HYPERFINITENESS OF BOUNDARY ACTIONS OF HYPERBOLIC GROUPS

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ABSTRACT. We prove that for every finitely generated hyperbolic group G, the action of G on its Gromov boundary induces a hyperfinite equivalence relation.

1. INTRODUCTION

The complexity theory of Borel equivalence relations has been developed in the last forty years as an attempt to measure the difficulty of classification problems ([Kan08]). A particularly active part of the theory is concerned with the structure of countable Borel equivalence relations. As all countable Borel equivalence relations are induced by Borel actions of countable groups, this has a close relationship with the study of Borel or measurable group actions on standard Borel spaces.

The interplay between the structure of measurable actions of countable (or finitely generated) groups and the algebraic properties of the groups is one of the main themes appearing at the intersection of group theory, ergodic theory and descriptive set theory. The classical result [Gao09, Theorem 7.2.4] of Slaman–Steel ([SS88]) and Weiss ([Wei84]) characterises the equivalence relations induced by the actions of $\mathbb{Z}$ (i.e. by a single automorphism) as the **hyperfinite** equivalence relations: those which can be written as an increasing union of finite equivalence relations. This notion has been studied both from the Borel and measurable point of view.

The above characterisation holds in a pure Borel context, where there is no probability measure around. Given a Borel probability measure μ, an equivalence relation is μ**-hyperfinite** if it is hyperfinite restricted to a certain subset of measure 1. In the presence of an invariant probability measure μ, every amenable group action induces a μ-hyperfinite equivalence relation. It is one of the notorious open problems whether the latter holds in the pure Borel setting.

Hyperbolic groups and spaces were introduced and studied by Gromov (see [Gro87]) and have attracted a lot of attention in geometric group theory. To every (geodesic) proper hyperbolic metric space X one associates a compact metric space ∂X, called its **Gromov boundary**, which is a quasi-isometry invariant of X. One then also defines the Gromov boundary ∂G of a finitely generated hyperbolic group G as the Gromov boundary of its Cayley graph (with respect to

a finite generating set), and the G-action on its Cayley graph naturally induces a G-action by homeomorphisms on ∂G.

Boundary actions of hyperbolic groups have also been studied from the point of view of their complexity. In the case of the free group G, Connes, Feldman and Weiss [CFW81, Corollary 13] and Vershik [Ver78] showed that the action of G on its Gromov boundary ∂G is μ-hyperfinite for every Borel quasi-invariant probability measure μ on ∂G. This was later generalised by Adams in [Ada94] to all finitely generated hyperbolic groups.

On the other hand, in [DJK94, Corollary 8.2] Dougherty, Jackson and Kechris proved that the boundary action of the free group F_2 is hyperfinite (in the pure Borel sense) by studying the so-called **tail equivalence relation**. More recently, Huang, Sabok and Shinko showed in [HSS19] that every cubulated hyperbolic group G (i.e. G acts geometrically on a hyperbolic CAT(0) cube complex X) has a hyperfinite boundary action. Their proof is based on an analysis of geodesic ray bundles in hyperbolic CAT(0) cube complexes. More precisely, given a point $\eta \in \partial X$ and a vertex x of the 1-skeleton $X^{(0)}$ of X, the **geodesic ray bundle** $\mathrm{Geo}(x, \eta)$ consists of all vertices that appear on a geodesic ray starting at x and converging to η. The hyperfiniteness of the G-action on ∂X (or equivalently, on ∂G since X and G are quasi-isometric) is then established as a consequence of the following geometric condition (see [HSS19, Theorem 1.4]): for every $x, y \in X^{(0)}$ and $\eta \in \partial X$, the sets $\mathrm{Geo}(x, \eta)$ and $\mathrm{Geo}(y, \eta)$ have a finite symmetric difference.

In [HSS19, Question 1.5] it was asked whether this condition holds in Cayley graphs of arbitrary hyperbolic groups. However, Touikan constructed in [Tou18] hyperbolic groups (or rather, appropriate sets of generators for the free group) with an associated Cayley graph in which this condition does not hold. On the other hand, even though this condition turned out to be quite restrictive, Marquis provided in [Mar18] a large class of examples (including groups having the Kazhdan property) which act geometrically on hyperbolic graphs where the condition does hold, thereby establishing the hyperfiniteness of the corresponding boundary actions.

In this paper we solve the problem of hyperfiniteness of boundary actions of hyperbolic groups in full generality, by proving the following unconditional theorem.

Theorem A. *Let G be a finitely generated hyperbolic group. Then the action of G on its Gromov boundary ∂G is hyperfinite.*

Note that for a finitely generated hyperbolic group G, any geometric action of G on a space X induces a boundary action on ∂X and for all such actions there exists a G-equivariant homeomorphism of ∂X and ∂G [Gro87]. Therefore, by Theorem A, all such boundary actions of G also induce hyperfinite equivalence relations.

To prove Theorem A, we establish a new hyperfiniteness criterion of geometric nature and we show that it holds in every (uniformly) locally finite hyperbolic graph. More precisely, given such a graph X with set of vertices $X^{(0)}$, we construct for each $x \in X^{(0)}$ and $\eta \in \partial X$ a subset $\mathrm{Geo}_1(x, \eta)$ of $\mathrm{Geo}(x, \eta)$ containing a sub-geodesic ray of every geodesic ray from x to η, and such that $\mathrm{Geo}_1(x, \eta)$ and $\mathrm{Geo}_1(y, \eta)$ have a finite symmetric difference for every $x, y \in X^{(0)}$ (see Proposition 5.8 and Theorem 5.9). We then show, in §6, that if X is the Cayley graph

of a finitely generated hyperbolic group G, then this property implies the hyperfiniteness of the boundary action of G.

As illustrated by Touikan's examples [Tou18] (see also Examples 3.6 and 5.7 below), establishing the finite symmetric difference property for the sets $\mathrm{Geo}_1(x, \eta)$ in *arbitrary* (hyperbolic, uniformly locally finite) graphs X is a rather subtle problem. The key idea to tackle this problem is to consider the *horoboundary* of X, which is a refinement of ∂X allowing for a better control of geodesic rays (and generalising the *combinatorial compactification* of the chamber graph of a building introduced in [CL11]), and to prove the existence of so-called *straight* geodesic rays (see Definition 4.9).

Acknowledgement. We would like to thank the referee for many valuable comments.

2. Preliminaries

2.1. **Graphs.** Throughout this paper, X will denote a connected locally finite graph, with vertex set $X^{(0)}$ and edge set $E(X) \subseteq X \times X$.

Two distinct vertices $x, y \in X^{(0)}$ are **adjacent** if $\{x, y\} \in E(X)$. A **path** in X is a (possibly infinite) ordered sequence $x_0, x_1, \ldots$ of vertices of X such that x_i and x_{i+1} are distinct and adjacent for each i. The path metric on $X^{(0)}$ will be denoted $\mathrm{d} \colon X^{(0)} \times X^{(0)} \to \mathbb{N}$. Given two vertices $x, y \in X^{(0)}$, we let

$$\Gamma(x, y) := \{z \in X^{(0)} \mid \mathrm{d}(x, y) = \mathrm{d}(x, z) + \mathrm{d}(z, y)\}$$

be the union of all geodesic paths from x to y. Given a (finite) path Γ_1 ending at some vertex $x \in X^{(0)}$, and a (possibly infinite) path Γ_2 starting at x, we denote by $\Gamma_1 \cdot \Gamma_2$ the path obtained by concatenating Γ_1 with Γ_2.

A **geodesic ray** is an infinite geodesic path $(x_n)_{n \in \mathbb{N}} \subseteq X^{(0)}$. Two geodesic rays r, r' are **asymptotic** if they are at bounded Hausdorff distance $\mathrm{d}_H(r, r')$. The **visual boundary** ∂X of X is the set of equivalence classes of asymptotic geodesic rays. A **CGR from $x \in X^{(0)}$ to** $\eta \in \partial X$ is a geodesic ray $(x_n)_{n \in \mathbb{N}} \subseteq X^{(0)}$ starting at x and pointing towards η (i.e. belonging to the equivalence class η); their set is denoted $\mathrm{CGR}(x, \eta)$. [The letters CGR stand for "combinatorial geodesic ray", emphasising the combinatorial nature of the metric.] We also let $\mathrm{Geo}(x, \eta) \subseteq X^{(0)}$ denote the union of all CGR from $x \in X^{(0)}$ to $\eta \in \partial X$.

2.2. **Cayley graphs.** If X is the Cayley graph of a finitely generated group G with respect to a finite symmetric generating set S (so that $X^{(0)} = G$), then given a path $\Gamma = (x_n)_{0 \le n \le \ell}$ (resp. $\Gamma = (x_n)_{n \in \mathbb{N}}$) in X, we define the **type** $\mathrm{typ}(\Gamma)$ of Γ as

$$\mathrm{typ}(\Gamma) := (x_n^{-1} x_{n+1})_{0 \le n \le \ell-1} \in S^\ell, \quad \text{resp. } \mathrm{typ}(\Gamma) := (x_n^{-1} x_{n+1})_{n \in \mathbb{N}} \in S^{\mathbb{N}}.$$

2.3. **Hyperbolicity.** The graph X is called **(Gromov) hyperbolic** if there is some $\delta > 0$ such that every geodesic triangle Δ in X is δ-slim, that is, such that each side of Δ is contained in the δ-neighbourhood of the other two sides. In that case, we also call X δ**-hyperbolic**. A finitely generated group G is called **hyperbolic** if it has a hyperbolic Cayley graph (with respect to a finite generating set S); since hyperbolicity is a quasi-isometry invariant, this does not depend on the choice of S.

The key property of hyperbolic graphs that we will need is the following.

Lemma 2.1. *Assume that X is δ-hyperbolic for some $\delta > 0$. Let $x \in X^{(0)}$, $\eta \in \partial X$ and $\Gamma = (x_n)_{n\in\mathbb{N}}, \Gamma' = (x'_n)_{n\in\mathbb{N}} \in \mathrm{CGR}(x,\eta)$. Then $\mathrm{d}(x_n, x'_n) \leq 2\delta$ for all $n \in \mathbb{N}$. In particular, $\mathrm{d}_H(\Gamma, \Gamma') \leq 2\delta$.*

Proof. This follows from [BH99, Lemma III.3.3]. $\hspace{2cm}$ $\square$

If X is hyperbolic, one can equip its visual boundary ∂X with a compact (metrisable) topology, defined as follows: given a base point $x \in X$, a sequence $\eta_n \in \partial X$ converges to some $\eta \in \partial X$ if and only if there exist CGR $\Gamma_n \in \mathrm{CGR}(x, \eta_n)$ such that every subsequence of $(\Gamma_n)_{n\in\mathbb{N}}$ subconverges (i.e. admits a subsequence that converges) to a CGR $\Gamma \in \mathrm{CGR}(x, \eta)$. The resulting topological space, which we again denote by ∂X, is called the **Gromov boundary** of X (see [BH99, Section III.3]).

2.4. **Horoboundary.** Throughout this paper, we fix a base point $z_0 \in X^{(0)}$. Set

$$\mathcal{F}(X, z_0) := \{f\colon X^{(0)} \to \mathbb{R} \mid |f(x) - f(y)| \leq \mathrm{d}(x, y) \ \forall x, y \in X^{(0)}, \quad f(z_0) = 0\}.$$

We equip $\mathcal{F}(X, z_0)$ with the topology of pointwise convergence. To each $x \in X^{(0)}$, we attach the function

$$f_x = f_{x,z_0}\colon X^{(0)} \to \mathbb{R} : y \mapsto \mathrm{d}(x, y) - \mathrm{d}(x, z_0),$$

so that $f_{x,z_0} \in \mathcal{F}(X, z_0)$. The map

$$\iota\colon X^{(0)} \to \mathcal{F}(X, z_0) : x \mapsto f_{x,z_0}$$

is then continuous and injective, and we identify $X^{(0)}$ with its image (see e.g. [CL11, §3]). The **horofunction compactification** $\mathcal{C}_{\mathrm{horo}}(X)$ of X is the closure of $X^{(0)}$ in $\mathcal{F}(X, z_0)$; it is independent of the base point z_0 (as $\mathcal{F}(X, z_0)$ can be canonically identified with the space of 1-Lipschitz functions $X^{(0)} \to \mathbb{R}$, modulo the constant functions). The **horoboundary** of X is $\mathcal{C}_{\mathrm{hb}}(X) := \mathcal{C}_{\mathrm{horo}}(X) \setminus X^{(0)}$.

2.5. **Descriptive set theory.** A **standard Borel space** is a set Z equipped with a σ-algebra which can be obtained as the σ-algebra of Borel sets from some Polish topology on Z. Examples of standard Borel spaces include the discrete finite sets, the discrete infinite countable set or the Cantor set $2^{\mathbb{N}}$. Given a standard Borel space Z, by a **Borel** set in Z we mean any set which belongs to the σ-algebra. If Z, Y are standard Borel spaces, there is a canonical standard Borel space structure on $Z \times Y$, as well as on $Z^{<\mathbb{N}} := \bigcup_{n\in\mathbb{N}} Z^n$ and $Z^{\mathbb{N}}$.

We will use the simple observation that the sets of the form $\{x \in Z \mid \phi(z)\}$ are Borel if $\phi(x)$ is a first-order formula where the predicates correspond to closed or open subsets of Polish spaces and all the quantifiers range over finite or countable sets. We provide the following example for the benefit of readers not familiar with descriptive set theory.

Example 2.2. Let G be a finitely generated group with a finite symmetric generating set S. Fix a total order on S and consider the induced lexicographical order $<$ on S^n. Let $n \in \mathbb{N}$ and $L_n \subseteq G^{\mathbb{N}} \times G^n$ be the set of pairs (Γ, s) such that Γ is a geodesic ray starting at e (the neutral element of G) and $s \in S^n$ is the lexicographically least string which appears infinitely often as a substring of $\mathrm{typ}(\Gamma) \in S^{\mathbb{N}}$. Then the set L_n is Borel in $G^{\mathbb{N}} \times G^n$ as witnessed by the following

formula ϕ: writing $\Gamma = (\Gamma_m)_{m \in \mathbb{N}}$ and $s = (s_m)_{0 \le m < n}$, we have $(\Gamma, s) \in L_n$ if and only if

$$\phi(\Gamma, s) \equiv \Gamma \text{ is a geodesic path from } e \text{ and } \forall k \in \mathbb{N} \, \exists l > k \, \forall i < n \, \Gamma_{l+i}^{-1} \Gamma_{l+i+1} = s_i$$
$$\text{and } \forall t \in S^n \text{ if } t < s \text{ then } \exists k \in \mathbb{N} \, \forall l > k \, \exists i < n \, \Gamma_{l+i}^{-1} \Gamma_{l+i+1} \ne t_i.$$

Note that the predicate stating that Γ is a geodesic path from e corresponds to a closed set A in $G^{\mathbb{N}}$, the conditions $\Gamma_{l+i}^{-1} \Gamma_{l+i+1} = s_i$ on (Γ, s) describe open and closed sets $B_{l,i}$ in $G^{\mathbb{N}} \times S^n$, and the conditions $\Gamma_{l+i}^{-1} \Gamma_{l+i+1} \ne t_i$ on Γ describe open and closed sets $C_{l,i,t}$ in $G^{\mathbb{N}}$. Hence, setting $D_t := \{s \in S^n \mid s \le t\} \subseteq G^n$ for each $t \in S^n$, and denoting by $\mathrm{proj}_{G^{\mathbb{N}}}$ and proj_{G^n} the natural projections from $G^{\mathbb{N}} \times G^n$ to $G^{\mathbb{N}}$ and G^n, respectively, we have an explicit Borel description of L_n as

$$L_n = \mathrm{proj}_{G^{\mathbb{N}}}^{-1}(A) \cap \bigcap_{k \in \mathbb{N}} \bigcup_{l > k} \bigcap_{i < n} B_{l,i} \cap \bigcap_{t \in S^n} \left(\mathrm{proj}_{G^n}^{-1}(D_t) \cup \bigcup_{k \in \mathbb{N}} \bigcap_{l > k} \bigcup_{i < n} \mathrm{proj}_{G^{\mathbb{N}}}^{-1}(C_{l,i,t}) \right).$$

In the above example we refer to the formula ϕ as a **Borel definition** of L_n. In Section 6 we will use more sophisticated computations similar to the one in the example above.

An **analytic** subset of a standard Borel space Z is an image of a Borel set by a Borel function (or equivalently, a projection to Z of a Borel set $B \subseteq Z \times Y$ for some standard Borel space Y). In other words, a subset A of Z is analytic if it can be written as $\{z \in Z \mid \exists y \in Y \, (z, y) \in B\}$ for a Borel set $B \subseteq Z \times Y$. So analytic subsets are those sets which are definable by formulas which have at most one existential quantifier whose range is an uncountable standard Borel space. Analytic sets are closed under countable unions and intersections but not under complements. In other words, if $A_n \subseteq Z$ are analytic for $n \in \mathbb{N}$, then $\{z \in Z \mid \exists n \in \mathbb{N} \, z \in A_n\}$ and $\{z \in Z \mid \forall n \in \mathbb{N} \, z \in A_n\}$ are also analytic. A subset of a standard Borel space is **coanalytic** if its complement is analytic. In other words, coanalytic sets are those sets which are definable by formulas which have at most one universal quantifer whose range is an uncountable Borel space. A classical result of Souslin [Kec95, Theorem 14.11] states that a set is Borel if and only if it is both analytic and coanalytic.

Given a Borel set $B \subseteq Z \times Y$ with countable vertical sections $B_z = \{y \in Y \mid (z, y) \in B\}$, the projection $\mathrm{proj}_Z(B) = \{z \in Z \mid \exists y \in Y \, (z, y) \in B\}$ is still Borel by the **Lusin–Novikov theorem** [Kec95, Theorem 18.10]. The same theorem says that in that case B contains the graph of a Borel function from $\mathrm{proj}_Z(B)$ to Y. This implies that B is the union of countably many graphs of Borel functions from $\mathrm{proj}_Z(B)$ to Y, and if the sections B_z ($z \in Z$) are of size at most r, then B is the union of r Borel functions. This is the version of the Lusin–Novikov theorem that we will use in Section 6.

A **Borel equivalence relation** is an equivalence relation E on a standard Borel space Z such that E is Borel as a subset of $Z \times Z$. If $A \subseteq Z$, we will write $E|A$ for the restriction of E to A. An equivalence relation is **countable** (resp. **finite**) if all of its equivalence classes are countable (resp. finite). An equivalence relation is **hyperfinite** if it can be written as an increasing union of finite equivalence relations.

Given two Borel equivalence relations E on Z and F on Y, a Borel function $f : Z \to Y$ is a **homomorphism** if $z_1 \, E \, z_2 \implies f(z_1) \, F \, f(z_2)$ for all $z_1, z_2 \in Z$. The Borel map f is a **Borel reduction** from E to F if $z_1 \, E \, z_2 \iff f(z_1) \, F \, f(z_2)$

for all $z_1, z_2 \in Z$. We say that a Borel equivalence relation E is **Borel reducible** to a Borel equivalence relation F if there exists a Borel reduction from E to F.

A Borel equivalence relation E is **smooth** if it is Borel reducible to the identity relation on a standard Borel space. Every finite Borel equivalence relation is smooth. The simplest non-smooth countable Borel equivalence relation is E_0 defined on $2^{\mathbb{N}}$ by $x \ E_0 \ y$ if $\exists n \ \forall m > n \ x(m) = y(m)$. The relation E_0 is hyperfinite. A countable Borel equivalence relation is hyperfinite if and only if it is Borel reducible to E_0.

A Borel equivalence relation is **hypersmooth** if it can be written as an increasing union of smooth Borel equivalence relations. For instance, the relation E_1, defined on $(2^{\mathbb{N}})^{\mathbb{N}}$ by $x \ E_1 \ y$ if $\exists n \ \forall m > n \ x(m) = y(m)$, is hypersmooth. A Borel equivalence relation is hypersmooth if and only if it is Borel reducible to E_1.

In the realm of countable Borel equivalence relations, the classes of hyperfinite and hypersmooth equivalence relations coincide: if a countable equivalence relation is hypersmooth, then it is hyperfinite.

An **analytic equivalence relation** E on Z is an equivalence relation such that E is analytic as a subset of $Z \times Z$.

We will use the following consequence of the Second Reflection theorem [Kec95, Theorem 35.16]

Lemma 2.3. [HSS19, Lemma 4.1] *Let Z be a standard Borel space, $A \subseteq Z$ be analytic and let E be an analytic equivalence relation on Z such that there is some $K > 1$ such that every $E|A$-class has size less than K. Then there is a Borel equivalence relation F on Z with $E|A \subseteq F$ such that every F-class has size less than K.*

For more details regarding notation and standard facts in descriptive set theory we refer the reader to [Kec95] or [Kan08].

3. Preliminary lemmas and basic definitions

Lemma 3.1. *Let $x \in X^{(0)}$, and let $\Gamma = (x_n)_{n\in\mathbb{N}}$ be a CGR. Then there exists some $k \in \mathbb{N}$ such that $\Gamma_{xx_k} \cdot (x_n)_{n \geq k}$ is a CGR for any geodesic path Γ_{xx_k} from x to x_k.*

Proof. Note that the function $g \colon \mathbb{N} \to \mathbb{Z} : n \mapsto \mathrm{d}(x, x_n) - n$ is non-increasing, as $\mathrm{d}(x, x_n) \geq \mathrm{d}(x, x_{n+1}) - 1$, and bounded from below, as $g(n) = \mathrm{d}(x, x_n) - \mathrm{d}(x_0, x_n) \geq -\mathrm{d}(x, x_0)$. Hence g is eventually constant, that is, there is some $k \in \mathbb{N}$ such that $\mathrm{d}(x, x_n) = \mathrm{d}(x, x_k) + n - k$ for all $n \geq k$, as desired. $\square$

Lemma 3.2. *If $\Gamma = (x_n)_{n\in\mathbb{N}} \subseteq X^{(0)}$ is a CGR, then $(x_n)_{n\in\mathbb{N}}$ converges in $\mathcal{C}_{\mathrm{horo}}(X)$. We denote its limit by $\xi_\Gamma \in \mathcal{C}_{\mathrm{hb}}(X)$ and we say that Γ **converges** to ξ_Γ.*

Proof. Let $y \in X^{(0)}$, and let us show that $(f_{x_n}(y))_{n\in\mathbb{N}}$ is eventually constant, as desired. By Lemma 3.1, there is some $N \in \mathbb{N}$ such that $\Gamma_{yx_N} \cdot (x_n)_{n \geq N}$ and $\Gamma_{z_0 x_N} \cdot (x_n)_{n \geq N}$ are CGR for some geodesic path Γ_{yx_N} from y to x_N and some geodesic path $\Gamma_{z_0 x_N}$ from z_0 to x_N. In particular, if $n \geq N$, then

$$\begin{aligned}
f_{x_n}(y) &= \mathrm{d}(x_n, y) - \mathrm{d}(x_n, z_0) \\
&= \mathrm{d}(y, x_N) + \mathrm{d}(x_N, x_n) - \mathrm{d}(z_0, x_N) - \mathrm{d}(x_N, x_n) = f_{x_N}(y),
\end{aligned}$$
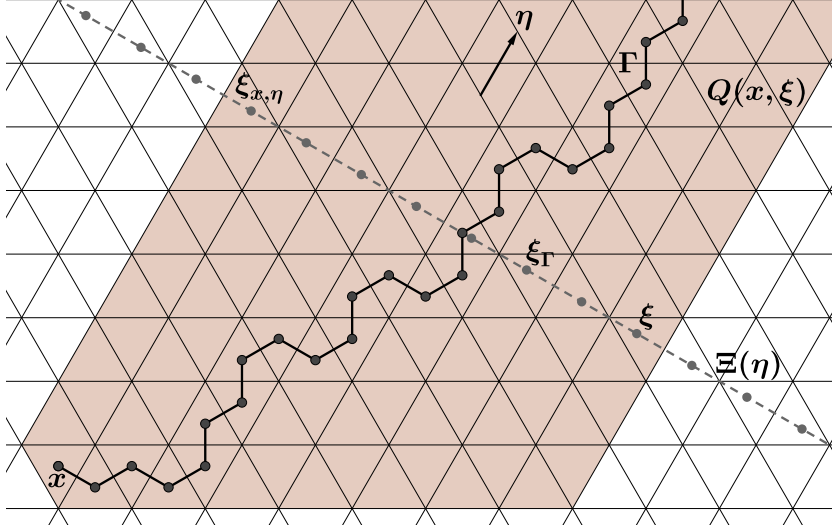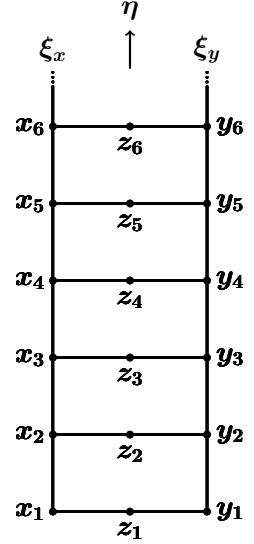
yielding the claim. $\square$

FIG. 1. Coxeter complex of type $\widetilde{A}_2$

FIG. 2. Bad ladder

Definition 3.3. Given $\eta \in \partial X$, we set

$$\Xi(\eta) := \{\xi \in \mathcal{C}_{\mathrm{hb}}(X) \mid \xi = \xi_\Gamma \text{ for some } \Gamma \in \mathrm{CGR}(z_0, \eta)\}.$$

Note that $\Xi(\eta)$ is independent of the choice of z_0 by Lemma 3.1: if $z'_0 \in X^{(0)}$ and $\xi \in \mathcal{C}_{\mathrm{hb}}(X)$, then $\xi = \xi_\Gamma$ for some $\Gamma \in \mathrm{CGR}(z_0, \eta)$ if and only if $\xi = \xi_{\Gamma'}$ for some $\Gamma' \in \mathrm{CGR}(z'_0, \eta)$.

Definition 3.4. For $x \in X^{(0)}$ and $\xi \in \Xi(\eta)$, define the **combinatorial sector**

$$Q(x, \xi) := \{y \in X^{(0)} \mid y \text{ is on a CGR from } x \text{ to } \eta \text{ that converges to } \xi\}.$$

Note that $Q(x, \xi)$ is nonempty by Lemma 3.1.

Example 3.5. Consider the Coxeter complex of type $\widetilde{A}_2$ (see e.g. [AB08, §3]), that is, the tiling of the Euclidean plane by congruent equilateral triangles (see Figure 1). Let X be the associated chamber graph, namely, the graph with vertex set the barycenters of these triangles, and with an edge between two barycenters if the corresponding triangles share a common edge.

In this situation, the horofunction compactification of X coincides with the combinatorial compactification of X introduced in [CL11, §2], and $\mathcal{C}_{\mathrm{hb}}(X)$ can be thought of as a refinement of ∂X (see [CL11, §3], and also [Mar18, Example 3.1]).

An example of a CGR $\Gamma \in \mathrm{CGR}(x, \eta)$ for some vertex $x \in X^{(0)}$ and some direction $\eta \in \partial X$ is depicted on Figure 1. The set $\Xi(\eta)$ can be viewed as the set of "strips" delimited by two adjacent lines in the direction of η, or else as the set of vertices of a simplicial line "transversal" to the direction η (the dashed line on Figure 1) — see also [Mar18, Appendix A].

An example of combinatorial sector $Q(x, \xi)$ for some $\xi \in \Xi(\eta)$ is depicted as the coloured area in Figure 1 (note that in this example, the notion of combinatorial sector coincides with that introduced in [CL11, §2.3] — see [Mar18, Theorem 3.11]).

Example 3.6. Consider the graph X pictured in Figure 2, with vertex set $X^{(0)} = \{x_n, y_n, z_n \mid n \geq 1\}$. Then ∂X has a unique element $\eta \in \partial X$, whereas $\mathcal{C}_{\mathrm{hb}}(X) =$

$\Xi(\eta)$ has two elements ξ_x, ξ_y, respectively corresponding to the (limits of the) CGR $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$. Here are some examples of combinatorial sectors:

$$Q(x_1, \xi_x) = \{x_n \mid n \geq 1\}, \quad Q(x_1, \xi_y) = X^{(0)}, \quad \text{and} \quad Q(z_1, \xi_y) = \{z_1, y_n \mid n \geq 1\}.$$

Note also that $\mathrm{Geo}(x_1, \eta) = X^{(0)}$ and $\mathrm{Geo}(z_1, \eta) = \{z_1, x_n, y_n \mid n \geq 1\}$ have infinite symmetric difference, and hence X does not satisfy the hyperfiniteness criterion given in [HSS19, Theorem 1.4]. This "bad ladder" is at the basis of the counter-example to [HSS19, Question 1.5] described in [Tou18].

4. Special vertices

Throughout this section, we fix some $\eta \in \partial X$.

Lemma 4.1. *Let $x \in X^{(0)}$ and $\xi \in \Xi(\eta)$. Let $(x_n)_{n\in\mathbb{N}} \in \mathrm{CGR}(x, \eta)$ be converging to ξ and let $y \in Q(x, \xi)$. Then there is some $N \in \mathbb{N}$ such that $\mathrm{d}(x, y) + \mathrm{d}(y, x_n) = \mathrm{d}(x, x_n)$ for all $n \geq N$.*

Proof. Let $(y_n)_{n\in\mathbb{N}} \in \mathrm{CGR}(x, \eta)$ be converging to ξ and passing through y (say $y = y_m$ for some $m \in \mathbb{N}$). Since $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ both converge to ξ, there exists some $N \in \mathbb{N}$ with $N \geq m$ such that

$$f_{x_n}(y) = \xi(y) = f_{y_n}(y) \quad \text{and} \quad f_{x_n}(x) = \xi(x) = f_{y_n}(x)$$

for all $n \geq N$. Hence, for each $n \geq N$, we have

$$\mathrm{d}(x, y) = \mathrm{d}(y_n, x) - \mathrm{d}(y_n, y) = f_{y_n}(x) - f_{y_n}(y) = f_{x_n}(x) - f_{x_n}(y)$$
$$= \mathrm{d}(x_n, x) - \mathrm{d}(x_n, y),$$

yielding the claim. $\square$

Lemma 4.2. *Let $x \in X^{(0)}$ and $\xi \in \Xi(\eta)$. Let $\Gamma = (x_n)_{n\in\mathbb{N}} \in \mathrm{CGR}(x, \eta)$ be converging to ξ. Then*

$$Q(x, \xi) = \bigcup_{n\in\mathbb{N}} \Gamma(x, x_n).$$

Proof. The inclusion $\supseteq$ is clear. Conversely, let $y \in Q(x, \xi)$. Then Lemma 4.1 yields some $N \in \mathbb{N}$ such that $\mathrm{d}(x, y) + \mathrm{d}(y, x_N) = \mathrm{d}(x, x_N)$, so that $y \in \Gamma(x, x_N)$.
                                                                                        $\square$

Lemma 4.3. *Let $x \in X^{(0)}$ and $\xi \in \Xi(\eta)$. If $y \in Q(x, \xi)$, then $Q(y, \xi) \subseteq Q(x, \xi)$.*

Proof. Let Γ_{xy} be a geodesic path from x to y and let $(y_n)_{n\in\mathbb{N}} \in \mathrm{CGR}(y, \eta)$ be converging to ξ and such that $\Gamma_{xy} \cdot (y_n)_{n\in\mathbb{N}}$ is a CGR. Since $\Gamma(y, y_n) \subseteq \Gamma(x, y_n)$ for each $n \in \mathbb{N}$, the lemma follows from Lemma 4.2. $\square$

Lemma 4.4. *Let $x \in X^{(0)}$ and $\xi \in \Xi(\eta)$. Let $y \in Q(x, \xi)$, and let $\Gamma = (y_n)_{n\in\mathbb{N}} \in \mathrm{CGR}(y, \eta)$ be such that $\Gamma \subseteq Q(y, \xi)$. Then the concatenation of any geodesic path from x to y with Γ is a CGR.*

Proof. Let $\Gamma' = (x_n)_{n\in\mathbb{N}}$ be a CGR from x to η passing through y and converging to ξ (see Figure 3). Let $r \in \mathbb{N}$. We have to show that

$$\mathrm{d}(x, y_r) = \mathrm{d}(x, y) + \mathrm{d}(y, y_r).$$

Since $y_r \in \Gamma \subseteq Q(y, \xi)$, we find by Lemma 4.1 some $N_1 \in \mathbb{N}$ such that

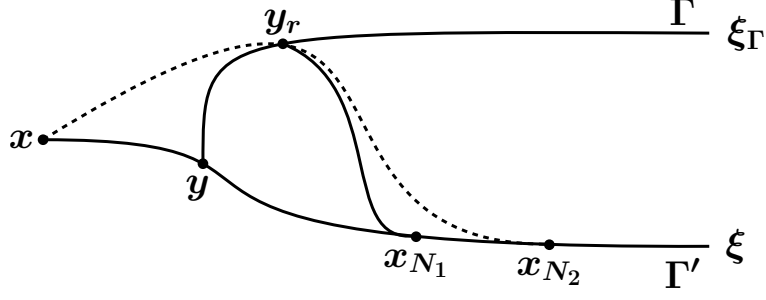$$\mathrm{d}(y, y_r) + \mathrm{d}(y_r, x_n) = \mathrm{d}(y, x_n) \quad \text{for all } n \geq N_1.$$

FIG. 3. Lemma 4.4

Similarly, since $y \in Q(x, \xi)$, we have $y_r \in Q(y, \xi) \subseteq Q(x, \xi)$ by Lemma 4.3, and hence we find by Lemma 4.1 some $N_2 \in \mathbb{N}$ such that

$$\mathrm{d}(x, y_r) + \mathrm{d}(y_r, x_n) = \mathrm{d}(x, x_n) \quad \text{for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then

$$\mathrm{d}(x, y_r) - \mathrm{d}(y, y_r) = \mathrm{d}(x, x_N) - \mathrm{d}(y_r, x_N) - \mathrm{d}(y, x_N) + \mathrm{d}(y_r, x_N)$$
$$= \mathrm{d}(x, y),$$

as desired. $\qquad \square$

Lemma 4.5. *Let* $x \in X^{(0)}$ *and* $\xi \in \Xi(\eta)$, *and let* $\Gamma \in \mathrm{CGR}(x, \eta)$. *If* $\Gamma \subseteq Q(x, \xi)$, *then* $Q(x, \xi_\Gamma) \subseteq Q(x, \xi)$.

Proof. Write $\Gamma = (x_n)_{n \in \mathbb{N}}$, and let $y \in Q(x, \xi_\Gamma)$. By Lemma 4.1, we find some $N \in \mathbb{N}$ such that

$$\mathrm{d}(x, y) + \mathrm{d}(y, x_N) = \mathrm{d}(x, x_N).$$

Let $(y_n)_{n \in \mathbb{N}} \in \mathrm{CGR}(x, \eta)$ converge to ξ. Since $x_N \in \Gamma \subseteq Q(x, \xi)$, we find by Lemma 4.1 some $M \in \mathbb{N}$ such that

$$\mathrm{d}(x, x_N) + \mathrm{d}(x_N, y_M) = \mathrm{d}(x, y_M).$$

Hence

$$\mathrm{d}(x, y) + \mathrm{d}(y, x_N) + \mathrm{d}(x_N, y_M) = \mathrm{d}(x, y_M),$$

so that $y \in \Gamma(x, y_M) \subseteq Q(x, \xi)$, as desired. $\qquad \square$

Lemma 4.6. *Let* $x \in X^{(0)}$. *If* $Q(x, \xi) = Q(x, \xi')$ *for some* $\xi, \xi' \in \Xi(\eta)$, *then* $\xi = \xi'$.

Proof. For any two vertices $a, b \in X^{(0)}$, we fix a geodesic path Γ_{ab} from a to b. Let $(x_n)_{n \in \mathbb{N}} \in \mathrm{CGR}(x, \eta)$ be converging to ξ, and $(y_n)_{n \in \mathbb{N}} \in \mathrm{CGR}(x, \eta)$ be converging to ξ' (see Figure 4). We claim that for each $z \in X^{(0)}$, there exists some $N = N_z \in \mathbb{N}$ such that

(1) $\quad \xi(z) = f_{x_n}(z), \quad \xi'(z) = f_{y_n}(z) \quad \text{and} \quad \mathrm{d}(z, x_n) = \mathrm{d}(z, y_n) \quad \text{for all } n \geq N.$

This will imply that for each $z \in X^{(0)}$,

$$\xi(z) = f_{x_n}(z) = \mathrm{d}(x_n, z) - \mathrm{d}(x_n, z_0) = \mathrm{d}(y_n, z) - \mathrm{d}(y_n, z_0) = f_{y_n}(z) = \xi'(z)$$

for all $n \geq \max\{N_z, N_{z_0}\}$, yielding the lemma.

Let thus $z \in X^{(0)}$. Let $N \in \mathbb{N}$ be such that $\xi(z) = f_{x_n}(z)$ and $\xi'(z) = f_{y_n}(z)$ for all $n \geq N$, and such that

(2) $\qquad \qquad \Gamma_{zx_N} \cdot (x_n)_{n \geq N} \text{ and } \Gamma_{zy_N} \cdot (y_n)_{n \geq N} \text{ are CGR}$
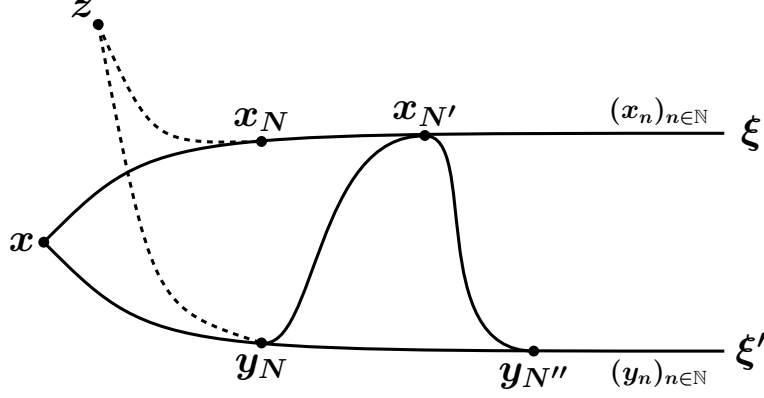
Fig. 4. Lemma 4.6

(see Lemma 3.1). We will prove that $\mathrm{d}(z, x_N) = \mathrm{d}(z, y_N)$, yielding (1).

Since $y_N \in Q(x, \xi') = Q(x, \xi)$, we find by Lemma 4.1 some $N' > N$ such that

$$(3) \qquad \mathrm{d}(x, y_N) + \mathrm{d}(y_N, x_{N'}) = \mathrm{d}(x, x_{N'}).$$

In particular,

$$(4) \qquad \mathrm{d}(y_N, x_{N'}) = \mathrm{d}(x, x_{N'}) - \mathrm{d}(x, y_N) = \mathrm{d}(x, x_{N'}) - \mathrm{d}(x, x_N) = \mathrm{d}(x_N, x_{N'}).$$

Similarly, since $x_{N'} \in Q(x, \xi) = Q(x, \xi')$, we find by Lemma 4.1 some $N'' > N'$ such that

$$(5) \qquad \mathrm{d}(x, x_{N'}) + \mathrm{d}(x_{N'}, y_{N''}) = \mathrm{d}(x, y_{N''}).$$

It then follows from (3) and (5) that

$$\mathrm{d}(y_N, x_{N'}) + \mathrm{d}(x_{N'}, y_{N''}) = -\mathrm{d}(x, y_N) + \mathrm{d}(x, y_{N''}) = \mathrm{d}(y_N, y_{N''}).$$

In particular, $x_{N'}$ is on the CGR $\Gamma_{y_N \xi'} := \Gamma_{y_N x_{N'}} \cdot \Gamma_{x_{N'} y_{N''}} \cdot (y_n)_{n \geq N''}$, so that $x_{N'} \in Q(y_N, \xi')$. As $y_N \in Q(z, \xi')$ by (2), it then follows from Lemma 4.4 (applied to $x := z$, $\xi := \xi'$, $y := y_N$ and $\Gamma := \Gamma_{y_N \xi'}$) that

$$(6) \qquad \mathrm{d}(z, y_N) + \mathrm{d}(y_N, x_{N'}) = \mathrm{d}(z, x_{N'}).$$

As

$$\mathrm{d}(z, x_N) + \mathrm{d}(x_N, x_{N'}) = \mathrm{d}(z, x_{N'})$$

by (2), we deduce from (6) that

$$\begin{aligned}
\mathrm{d}(z, y_N) &= \mathrm{d}(z, x_{N'}) - \mathrm{d}(y_N, x_{N'}) = \mathrm{d}(z, x_N) + \mathrm{d}(x_N, x_{N'}) - \mathrm{d}(y_N, x_{N'}) \\
&= \mathrm{d}(z, x_N),
\end{aligned}$$

where the last equality follows from (4), as desired. $\qquad \square$

Lemma 4.7. *Let $x \in X^{(0)}$. If $\bigcap_{\xi \in \Xi(\eta)} Q(x, \xi)$ contains a CGR from x to η, then there exists a unique $\xi_{x,\eta} \in \Xi(\eta)$ such that $Q(x, \xi_{x,\eta}) = \bigcap_{\xi \in \Xi(\eta)} Q(x, \xi)$.*

Proof. Let $\Gamma \in \mathrm{CGR}(x, \eta)$ be contained in $\bigcap_{\xi \in \Xi(\eta)} Q(x, \xi)$. Then $Q(x, \xi_\Gamma) \subseteq \bigcap_{\xi \in \Xi(\eta)} Q(x, \xi)$ by Lemma 4.5, and hence $Q(x, \xi_\Gamma) = \bigcap_{\xi \in \Xi(\eta)} Q(x, \xi)$. The uniqueness statement follows from Lemma 4.6. $\qquad \square$

Definition 4.8. Call $x \in X^{(0)}$ η**-special** if $\bigcap_{\xi \in \Xi(\eta)} Q(x, \xi)$ contains a CGR from x to η. We let $X_{\mathrm{s}, \eta}^{(0)}$ denote the set of η-special vertices. If $x \in X_{\mathrm{s}, \eta}^{(0)}$, we let $\xi_{x, \eta}$ be the unique element of $\Xi(\eta)$ such that $Q(x, \xi_{x, \eta}) = \bigcap_{\xi \in \Xi(\eta)} Q(x, \xi)$ (see Lemma 4.7).

Definition 4.9. Let $x \in X^{(0)}$. We call $\Gamma \in \mathrm{CGR}(x, \eta)$ **straight** if $\Gamma \subseteq \bigcup_{n \in \mathbb{N}} \Gamma(x, y_n)$ for all $(y_n)_{n \in \mathbb{N}} \in \mathrm{CGR}(x, \eta)$.

Example 4.10. In the context of Example 3.5, every vertex $x \in X^{(0)}$ is η-special (see [Mar18, Lemma 3.3]). The element $\xi_{x,\eta} \in \Xi(\eta)$ is represented on Figure 1; the combinatorial sector $Q(x, \xi_{x,\eta})$ in this case coincides with the unique straight CGR from x to η (contained in the "stripe" between the two adjacent lines in the direction of η that are on each side of x).

Example 4.11. In the context of Example 3.6 (see Figure 2), the vertices x_n and y_n ($n \geq 1$) are η-special: we have $\xi_{x_n,\eta} = \xi_x$ and $\xi_{y_n,\eta} = \xi_y$ for all $n \geq 1$. The vertices z_n ($n \geq 1$), on the other hand, are not η-special, as $Q(z_n, \xi_x) \cap Q(z_n, \xi_y) = \{z_n\}$ for each $n \geq 1$.

Lemma 4.12. *Let $x \in X^{(0)}$ and $\Gamma = (x_n)_{n \in \mathbb{N}} \in \mathrm{CGR}(x, \eta)$. Then the following assertions are equivalent:*

(1) Γ *is straight.*
(2) $\Gamma \subseteq \bigcap_{\xi \in \Xi(\eta)} Q(x, \xi)$.
(3) *For any $(y_n)_{n \in \mathbb{N}} \in \mathrm{CGR}(y_0, \eta)$ and any $n \in \mathbb{N}$, there is some $M \in \mathbb{N}$ such that $\mathrm{d}(x, x_n) + \mathrm{d}(x_n, y_m) = \mathrm{d}(x, y_m)$ for all $m \geq M$.*
(4) *Every sub-CGR of Γ (i.e. CGR of the form $(x_n)_{n \geq m}$ for some $m \in \mathbb{N}$) is straight.*

Proof. (1) $\implies$ (2): Let $\xi \in \Xi(\eta)$, and let $(y_n)_{n \in \mathbb{N}} \in \mathrm{CGR}(x, \eta)$ be converging to ξ. Then $\Gamma \subseteq \bigcup_{n \in \mathbb{N}} \Gamma(x, y_n)$ by assumption and $\bigcup_{n \in \mathbb{N}} \Gamma(x, y_n) = Q(x, \xi)$ by Lemma 4.2, yielding (2).

(2) $\implies$ (3): Let $\Gamma' = (y_n)_{n \in \mathbb{N}} \in \mathrm{CGR}(y_0, \eta)$ and let $n \in \mathbb{N}$. By assumption, $x_n \in \Gamma \subseteq Q(x, \xi_{\Gamma'})$. By Lemma 3.1, there is some $M \in \mathbb{N}$ such that the concatenation $\widetilde{\Gamma} := \Gamma_{x_n y_M} \cdot (y_m)_{m \geq M}$ of a geodesic $\Gamma_{x_n y_M}$ from x_n to y_M with the CGR $(y_m)_{m \geq M}$ is again a CGR. Hence (3) follows from Lemma 4.4 (with $x := x$, $\xi := \xi_{\Gamma'}$, $y := x_n$ and $\Gamma := \widetilde{\Gamma}$).

(3) $\implies$ (1): Let $(y_n)_{n \in \mathbb{N}} \in \mathrm{CGR}(x, \eta)$. By assumption, we find for any $n \in \mathbb{N}$ some $M \in \mathbb{N}$ such that $x_n \in \Gamma(x, y_M)$. Hence $\Gamma \subseteq \bigcup_{M \in \mathbb{N}} \Gamma(x, y_M)$, yielding (1).

(4) $\iff$ (1): The implication (4) $\implies$ (1) is trivial, and the converse is clear in view of the equivalence (1) $\iff$ (3) that we have just established. $\square$

Lemma 4.13. *Let $x \in X^{(0)}$. Then the following assertions are equivalent.*

(1) x *is η-special.*
(2) *There exists a straight $\Gamma \in \mathrm{CGR}(x, \eta)$.*

If the above assertions hold, then $\xi_{x,\eta} = \xi_\Gamma$.

Proof. This readily follows from the equivalence (1) $\iff$ (2) in Lemma 4.12. $\square$

Lemma 4.14. *Let $x \in X^{(0)}$ and let $y \in \mathrm{Geo}(x, \eta)$ be η-special. Then $\Gamma_{xy} \cdot \Gamma_{y\xi_{y,\eta}} \in \mathrm{CGR}(x, \eta)$ for any geodesic path Γ_{xy} from x to y and any CGR $\Gamma_{y\xi_{y,\eta}}$ from y converging to $\xi_{y,\eta}$.*

Proof. Since $y \in Q(x, \xi_\Gamma)$ where Γ is a CGR from x to η passing through y, and since $\Gamma_{y\xi_{y,\eta}} \subseteq Q(y, \xi_{y,\eta}) \subseteq Q(y, \xi_\Gamma)$, this follows from Lemma 4.4. $\square$

Lemma 4.15. *Let $x \in X_{s,\eta}^{(0)}$. Then $Q(x, \xi_{x,\eta}) \subseteq X_{s,\eta}^{(0)}$ and $\xi_{y,\eta} = \xi_{x,\eta}$ for all $y \in Q(x, \xi_{x,\eta})$.*

Proof. Let $y \in Q(x, \xi_{x,\eta})$, and let Γ be a CGR from x converging to $\xi_{x,\eta}$ and passing through y. Thus $\Gamma \subseteq Q(x, \xi_{x,\eta})$ and hence Γ is straight by Lemma 4.12(1)⇔(2). In particular, the sub-CGR of Γ starting at y is straight by Lemma 4.12(1)⇔(4), so that $y \in X_{s,\eta}^{(0)}$ and $\xi_{y,\eta} = \xi_{x,\eta}$ by Lemma 4.13. □

Definition 4.16. For $x \in X^{(0)}$ and $\xi \in \Xi(\eta)$, we define the function
$$d_{x,\xi} \colon X^{(0)} \to \mathbb{N} : a \mapsto d(x, a) + \xi(a) - \xi(x).$$

Lemma 4.17. *Let $x \in X^{(0)}$ and $\xi \in \Xi(\eta)$. Let also $\Gamma = (x_n)_{n \in \mathbb{N}} \in CGR(x, \eta)$.*
  (1) *The sequence $(d_{x,\xi}(x_n))_{n \in \mathbb{N}}$ is non-decreasing.*
  (2) $d_{x,\xi}(x_n) \leq 2 \cdot d_H(\Gamma, \Gamma')$ *for any $n \in \mathbb{N}$ and for any $\Gamma' \in CGR(x, \eta)$ converging to ξ.*
  (3) *There is some $N \in \mathbb{N}$ such that $d_{x,\xi}(x_n) = d_{x,\xi}(x_N)$ for all $n \geq N$.*
  (4) *If N is as in (3), then for any $\Gamma' = (y_n)_{n \in \mathbb{N}} \in CGR(y_0, \eta)$ converging to ξ, and for any $n \geq N$, there exists some $M \in \mathbb{N}$ such that $d(x_N, x_n) + d(x_n, y_m) = d(x_N, y_m)$ for all $m \geq M$.*

Proof. (1) Let $\Gamma' = (y_n)_{n \in \mathbb{N}} \in CGR(x, \eta)$ be converging to ξ. For all large enough $M \in \mathbb{N}$, we have
$$d_{x,\xi}(x_{n+1}) - d_{x,\xi}(x_n) = d(x, x_{n+1}) + d(x_{n+1}, y_M) - d(x, x_n) - d(x_n, y_M)$$
$$= 1 + d(x_{n+1}, y_M) - d(x_n, y_M) \geq 0.$$

(2) Let $\Gamma' = (y_n)_{n \in \mathbb{N}} \in CGR(x, \eta)$ be converging to ξ. Let $n \in \mathbb{N}$. Then for all large enough $M \in \mathbb{N}$, we have
$$d_{x,\xi}(x_n) = d(x, x_n) + d(x_n, y_M) - d(x, y_M) \leq 2 \cdot d(x_n, \Gamma') \leq 2 \cdot d_H(\Gamma, \Gamma'),$$
as desired.

(3) Let $\Gamma' \in CGR(x, \eta)$ be converging to ξ. Since $d_H(\Gamma, \Gamma') < \infty$, it follows from (2) that the set $\{d_{x,\xi}(x_n) \mid n \in \mathbb{N}\}$ is finite. In view of (1), this implies that $(d_{x,\xi}(x_n))_{n \in \mathbb{N}}$ is eventually constant, as desired.

(4) Let $(y_n)_{n \in \mathbb{N}} \in CGR(y_0, \eta)$ be converging to ξ and $n \geq N$. Let $M \in \mathbb{N}$ be such that $\xi(x_n) - \xi(x) = d(y_m, x_n) - d(y_m, x)$ and $\xi(x_N) - \xi(x) = d(y_m, x_N) - d(y_m, x)$ for all $m \geq M$. Then for all $m \geq M$, we have
$$d(x, x_n) + d(y_m, x_n) = d_{x,\xi}(x_n) + d(y_m, x) = d_{x,\xi}(x_N) + d(y_m, x)$$
$$= d(x, x_N) + d(y_m, x_N),$$
and hence
$$d(x_N, x_n) + d(x_n, y_m) = d(x_N, x_n) + d(x, x_N) - d(x, x_n) + d(y_m, x_N)$$
$$= d(x_N, y_m),$$
yielding the claim. □

Lemma 4.18. *Let $x \in X^{(0)}$ and $(x_n)_{n \in \mathbb{N}} \in CGR(x, \eta)$. If $\Xi(\eta)$ is finite, then there is some $N \in \mathbb{N}$ such that $(x_n)_{n \geq N}$ is straight.*

Proof. Assume that $\Xi(\eta)$ is finite. For each $\xi \in \Xi(\eta)$, let $N_\xi \in \mathbb{N}$ be such that $d_{x,\xi}(x_n) = d_{x,\xi}(x_{N_\xi})$ for all $n \geq N_\xi$ (see Lemma 4.17(3)). Set $N := \max\{N_\xi \mid \xi \in \Xi(\eta)\}$. Then $\Gamma_N := (x_n)_{n \geq N}$ is straight, as follows from Lemma 4.12(1)⇔(3) and Lemma 4.17(4). □
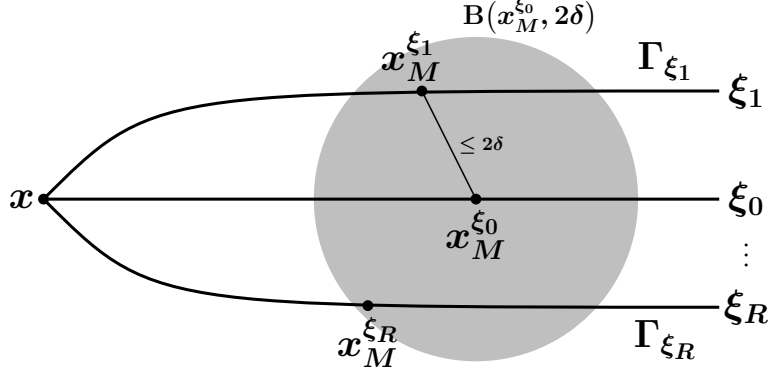
FIG. 5. Proposition 5.2

5. CONSEQUENCES OF HYPERBOLICITY

Throughout this section, we fix some $\eta \in \partial X$, and we assume that X is hyperbolic.

Lemma 5.1. *Let $x \in X^{(0)}$, and let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of CGR from x to η. Then $(\Gamma_n)_{n\in\mathbb{N}}$ subconverges to some $\Gamma \in \mathrm{CGR}(x, \eta)$.*

Proof. Since X is locally finite, $(\Gamma_n)_{n\in\mathbb{N}}$ subconverges to some CGR Γ from x. Moreover, Γ is contained in a tubular neighbourhood of Γ_0 by Lemma 2.1, as desired. $\square$

X is called **uniformly locally finite** if there exists a constant R such that each $x \in X^{(0)}$ is contained in at most R edges.

Proposition 5.2. *Assume that X is uniformly locally finite. Then $\Xi(\eta)$ is finite. Moreover, there is a constant $R \in \mathbb{N}$ independent of η such that $\Xi(\eta)$ has at most R elements.*

Proof. Let $\delta > 0$ be such that X is δ-hyperbolic, and let $R \in \mathbb{N}$ be such that each ball of radius 2δ in $X^{(0)}$ contains at most R vertices. We claim that $\Xi(\eta)$ has at most R elements. Indeed, assume for a contradiction that there is some subset $\Xi = \{\xi_0, \xi_1, \ldots, \xi_R\} \subseteq \Xi(\eta)$ with $R + 1$ elements. Let $x \in X^{(0)}$. For each $\xi \in \Xi$, let $\Gamma_\xi = (x_n^\xi)_{n\in\mathbb{N}} \in \mathrm{CGR}(x, \eta)$ converge to ξ (see Figure 5). By Lemma 4.17(3), there is some $N \in \mathbb{N}$ such that

$$(7) \qquad \mathrm{d}_{x,\xi'}(x_n^\xi) = \mathrm{d}_{x,\xi'}(x_N^\xi) \quad \text{for all } \xi, \xi' \in \Xi \text{ and all } n \geq N.$$

Set $M := N + 2\delta$. Then by Lemma 2.1, the ball $B(x_M^{\xi_0}, 2\delta) \subseteq X^{(0)}$ centered at $x_M^{\xi_0}$ and of radius 2δ contains the vertex x_M^ξ of Γ_ξ for each $\xi \in \Xi$. We claim that the vertices x_M^ξ for $\xi \in \Xi$ are pairwise distinct, yielding the desired contradiction. Assume that $x_M^\xi = x_M^{\xi'} =: y$ for some $\xi, \xi' \in \Xi$, and let us show that $\xi = \xi'$. By Lemma 4.6, it is sufficient to show that $Q(y, \xi) = Q(y, \xi')$. By Lemma 4.5, it is then sufficient to show that $(x_n^\xi)_{n\geq M} \subseteq Q(y, \xi')$ (the converse inclusion $Q(y, \xi) \supseteq Q(y, \xi')$ being obtained by exchanging the roles of ξ and ξ'). Let thus $n \geq M \geq N$. In view of (7), we can apply Lemma 4.17(4) (with $x := x$, $\xi := \xi'$, $\Gamma := \Gamma_\xi$ and $\Gamma' := \Gamma_{\xi'}$) to conclude that there exists some $M' \in \mathbb{N}$ such that $\mathrm{d}(y, x_n^\xi) + \mathrm{d}(x_n^\xi, x_m^{\xi'}) = \mathrm{d}(y, x_m^{\xi'})$ for all $m \geq M'$. Thus, $x_n^\xi \in Q(y, \xi')$, as desired. $\square$
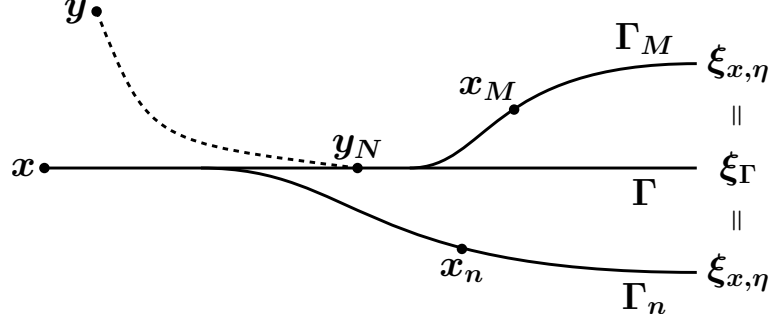
Fig. 6. Lemma 5.4

Corollary 5.3. *Assume that X is uniformly locally finite. Let $\Gamma = (x_n)_{n\in\mathbb{N}} \in \mathrm{CGR}(x_0, \eta)$. Then there is some $N \in \mathbb{N}$ such that $(x_n)_{n\geq N}$ is a straight CGR. In particular, $\Gamma \setminus \mathrm{X}_{s,\eta}^{(0)}$ is finite.*

Proof. By Proposition 5.2, the set $\Xi(\eta)$ is finite. Hence Lemma 4.18 yields some $N \in \mathbb{N}$ such that $(x_n)_{n\geq N}$ is straight. The second claim then follows from Lemma 4.13. $\qquad\square$

Lemma 5.4. *Let $x \in \mathrm{X}_{s,\eta}^{(0)}$ and $y \in X^{(0)}$. Then $Q(x, \xi_{x,\eta}) \setminus Q(y, \xi_{x,\eta})$ is finite.*

Proof. Assume for a contradiction that there exists an (unbounded) sequence $(x_n)_{n\in\mathbb{N}} \subseteq Q(x, \xi_{x,\eta}) \setminus Q(y, \xi_{x,\eta})$. In particular, $x_n \in \mathrm{X}_{s,\eta}^{(0)}$ and $\xi_{x_n,\eta} = \xi_{x,\eta}$ for all $n \in \mathbb{N}$ by Lemma 4.15. For each $n \in \mathbb{N}$, let $\Gamma_n \in \mathrm{CGR}(x, \eta)$ converge to $\xi_{x,\eta}$ and passing through x_n (see Figure 6). By Lemma 5.1, we may assume, up to extracting a subsequence, that $(\Gamma_n)_{n\in\mathbb{N}}$ converges to a CGR $\Gamma = (y_n)_{n\in\mathbb{N}}$ from x to η. Note that $\Gamma \subseteq Q(x, \xi_{x,\eta})$, so that $Q(x, \xi_\Gamma) \subseteq Q(x, \xi_{x,\eta})$ by Lemma 4.5. Hence $Q(x, \xi_\Gamma) = Q(x, \xi_{x,\eta})$ by definition of $\xi_{x,\eta}$, and $\xi_\Gamma = \xi_{x,\eta}$ by Lemma 4.6. Let $N \in \mathbb{N}$ be such that $y_N \in Q(y, \xi_\Gamma)$ (see Lemma 3.1), and let $M \in \mathbb{N}$ be such that $y_N \in \Gamma_M \cap \Gamma(x, x_M)$. Thus $x_M \in Q(y_N, \xi_{x,\eta})$. On the other hand, Lemma 4.3 yields $Q(y_N, \xi_\Gamma) \subseteq Q(y, \xi_\Gamma)$. Therefore, $x_M \in Q(y, \xi_{x,\eta})$, yielding the desired contradiction. $\qquad\square$

Definition 5.5. Assume that X is uniformly locally finite. Let $x \in X^{(0)}$. For $\xi \in \Xi(\eta)$, we let $Y(x, \xi)$ denote the set of $y \in \mathrm{Geo}(x, \eta) \cap \mathrm{X}_{s,\eta}^{(0)}$ with $\xi_{y,\eta} = \xi$ and such that $\mathrm{d}(x, y)$ is minimal for these properties. Note that $Y(x, \xi)$ is finite (because X is locally finite) and nonempty by Corollary 5.3. We set

$$\mathrm{Geo}_1(x, \eta) := \bigcup_{\xi\in\Xi(\eta)} \bigcup_{y\in Y(x,\xi)} Q(y, \xi).$$

Example 5.6. In the context of Example 3.6 (see Figure 2), if $x \in \{x_m, y_m, z_m\}$ for some $m \geq 1$, then $\mathrm{Geo}_1(x, \eta) = \{x, x_n, y_n \mid n \geq m\}$.

Example 5.7. Consider the graph X depicted on Figure 7, with vertex set $X^{(0)} = \{x_n, y_n, z_n \mid n \geq 1\}$. As in Example 3.6, ∂X has a unique element $\eta \in \partial X$, while $\mathcal{C}_{\mathrm{hb}}(X) = \Xi(\eta)$ has two elements ξ_x, ξ_y, respectively corresponding to the (limits of the) CGR $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$. However, in this case, all vertices are η-special: if $m \geq 1$, then $(x_n)_{n\geq m} \in \mathrm{CGR}(x_m, \eta)$, $(y_n)_{n\geq m} \in \mathrm{CGR}(y_m, \eta)$ and

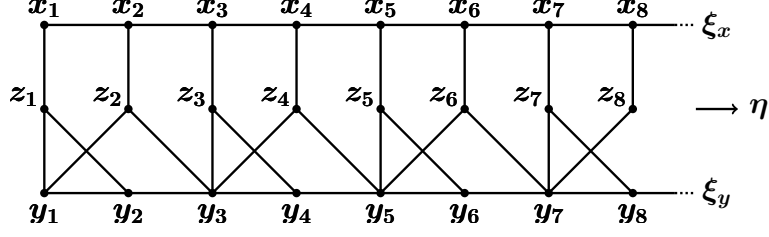$$(z_m, y_{m+1}) \cdot (y_n)_{n\geq m+1} \in \mathrm{CGR}(z_m, \eta)$$

FIG. 7. Another "bad ladder"

are straight CGR. On the other hand, the sets $\mathrm{Geo}(x_1, \eta) = X^{(0)} \setminus \{y_1\}$ and $\mathrm{Geo}(y_1, \eta) = \{x_{n+1}, y_n, z_{2n} \mid n \geq 1\}$ have an infinite symmetric difference. Note, however, that the sets $\mathrm{Geo}_1(x_1, \eta) = \{x_n, y_{n+1}, z_1, z_{2n+2} \mid n \geq 1\}$ and $\mathrm{Geo}_1(y_1, \eta) = \{x_{n+1}, y_n, z_{2n} \mid n \geq 1\}$ have finite symmetric difference.

Proposition 5.8. *Assume that X is uniformly locally finite. Let $x \in X^{(0)}$. Then $\mathrm{Geo}_1(x, \eta) \subseteq \mathrm{Geo}(x, \eta) \cap \mathrm{X}^{(0)}_{\mathrm{s}, \eta}$, and for any $\Gamma \in \mathrm{CGR}(x, \eta)$, the set $\Gamma \setminus \mathrm{Geo}_1(x, \eta)$ is finite.*

Proof. The inclusion $\mathrm{Geo}_1(x, \eta) \subseteq \mathrm{Geo}(x, \eta)$ readily follows from Lemma 4.14, and the inclusion $\mathrm{Geo}_1(x, \eta) \subseteq \mathrm{X}^{(0)}_{\mathrm{s}, \eta}$ from Lemma 4.15. Let now $\Gamma = (x_n)_{n \in \mathbb{N}} \in \mathrm{CGR}(x, \eta)$. By Corollary 5.3, there is some $N \in \mathbb{N}$ such that $(x_n)_{n \geq N}$ is a straight CGR. Note that $x_N \in \mathrm{X}^{(0)}_{\mathrm{s}, \eta}$ and that $\xi := \xi_\Gamma = \xi_{x_N, \eta}$ by Lemma 4.13. Moreover, $(x_n)_{n \geq N} \subseteq Q(x_N, \xi)$. Let $y \in Y(x, \xi)$. Then $Q(x_N, \xi) \setminus Q(y, \xi)$ is finite by Lemma 5.4, and hence $(x_n)_{n \geq N} \setminus \mathrm{Geo}_1(x, \eta)$ is finite, yielding the claim. $\square$

Theorem 5.9. *Assume that X is hyperbolic and uniformly locally finite. Let $x, y \in X^{(0)}$. Then $\mathrm{Geo}_1(x, \eta) \setminus \mathrm{Geo}_1(y, \eta)$ is finite.*

Proof. Assume for a contradiction that there exists an infinite sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathrm{Geo}_1(x, \eta) \setminus \mathrm{Geo}_1(y, \eta)$. Since $\Xi(\eta)$ is finite by Proposition 5.2, and since $Y(x, \xi)$ is finite for each $\xi \in \Xi(\eta)$ (because X is locally finite), we may assume, up to extracting a subsequence, that $(x_n)_{n \in \mathbb{N}} \subseteq Q(x', \xi)$ for some $\xi \in \Xi(\eta)$ and some $x' \in \mathrm{X}^{(0)}_{\mathrm{s}, \eta}$ with $\xi_{x', \eta} = \xi$. Let $y' \in Y(y, \xi)$, so that $Q(y', \xi) \subseteq \mathrm{Geo}_1(y, \eta)$. Then Lemma 5.4 implies that $Q(x', \xi) \setminus Q(y', \xi)$ is finite, and hence $(x_n)_{n \in \mathbb{N}} \setminus \mathrm{Geo}_1(y, \eta)$ is finite, yielding the desired contradiction. $\square$

We conclude this section with an easy observation that will be used in §6.

Lemma 5.10. *Assume that X is uniformly locally finite. Let G be a subgroup of $\mathrm{Aut}(X)$. Then $g \cdot \mathrm{Geo}_1(x, \eta) = \mathrm{Geo}_1(gx, g\eta)$ for all $g \in G$ and $x \in X^{(0)}$.*

Proof. Let $g \in G$. Since g preserves geodesic paths and CGR, it acts on ∂X and $\mathcal{C}_{\mathrm{hb}}(X)$, and we have $g \cdot \Xi(\eta) = \Xi(g\eta)$, $g \cdot Q(x, \xi) = Q(gx, g\xi)$ ($x \in X^{(0)}$, $\xi \in \Xi(\eta)$), $g \cdot \mathrm{X}^{(0)}_{\mathrm{s}, \eta} = \mathrm{X}^{(0)}_{\mathrm{s}, g\eta}$, $g \cdot \xi_{x, \eta} = \xi_{gx, g\eta}$ ($x \in \mathrm{X}^{(0)}_{\mathrm{s}, \eta}$), $g \cdot \mathrm{Geo}(x, \eta) = \mathrm{Geo}(gx, g\eta)$ ($x \in X^{(0)}$), $g \cdot Y(x, \xi) = Y(gx, g\xi)$ ($x \in X^{(0)}$, $\xi \in \Xi(\eta)$), and hence $g \cdot \mathrm{Geo}_1(x, \eta) = \mathrm{Geo}_1(gx, g\eta)$ ($x \in X^{(0)}$). $\square$

6. Endgame

This section is devoted to the proof of Theorem A.

Let G be a finitely generated hyperbolic group. Write e for the neutral element of G. Let S be a finite symmetric set of generators of G, and let X be the Cayley

graph of G with respect to S. Thus X is a uniformly locally finite δ-hyperbolic graph (for some $\delta > 0$) with $X^{(0)} = G$, and we keep the notions and notations relative to X that we developed in the previous sections, choosing $z_0 := e$ as base point (we will also write ∂G and $\mathcal{C}_{\mathrm{hb}}(G)$ instead of ∂X and $\mathcal{C}_{\mathrm{hb}}(X)$). In addition, given a path $\Gamma = (x_m)_{0 \le m \le \ell}$ (resp. $\Gamma = (x_m)_{m \in \mathbb{N}}$) in X, we denote the n-th entry x_n of Γ by Γ_n. Fix a total order $\le$ on G (and hence, in particular, on S) such that

$$v \le w \implies \mathrm{d}(e,v) \le \mathrm{d}(e,w) \quad \text{for all } v, w \in G.$$

Definition 6.1. Given a boundary point $\eta \in \partial G$, we define $C^\eta \subseteq G \times S^{<\mathbb{N}}$ as

$$C^\eta := \{(\Gamma_0, \mathrm{typ}(\Gamma_0, \dots, \Gamma_n)) \in G \times S^{<\mathbb{N}} \mid \Gamma \in \mathrm{CGR}(\Gamma_0, \eta), \Gamma_0 \in \mathrm{Geo}_1(e, \eta), n \in \mathbb{N}\}.$$

Write $s_n^\eta \in S^{<\mathbb{N}}$ for the lexicographically least string of length n which appears infinitely many times in C^η, i.e. such that $(g, s_n^\eta) \in C^\eta$ for infinitely many $g \in \mathrm{Geo}_1(e, \eta)$ (note that C^η contains infinitely many elements of $G \times S^n$ for every $n \in \mathbb{N}$ by Proposition 5.8). Note also that for each $n \in \mathbb{N}$, the sequence s_n^η is an initial segment of the sequence s_{n+1}^η (because if $(\Gamma_0, \mathrm{typ}(\Gamma_0, \dots, \Gamma_n)) \in C^\eta$, then also $(\Gamma_0, \mathrm{typ}(\Gamma_0, \dots, \Gamma_{n+1})) \in C^\eta$). In particular, $s^\eta := \lim_n s_n^\eta \in S^{\mathbb{N}}$ is well-defined. Write

$$T_n^\eta := \{g \in \mathrm{Geo}_1(e, \eta) \mid (g, s_n^\eta) \in C^\eta\}$$

and let $g_n^\eta := \min T_n^\eta$ (with respect to the total order on G). Finally, set

$$k_n^\eta := \mathrm{d}(e, g_n^\eta)$$

and note that the sequence k_n^η is nondecreasing in n.

We will now establish the Borelness of a number of subsets of standard Borel spaces constructed from the standard Borel spaces G, ∂G and $\mathcal{C}_{\mathrm{hb}}(G)$. We recall that G has the discrete topology, ∂G the topology defined in §2.3, and $\mathcal{C}_{\mathrm{hb}}(G)$ the topology of pointwise convergence.

Claim 6.2. The set

$$C := \{\Gamma \in G^{\mathbb{N}} \mid \Gamma \text{ is a CGR}\}$$

is closed in $G^{\mathbb{N}}$, and each set $C_g := \{\Gamma \in C \mid \Gamma_0 = g\}$ ($g \in G$) is compact.

Proof. Each C_g is compact by Lemma 5.1, and hence C is closed, as its intersection with each clopen subset $\{\Gamma \in G^{\mathbb{N}} \mid \Gamma_0 = g\}$ ($g \in G$) of $G^{\mathbb{N}}$ is compact (hence closed). $\square$

Claim 6.3. The set

$$R := \{(\eta, g, \Gamma) \in \partial G \times G^{\mathbb{N}} \mid \Gamma \in \mathrm{CGR}(g, \eta)\}$$

is closed in $\partial G \times G^{\mathbb{N}}$.

Proof. Indeed, if $\Gamma_n \in \mathrm{CGR}(g, \eta_n)$ converges to $\Gamma \in \mathrm{CGR}(g, \eta)$ and η_n converges to $\eta' \in \partial G$, then by definition of the topology on ∂X, there is a sequence of CGR $\Gamma_n' \in \mathrm{CGR}(g, \eta_n)$ subconverging to a CGR $\Gamma' \in \mathrm{CGR}(g, \eta')$. As $\mathrm{d}_H(\Gamma_n, \Gamma_n') \le 2\delta$ by Lemma 2.1, this implies that $\mathrm{d}_H(\Gamma, \Gamma') \le 2\delta$, and hence that $\eta = \eta'$, as desired. $\square$

Claim 6.4. The set

$$F := \{(\eta, g, (\Gamma_0, \dots, \Gamma_n)) \in \partial G \times G^{<\mathbb{N}} \mid \Gamma \in \mathrm{CGR}(g, \eta)\}$$

is Borel in $\partial G \times G^{<\mathbb{N}}$.

Proof. Since $F = \bigcup_{g \in G} F_g$, where $F_g := \{(\eta, h, (\Gamma_0, \ldots, \Gamma_n)) \in F \mid h = g\}$, it is sufficient to check that F_g is closed for each $g \in G$. Let thus $g \in G$. Note that F_g is the projection of the subset

$$F'_g = \{(\eta, g, (\Gamma_0, \ldots, \Gamma_n), \Gamma') \in \partial G \times G^{<\mathbb{N}} \times C_g \mid (\eta, g, \Gamma') \in R \text{ and } \forall i \leq n \ \Gamma'_i = \Gamma_i\}$$

of $\partial G \times G^{<\mathbb{N}} \times C_g$ to $\partial G \times G^{<\mathbb{N}}$. As F'_g is closed by Claim 6.3, and as the projection of a closed set along a compact space is also closed, F_g is indeed closed by Claim 6.2. $\qquad\square$

Claim 6.5. The set

$$M := \{(\eta, \xi) \in \partial G \times \mathcal{C}_{\mathrm{hb}}(G) \mid \xi \in \Xi(\eta)\}$$

is Borel in $\partial G \times \mathcal{C}_{\mathrm{hb}}(G)$.

Proof. We will show that M is both analytic and coanalytic, hence Borel.

Note first that if $\eta \in \partial G$ and $\xi \in \mathcal{C}_{\mathrm{hb}}(G)$, then $(\eta, \xi) \in M$ if and only if

$$\exists \Gamma \in G^{\mathbb{N}} \ (\eta, \Gamma_0, \Gamma) \in R \text{ and } \xi_\Gamma = \xi.$$

As $\xi_\Gamma = \xi$ if and only if $\forall g \in G \ \exists n \in \mathbb{N} \ \forall m \geq n \ f_{\Gamma_m}(g) = \xi(g)$, which is a Borel definition of the subset $\{(\xi, \Gamma) \in \mathcal{C}_{\mathrm{hb}}(G) \times C \mid \xi_\Gamma = \xi\}$ of $\mathcal{C}_{\mathrm{hb}}(G) \times G^{\mathbb{N}}$, we deduce from Claim 6.3 that M is analytic (because the above formula defining M has only one existential quantifier, $\exists \Gamma \in G^{\mathbb{N}}$, ranging over an uncountable standard Borel space).

Now, to see that M is coanalytic, note that $(\eta, \xi) \in M$ if and only if for every $\Lambda \in \mathrm{CGR}(e, \eta)$, we can find a sequence of geodesic paths $\Gamma^k \in G^k$ starting at e and contained in the 2δ-neighbourhood $N_{2\delta}(\Lambda)$ of Λ (such a sequence subconverges to some $\Gamma \in \mathrm{CGR}(e, \eta)$ by local finiteness of X), and such that $f_{\Gamma_i^j}(g) = \xi(g)$ for all vertices $g \in G$ and all large enough i, j (i.e. so that $\xi_\Gamma = \xi$). (The forward implication follows from Lemma 2.1.) In other words, $(\eta, \xi) \in M$ if and only if

$$\forall \Lambda \in G^{\mathbb{N}} \text{ if } (\eta, e, \Lambda) \in R, \text{ then } \forall k \in \mathbb{N} \ \exists \Gamma^k \in G^k \text{ a geodesic path with } \Gamma_0^k = e$$

$$\text{such that } \Gamma^k \subseteq N_{2\delta}(\Lambda) \text{ and such that } \forall g \in G \ \exists n_g \in \mathbb{N} \ \forall i, j > n_g \ f_{\Gamma_i^j}(g) = \xi(g).$$

Note that the three subformulas "$(\eta, e, \Lambda) \in R$", "$\Gamma^k \subseteq N_{2\delta}(\Lambda)$" and "$\forall i, j > n_g \ f_{\Gamma_i^j}(g) = \xi(g)$" in the above formula correspond to closed or closed and open sets (in appropriate standard Borel spaces) and all the quantifiers except for one range over countable sets. This single quantifier is the universal one $\forall \Lambda \in G^{\mathbb{N}}$, and thus the above formula defines a coanalytic set. $\qquad\square$

Definition 6.6. By Proposition 5.2, each $\Xi(\eta)$ is finite and there exists $r > 0$ such each $\Xi(\eta)$ has at most r elements. Since M is Borel in $\partial G \times \mathcal{C}_{\mathrm{hb}}(G)$ by Claim 6.5 and has finite sections $M_\eta \subseteq \mathcal{C}_{\mathrm{hb}}(G)$ of size at most r, the Lusin–Novikov theorem yields Borel functions $\xi_1, \ldots, \xi_r \colon \partial G \to \mathcal{C}_{\mathrm{hb}}(G)$ such that M is the union of their graphs $G_{\xi_i} := \{(\eta, \xi_i(\eta)) \in \partial G \times \mathcal{C}_{\mathrm{hb}}(G)\}$.

Claim 6.7. For each $i \in \{1, \ldots, r\}$, the set

$$Q_i := \{(\eta, g, h) \in \partial G \times G^2 \mid h \in Q(g, \xi_i(\eta))\}$$

is Borel in $\partial G \times G^2$.

Proof. By Lemma 4.2, $Q(g, \xi_i(\eta)) = \bigcup_{n \in \mathbb{N}} \Gamma(g, x_n)$ for some, or equivalently every CGR $(x_n)_{n \in \mathbb{N}}$ converging to $\xi_i(\eta)$. This easily gives both analytic and coanalytic definitions of Q_i, because $h \in Q(g, \xi_i(\eta))$ if and only if $\exists \Lambda \in C$ (resp. $\forall \Lambda \in C$) with $\Lambda_0 = g$ and $\xi_\Lambda = \xi_i(\eta)$ and $\exists n \in \mathbb{N}$ $h \in \Gamma(g, \Lambda_n)$. $\qquad \square$

Claim 6.8. The set
$$P := \{(\eta, h) \in \partial G \times G \mid h \text{ is } \eta\text{-special}\}$$
is Borel in $\partial G \times G$.

Proof. This follows from the definition of special vertices (see Definition 4.8) and the fact that the sets F and Q_i are Borel (see Claims 6.4 and 6.7). Indeed, h is η-special if and only if
$$\forall n \in \mathbb{N} \ \exists \Gamma^n \in G^n \ (\eta, h, \Gamma^n) \in F \text{ and } \forall i \leq r \ \forall k < n \ \Gamma_k^n \in Q(h, \xi_i(\eta)).$$
The "if" in the above equivalence follows by local finiteness of X, since the sequence $(\Gamma^n)_{n \in \mathbb{N}}$ with $(\eta, h, \Gamma^n) \in F$ will subconverge to some $\Gamma \in \mathrm{CGR}(h, \eta)$. $\qquad \square$

Claim 6.9. The set
$$P_1 := \{(\xi, \eta, h) \in \mathcal{C}_{\mathrm{hb}}(G) \times \partial G \times G \mid h \text{ is } \eta\text{-special and } \xi = \xi_{h,\eta}\}$$
is Borel in $\mathcal{C}_{\mathrm{hb}}(G) \times \partial G \times G$.

Proof. Note that $(\xi, \eta, h) \in P_1$ if and only if $(\eta, h) \in P$ and $\exists i \leq r \ (\eta, \xi) \in G_{\xi_i}$ and $\forall j \leq r \ Q(h, \xi_i(\eta)) \subseteq Q(h, \xi_j(\eta))$. Hence P_1 is Borel by Claims 6.7 and 6.8. $\qquad \square$

Claim 6.10. The set
$$L := \{(h, \xi, \eta) \in G \times \mathcal{C}_{\mathrm{hb}}(G) \times \partial G \mid h \in Y(e, \xi), \ \xi \in \Xi(\eta)\}$$
is Borel in $G \times \mathcal{C}_{\mathrm{hb}}(G) \times \partial G$.

Proof. Note that $(h, \xi, \eta) \in L$ if and only if $(\eta, \xi) \in M$ and $h \in \mathrm{Geo}(e, \eta)$ and $(\xi, \eta, h) \in P_1$ and $\mathrm{d}(e, h)$ is minimal for these properties. Hence L is Borel by Claims 6.4, 6.5 and 6.9. $\qquad \square$

Claim 6.11. The set
$$B := \{(g, h, \xi, \eta) \in G^2 \times \mathcal{C}_{\mathrm{hb}}(G) \times \partial G \mid g \in Q(h, \xi), \ h \in Y(e, \xi), \ \xi \in \Xi(\eta)\}$$
is Borel in $G^2 \times \mathcal{C}_{\mathrm{hb}}(G) \times \partial G$.

Proof. Note that $(g, h, \xi, \eta) \in B$ if and only if $\exists i \leq r \ \xi = \xi_i(\eta)$ and $(\eta, h, g) \in Q_i$ and $(h, \xi, \eta) \in L$. Hence B is Borel by Claims 6.7 and 6.10. $\qquad \square$

Claim 6.12. The set
$$A := \{(\eta, g) \in \partial G \times G \mid g \in \mathrm{Geo}_1(e, \eta)\}$$
is Borel in $\partial G \times G$.

Proof. Recall that
$$\mathrm{Geo}_1(e, \eta) := \bigcup_{\xi \in \Xi(\eta)} \bigcup_{h \in Y(e, \xi)} Q(h, \xi),$$
so that $(\eta, g) \in A$ if and only if $\exists \xi \in \Xi(\eta) \ \exists h \in Y(e, \xi) \ (g, h, \xi, \eta) \in B$. In other words, A is obtained as the projection on $\partial G \times G$ of the Borel set B (see Claim 6.11). However, vertical sections of B are finite by Proposition 5.2. Thus, A is Borel by the Lusin–Novikov theorem. $\qquad \square$

Claim 6.13. The set
$$D := \{(\eta, (\Gamma_0, \ldots, \Gamma_n)) \in \partial G \times G^{<\mathbb{N}} \mid \Gamma_0 \in \mathrm{Geo}_1(e, \eta) \text{ and } \Gamma \in \mathrm{CGR}(\Gamma_0, \eta)\}$$
is Borel in $\partial G \times G^{<\mathbb{N}}$.

Proof. Note that $(\eta, (\Gamma_0, \ldots, \Gamma_n)) \in D$ if and only if $(\eta, \Gamma_0, (\Gamma_0, \ldots, \Gamma_n)) \in F$ and $(\eta, \Gamma_0) \in A$. Hence D is Borel by Claims 6.4 and 6.12. $\square$

Recall from Definition 6.1 the definition of the sequences s_n^η.

Claim 6.14. For each $n \in \mathbb{N}$, the set
$$S_n := \{(\eta, s^n) \in \partial G \times S^n \mid s^n = s_n^\eta\}$$
is Borel in $\partial G \times S^n$.

Proof. Note that $(\eta, s^n) \in S_n$ if and only if $\forall m \in \mathbb{N} \ \exists \Gamma^n \in G^n \ \mathrm{d}(\Gamma_0^n, e) \geq m$ and $(\eta, \Gamma^n) \in D$ and $\mathrm{typ}(\Gamma^n) = s^n$ and s^n is minimal in S^n for these properties. Hence S_n is Borel by Claim 6.13. $\square$

Let now $E := \{(\eta, g\eta) \mid g \in G\} \subseteq \partial G \times \partial G$ denote the orbital equivalence relation for the G-action on ∂G. To prove Theorem A, we have to show that E is hyperfinite.

Definition 6.15. Set $Z := \{\eta \in \partial G \mid k_n^\eta \not\to \infty\}$. In other words, since X is locally finite, Z is the set of $\eta \in \partial G$ such that there exists some $g_\eta \in \mathrm{Geo}_1(e, \eta)$ belonging to T_n^η for all $n \in \mathbb{N}$, i.e. for which there exists some CGR $\Gamma^\eta \in \mathrm{CGR}(g_\eta, \eta)$ of type $s^\eta = \lim_n s_n^\eta$.

Lemma 6.16. *The map $\alpha \colon (Z, E|Z) \to (\partial G, =) : \eta \mapsto g_\eta^{-1}\eta$ is a Borel reduction.*

Proof. Note first that
$$(8) \qquad\qquad s_n^\eta = s_n^{g\eta} \quad \text{for all } g \in G \text{ and } \eta \in \partial G.$$
Indeed, if there are infinitely many couples $(h, s_n^\eta) \in C^\eta$, then there are infinitely many couples $(gh, s_n^\eta) \in G \times S^n$ with s_n^η the type of the initial segment of length n of a CGR $\Gamma \in \mathrm{CGR}(\Gamma_0, g\eta)$ with $\Gamma_0 \in g\,\mathrm{Geo}_1(e, \eta) = \mathrm{Geo}_1(g, g\eta)$ (see Lemma 5.10), and hence there are infintely many couples (gh, s_n^η) in $C^{g\eta}$ as the symmetric difference between $\mathrm{Geo}_1(g, g\eta)$ and $\mathrm{Geo}_1(e, g\eta)$ is finite by Theorem 5.9.

In particular, α is constant on G-orbits, as it maps the class of Γ^η to the class of $g_\eta^{-1}\Gamma^\eta$, and $g_\eta^{-1}\Gamma^\eta$ is the unique CGR Γ_{e, s^η} from e of type $s^\eta = s^{g_\eta^{-1}\eta}$. This shows that α is a reduction from $E|Z$ to the identity, and it remains to see that α is Borel. But this boils down to the Borelness of the set $\{(\eta, s) \in \partial G \times S^{\mathbb{N}} \mid s = s^\eta\}$, which follows from Claim 6.14 as $s = s^\eta$ if and only if $\forall n \in \mathbb{N} \ (\eta, (s_0, \ldots, s_n)) \in S_{n+1}$. $\square$

Lemma 6.17. *E is smooth on the E-saturation $[Z]_E := \{\eta \in \partial G \mid \exists \theta \in Z \ \theta E \eta\}$ of Z.*

Proof. Lemma 6.16 implies that E is smooth on Z, whence the claim. $\square$

It now remains to check that E is hyperfinite on the complement of $[Z]_E$ in ∂G.

Definition 6.18. Let Y be the complement of $[Z]_E$ in ∂G, i.e.
$$Y = \{\eta \in \partial G \mid \forall \theta \in \partial G : \ \theta E \eta \implies k_n^\theta \to \infty\}.$$
For each $n \in \mathbb{N}$, we define the function $H_n : \partial G \to 2^G$ by
$$H_n(\eta) := (g_n^\eta)^{-1} T_n^\eta.$$

Let F_n be the equivalence relation on $\operatorname{im} H_n$ which is the restriction of the shift action of G on 2^G.

The following lemma has exactly the same proof as [HSS19, Lemma 5.2] and we reproduce it here for completeness.

Lemma 6.19. *There exists a constant $K > 0$ such that for each $n \in \mathbb{N}$, the relation F_n on $\operatorname{im} H_n$ has equivalence classes of size at most K.*

Proof. We will prove the lemma for K the size of the ball centred at e and of radius 8δ in X. Let $n \in \mathbb{N}$. Let $\eta, \theta \in \partial G$ and suppose $H_n(\eta) = gH_n(\theta)$ are F_n-related as witnessed by $g \in G$. Then $(g_n^\eta)^{-1} T_n^\eta = g(g_n^\theta)^{-1} T_n^\theta$. We will show that $\mathrm{d}(e, g) \leq 8\delta$, as desired.

Note that since T_n^η (resp. T_n^θ) is an infinite subset of $\operatorname{Geo}(e, \eta)$ (resp. $\operatorname{Geo}(e, \theta)$), it uniquely determines the boundary point η (resp. θ). In particular,

$$\sigma := (g_n^\eta)^{-1}\eta = g(g_n^\theta)^{-1}\theta.$$
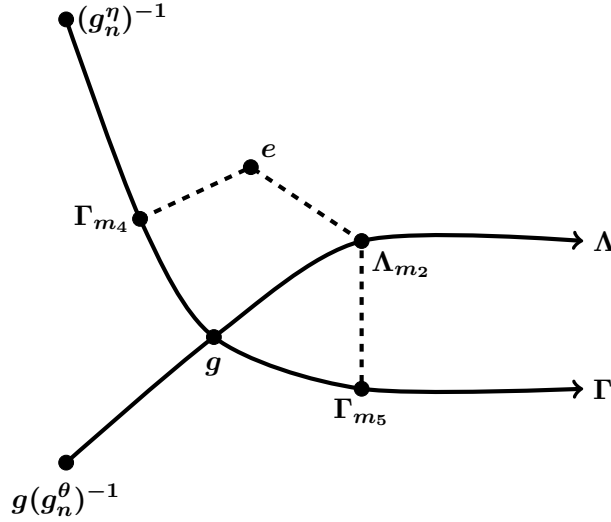
As $g, e \in (g_n^\eta)^{-1} T_n^\eta = g(g_n^\theta)^{-1} T_n^\theta \subseteq \operatorname{Geo}(g(g_n^\theta)^{-1}, \sigma)$, there exist a CGR $\Lambda \in \operatorname{CGR}(g(g_n^\theta)^{-1}, \sigma)$ passing through g, say $g = \Lambda_{m_1}$ for some $m_1 \in \mathbb{N}$, and a CGR $\Lambda' \in \operatorname{CGR}(g(g_n^\theta)^{-1}, \sigma)$ passing through e, say $e = \Lambda'_{m_2}$ for some $m_2 \in \mathbb{N}$. Note that $\mathrm{d}(e, \Lambda_{m_2}) \leq 2\delta$ by Lemma 2.1. Moreover, $m_2 \geq m_1$: indeed, as $g_n^\theta g^{-1} \in g_n^\theta g^{-1}(g_n^\eta)^{-1} T_n^\eta = T_n^\theta$, the minimality of g_n^θ in T_n^θ implies that

$$m_2 = \mathrm{d}(e, g(g_n^\theta)^{-1}) = \mathrm{d}(e, g_n^\theta g^{-1}) \geq \mathrm{d}(e, g_n^\theta) = \mathrm{d}(e, (g_n^\theta)^{-1}) = \mathrm{d}(g, g(g_n^\theta)^{-1}) = m_1.$$

Similarly, $g, e \in \operatorname{Geo}((g_n^\eta)^{-1}, \sigma)$, and hence there exist $\Gamma \in \operatorname{CGR}((g_n^\eta)^{-1}, \sigma)$ passing through g, say $g = \Gamma_{m_3}$ for some $m_3 \in \mathbb{N}$, and $\Gamma' \in \operatorname{CGR}((g_n^\eta)^{-1}, \sigma)$ passing through e, say $e = \Gamma'_{m_4}$ for some $m_4 \in \mathbb{N}$. Again, $\mathrm{d}(e, \Gamma_{m_4}) \leq 2\delta$ by Lemma 2.1, and $m_4 \leq m_3$ because $g_n^\eta g \in g_n^\eta g(g_n^\theta)^{-1} T_n^\theta = T_n^\eta$ and hence the minimality of g_n^η in T_n^η implies that

$$m_3 = \mathrm{d}((g_n^\eta)^{-1}, g) = \mathrm{d}(e, g_n^\eta g) \geq \mathrm{d}(e, g_n^\eta) = \mathrm{d}(e, (g_n^\eta)^{-1}) = m_4.$$

Applying Lemma 2.1 one more time to the sub-CGR of Γ and Λ starting at g, and using the fact that $m_2 \geq m_1$, we find some $m_5 \geq m_3$ such that $\mathrm{d}(\Lambda_{m_2}, \Gamma_{m_5}) \leq 2\delta$.

Note that

$$\mathrm{d}(\Gamma_{m_4}, \Gamma_{m_5}) \leq \mathrm{d}(\Gamma_{m_4}, e) + \mathrm{d}(e, \Lambda_{m_2}) + \mathrm{d}(\Lambda_{m_2}, \Gamma_{m_5}) \leq 6\delta.$$

Thus,

$$\mathrm{d}(e, g) = \mathrm{d}(e, \Gamma_{m_3}) \leq \mathrm{d}(e, \Gamma_{m_4}) + \mathrm{d}(\Gamma_{m_4}, \Gamma_{m_3}) \leq 2\delta + \mathrm{d}(\Gamma_{m_4}, \Gamma_{m_5}) \leq 8\delta,$$

where the inequality $\mathrm{d}(\Gamma_{m_4}, \Gamma_{m_3}) \leq \mathrm{d}(\Gamma_{m_4}, \Gamma_{m_5})$ follows from $m_5 \geq m_3 \geq m_4$. □

Lemma 6.20. *Let $n \in \mathbb{N}$. Then the map H_n is Borel. In particular, the set $\mathrm{im}\, H_n$ is analytic.*

Proof. The sets $S_n = \{(\eta, s_n^\eta) \in \partial G \times S^n\}$, $\{(\eta, g_n^\eta) \in \partial G \times G\}$, $\{(\eta, T_n^\eta) \in \partial G \times 2^G\}$, and $G_{H_n} = \{(\eta, H_n(\eta)) \in \partial G \times 2^G\}$ are easily seen to be definable using only formulas with countable quantifiers and references to the Borel set D (see Claims 6.13 and 6.14). In particular, these sets are Borel. Hence H_n is Borel (as its graph G_{H_n} is Borel) and $\mathrm{im}\, H_n = \mathrm{proj}_{2^G}(G_{H_n})$ is analytic, as desired. □

By Lemma 2.3 (applied to $Z := 2^G$, $A := \mathrm{im}\, H_n$, E the shift action of G on 2^G, and K the constant from Lemma 6.19), we find a finite Borel equivalence relation F_n' on 2^G with $F_n \subseteq F_n'$. Let $f_n : 2^G \to 2^{\mathbb{N}}$ be a reduction from F_n' to E_0 (such a reduction exists as every finite Borel equivalence relation is smooth) and define $f : \partial G \to (2^{\mathbb{N}})^{\mathbb{N}}$ by $f(\eta) = (f_n(H_n(\eta)))_{n \in \mathbb{N}}$. Let $E' := f^{-1}(E_1)$ (i.e. $\eta\, E'\, \theta \iff f(\eta)\, E_1\, f(\theta)$ for all $\eta, \theta \in \partial G$).

Lemma 6.21. *The relation E' is hyperfinite.*

Proof. Note first that E' is Borel, because each H_n is Borel by Lemma 6.20. Note next that E' is a countable equivalence relation. To see this, let E_n' ($n \in \mathbb{N}$) be the relation on ∂G defined by $\eta\, E_n'\, \theta$ if $f_m(H_m(\eta)) = f_m(H_m(\theta))$ for all $m \geq n$. Note that each E_n' is a countable equivalence relation. Indeed, if $\eta\, E_n'\, \theta$, then in particular $f_n(H_n(\eta)) = f_n(H_n(\theta))$. However, the function $f_n \circ H_n$ is countable-to-one as H_n is countable-to-one (namely, if $H_n(\eta) = H_n(\theta)$ then $(g_n^\eta)^{-1}\eta = (g_n^\theta)^{-1}\theta$, and hence $\eta \in G \cdot \theta$) and f_n is finite-to-one, yielding the claim. Thus, $E' = \bigcup_{n \in \mathbb{N}} E_n'$ is indeed countable. Also, E' is by definition hypersmooth. Thus, by [Gao09, Theorem 8.1.5], the relation E' is hyperfinite. □

Lemma 6.22. *The function f is a homomorphism from $E|Y$ to E_1.*

Proof. Suppose $\eta, \theta \in Y$ are such that $g\eta = \theta$ for some $g \in G$. By Theorem 5.9 (and Lemma 5.10), the sets $g\mathrm{Geo}_1(e, \eta)$ and $\mathrm{Geo}_1(e, \theta)$ differ by a finite set. Since both η, θ are in Y and X is locally finite, there is some $N \in \mathbb{N}$ such that $gT_n^\eta \subseteq \mathrm{Geo}_1(e, \theta)$ for all $n \geq N$. Since $s_n^\eta = s_n^\theta$ by (8), we then have $gT_n^\eta = T_n^\theta$ and hence $(g_n^\theta)^{-1}gg_n^\eta H_n^\eta = H_n^\theta$ for all $n \geq N$. Thus $H_n(\eta)\, F_n\, H_n(\theta)$, and hence $H_n(\eta)\, F_n'\, H_n(\theta)$ for all $n \geq N$. Therefore, $f_n(H_n(\eta)) = f_n(H_n(\theta))$ for all $n \geq N$, that is, $f(\eta)\, E_1\, f(\theta)$. □

Proof of Theorem A: By Lemma 6.22, the relation $E|Y$ is a subrelation of E'. A subrelation of a hyperfinite equivalence relation is also hyperfinite, so $E|Y$ is hyperfinite by Lemma 6.21. On the other hand, E is smooth, and hence also hyperfinite, on $\partial G \setminus Y = [Z]_E$ by Lemma 6.17. Therefore, E is hyperfinite on ∂G, thus concluding the proof of Theorem A. □

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