j\leq \ell}$ is a **generalised Cartan matrix** (GCM), in the sense that, for each $i, j \in \{1, \dots, \ell\}$,

- (C1) $a_{ii} = 2$ (to ensure that $e_i, f_i, \alpha_i^\vee$ span a copy of $\mathfrak{sl}_2(\mathbb{C})$),
- (C2) a_{ij} is a nonpositive integer if $i \neq j$ (to ensure that the Serre relations (4) make sense),
- (C3) $a_{ij} = 0$ implies $a_{ji} = 0$ (because of the Serre relations $(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$ and $(\operatorname{ad} e_j)^{1-a_{ji}} e_i = 0$).

The resulting Lie algebra $\mathfrak{g}_A$ is the **Kac–Moody algebra** associated to A (or rather, its derived Lie algebra, see below).

It admits a grading by the free abelian group $Q := \bigoplus_{1\leq i\leq \ell} \mathbb{Z}\alpha_i$ on the symbols $\alpha_1, \dots, \alpha_\ell$, obtained by setting $\deg(e_i) := \alpha_i$, $\deg(f_i) := -\alpha_i$ and $\deg(\alpha_i^\vee) = 0$ for each i (this defines a grading of the free Lie algebra on $e_i, f_i, \alpha_i^\vee$, which passes to the quotient $\mathfrak{g}_A$ since the relations (3)–(4) are homogeneous):

$$\mathfrak{g}_A = \mathfrak{h}' \oplus \bigoplus_{\alpha\in Q\setminus\{0\}} \mathfrak{g}_\alpha \quad \text{where } \mathfrak{h}' := \sum_{i=1}^{\ell} \mathbb{C}\alpha_i^\vee.$$

If we define the $\mathbb{Z}$-linear map

$$c\colon\ Q \to (\mathfrak{h}')^* : \alpha \mapsto c_\alpha, \quad \text{where } c_{\alpha_j}(\alpha_i^\vee) = a_{ij},$$

the relations (3) then imply that

$$\mathfrak{g}_\alpha \subseteq \{x \in \mathfrak{g}_A \mid [h, x] = c_\alpha(h)x \ \forall h \in \mathfrak{h}'\}.$$

If A is invertible, then c is injective (we may then identify α_i with c_{α_i} and Q with a subset of $\mathfrak{h}^*$) and the above inclusions are equalities. If A is singular, however, c is not injective and the above inclusion is in general proper; in other words, the Q-grading of $\mathfrak{g}_A$ is then finer than its eigenspace decomposition with respect to the adjoint action of $\mathfrak{h}'$. To remedy this, we can enlarge $\mathfrak{h}'$ to a **Cartan subalgebra** $\mathfrak{h}$ in such a way that $\Pi^\vee := \{\alpha_i^\vee \mid 1 \leq i \leq \ell\}$ and $\Pi := \{\alpha_i \mid 1 \leq i \leq \ell\}$ are linearly independent subsets of $\mathfrak{h}$ and $\mathfrak{h}^*$ paired by $\alpha_j(\alpha_i^\vee) = a_{ij}$, and $\dim \mathfrak{h}$ is minimal for these properties. Such a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is essentially unique and called a **realisation of** A.

Example 3.1. Consider the GCM $A = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. Then $c_{\alpha_1+\alpha_2} = 0$. If we enlarge $\mathfrak{h}'$ to a 3-dimensional vector space $\mathfrak{h}$ with basis $\{\alpha_1^\vee, \alpha_2^\vee, d\}$, then denoting by $\{u_1, u_2, u_3\} \subseteq \mathfrak{h}^*$ the corresponding dual basis, one can identify α_1 with $2u_1 - 2u_2$ and α_2 with $-2u_1 + 2u_2 + u_3$.

The Lie algebra $\mathfrak{g}(A) = \mathfrak{h} + \mathfrak{g}_A = \mathfrak{h} \oplus \bigoplus_{\alpha \in Q\setminus\{0\}} \mathfrak{g}_\alpha$ where

$$[h, h'] = 0, \quad [h, e_i] = \alpha_i(h) e_i \quad \text{and} \quad [h, f_i] = -\alpha_i(h) f_i$$

for all $h, h' \in \mathfrak{h}$ and all $i \in \{1, \ldots, \ell\}$ is called the **Kac–Moody algebra** associated to A, and we have $\mathfrak{g}_A = [\mathfrak{g}(A), \mathfrak{g}(A)]$.

Kac–Moody algebras share with finite-dimensional simple Lie algebras a number of properties:

(1) The set $\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ of **roots** is a disjoint union of the sets $\Delta^+ := \{\alpha = \sum_{i=1}^{\ell} n_i \alpha_i \in \Delta \mid n_i \in \mathbb{N}\}$ and $\Delta^- = -\Delta^+$ of **positive** and **negative** roots. In particular, we have a **triangular decomposition**

$$\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \quad \text{(direct sum of vector spaces)}$$

where $\mathfrak{n}^\pm := \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$ is the subalgebra of $\mathfrak{g}(A)$ generated by the e_i/f_i.

(2) The **Weyl group** $W := \langle S \rangle \subseteq \mathrm{GL}(\mathfrak{h}^*)$ of $\mathfrak{g}(A)$, generated by the set $S := \{s_i \colon \alpha \mapsto \alpha - \alpha(\alpha_i^\vee)\alpha_i \mid 1 \leq i \leq \ell\}$ of **simple reflections**, stabilises Δ. The pair (W, S) is a Coxeter system.

Example 3.2. If $A = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, then $\mathfrak{g}(A) \cong \mathfrak{sl}_3(\mathbb{C})$.

Example 3.3. If $A = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$, then $\mathfrak{g}_A \cong \mathfrak{sl}_2(\mathbb{C}[t, t^{-1}]) \rtimes \mathbb{C}K$ is a one-dimensional (nontrivial) central extension of $\mathfrak{sl}_2(\mathbb{C}[t, t^{-1}])$, with

$$e_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad f_1 = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right), \quad \alpha_1^\vee = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

and

$$e_2 = \left(\begin{smallmatrix} 0 & 0 \\ -t & 0 \end{smallmatrix}\right), \quad f_2 = \left(\begin{smallmatrix} 0 & t^{-1} \\ 0 & 0 \end{smallmatrix}\right), \quad \alpha_2^\vee = -\alpha_1^\vee + K.$$

The associated Weyl group is isomorphic to the infinite dihedral group D_∞.

Appendix A. Coxeter groups and complexes

A **Coxeter group** is a group W generated by a subset S of involutions such that W admits a presentation

$$W = \langle s \in S \mid s^2 = 1 = (st)^{m_{st}} \text{ for all } s, t \in S \text{ with } s \neq t \rangle$$

for some $m_{st} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ (where the relation $1 = (st)^{m_{st}}$ is omitted in case $m_{st} = \infty$). The pair (W, S) is a **Coxeter system**.

Example A.1. The **infinite dihedral group** is the Coxeter group $W = D_\infty$ with presentation $W = \langle s, t \mid s^2 = t^2 = 1 \rangle$. It is the group of simplicial isometries of the simplicial line generated by the orthogonal reflections s, t with respect to the endpoints of a fixed edge C_0.

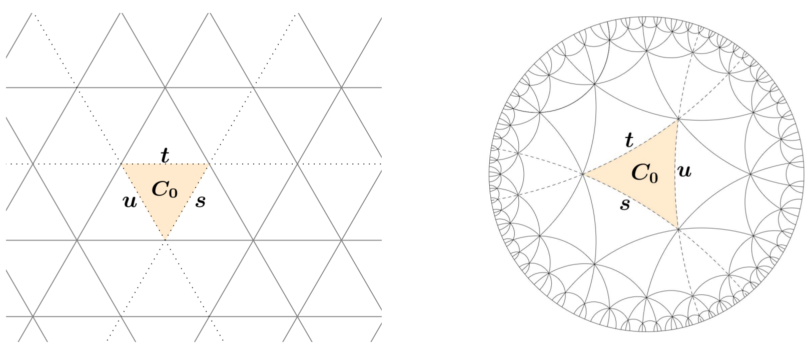
Example A.2. Consider the Coxeter group

$$W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^m = (su)^m = (tu)^m = 1 \rangle$$

for some $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$.

If $m = 3$, then W is the group of simplicial isometries of the tesselation of the Euclidean plane by congruent equilateral triangles, generated by the orthogonal reflections s, t, u on the sides of a fixed triangle C_0.

If $m = 4$, then W is the group of simplicial isometries of the tesselation of the hyperbolic plane by congruent equilateral triangles with interior angles $\pi/4$, generated by the orthogonal reflections s, t, u on the sides of a fixed triangle C_0.



$m = 3$ $m = 4$

The simplicial complex $\Sigma = \Sigma(W, S)$ induced by the tesselations in the above examples is called the **Coxeter complex** of W. The hyperplanes of the tesselation are called **walls** and are in bijection with the set of **reflections** $S^W := \{wsw^{-1} \mid w \in W,\ s \in S\}$ of W. Each wall m determines two half-spaces m^+ and m^-, where m^+ is the half-space containing C_0. The maximal simplices are called **chambers**. The function d_{Ch} assigning to a pair of chambers the number of walls that separate them is a metric, called the **chamber distance**.

Example A.3. Let W be the Weyl group of a Kac–Moody algebra $\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, with set of simple reflections S and set of simple roots $\Pi \subseteq \mathfrak{h}^*$. Set

$\Delta^{re} := W(\Pi) \subseteq \Delta$. Then there is a W-equivariant bijection

$$\Delta^{re} \xrightarrow{\sim} \{\text{half-spaces of } \Sigma(W,S)\}$$

mapping each simple root α_i to the half-space m_i^+ where m_i is the wall fixed by s_i. Under this bijection, the positive roots in Δ^{re} correspond to the half-spaces containing C_0.

Appendix B. Buildings and BN-pairs

A building is a simplicial complex X obtained by glueing copies of a given Coxeter complex $\Sigma(W,S)$ (the **type** of the building) in such a way that the chamber distances on the various Coxeter complexes can be patched together to get a global metric on X.

Example B.1. Buildings of type D_∞ are precisely the trees without leaves (i.e. without endpoints).

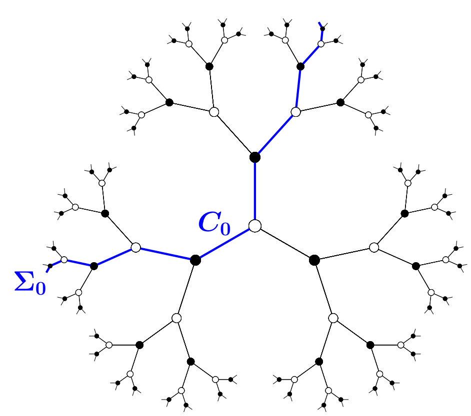
A **BN-pair** for a group G is a pair of subgroups (B,N) of G satisfying certain axioms. Whenever a group G has a BN-pair (B,N), one can construct a building $X = X(G,B)$ on which G acts *strongly transitively* by simplicial isometries, with B the stabiliser of a chamber C_0 and N the stabiliser of one of the Coxeter complexes $\Sigma_0 \subseteq X$ containing C_0. The associated Coxeter group W is isomorphic to N/T where $T := B \cap N$ is the pointwise fixer in G of Σ_0 (at least when the BN-pair is *saturated*). Finally, the group G admits the following decomposition (called **Bruhat decomposition**) into double B-cosets:

$$G = \coprod_{w \in W} BwB,$$

where $BwB := BnB$ for any $n \in N$ mapped to w under the quotient map $N \twoheadrightarrow N/T = W$.

Example B.2. Consider the group $G = \mathrm{SL}_2(\mathbb{K}[t,t^{-1}])$. Let N be the subgroup of monomial matrices in G (i.e. those with exactly one nonzero entry in each row and each column). Let B^+ denote the inverse image under the canonical projection $\mathrm{SL}_2(\mathbb{K}[t]) \twoheadrightarrow \mathrm{SL}_2(\mathbb{K})$ of the group of upper triangular matrices in $\mathrm{SL}_2(\mathbb{K})$. Similarly, let B^- denote the inverse image under the canonical projection $\mathrm{SL}_2(\mathbb{K}[t^{-1}]) \twoheadrightarrow \mathrm{SL}_2(\mathbb{K})$ of the group of lower triangular matrices in $\mathrm{SL}_2(\mathbb{K})$.

Then (B^+,N) and (B^-,N) are BN-pairs for G. If $B \in \{B^\pm\}$, then the buildings $X^+ = X(G,B^+)$ and $X^- = X(G,B^-)$ are of type $W \cong N/T \cong D_\infty$. More precisely, $X^\pm$ is a regular tree, in which each edge is adjacent to exactly $|\mathbb{K}|$ other edges. Here is for instance $X^\pm$ for $G = \mathrm{SL}_2(\mathbb{F}_2[t,t^{-1}])$:



Appendix C. Chevalley groups

To each finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ with Cartan matrix A and each field $\mathbb{K}$, one can associate a group $\mathfrak{G}_A(\mathbb{K})$ called the (simply connected) **Chevalley group** of type A: for instance, if $\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbb{C})$, then $\mathfrak{G}_A(\mathbb{K}) = \mathrm{SL}_{\ell+1}(\mathbb{K})$. This group can be defined by a presentation, as follows. We keep the notations from Section 2.

Consider first the dual action of W on $\mathfrak{h}$, defined by

$$s_i(h) = h - \alpha_i(h)\alpha_i^\vee \quad \text{for all } h \in \mathfrak{h} \text{ and } i \in I := \{1, \dots, \ell\}.$$

For each $i \in I$, define the automorphism

$$s_i^* := \exp(\operatorname{ad} e_i)\exp(\operatorname{ad} f_i)\exp(\operatorname{ad} e_i) \in \operatorname{Aut}(\mathfrak{g})$$

of $\mathfrak{g}$, where $\exp(\operatorname{ad} x)y := \sum_{n\geq 0} \frac{1}{n!}(\operatorname{ad} x)^n y$ for $x \in \{e_i, f_i\}$ and $y \in \mathfrak{g}$.

Example C.1. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then $\exp(e_1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\exp(f_1) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, so that $\exp(e_1)\exp(f_1)\exp(e_1) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Hence s_1^* is the conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Lemma C.2. *Let $i \in I$. Then:*

1. $s_i^*|_{\mathfrak{h}} = s_i \in \mathrm{GL}(\mathfrak{h})$.
2. $s_i^*\mathfrak{g}_\alpha = \mathfrak{g}_{s_i\alpha}$ *for all* $\alpha \in \Delta$.
3. *There is a surjective group morphism* $\nu\colon W^* \to W : s_i^* \mapsto s_i$, *where* $W^* := \langle s_i^* \mid i \in I\rangle \subseteq \operatorname{Aut}(\mathfrak{g})$.
4. *For each* $w^* \in W^*$ *and* $i \in I$, *the couple* $E_\alpha = \{\pm w^* e_i\}$ *depends only on* $\alpha = \nu(w^*)\alpha_i \in \Delta$.

For each $\alpha \in \Delta$, we choose an element $e_\alpha \in E_\alpha$ so that $e_{\alpha_i} = e_i$ and $e_{-\alpha_i} = f_i$ for each $i \in I$, and $[e_\alpha, e_{-\alpha}] = -\alpha^\vee$ for all $\alpha \in \Delta$.

Example C.3. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. Then $\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$ and we can choose $e_{\alpha_1+\alpha_2} = E_{13}$ and $e_{-\alpha_1-\alpha_2} = -E_{31}$.

Let $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. For $\alpha, \beta \in \Delta$ with $\alpha \neq -\beta$, we define the (finite) sets

$$[\alpha, \beta]_{\mathbb{N}} := (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta \quad \text{and} \quad]\alpha, \beta[_{\mathbb{N}} := [\alpha, \beta]_{\mathbb{N}} \setminus \{\alpha, \beta\}.$$

Theorem C.4. *Let $\alpha, \beta \in \Delta$ with $\alpha \neq -\beta$. Fix an arbitrary order on $]\alpha, \beta[_{\mathbb{N}}$. Then there exist integers $C_{ij}^{\alpha\beta}$ depending only on α, β and the chosen order, such that in the ring $\mathcal{U}[\![t, u]\!]$ of formal power series in two indeterminates t, u and with coefficients in $\mathcal{U}$ we have the group commutator formula*

$$[\exp(te_\alpha), \exp(ue_\beta)] = \prod_\gamma \exp(t^i u^j C_{ij}^{\alpha\beta} e_\gamma),$$

where $\gamma = i\alpha + j\beta$ runs through $]\alpha, \beta[_{\mathbb{N}}$ in the prescribed order.

Theorem C.5. *Suppose $\mathfrak{g} \not\cong \mathfrak{sl}_2(\mathbb{C})$. Let $\mathbb{K}$ be a field. For every root α and every $t \in \mathbb{K}$, we introduce the symbol $x_\alpha(t)$. Let $\mathfrak{G}_A(\mathbb{K})$ be the abstract group defined by the generators $x_\alpha(t)$ and the relations*

$$x_\alpha(t) \cdot x_\alpha(u) = x_\alpha(t+u) \quad (t, u \in \mathbb{K}),$$

$$[x_\alpha(t), x_\beta(u)] = \prod_\gamma x_\gamma(C_{ij}^{\alpha\beta} t^i u^j) \quad \text{where } \gamma = i\alpha + j\beta \text{ runs through }]\alpha, \beta[_{\mathbb{N}} \ (\alpha \neq \pm\beta),$$

$$h_\alpha(t) \cdot h_\alpha(u) = h_\alpha(tu) \quad (t, u \in \mathbb{K}^\times),$$

where the integers $C_{ij}^{\alpha\beta}$ are as in Theorem C.4, and where $h_\alpha(t) := n_\alpha(t) \cdot n_\alpha(-1)$ with $n_\alpha(t) := x_\alpha(t) \cdot x_{-\alpha}(t^{-1}) \cdot x_\alpha(t)$. Finally, let Z denote the center of $\mathfrak{G}_A(\mathbb{K})$. Then $\mathfrak{G}_A(\mathbb{K})/Z$ is the simple quotient of the Chevalley group of type A over $\mathbb{K}$ (up to five exceptions for $\mathbb{K} = \mathbb{F}_2$ or $\mathbb{F}_3$).

Exercise C.6. Let $\mathfrak{g}_{\mathbb{K}} = \mathfrak{sl}_{\ell+1}(\mathbb{K})$ for $\mathbb{K}$ a field. For each $i \in I$, we identify as before the subalgebra generated by $e_i, f_i, \alpha_i^\vee$ with $\mathfrak{sl}_2(\mathbb{K})$ via $e_i = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f_i = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)$ and $\alpha_i^\vee = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, embedded in position $\{i, i+1\}$. We define the following elements of $\mathrm{SL}_2(\mathbb{K})$: for $r \in \mathbb{K}$ and $i \in I$, we set

$$x_i(r) := \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad x_{-i}(r) := \begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix},$$

and for $r \in \mathbb{K}^\times$ (an invertible element) we set

$$r^{\alpha_i^\vee} := \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \quad \text{and} \quad \widetilde{s}_i(r) := x_i(r) x_{-i}(r^{-1}) x_i(r) = \begin{pmatrix} 0 & r \\ -r^{-1} & 0 \end{pmatrix}.$$

(1) Show that if $\mathbb{K} \subseteq \mathbb{C}$, then for all $r \in \mathbb{K}$, we have $\exp(re_i) = x_i(r)$, $\exp(rf_i) = x_{-i}(r)$ and $\exp(r\alpha_i^\vee) = (e^r)^{\alpha_i^\vee}$.
(2) Show that $\widetilde{s}_i(1) = \exp(e_i)\exp(f_i)\exp(e_i)$ and that $r^{\alpha_i^\vee} = \widetilde{s}_i(1)^{-1}\widetilde{s}_i(r^{-1})$ for $r \in \mathbb{K}^\times$.
(3) Note that, for $r \in \mathbb{K}^\times$ and $\alpha = \alpha_i$ a simple root, the elements $x_\alpha(r)$, $n_\alpha(r)$ and $h_\alpha(r)$ in Theorem C.5 respectively correspond to the elements $x_i(r)$, $\widetilde{s}_i(r)$ and $r^{\alpha_i^\vee}$ (with respect to the corresponding embedding of $\mathrm{SL}_2(\mathbb{K})$ in $\mathfrak{G}_A(\mathbb{K}) = \mathrm{SL}_{\ell+1}(\mathbb{K})$, in position $\{i, i+1\}$).

Exercise C.7. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, with the fundamental copies $\mathbb{C}f_i \oplus \mathbb{C}\alpha_i^\vee \oplus \mathbb{C}e_i$ of $\mathfrak{sl}_2(\mathbb{C})$ $(i = 1, 2)$ respectively embedded in the upper left and lower right corners. Identify elements $x_\alpha(t)$, $h_\alpha(t)$ and $n_\alpha(t)$ (for $\alpha \in \Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$ and $t \in \mathbb{K}^\times$) of $\mathfrak{G}_A(\mathbb{K}) = \mathrm{SL}_3(\mathbb{K})$ as in Theorem C.5, and compute the integers $C_{ij}^{\alpha\beta}$.