Université Catholique de Louvain
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IRMP

Avril 2013

Topological Kac–Moody groups and their subgroups

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Thèse présentée en vue
de l'obtention du grade de
Docteur en Sciences

Directeur de thèse:
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11 August 2015: Minor corrections in §4.2.1 and in the statement of Proposition 6.29.

Acknowledgement

First of all, I am (at least) infinitely thankful to my advisor, Pierre-Emmanuel Caprace. The genesis of virtually all of the original results presented in this thesis began with a most stimulating discussion in Pierre-Emmanuel's office (or elsewhere...). No less important were his repeated encouragements as well as his total trust in me and in my research, and it is not a figure of speech to say that this thesis could simply not have happened without him.

I would also like to thank Ralf Köhl for welcoming me in Giessen in March–April 2012. It is during this period that I obtained the results of the paper [Mar12b], and I am grateful to Ralf as well as to Tobias Hartnick for their useful comments on that paper.

I would like to thank my mother as well, for her continuous support, particularly in the first years of my thesis.

Finally, I would like to thank Patricia, for her support, and above all for her much needed love.

Contents

Introduction

As suggested by the title, the main objects of this thesis are the so-called Kac–Moody groups. These are, roughly speaking, infinite-dimensional analogues of semi-simple Lie groups. More precisely, one first considers **Kac–Moody algebras**, which are – usually infinite-dimensional – complex Lie algebras, and which may be viewed as generalisations of the finite-dimensional semi-simple complex Lie algebras (see [Kac90]). To a Kac–Moody algebra $\mathfrak{g}$, one can then associate a so-called **Tits functor**, which is a group functor $\mathfrak{G}$ defined over the category of fields. This functor is characterised by a small number of properties; one of them ensures that the group $\mathfrak{G}(\mathbb{C})$ possesses an adjoint action by automorphisms on the Lie algebra $\mathfrak{g}$. By definition, any group obtained by evaluating a Tits functor over a field is called a **(minimal) Kac–Moody group** over that field (see [Tit87] and [Rém02]).

The denotation "minimal" for a Kac–Moody group $\mathfrak{G}(\mathbb{K})$ over a field $\mathbb{K}$ is justified by the fact that one can also construct "maximal" versions $\widehat{\mathfrak{G}}(\mathbb{K})$ of $\mathfrak{G}(\mathbb{K})$, which are obtained from $\mathfrak{G}(\mathbb{K})$ by completing it with respect to some suitable topology, and thus contain $\mathfrak{G}(\mathbb{K})$ as a dense subgroup. An example of this is given by the affine Kac–Moody group $\mathrm{SL}_n(\mathbb{K}[t, t^{-1}])$ of type $\tilde{A}_{n-1}$ and its completion $\mathrm{SL}_n(\mathbb{K}(\!(t)\!))$. Such a group $\widehat{\mathfrak{G}}(\mathbb{K})$ will be called a **maximal** or else **complete Kac–Moody group** over $\mathbb{K}$. One finds in fact several constructions of $\widehat{\mathfrak{G}}(\mathbb{K})$ in the literature, from very different points of view, corresponding to completions of $\mathfrak{G}(\mathbb{K})$ with respect to different topologies. While this diversity accounts for the richness of the theory, it would be nice to have a unified approach to maximal Kac–Moody groups, as in the minimal case. Fortunately, the different constructions are strongly related, and hopefully equivalent (see [Rou12]).

The main goal of this thesis is to give structure results for topological Kac–Moody groups, that is, for Kac–Moody groups (minimal or maximal) admitting a topological group structure.

As mentioned above, several objects deserving the name of "Kac–Moody groups" were introduced in the literature, but unfortunately there is at present no "Introduction to Kac–Moody algebras and groups" handbook presenting them all. Along the way to our structure results for topological Kac–Moody groups, we include a synthesis of the constructions of these objects, with the hope it will provide anyone willing to learn about Kac–Moody theory with an accessible introduction.

The structure results we will establish will be exposed in two separate chapters, one dealing with Kac–Moody groups over fields of characteristic zero (see Chapter 7) and the other dealing with Kac–Moody groups over fields of positive characteristic (see Chapter 8), since each of these two cases has its own specificities.

Minimal Kac–Moody groups $\mathfrak{G}(\mathbb{K})$ over a field $\mathbb{K}$ of characteristic zero, say $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, are interesting in themselves as infinite-dimensional generalisations of semi-simple real or complex Lie groups. They also seem to appear more and more in connection with theoretical physics. Such a Kac–Moody group $\mathfrak{G}(\mathbb{K})$ can be equipped with a topology coming from the field $\mathbb{K}$ – the so-called Kac–Peterson topology – which turns $\mathfrak{G}(\mathbb{K})$ into a connected Hausdorff topological group (see [PK83] and [HKM12]). Note, however, that $\mathfrak{G}(\mathbb{K})$ is not locally compact as soon as its Kac–Moody algebra is infinite-dimensional.

Maximal Kac–Moody groups $\widehat{\mathfrak{G}}(\mathbb{K})$ are naturally topological groups, as completions of $\mathfrak{G}(\mathbb{K})$ with respect to some suitable filtration. In positive characteristic, and more precisely when the field $\mathbb{K}$ is finite, the topological group $\widehat{\mathfrak{G}}(\mathbb{K})$ is locally compact and totally disconnected. The study of these groups is then motivated by the fact that they yield a prominent class – one of the very few that are known in fact – of simple, non-linear, totally disconnected and compactly generated locally compact groups. The latter class of groups plays a fundamental role in the structure theory of general (compactly generated) locally compact groups (see [MZ55] and [CM11b]).

A fundamental tool in the study of Kac–Moody groups is their natural actions on geometrical objects called **buildings** (see [AB08]). Roughly speaking, these are simplicial complexes obtained by glueing copies of a given **Coxeter complex**, the latter being the natural geometrical object associated with a "discrete reflection group", namely, with a **Coxeter group**; typical examples of Coxeter complexes are tilings of the Euclidean space $\mathbb{E}^n$ or of the real hyperbolic space $\mathbb{H}^n$. In particular, in the course of our study of the structure of Kac–Moody groups, we will have to establish several results, of totally independent interest, within Coxeter group and building theory.

The first part of this thesis is devoted to the exposition and proof of these results, and does not require any knowledge of Kac–Moody theory whatsoever. The structure results about topological Kac–Moody groups are then presented in the second part.

We now proceed to describe, chapter by chapter, how this thesis is organised, highlighting along the way the key original results that we established. We invite the reader to go through the introductions of each of the sections containing original results, as they also contain additional material of independent interest, which we will not mention in this introduction. The sections containing original results will be clearly identified at the beginning of each chapter.

In Chapter 1, we introduce Coxeter groups and complexes, and their realisations as CAT(0) spaces. Given a Coxeter group W with finite Coxeter generating set S, we recall that a **parabolic subgroup** of W is a subgroup of W of the form wW_Jw^{-1} for some $w \in W$ and some subset J of S, where W_J is generated by J. Since any intersection of parabolic subgroups is itself a parabolic subgroup (see [Tit74]), it makes sense to define the **parabolic closure** $\mathrm{Pc}(E)$ of a subset $E \subset W$ as the smallest parabolic subgroup of W containing E. Our first key result, presented in Section 1.4, and which is a joint work with Pierre-Emmanuel Caprace, is the following.

Theorem A. *Let H be a subgroup of W. Then there exists $h \in H$ such that the parabolic closure of h is normal and of finite index in the parabolic closure of H.*

This result will be of fundamental importance in studying the topological group structure of complete Kac–Moody groups over finite fields (see Theorem D below). It will be a consequence of a more general theorem relating the parabolic closure of a product of two elements of W with the respective parabolic closures of these elements.

In Chapter 2, we introduce buildings and their realisations as CAT(0) spaces, and we describe the nice interplay between group theory and geometry arising when a group G admits a **strongly transitive** action on a building Δ. As we will see, this is formalised through the notion of **BN-pair** for G, namely of a pair of subgroups (B, N) of G satisfying certain axioms. In particular, any group G with a strongly transitive action on a building Δ will be equipped with such a BN-pair (B, N), and conversely the building Δ will then be possible to reconstruct purely in terms of G, B and N. Moreover, one of the axioms for a BN-pair (B, N) is that the quotient $N/(B \cap N)$ is a group isomorphic to a Coxeter group W. The corresponding building Δ is then obtained by glueing copies of the Coxeter complex associated to this W.

In Chapter 3, we investigate a fixed point property for almost connected locally compact groups G acting on buildings Δ by simplicial isometries. This fixed point property may be viewed as a "higher rank" analogue of Jean-Pierre Serre's property (FA), with trees replaced by buildings. We recall that a group G has property (FA) if every action by simplicial isometries, without inversion, of G on a tree has a fixed point. Since, as we will see, not much can be expected from connected locally compact groups without any restriction on the actions involved, we will define a **measurable action** of a locally compact group G on a building Δ as a type-preserving simplicial isometric action of G on Δ, such that the point stabilisers in G for this action (or rather for the induced action of G on the Davis CAT(0) realisation of Δ) are Haar measurable. Our second key result is then the following.

Theorem B. *Let G be an almost connected Lie group. Then every measurable action of G on a finite rank building has a fixed point.*

This will be an essential tool in studying the infinitesimal structure of minimal Kac–Moody groups over $\mathbb{R}$ or $\mathbb{C}$ (see Theorem C below). Theorem B has also already been used by L. Kramer as a tool to complete the classification of (closed) BN-pairs in connected Lie groups (see [Kra12]).

In Chapter 4, we review the basic theory of Kac–Moody algebras, and we introduce for a given Kac–Moody algebra $\mathfrak{g}$ a $\mathbb{Z}$-form $\mathcal{U}$ of its enveloping algebra $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$, which will be at the basis of the construction of Kac–Moody groups. As mentioned earlier, Kac–Moody algebras generalise semi-simple complex Lie algebras to infinite dimension. In particular, a Kac–Moody algebra $\mathfrak{g}$ admits, as in the classical case, a root space decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\Delta\cup\{0\}}$ with respect to some Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$. The main difference in this infinite-dimensional setting is that the set of roots Δ not only contains *real* roots but also *imaginary* roots.
In section 4.3, we extend a result of V. Kac [Kac90, Corollary 9.12] stating that in the symmetrisable case, subalgebras of $\mathfrak{g}$ of the form $\bigoplus_{n\geq 1}\mathfrak{g}_{n\beta}$ for β an imaginary root are free Lie algebras.
In section 4.4, we establish that Kac–Moody algebras $\mathfrak{g}_k := (\mathfrak{g}\cap\mathcal{U})\otimes_{\mathbb{Z}} k$ over a field k of positive characteristic p are restricted, and we give explicit formulas allowing to compute the corresponding p-operation.

In Chapter 5, we present Tits' construction of minimal Kac–Moody groups.

In Chapter 6, we review the different approaches to complete Kac–Moody groups and explain how they are related.

In Chapter 7, we study Kac–Moody groups over fields of characteristic zero.
In Section 7.1, we investigate the infinitesimal structure of a minimal Kac–Moody group $\mathfrak{G}(\mathbb{K})$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ equipped with the Kac–Peterson topology. More precisely, recall that there is a nice *Lie correspondence* between (finite-dimensional) Lie algebras $\mathfrak{g}$ and real or complex Lie groups G, where the Lie group G is obtained from its Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ using the exponential map, and where the Lie algebra $\mathfrak{g}$ can be reconstructed from the topological group structure of G as the set of continuous one-parameter subgroups $\mathrm{Hom}_c(\mathbb{R}, G)$ of G (see e.g. [HM07]). As Kac–Moody groups are infinite-dimensional analogues of Lie groups, it is natural to ask whether this Lie correspondence extends to $\mathfrak{G}(\mathbb{K})$ and its associated Kac–Moody algebra $\mathfrak{g} = \mathrm{Lie}\,\mathfrak{G}(\mathbb{K})$. As mentioned at the beginning of this introduction, $\mathfrak{G}(\mathbb{K})$ admits an adjoint representation $\mathrm{Ad}\colon\ \mathfrak{G}(\mathbb{K}) \to \mathrm{Aut}(\mathfrak{g})$ in the automorphism group of its Lie algebra. As it turns out, the kernel of this action is quite small (in fact, contained in the center of $\mathfrak{G}(\mathbb{K})$), and one does not loose much information by considering the **adjoint form** $G = \mathrm{Ad}(\mathfrak{G}(\mathbb{K})) \leq \mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{G}(\mathbb{K})$. This allows to consider an *exponential map* associating to an element $x \in \mathfrak{g}$ the element $\exp\mathrm{ad}(x) = \sum_{n\geq 0}(\mathrm{ad}(x))^n/n!$, where ad is the usual adjoint representation of the Lie algebra $\mathfrak{g}$; note, however, that in this infinite-dimensional setting, this exponential map is not defined everywhere, that is, $\exp\mathrm{ad}(x)$ is an element of $\mathrm{Aut}(\mathfrak{g})$ – and in fact, of G – only when x is ad-**locally finite**, that is, when the above sum

can be realised as the exponential of a (finite) matrix whenever it acts on a vector v of $\mathfrak{g}$. Still, the group G is generated by the exponentials of the ad-locally finite elements of its Lie algebra. Conversely, if one wants to get back the Kac–Moody algebra $\mathfrak{g}$ from the topological group G, the classical theory suggests to study the set of continuous one-parameter subgroups of G. One then has, as in the classical case, a continuous one-parameter subgroup $\alpha\colon \mathbb{R} \to G : t \mapsto \exp \mathrm{ad}(tx)$ of G for each ad-locally finite $x \in \mathfrak{g}$. Our third key result states that these are in fact the only examples of continuous one-parameter subgroups of G.

Theorem C. *Let $G = \mathrm{Ad}(\mathfrak{G}(\mathbb{K}))$ be the adjoint form of a real or complex Kac–Moody group $\mathfrak{G}(\mathbb{K})$. Then every continuous one-parameter subgroup α of G is of the form $\alpha(t) = \exp \mathrm{ad}(tx)$ for some* ad-*locally finite $x \in \mathrm{Lie}(G)$.*

This result should then allow for a recovering of the Kac–Moody algebra $\mathrm{Lie}(G)$ uniquely from the topological group structure of G, by generators and relations.
In Section 7.2, we translate the results from Section 4.3 at the group level, namely, we exponentiate the free Lie subalgebras of a Kac–Moody algebra to free groups inside the corresponding complete Kac–Moody group over a field of characteristic zero.

In Chapter 8, we study complete Kac–Moody groups over fields of positive characteristic, and more specifically over finite fields. Since, as mentioned earlier, these groups are locally compact, they will be called **locally compact Kac–Moody groups** in this thesis.
In Section 8.1, we investigate what the open subgroups of a locally compact Kac–Moody group G are. As alluded to at the beginning of this introduction, G acts strongly transitively on a building Δ, and therefore possesses a BN-pair (B, N) with associated Coxeter group $W \cong N/(B \cap N)$. In the same way as W possesses parabolic subgroups wW_Jw^{-1}, one can then define **parabolic subgroups** of G, which are subgroups of G of the form $gBW_JBg^{-1} = \bigcup_{w \in W_J} gBwBg^{-1}$ for $g \in G$, where each $w \in W$ is identified with one of its representatives in N. The definition of the topology on G implies that these parabolic subgroups of G are open. Our fourth key result, which is a joint work with Pierre-Emmanuel Caprace, is that parabolic subgroups of G are essentially the only source of open subgroups.

Theorem D. *Every open subgroup of a complete Kac–Moody group G over a finite field has finite index in some parabolic subgroup.*

Moreover, given an open subgroup O, there are only finitely many distinct parabolic subgroups of G containing O as a finite index subgroup.

In Section 8.2, we investigate the question of the abstract simplicity of complete Kac–Moody groups $\widehat{\mathfrak{G}}(\mathbb{K})$ over finite fields $\mathbb{K}$, that is, the question whether $\widehat{\mathfrak{G}}(\mathbb{K})$ possesses no non-trivial (abstract) normal subgroup. This question only makes sense for $\widehat{\mathfrak{G}}(\mathbb{K})$ of indecomposable type. Moreover, as the kernel $Z' = Z'(\widehat{\mathfrak{G}}(\mathbb{K}))$

of the $\widehat{\mathfrak{G}}(\mathbb{K})$-action on its associated building is a normal subgroup of $\widehat{\mathfrak{G}}(\mathbb{K})$, the group to consider for simplicity questions is $G = \widehat{\mathfrak{G}}(\mathbb{K})/Z'$. The question whether the group G of indecomposable type is (abstractly) simple for $\mathbb{K}$ an arbitrary field was explicitly addressed by J. Tits [Tit89]. R. Moody [Moo82] and G. Rousseau [Rou12] could respectively deal with the case of a field $\mathbb{K}$ of characteristic zero, and with the case of a field $\mathbb{K}$ of positive characteristic p that is not algebraic over $\mathbb{F}_p$. The abstract simplicity of G when $\mathbb{K}$ is a finite field was shown in [CER08] in some important special cases, including groups of 2-spherical type over fields of order at least 4, as well as some other hyperbolic types under additional restrictions on the order of the ground field. Our fifth key result is to establish the abstract simplicity of G over arbitrary finite fields, without any restriction. Our proof relies on an approach which is completely different from the one used in [CER08].

Theorem E. *Let $\widehat{\mathfrak{G}}(\mathbb{K})$ be a complete Kac–Moody group of indecomposable indefinite type over a finite field $\mathbb{K}$. Then $\widehat{\mathfrak{G}}(\mathbb{K})/Z'(\widehat{\mathfrak{G}}(\mathbb{K}))$ is abstractly simple.*

After completion of this work, I was informed by Bertrand Rémy that, in a recent joint work [CR13] with I. Capdeboscq, they obtained independently a special case of this theorem, namely the abstract simplicity of $\widehat{\mathfrak{G}}(\mathbb{K})$ over finite fields $\mathbb{K}$ of order at least 4 and of characteristic p in case p is greater than the maximum in absolute value of the non-diagonal entries of the generalised Cartan matrix associated to $\widehat{\mathfrak{G}}(\mathbb{K})$. Their approach is similar to the one used in [CER08].

Conventions All the Coxeter groups, buildings, BN-pairs in this thesis are always assumed to be of finite rank.

Part I

Groups acting on buildings

Chapter 1

Coxeter groups

The original results of this chapter, which are a joint work with Pierre-Emmanuel Caprace, are presented in Section 1.4.

1.1 Coxeter groups and complexes

We begin by briefly reviewing the *finite* Coxeter groups and their associated complexes. This will motivate the definitions and terminology for the corresponding infinite objects, hopefully giving some geometric intuition about them. The general reference for this section, which contains all stated results, is [AB08, Chapters 1–3 and 10].

1.1.1 Finite reflection groups and their associated poset. — Let V be a Euclidean space, that is, a finite-dimensional real vector space endowed with a scalar product. For each (linear) hyperplane H of V, let $s_H \colon V \to V$ denote the orthogonal reflection of V with fixed point set H. A **finite reflection group** is a finite subgroup W of the orthogonal group $\mathrm{O}(V)$ generated by reflections s_H for H in a (finite) set $\mathcal{H}$ of hyperplanes. Up to enlarging $\mathcal{H}$ by setting $\mathcal{H} = \{H \mid s_H \in W\}$, we may assume that it is W-invariant. Up to quotient out V by $\bigcap_{H\in\mathcal{H}} H$, we may also assume that the pair (W, V) is **essential**, that is, $\bigcap_{H\in\mathcal{H}} H = \{0\}$.

Let $\dim V = n$ and let $\mathbb{S}^{n-1}$ denote the unit sphere in V. Consider the trace on $\mathbb{S}^{n-1}$ of the polyhedral structure on V induced by the hyperplanes of $\mathcal{H} = \{H_1, \dots, H_k\}$. For each $i = 1, \dots, k$, let $f_i \colon V \to \mathbb{R}$ be a linear form defining H_i. Define a **cell** to be a nonempty subset A of $\mathbb{S}^{n-1}$ obtained by choosing for each i a sign $\sigma_i \in \{+, -, 0\}$ and by specifying $f_i = \sigma_i$ (where "$f_i = +$" means "$f_i > 0$", and similarly for "$f_i = -$"). A cell B is called a **face** of A, which we write $B \leq A$, if its description is obtained from the description of A by replacing some (maybe none) inequalities by equalities. The partially ordered set (or **poset**) of cells ordered by the face relation $\leq$ is denoted by Σ. Maximal faces of Σ are called **chambers**.

Each chamber is delimited by its walls, where the **walls** of Σ are the traces on $\mathbb{S}^{n-1}$ of the hyperplanes of $\mathcal{H}$. By abuse of language, we will also call walls the elements of $\mathcal{H}$. In turn, each wall m determines two **half-spaces** of Σ, which are the two connected components of $\mathbb{S}^{n-1} \setminus m$.

1.1.2 Simplicial complexes. — We recall that a **simplicial complex** on a vertex set $\mathcal{V}$ is a collection Δ of finite subsets of $\mathcal{V}$ (called **simplices**) such that every singleton $\{v\}$ is a simplex and every subset of a simplex A is again a simplex (called a **face** of A). The cardinality r of A is called its **rank**, while $r-1$ is its **dimension**. The **dimension** (respectively, the **rank**) of Δ is the dimension (respectively, the rank) of a maximal simplex. Note that the empty set is also a simplex. A **subcomplex** of Δ is a subset Δ' containing, for each element A, all the faces of A; in particular subcomplexes are simplicial complexes in their own right.

Alternatively, a simplicial complex Δ may be defined as a poset, ordered with the face relation, and possessing the two following properties:

(a) Two elements $A, B \in \Delta$ always admit a greater lower bound $A \cap B \in \Delta$.

(b) For all $A \in \Delta$, the poset $\Delta_{\leq A}$ of faces of A is isomorphic to the poset of subsets of $\{1, \ldots, r\}$ for some $r \geq 0$ (the **rank** of A).

The vertex set $\mathcal{V}$ of such a poset Δ is then the set of elements of rank 1, and an element A of Δ identifies with the simplex $\{v \in \mathcal{V} \mid v \leq A\}$.

Example 1.1. The poset Σ associated to a finite reflection group as constructed above is a simplicial complex.

We visualise a simplex A of rank r in the simplicial complex Δ as a geometric $(r-1)$-simplex, the convex hull of its r vertices. To make this precise, one defines the **geometric realisation** $|\Delta|$ of Δ. This is a topological space partitioned into (open) simplices $|A|$, one for each nonempty $A \in \Delta$, which is constructed as follows. Start with an abstract real vector space E with $\mathcal{V}$ as a basis. Let $|A|$ be the interior of the simplex in this vector space spanned by the vertices of A:

$$|A| = \{\sum_{v \in A} \lambda_v v \mid \lambda_v > 0 \text{ for all } v \text{ and } \sum_{v \in A} \lambda_v = 1\}.$$

We then set

$$|\Delta| = \bigcup_{A \in \Delta} |A|.$$

If Δ is finite, then $E = \mathbb{R}^N$ with N the cardinality of $\mathcal{V}$, and we topologise Δ as a subspace of $\mathbb{R}^N$. If $|\Delta|$ is infinite, one first topologises each closed simplex as a subspace of Euclidean space, and one then declares a subset of $|\Delta|$ to be closed if and only if its intersection with each closed simplex is closed.

1.1.3 A group theoretic reconstruction of Σ. — Since the finite reflection group W is defined in terms of $\mathcal{H}$, it acts on the simplicial complex Σ by **simplicial automorphisms**, that is, by preserving the poset structure of Σ.

Proposition 1.2. *Let C be a fixed chamber of Σ. Then:*

(1) The W-action is simply transitive on the set of chambers.

(2) W is generated by the set $S := \{s_H \mid H$ is a wall of $C\}$.

(3) $\mathcal{H}$ necessarily consists of the hyperplanes H of V such that $s_H \in W$. In particular, Σ only depends on (W, V).

(4) The closed chamber $\overline{C} := \bigcup_{A \leq C} A \subset \mathbb{S}^{n-1}$ is a fundamental domain for the W-action on $\mathbb{S}^{n-1}$. Moreover, the stabiliser W_x of a point $x \in \overline{C}$ is the subgroup generated by $S_x := \{s \in S \mid sx = x\}$.

One can then reconstruct the poset Σ purely group theoretically, starting from the abstract group W and a system $S = \{s_1, \ldots, s_n\}$ of generators of W as above. Indeed, the fourth point of Proposition 1.2 shows that one can identify the poset $\Sigma_{\leq C}$ of faces of a given chamber C to the poset of their stabilisers $\{W_J := \langle J \rangle \leq W \mid J \subseteq S\}$, ordered by the opposite of the inclusion relation. Defining a W-action on this last poset by left translation, the first point of Proposition 1.2 then allows to extend this identification to an isomorphism of posets

$$\Sigma \cong \{wW_J \mid w \in W,\ J \subseteq S\}^{\mathrm{op}},$$

where “op” indicates that the inclusion order should be reversed.

1.1.4 Coxeter groups and complexes. — With the previous paragraphs in mind, we are now ready to motivate the definitions and terminology related to (general) Coxeter groups and complexes. Let thus W be an abstract group (maybe infinite), generated by a finite subset S of order 2 elements. Define as above the poset $\Sigma = \Sigma(W, S)$ of cosets of the form wW_J, with $w \in W$ and $J \subseteq S$, ordered by the opposite of the inclusion relation (called the **face** relation). Then W acts on Σ by left translation.

When W is finite and as in §1.1.1, we know that Σ possesses a rich geometry (with walls, half-spaces, etc.) and that the elements of S act on Σ as reflections. We now extend this terminology to the infinite case, and then give a condition on W for it to deserve the name of “reflection group”.

First, we need a replacement for the set $\mathcal{H}$ of hyperplanes. We define the set S^W of **reflections** of W as the set of W-conjugates of elements of S. This is of course motivated by the formula $s_{wH} = ws_Hw^{-1}$, in the notations of §1.1.1. Let now $\mathcal{H}$ be an abstract set of **walls** in bijection with S^W. Denote this bijection by $H \in \mathcal{H} \mapsto s_H \in S^W$. Then W acts on $\mathcal{H}$ following the formula

$$s_{wH} = w s_H w^{-1} \quad \text{for all } w \in W \text{ and } H \in \mathcal{H}.$$

We now turn to the geometry of Σ. A **chamber** should be a maximal element of Σ. Thus chambers correspond to cosets of the form wW_J with $J = \emptyset$, or else to elements of W. Singling out a **fundamental chamber**, say $C_0 = \{1_W\}$, a typical chamber $\{w\}$ can then be written as wC_0. Elements $A \in \Sigma$ that are not chambers and that are maximal for this property are called **panels**. Such a panel A is of the form $w\langle s\rangle = \{w, ws\}$ for some $w \in W$ and $s \in S$; it is a face of precisely two chambers, namely $\{w\}$ and $\{ws\}$. Two chambers $\{w\}$ and $\{w'\}$ are called **adjacent** if they share a common panel, that is, if either $w = w'$ or $w' = ws$ for some $s \in S$. In this case, they are said to be s**-adjacent**.

For all these notions to correspond to the geometric intuition, there should also be a nice action of W on a set of "half-spaces" (see §1.1.5 for a precise definition). To account for this fact, (W, S) must satisfy an additional condition, and as it turns out, this condition amounts to require that (W, S) be a **Coxeter system**.

Definition 1.3. Let W be a group generated by a finite[1] subset S of involutions. W is called a **Coxeter group** if it admits a presentation of the form

$$W = \langle S \mid ((st)^{m_{st}})_{s,t\in S}\rangle,$$

where $m_{st} \in \mathbb{N}^* \cup \{\infty\}$ is the order of st in W. In this case, S is called a **Coxeter generating set** for W. The couple (W, S) is called a **Coxeter system** and is completely determined by the **Coxeter matrix** $M = (m_{st})_{s,t\in S}$. The cardinality of S is called the **rank** of W. The poset $\Sigma = \Sigma(W, S)$ constructed above is a simplicial complex, called the **Coxeter complex** of (W, S).

Note that finite Coxeter groups indeed coincide with the finite reflection groups defined in §1.1.1. These groups, which are said to be **of spherical type** for obvious reasons, are completely classified (see [AB08, Section 1.3] for a list).

The Coxeter group W is said to be **irreducible**, or **of irreducible type**, if there is no nontrivial decomposition $S = I \sqcup J$ such that W_I and W_J commute. Note that any Coxeter group can be decomposed as a direct product of its **irreducible components**. By abuse of language, we will also say that a subset I of S is spherical (respectively, irreducible, an irreducible component of S, and so on) if the corresponding property for W_I holds.

[1] Technically, a Coxeter group need not be of finite rank, that is, the generating set S need not be finite. However, since this finite rank assumption will be made throughout this thesis, we prefer to include it as a definition. We will recall this assumption at the beginning of each section presenting original results, if relevant.

Define a **type function** $\lambda\colon \Sigma \to S$ which associates to a simplex wW_J of Σ its **type** $\lambda(wW_J) := S \setminus J$. Then the Coxeter group W in fact coincides with the group $\mathrm{Aut}_0 \Sigma$ of **type-preserving simplicial automorphisms** of Σ. Moreover, it acts simply transitively on the set of chambers, and the stabilisers of simplices correspond to the **parabolic subgroups** of W, that is, the subgroups of W of the form wW_Jw^{-1} for some $w \in W$ and $J \subseteq S$. Parabolic subgroups of the form W_J are called **standard**. Note that parabolic subgroups are Coxeter groups in their own right.

1.1.5 The geometry of Coxeter complexes. — The simplicial complex $\Sigma = \Sigma(W, S)$ turns out to be completely determined by its **chamber system**, that is, by the set $\mathrm{Ch}\,\Sigma$ of its chambers together with the s-adjacency relations between chambers ($s \in S$), as defined in §1.1.4. Let C, D be two chambers of Σ. A **gallery** from C to D is a sequence of chambers $\Gamma = (C = C_0, \dots, C_d = D)$ such that C_{i-1} is distinct from and adjacent to C_i for each $i = 1, \dots, d$. The integer d is called the **length** of Γ. A **minimal gallery** from C to D is a gallery from C to D of minimal length. The length of such a gallery is denoted $\mathrm{d}(C, D)$ and is called the **(chamber) distance** between C and D.

Let C, C' be two adjacent chambers of Σ. Then $\mathrm{Ch}\,\Sigma$ is the disjoint union of the two sets of chambers

$$\mathrm{Ch}(\Phi) := \{D \in \mathrm{Ch}\,\Sigma \mid \mathrm{d}(C, D) < \mathrm{d}(C', D)\}$$

and

$$\mathrm{Ch}(\Phi') := \{D \in \mathrm{Ch}\,\Sigma \mid \mathrm{d}(C, D) > \mathrm{d}(C', D)\}.$$

The subcomplexes Φ, Φ' of Σ with respective underlying chamber sets $\mathrm{Ch}(\Phi)$ and $\mathrm{Ch}(\Phi')$ are called **half-spaces** or **roots** of Σ. Their intersection is the **wall** associated to these roots. If C is the chamber $\{w\}$ and C' the chamber $\{ws\}$ for some $w \in W$ and $s \in S$, then this wall corresponds to the – previously defined – wall $H \in \mathcal{H}$ such that $s_H = wsw^{-1}$ (see §1.1.4). We denote by $\alpha_{wsw^{-1}}$ the associated root containing the fundamental chamber $\{1_W\}$, and by $-\alpha_{wsw^{-1}}$ the other root. As it turns out, these roots indeed only depend on wsw^{-1}, and not on a choice of s and w. Let $\Phi(\Sigma)$ denote the set of all roots of Σ. Thus, $\Phi(\Sigma)$ is just the W-orbit $W.\Pi$, where $\Pi = \{\alpha_s \mid s \in S\}$ is the set of **simple roots**. For each root α, we denote by $\partial\alpha$ its wall and by r_α, or else $r_{\partial\alpha}$, its **associated reflection**, that is, $r_\alpha = wsw^{-1}$ if $\alpha = w\alpha_s$.

There is a nice correspondence between the geometry of Σ and the decompositions of elements of W in terms of products of generators from S, which we now describe. Let C and D be two chambers of Σ. A wall m is said to **separate** C from D if C and D are contained in different roots associated to m. Let $\Gamma = (C = C_0, \dots, C_d = D)$ be a gallery from C to D. The d walls separating C_{i-1} from C_i for $i = 1, \dots, d$ are called the walls **crossed** by Γ. The **type** of Γ is the sequence $\mathbf{s} = (s_1, \dots, s_d)$ of elements of S such that C_{i-1} is s_i-adjacent to C_i for each

$i = 1, \dots, d$. To such a gallery of type $\mathbf{s}$ and starting at the fundamental chamber $C = C_0$, one can associate the element $w = s_1 \dots s_d$ of W. Such a decomposition of w in terms of the generators of S is said to be **reduced** if d is minimal, in which case d is called the **length** of w, denoted $\ell(w)$. Note that $\ell(w) = 0$ if and only if $w = 1_W$.

Proposition 1.4. *Let $D = \{w\}$ be a chamber of Σ.*

(1) (Minimal) galleries from C_0 to D correspond precisely to (reduced) decompositions of w.

(2) The distance $\mathrm{d}(C_0, D)$ between C_0 and D coincides with the number of walls crossed by a minimal gallery from C_0 to D, or else with the length $\ell(w)$ of w.

Finally, given a simplex A of Σ, one defines the **residue** R_A as the subcomplex of Σ with underlying chamber set the set of chambers admitting A as a face. It is **convex** in the sense that any minimal gallery between two chambers of R_A lies entirely in R_A. If A is of type $S \setminus J$, then the **type** of R_A is defined to be J. Given any chamber C in R_A, the residue R_A can also be described as the subcomplex with set of chambers the chambers that are connected to C by a **J-gallery**, that is, a gallery whose type is contained in J^d where d is the length of the gallery. For this reason, R_A will also be denoted by $R_J(C)$ and called the **J-residue containing C**. If $C = \{w\}$, it is the Coxeter complex associated to the Coxeter system (wW_Jw^{-1}, wJw^{-1}). A residue of the form $R_J(C_0)$ is called **standard**.

1.1.6 The geometric linear representation of W. — As in the case of finite reflection groups, a Coxeter group of rank n also admits a linear representation in $\mathbb{R}^n$, with the elements of the generating set S acting as (this time, non-necessarily orthogonal) reflections.

Let thus W be a Coxeter group with generating set $S = \{s_1, \dots, s_n\}$ and set $V = \mathbb{R}^n$. Let $\{e_1, \dots, e_n\}$ be the canonical basis of V. Endow V with the symmetric bilinear form $B\colon V \times V \to \mathbb{R}$ defined on the basis vectors by

$$B(e_i, e_j) = -\cos(\pi/m_{s_is_j}),$$

where $m_{s_is_j}$ is as in Definition 1.3. For each $i = 1, \dots, n$, let $\sigma_i\colon V \to V$ be the reflection defined by the usual formula

$$\sigma_i(x) = x - 2B(e_i, x)e_i.$$

It takes e_i to $-e_i$ and it fixes the hyperplane $e_i^{\perp} = \{x \in V \mid B(e_i, x) = 0\}$. The map $W \to \mathrm{O}(V) : s_i \mapsto \sigma_i$ is called the **geometric linear representation of W**.

Proposition 1.5. *The geometric linear representation of W is faithful. Hence W is isomorphic to a group of linear transformations generated by reflections.*

1.1.7 Affine Coxeter groups. — We conclude this section by reviewing the class of **affine Coxeter groups**. These correspond to the Coxeter groups for which the bilinear form B introduced in §1.1.6 is positive semi-definite and degenerate. As for spherical Coxeter groups, affine Coxeter groups have been completely classified (see [Bou68, VI §4 Théorème 4 p199] for a list). We give here a geometric, more illuminating approach to these groups, which will be needed in Section 1.4.

Let V be a Euclidean space and let $\mathrm{Aff}(V) = V \rtimes \mathrm{GL}(V)$ be the group of affine automorphisms of V. Let W be a subgroup of $\mathrm{Aff}(V)$ generated by **affine reflections** s_H, where H runs through a set $\mathcal{H}$ of affine hyperplanes: if $H = x + H_0$ with H_0 a linear hyperplane, then $s_H = \tau_x s_{H_0} \tau_{-x}$ where $\tau_x \colon V \to V : v \mapsto v + x$. Up to enlarging $\mathcal{H}$, we may assume that it is W-invariant. We assume moreover that $\mathcal{H}$ is **locally finite**, in the sense that every point of V admits a neighbourhood that meets only finitely many hyperplanes of $\mathcal{H}$.

The hyperplanes $H \in \mathcal{H}$ yield a partition of V into convex sets and one can define, as in §1.1.1, the induced cellular complex $\Sigma = \Sigma(W, V)$ and all the related notions (note that this time, we do not restrict our attention to the trace of this partition on the unit sphere). Proposition 1.2 then remains valid in this more general context.

Proposition 1.6. *Let C be a chamber of Σ, and let S be the set of (affine) reflections through the walls of C. Then:*

(1) S is finite and (W, S) is a Coxeter system.

(2) W acts simply transitively on the chambers of Σ.

(3) $\mathcal{H}$ necessarily consists of the hyperplanes H of V such that $s_H \in W$.

(4) $\overline{C}$ is a fundamental domain for the W-action on V and the stabiliser of a point $x \in \overline{C}$ is the parabolic subgroup generated by $\{s \in S \mid sx = x\}$.

(5) The reflection part $\overline{W}$ of W, that is, the image of W under the projection $\mathrm{Aff}(V) \twoheadrightarrow \mathrm{GL}(V)$, is a finite reflection group.

Call W **essential** if its reflection part $\overline{W}$ is essential. An **affine Coxeter group**, or else a **Coxeter group of affine type**, is then a Coxeter group W as above which is moreover essential, irreducible and infinite.

Proposition 1.7. *Let W be an affine Coxeter group, and let C be a chamber of $\Sigma(W, V)$. Then:*

(1) C is a simplex possessing exactly $n + 1$ walls, where $n = \dim V$.

(2) W decomposes as a semidirect product $W \cong \mathbb{Z}^n \rtimes \overline{W}$.

(3) The complex $\Sigma(W, V)$ is isomorphic to the Coxeter complex $\Sigma(W, S)$.

(4) *The canonical bijection* $|\Sigma(W,V)| \approx V$ *induces a homeomorphism of the geometric realisation of* $\Sigma(W,S)$ *onto* V.

Example 1.8. The **infinite dihedral group**, denoted D_∞, is the affine Coxeter group $W = \langle s,t \mid s^2 = t^2 = 1\rangle$ generated by two reflections s,t whose product st has infinite order. W then acts by isometries on the real line $V = \mathbb{R}$, with s and t respectively acting as the reflections about 0 ($x \mapsto -x$) and 1 ($x \mapsto 2-x$). The triangulation of V by the integers yields a simplicial complex of rank 2 (the simplicial line), which is the Coxeter complex $\Sigma(W,S) \approx \Sigma(W,V)$. The fundamental chamber of $\Sigma(W,S)$ identifies with the interval $[0,1]$ of V, and its walls are then the singletons $\{0\}$ and $\{1\}$ in V.

Example 1.9. The affine Coxeter group of **type** $\tilde{A}_2$ is the Coxeter group $W = \langle s,t,u \mid (st)^3 = (tu)^3 = (su)^3 = s^2 = t^2 = u^2 = 1\rangle$. It acts by isometries on the Euclidean plane $V = \mathbb{R}^2$, preserving the tiling of V by congruent equilateral triangles. The Coxeter complex $\Sigma = \Sigma(W,S)$ of W is the simplicial complex of rank 3 induced by this tiling. Singling out one of the triangles (or else a fundamental chamber C_0 of Σ), the elements s,t and u of W then act on V as reflections across the lines containing the edges of this triangle (or else, across the walls of C_0). Such affine Coxeter groups of rank 3 are called **Euclidean triangle groups**.

1.2 CAT(0) spaces

Coxeter complexes (and more generally buildings, as we will see in Chapter 2) can be realised as CAT(0) spaces, a fact which will be used extensively in all our results. In this section, we review some definitions and basic facts about these spaces. The general reference for this section, in which all results stated without a proof can be found, is [BH99].

1.2.1 Basic definitions. — Let (X,d) be a metric space. A **geodesic segment**, or simply **geodesic**, joining two points $x,y \in X$ is an isometry $f\colon [0,1] \subset \mathbb{R} \to X$ such that $f(0) = x$ and $f(1) = y$. A **geodesic ray** based at $x \in X$ is an isometry $f\colon \mathbb{R}_+ \to X$ such that $f(0) = x$. Finally, a **goedesic line** in X is just an isometry $f\colon \mathbb{R} \to X$. By abuse of language, we will often identify such geodesics with their image in X. One says that (X,d) is a **geodesic space** if every pair of points of X can be joined by a geodesic.

Definition 1.10. A geodesic metric space (X,d) is called a CAT(0)**-space** if it satisfies the following CAT(0)**-inequality**: For all $x,y \in X$, there exists a point $m \in X$ such that

$$\mathrm{d}^2(z,m) \leq \frac{1}{2}(\mathrm{d}^2(z,x) + \mathrm{d}^2(z,y)) - \frac{1}{4}\,\mathrm{d}^2(x,y) \quad \text{for all } z \in X.$$

This point m is uniquely determined by x,y and is called the **midpoint** of $\{x,y\}$.

Example 1.11. Any complete, simply connected Riemannian manifold X of negative curvature is CAT(0).

As it turns out, CAT(0) spaces are **uniquely geodesic**, that is, for every pair of points $x, y \in X$, there is a unique geodesic segment, denoted $[x, y]$, joining them.

Lemma 1.12. *Let* (X, d) *be a complete* CAT(0) *metric space, and let* C *be a nonempty closed convex subset of* X*. Then:*

(1) For all $x \in X$*, there exists a unique* $y \in C$ *achieving the infimum of the distances* $\mathrm{d}(x, z)$*, where* z *runs through* C*. This* y *is called the* **projection** *of* x *on* C*, which we denote by* $y = \mathrm{proj}_C\, x$*.*

(2) For all $x_1, x_2 \in X$*, one has* $\mathrm{d}(\mathrm{proj}_C\, x_1, \mathrm{proj}_C\, x_2) \leq \mathrm{d}(x_1, x_2)$*.*

1.2.2 Isometric actions on CAT(0) **spaces.** — Let G be a group acting by isometries on a CAT(0) space X. For every $g \in G$, let

$$|g| := \inf\{\mathrm{d}(x, gx) \mid x \in X\} \in [0, \infty)$$

denote its **translation length**, and set

$$\mathrm{Min}(g) := \{x \in X \mid \mathrm{d}(x, gx) = |g|\}.$$

An element $g \in G$ is said to be **semi-simple** if $\mathrm{Min}(g)$ is nonempty. It is **elliptic** if, moreover, $|g| = 0$, that is, if it fixes some point. Otherwise, if $|g| > 0$ (and $\mathrm{Min}(g)$ is nonempty), it is **hyperbolic**. For a subgroup H of G, we also write

$$\mathrm{Min}(H) := \bigcap_{h \in H} \mathrm{Min}(h),$$

and we say that the H-action on X is **locally elliptic** if each $h \in H$ is elliptic. Finally, for a hyperbolic element $g \in G$, we call a geodesic line D in X a **translation axis** (or simply **axis**) of g if it is $\langle g \rangle$-invariant.

Lemma 1.13. *Let* X *be a* CAT(0) *space and let* $g \in \mathrm{Isom}(X)$ *be semi-simple. Then:*

(1) $\mathrm{Min}(g)$ *is a nonempty closed convex subset of* X*.*

(2) If g *is elliptic, then* $\mathrm{Min}(g)$ *is its fixed point set.*

(3) If g *is hyperbolic, then* $\mathrm{Min}(g)$ *is the reunion of all its axes. Moreover,* g *acts on such an axis as a translation of step* $|g|$*.*

Lemma 1.14. *Let* X *be a complete* CAT(0) *space and let* g, h *be two commuting semi-simple isometries of* X*. Then* $\mathrm{Min}(g) \cap \mathrm{Min}(h)$ *is nonempty.*

Proof. Indeed, given an element $x \in \mathrm{Min}(g)$, one may consider by part (1) of Lemma 1.13 its projection y on $\mathrm{Min}(h)$. Using the second part of Lemma 1.12 and the fact that g and h commute, one easily concludes that $y \in \mathrm{Min}(g) \cap \mathrm{Min}(h)$, as desired. $\square$

Lemma 1.15. *Let X be a* CAT(0) *space and let $x \in X$. Let $g \in \mathrm{Isom}(X)$ be such that $\mathrm{d}(x, g^2x) = 2\,\mathrm{d}(x, gx) > 0$. Then g is a hyperbolic isometry and $D := \bigcup_{n\in\mathbb{Z}} [g^n x, g^{n+1}x] \subset X$ is an axis for g.*

Proof. Since D is g-invariant, we only have to check that it is a geodesic. Set $d := \mathrm{d}(x, gx)$. We prove by induction on $n + m$, $n, m \in \mathbb{N}$, that $D_{n,m} := \bigcup_{-n\leq l\leq m+1} [g^l x, g^{l+1}x] \subset D$ is a geodesic. For $n = m = 0$, this is the hypothesis. Let now $n, m \geq 0$ and let us prove that $D_{n,m+1}$ is a geodesic (the proof for $D_{n+1,m}$ being identical). By the CAT(0) inequality applied to the triangle $A = g^{-n}x$, $B = g^m x$, $C = g^{m+1}x$, we get that

$$\mathrm{d}^2(M, C) \leq \frac{1}{2}(\mathrm{d}^2(A, C) + \mathrm{d}^2(B, C)) - \frac{1}{4}\,\mathrm{d}^2(A, B) = \frac{1}{2}(\mathrm{d}^2(A, C) + d^2) - \frac{1}{4}(m + n)^2 d^2,$$

where M is the midpoint of $\{A, B\}$. Since by induction hypothesis $\mathrm{d}(M, C) = \frac{1}{2}(m + n)d + d$, we finally get that $\mathrm{d}^2(A, C) \geq d^2(m + n + 1)^2$, as desired.

We remark that this lemma is immediate using the notion of *angle* in a CAT(0) space; we prefer however to give a more elementary argument here. $\square$

We finally record the following result, which is known as the **Bruhat–Tits fixed point theorem**.

Proposition 1.16. *Let G be a group acting by isometries on a complete* CAT(0) *space X. Assume that G has a bounded orbit. Then G admits a global fixed point.*

1.2.3 Visual boundary of a CAT(0) **space.** — Let X be a CAT(0) space. Two geodesic rays $r, r' \colon [0, \infty) \to X$ are said to be **asymptotic** if there exists a constant K such that $\mathrm{d}(c(t), c'(t)) \leq K$ for all $t \geq 0$. The **visual boundary** ∂X of X is the set of equivalence classes of geodesic rays of X, where two geodesic rays are equivalent if and only if they are asymptotic. Note that any isometric action of a group G on X induces a G-action on ∂X.

1.3 The Davis complex of a Coxeter group

As mentioned at the beginning of the previous section, Coxeter complexes can be realised as CAT(0) spaces. We now explain how this can be achieved. The general reference for this section is [AB08].

1.3.1 Flag complexes and barycentric subdivision. — Let $(\Delta, \leq)$ be a simplicial complex of finite rank. A **flag** of Δ is a chain $A_1 \leq \cdots \leq A_k$ of pairwise distinct simplices of Δ. The **flag complex** of Δ, denoted $\Delta_{(1)}$, is the simplicial complex with vertex set Δ and with simplices the flags of Δ.

Note that the geometric realisation $|\Delta_{(1)}|$ of the flag complex of Δ is homeomorphic to the **barycentric subdivision** of $|\Delta|$, which is obtained from $|\Delta|$ by subdividing each geometrical simplex $|A|$ of dimension k into $(k+1)!$ geometrical simplices of the same dimension, adding the barycenters of $|A|$ and of its faces as vertices.

1.3.2 The Davis complex. — Let (W, S) be a Coxeter system and let $\Sigma = \Sigma(W, S)$ be the associated Coxeter complex. Thus $\Sigma_{(1)}$ is the simplicial complex with vertex set the cosets of the form wW_J for $w \in W$ and $J \subseteq S$ and with maximal simplices the flags $W = wW \leq wW_{S\setminus\{s_1\}} \leq wW_{S\setminus\{s_1,s_2\}} \leq \cdots \leq \{w\}$. Let $\Sigma^s_{(1)}$ denote the subcomplex of $\Sigma_{(1)}$ with vertex set the set of **spherical simplices** of Σ, namely those of the form wW_J with W_J spherical. The **Davis complex** X of W is the geometric realisation $|\Sigma^s_{(1)}|$ of $\Sigma^s_{(1)}$, together with a metric (whose precise definition we omit) extending the canonical Euclidean metrics on its simplices (or **cells**) and which turns it into a complete CAT(0) metric space.

Heuristically, the reason for passing from the complex Σ to the complex $\Sigma^s_{(1)}$ is that $|\Sigma|$ contains spheres (one for each spherical residue), which are of course not negatively curved. However, in $\Sigma^s_{(1)}$, a spherical simplex A is replaced by the (complete) flag complex with vertex set its faces, including the simplex A itself, which amounts in $|\Sigma^s_{(1)}|$ to "fill in" the corresponding sphere.

Since W permutes the spherical simplices, there is a natural induced W-action on X. This action is **cellular** in the sense that it preserves the cell complex structure on X. The following lemma from [Bri99] then implies that this action is semi-simple: more precisely, the elements of W of finite order are elliptic and those of infinite order are hyperbolic (see also Proposition 1.16).

Lemma 1.17. *Let Z be a locally Euclidean* CAT(0) *cell complex with only finitely many isometry classes of cells. Let G be a group acting on Z by cellular isometries. Then every element of G is semi-simple.*

Each point x of the Davis complex X of W determines a unique spherical simplex of Σ, called its **support**, as follows: let $A_1 \leq A_2 \leq \cdots \leq A_k = wW_J$ be the simplex of $\Sigma^s_{(1)}$ whose geometric realisation is the **cell supporting** x, that is, the unique (open) cell containing x. Then the support of x is the spherical simplex wW_J. In particular, an element $w \in W$ fixes a point $x \in X$ if and only if it stabilises its support, that is, if and only if it belongs to the corresponding finite parabolic subgroup.

Note that X may be viewed as a subcomplex of the barycentric subdivision of $|\Sigma|$. Note also that all the geometric notions which we introduced for Σ (such as chambers, walls, roots, etc.) admit realisations in X. For instance, a chamber $\{w\}$ in Σ may be viewed as the (open) chamber C of X consisting of the points $x \in X$ whose support is $\{w\}$. Similarly, a panel of type $s \in S$ connecting two adjacent chambers $\{w\}$ and $\{ws\}$ of Σ can be realised as the set of $x \in X$ whose support is $wW_{\{s\}}$, yielding a notion of s-adjacency for the corresponding chambers in X. This in turn yields notions of galleries, walls, roots, and so on. To comfort the intuition about how walls look like in X, we record the following result from [NV02, Lemma 3.4].

Lemma 1.18. *Let X be the Davis complex of a Coxeter group W. Then the intersection of a wall and any geodesic segment of X which is not entirely contained in that wall is either empty or consists of a single point.*

In particular, walls and roots are closed convex subspaces of X. We say that a wall is **transverse** to a geodesic line (or segment) if it intersects this line (or segment) in a single point. Given $x, y \in X$, we say that a wall **separates** x from y if it is transverse to the geodesic segment $[x, y]$. We say that two walls m, m' of X are **parallel** if either they coincide or they are disjoint. Given a root $\alpha \in \Phi = \Phi(\Sigma)$, we write as before r_α or $r_{\partial\alpha}$ for the unique reflection of W fixing the wall $\partial\alpha$ of α pointwise and exchanging the two opposite roots α and $-\alpha$. We then say that two walls m, m' of X are **perpendicular** if they are distinct and if the reflections r_m and $r_{m'}$ commute.

1.4 On walls and parabolic closures in Coxeter groups

In this section, we prove Theorem A from the introduction, as well as several other related results of independent interest (see in particular Corollary 1.30, Corollary 1.32, and Theorem 1.34 below). Theorem A will be of fundamental importance for establishing the results of Section 8.1. The content of the present section is part of the paper [CM12] and is a joint work with Pierre-Emmanuel Caprace.

Throughout this section, we let (W, S) be a Coxeter system of finite rank. We also let $\Sigma = \Sigma(W, S)$ be the associated Coxeter complex with set of roots (or half-spaces) $\Phi = \Phi(\Sigma)$, and we let X be the Davis complex of W.

For a subset $J \subseteq S$, we set $J^\perp := \{s \in S \setminus J \mid sj = js \ \forall j \in J\}$. We will call a parabolic subgroup of W of **essential type** if its irreducible components are all non-spherical. The subset $J \subseteq S$ will be called **essential** if W_J is of essential type.

The normaliser of a parabolic subgroup

Lemma 1.19. *Let $L \subseteq S$ be essential. Then $\mathscr{N}_W(W_L) = W_L \times \mathscr{Z}_W(W_L)$ and is again parabolic. Moreover, $\mathscr{Z}_W(W_L) = W_{L^\perp}$.*

Proof. See [Deo82, Proposition 5.5] and [Kra09, Chapter 3]. □

Preliminaries on parabolic closures Since any intersection of parabolic subgroups of W is itself a parabolic subgroup (see [Tit74]), it makes sense to define the **parabolic closure** $\mathrm{Pc}(E)$ of a subset $E \subset W$ as the smallest parabolic subgroup of W containing E. For $w \in W$, we will also write $\mathrm{Pc}(w)$ instead of $\mathrm{Pc}(\{w\})$.

Lemma 1.20. *Let G be a reflection subgroup of W, namely a subgroup of W generated by a set T of reflections. We have the following:*

(1) There is a set of reflections $R \subset G$, each conjugate to some element of T, such that (G, R) is a Coxeter system.

(2) If T has no nontrivial partition $T = T_1 \cup T_2$ such that $[T_1, T_2] = 1$, then (G, R) is irreducible.

(3) If (G, R) is irreducible (resp. spherical, affine of rank ≥ 3), then so is $\mathrm{Pc}(G)$.

(4) If G' is a reflection subgroup of irreducible type which centralises G and if G is of irreducible non-spherical type, then either $\mathrm{Pc}(G \cup G') \cong \mathrm{Pc}(G) \times \mathrm{Pc}(G')$ or $\mathrm{Pc}(G) = \mathrm{Pc}(G')$ is of irreducible affine type.

Proof. For (1) and (3), see [Cap09b, Lemma 2.1]. Assertion (2) is easy to verify. For (4), see [Cap09b, Lemma 2.3]. □

Lemma 1.21. *Let $\alpha_0 \subsetneq \alpha_1 \subsetneq \cdots \subsetneq \alpha_k$ be a nested sequence of half-spaces such that $A = \langle r_{\alpha_i} \mid i = 0, \dots, k\rangle$ is infinite dihedral. If $k \geq 7$, then for any wall m which meets every $\partial\alpha_i$, either r_m centralises $\mathrm{Pc}(A)$, or $\langle A \cup \{r_m\}\rangle$ is a Euclidean triangle group.*

Proof. This follows from [Cap06, Lemma 11] together with Lemma 1.20 (4). □

Parabolic closures and finite index subgroups

Lemma 1.22. *Let $H_1 < H_2$ be subgroups of W. If H_1 is of finite index in H_2, then $\mathrm{Pc}(H_1)$ is of finite index in $\mathrm{Pc}(H_2)$.*

Proof. For $i = 1, 2$, set $P_i := \mathrm{Pc}(H_i)$. Since the kernel N of the action of H_2 on the coset space H_2/H_1 is a finite index normal subgroup of H_2 that is contained in H_1, so that in particular $\mathrm{Pc}(N) \subseteq \mathrm{Pc}(H_1)$, we may assume without loss of generality that H_1 is normal in H_2. But then H_2 normalises P_1. Up to conjugating by an element of W, we may also assume that P_1 is standard, namely $P_1 = W_I$ for some $I \subseteq S$. Finally, it is sufficient to prove the lemma when I is essential, which we assume henceforth. Lemma 1.19 then implies that $P_2 < W_I \times W_{I^\perp}$. We thus have an action of H_2 on the residue $W_I \times W_{I^\perp}$, and since H_1 stabilises W_I and has finite index in H_2, the induced action of H_2 on $W_{I^\perp}$ possesses finite orbits. By the Bruhat–Tits fixed point theorem, it follows that H_2 fixes a point in the Davis complex of $W_{I^\perp}$, that is, it stabilises a spherical residue of $W_{I^\perp}$. This shows that $[P_2 : W_I] < \infty$. □

Parabolic closures and essential roots Our next goal is to present a description of the parabolic closure $\mathrm{Pc}(w)$ of an element $w \in W$, which is essentially due to D. Krammer [Kra09].

Let $w \in W$. A root $\alpha \in \Phi$ is called w**-essential** if either $w^n\alpha \subsetneq \alpha$ or $w^{-n}\alpha \subsetneq \alpha$ for some $n > 0$. A wall is called w-essential if it bounds a w-essential root. We denote by

$$\mathrm{Ess}(w)$$

the set of w-essential walls. Clearly $\mathrm{Ess}(w)$ is empty if w is of finite order. If w is of infinite order, then it acts on X as a hyperbolic isometry and thus possesses some translation axis.

Lemma 1.23. *Let $w \in W$ be of infinite order and let λ be a translation axis for w in X. Then* $\mathrm{Ess}(w)$ *coincides with those walls which are transverse to λ.*

The proof requires a subsidiary fact. Recall that Selberg's lemma ensures that any finitely generated linear group over $\mathbb{C}$ admits a finite index torsion-free subgroup. This is thus the case for Coxeter groups (see Proposition 1.5). The following lemma provides important combinatorial properties of those torsion-free subgroups of Coxeter groups. Throughout the rest of this section, we let $W_0 < W$ be a torsion-free finite index normal subgroup.

Lemma 1.24. *For all $w \in W_0$ and $\alpha \in \Phi$, either $w\alpha = \alpha$ or $w.\partial\alpha \cap \partial\alpha = \varnothing$.*

Proof. See Lemma 1 in [DJ99]. □

Proof of Lemma 1.23. It is clear that if $\alpha \in \Phi$ is w-essential, then $\partial\alpha$ is tranverse to any w-axis. To see the converse, let $n > 0$ be such that $w^n \in W_0$. Since λ is also a w^n-axis, we deduce from Lemma 1.24 that for all roots α such that $\partial\alpha$ is transverse to λ, we have either $w^n\alpha \subsetneq \alpha$ or $\alpha \subsetneq w^n\alpha$. The result follows. □

We also set

$$\mathrm{Pc}^\infty(w) = \langle r_\alpha \mid \alpha \text{ is a } w\text{-essential root}\rangle.$$

Notice that every nontrivial element of W_0 is hyperbolic. Moreover, in view of Lemma 1.24, we deduce that if $w \in W_0$, then a root α is w-essential if and only if $w\alpha \subsetneq \alpha$ or $w^{-1}\alpha \subsetneq \alpha$.

Lemma 1.25. *Let $w \in W$ be of infinite order, let λ be a translation axis for w in X and let $x \in \lambda$.*

Then we have the following.

(1) $$\begin{aligned} \mathrm{Pc}^\infty(w) &= \langle r_\alpha \mid \partial\alpha \text{ is a wall transverse to } \lambda\rangle \\ &= \langle r_\alpha \mid \partial\alpha \text{ is a wall transverse to } \lambda \text{ separating } x \text{ from } wx\rangle. \end{aligned}$$

(2) $\mathrm{Pc}^\infty(w)$ *coincides with the essential component of* $\mathrm{Pc}(w)$, i.e. *the product of its non-spherical components. In particular* $\mathrm{Pc}(w) = \mathrm{Pc}^\infty(w)$ *if and only if* $\mathrm{Pc}(w)$ *is of essential type.*

(3) If $w \in W_0$, then $\mathrm{Pc}(w) = \mathrm{Pc}^\infty(w)$.

Proof. The first equality in Assertion (1) follows from Lemma 1.23. To check the second, it suffices to remark that if $\partial\alpha$ is any wall transverse to λ, then there exists a power w^k of w such that $w^k\partial\alpha$ separates x from wx.

Assertion (2) follows from Corollary 5.8.7 in [Kra09] (notice that what we call *essential* roots here are called *odd* roots in *loc. cit.*). Assertion (3) follows from Lemma 1.24 and Theorem 5.8.3 from [Kra09]. □

The Grid Lemma The following lemma is an unpublished observation due to Pierre-Emmanuel Caprace and Piotr Przytycki.

Lemma 1.26 (Caprace–Przytycki)**.** *There exists a constant N, depending only on (W, S), such that the following property holds. Let $\alpha_0 \subsetneq \alpha_1 \subsetneq \dots \alpha_k$ and $\beta_0 \subsetneq \beta_1 \subsetneq \dots \subsetneq \beta_l$ be two nested families of half-spaces of X such that $\min\{k, l\} > 2N$. Set $A = \langle r_{\alpha_i} \mid i = 0, \dots, k\rangle$, $A' = \langle r_{\alpha_i} \mid i = N, N+1, \dots, k-N\rangle$, $B = \langle r_{\beta_j} \mid j = 0, \dots, l\rangle$ and $B' = \langle r_{\beta_j} \mid j = N, N+1, \dots, k-N\rangle$. If $\partial\alpha_i$ meets $\partial\beta_j$ for all i, j, then either of the following assertions holds:*

1. *The groups A and B are both infinite dihedral, their union generates a Euclidean triangle group and the parabolic closure* $\mathrm{Pc}(A \cup B)$ *coincides with* $\mathrm{Pc}(A)$ *and* $\mathrm{Pc}(B)$ *and is of irreducible affine type.*
2. *The parabolic closures* $\mathrm{Pc}(A)$, $\mathrm{Pc}(A')$, $\mathrm{Pc}(B)$ *and* $\mathrm{Pc}(B')$ *are all of irreducible type; furthermore we have*
$$\mathrm{Pc}(A' \cup B) \cong \mathrm{Pc}(A') \times \mathrm{Pc}(B) \qquad \text{and} \qquad \mathrm{Pc}(A \cup B') \cong \mathrm{Pc}(A) \times \mathrm{Pc}(B').$$

We shall use the following related result.

Lemma 1.27. *There exists a constant L, depending only on (W, S), such that the following property holds. Let $\alpha_0 \subsetneq \alpha_1 \subsetneq \dots \alpha_k$ be a nested sequence of half-spaces. Let m, m' be two walls meeting $\partial\alpha_i$ for each i, and such that their intersection $m \cap m'$ is nonempty and contained in $\partial\alpha_0$. If $k \geq L$, then $\langle r_m, r_{m'}, r_{\alpha_i} \mid i = 0, \dots, k\rangle$ is a Euclidean triangle group and $\langle r_{\alpha_i} \mid i = 0, \dots, k\rangle$ is infinite dihedral.*

Proof. See [Cap06, Theorem 8]. □

Proof of Lemma 1.26. We let $N = \max\{8, L\}$ where L is the constant appearing in Lemma 1.27.

Assume first that for some $i \in \{0, 1, \dots, k\}$ and some $j \in \{N, N+1, \dots, l-N\}$, the reflections r_{α_i} and r_{β_j} do not centralise one another. Let $\phi = r_{\alpha_i}(\beta_j)$; thus $\phi \notin \{\pm\alpha_i, \pm\beta_j\}$. Let $x_0 \in \partial\alpha_0 \cap \partial\beta_j$ and $x_k \in \partial\alpha_k \cap \partial\beta_j$. Then the geodesic segment $[x_0, x_k]$ lies entirely in $\partial\beta_j$ and crosses $\partial\alpha_i$. Since $\partial\alpha_i \cap \partial\beta_j$ is contained in $\partial\phi$, it follows that $[x_0, x_k]$ meets $\partial\phi$. This shows that the wall $\partial\phi$ separates x_0 from x_k.

Let now $p_0 \in \partial\alpha_0 \cap \partial\beta_0$ and $p_k \in \partial\alpha_k \cap \partial\beta_0$. Then the piecewise geodesic path $[x_0, p_0] \cup [p_0, p_k] \cup [p_k, x_k]$ is a continuous path joining x_0 to x_k. This path must therefore cross $\partial\phi$. Thus $\partial\phi$ meets either $\partial\alpha_0$ or $\partial\beta_0$ or $\partial\alpha_k$. We now deal with the case where $\partial\phi$ meets $\partial\alpha_0$. The other two cases may be treated with analogous arguments; the straightforward adaption will be omitted here.

Then $\partial\phi$ meets $\partial\alpha_m$ for each $m = 0, 1, \dots, i$. Therefore Lemma 1.27 may be applied, thereby showing that $A_i = \langle r_{\alpha_m} \mid m = 0, \dots, i\rangle$ is infinite dihedral and that the subgroup $T = \langle r_{\alpha_m}, r_{\beta_j} \mid m = 0, \dots, i\rangle$ is a Euclidean triangle group. Furthermore Lemma 1.20 (3) shows that $\mathrm{Pc}(T)$ is of irreducible affine type. Since $\mathrm{Pc}(A_i)$ is infinite (because A_i is infinite) and contained in $\mathrm{Pc}(T)$ (because A_i is contained in T), it follows that $\mathrm{Pc}(A_i) = \mathrm{Pc}(T)$ since any proper parabolic subgroup of $\mathrm{Pc}(T)$ is finite. We set $P := \mathrm{Pc}(A_i) = \mathrm{Pc}(T)$.

Let now $n \in \{0, 1, \dots, l\}$ with $n \neq j$. Then r_{β_n} does not centralise r_{β_i}; in particular it does not centralise T. On the other hand the wall $\partial\beta_n$ meets $\partial\alpha_m$ for all $m = 0, \dots, i$, which implies by Lemma 1.21 that $\langle A_i \cup \{r_{\beta_n}\}\rangle$ is a Euclidean triangle group. Therefore $r_{\beta_n} \in P$ by Lemma 1.20 (3).

We have already seen that P is of irreducible affine type. We have just shown that B is contained in $\mathrm{Pc}(A_i) = P$; in particular this shows that B is infinite dihedral since the walls $\partial\beta_0, \dots, \partial\beta_l$ are pairwise parallel. Moreover, the group $\langle B \cup \{r_{\alpha_i}\}\rangle$ must be a Euclidean triangle group since it is a subgroup of P. In particular we have $\mathrm{Pc}(B) = P$ by Lemma 1.20 (3). Since every $\partial\alpha_m$ meets every $\partial\beta_j$, the same arguments as before now show that $r_{\alpha_m} \in \mathrm{Pc}(B) = P$ for all $m = i+1, \dots, k$. Finally we conclude that $\mathrm{Pc}(A) = \mathrm{Pc}(B) = P$ in this case.

Notice that, in view of the symmetry between the α's and the β's, the previous arguments yield the same conclusion if one assumed instead that for some $i \in \{N, N+1, \dots, k-N\}$ and some $j \in \{0, 1, \dots l\}$, the reflections r_{α_i} and r_{β_j} do not centralise one another.

Assume now that for all $i \in \{0, \dots, k\}$ and all $j \in \{N, N+1, \dots, l-N\}$, the reflections r_{α_i} and r_{β_j} commute and that, furthermore, for all $i \in \{N, N+1, \dots, k-N\}$ and all $j \in \{0, 1, \dots l\}$, the reflections r_{α_i} and r_{β_j} commute. By Lemma 1.20 (2) the parabolic closures $\mathrm{Pc}(A)$, $\mathrm{Pc}(A')$ $\mathrm{Pc}(B)$ and $\mathrm{Pc}(B')$ are of irreducible type. By assumption A' centralises B. By Lemma 1.20 (4), either $\mathrm{Pc}(A') = \mathrm{Pc}(B)$ is of affine type or else $\mathrm{Pc}(A' \cup B) \cong \mathrm{Pc}(A') \times \mathrm{Pc}(B)$. In the former case, we may argue as before to conclude again that $\mathrm{Pc}(A) = \mathrm{Pc}(B)$ is of affine type and we are in case (1) of the alternative. Otherwise, we have $\mathrm{Pc}(A' \cup B) \cong \mathrm{Pc}(A') \times \mathrm{Pc}(B)$ and by similar arguments we deduce that $\mathrm{Pc}(A \cup B') \cong \mathrm{Pc}(A) \times \mathrm{Pc}(B')$. □

Orbits of essential roots: affine versus non-affine Using the Grid Lemma, we can now establish a basic description of the w-orbit of a w-essential wall for some fixed $w \in W$. As before, we let $W_0 < W$ be a torsion-free finite index normal subgroup. Recall from Lemma 1.23 that for all $n > 0$ we have $\mathrm{Ess}(w) = \mathrm{Ess}(w^n)$ and, moreover, the set $\mathrm{Ess}(w)$ has finitely many orbits under the action of $\langle w \rangle$ (and hence also under $\langle w^n \rangle$).

Proposition 1.28. *Let $w \in W$ be of infinite order, let $k > 0$ be such that $w^k \in W_0$ and let $\mathrm{Ess}(w) = \mathrm{Ess}(w^k) = M_1 \cup \cdots \cup M_t$ be the partition of $\mathrm{Ess}(w)$ into $\langle w^k \rangle$-orbits. For each $i \in \{1, \dots, t\}$, let also $P_i = \mathrm{Pc}(\{r_m \mid m \in M_i\})$.*

Then for all $i \in \{1, \dots, t\}$, the group P_i is an irreducible direct component of $\mathrm{Pc}(w)$. In particular, for all $j \neq i$, we have either $P_i = P_j$ or $\mathrm{Pc}(P_i \cup P_j) \cong P_i \times P_j$. More precisely, one of the following assertions holds.

(1) $P_i = P_j$ and each $m \in M_i$ meets finitely many walls in M_j.

(2) $P_i = P_j$ is irreducible affine.

(3) $\mathrm{Pc}(P_i \cup P_j) \cong P_i \times P_j$.

Proof. Let $i \in \{1, \dots, t\}$. Since M_i is $\langle w^k \rangle$-invariant, it follows that P_i is normalised by w^k. Moreover, as $\langle r_m \mid m \in M_i \rangle$ is an irreducible reflection group by Lemma 1.20 (2), P_i is of irreducible non-spherical type by Lemma 1.20 (3). It then follows from Lemma 1.19 that $\mathscr{N}(P_i) = P_i \times \mathscr{Z}(P_i)$ is itself a parabolic subgroup. In particular it contains $\mathrm{Pc}(w^k)$. Since on the other hand we have $P_i \leq \mathrm{Pc}(w^k)$ by Lemma 1.25, we infer that P_i is a direct component of $\mathrm{Pc}(w^k)$. Since $\mathrm{Pc}(w^k) = \mathrm{Pc}^\infty(w)$ is the essential component of $\mathrm{Pc}(w)$ by Lemma 1.25, we deduce that P_i is a direct component of $\mathrm{Pc}(w)$ as desired.

Let now $j \neq i$. Since we already know that P_i and P_j are irreducible direct components of $\mathrm{Pc}(w)$, it follows that either $P_i = P_j$ or (3) holds. So assume that $P_i = P_j$ and that there exists a wall $m \in M_i$ meeting infinitely many walls in M_j. We have to show that (2) holds.

Let λ be a w-axis. By Lemma 1.23, all walls in $M_i \cup M_j$ are transverse to λ. Moreover, by Lemma 1.24 the elements of M_i (resp. M_j) are pairwise parallel. Therefore, we deduce that infinitely many walls in M_i meet infinitely many walls in M_j. Since M_i and M_j are both $\langle w^k \rangle$-invariant, it follows that all walls in M_i meet all walls in M_j. Thus $M_i \cup M_j$ forms a grid and the desired conclusion follows from Lemma 1.26. □

We shall now deduce a rather subtle, but nevertheless important, difference between the affine and non-affine cases concerning the $\langle w\rangle$-orbit of a w-essential root α.

Let us start by considering a specific example, namely the Coxeter group $W = \langle r_a, r_b, r_c\rangle$ of type $\tilde{A}_2$, acting on the Euclidean plane. One verifies easily that W contains a nonzero translation t which preserves the r_a-invariant wall m_a. Let $w = tr_a$. Then w is of infinite order so that $\mathrm{Pc}(w) = W$. Moreover the walls m_b and m_c, respectively fixed by r_b and r_c, are both w-essential by Lemma 1.23. Now we observe that, for each even integer n the walls m_b and $w^n m_b$ are parallel, while for each odd integer the walls m_b and $w^n m_b$ have a non-empty intersection.

The following result (in the special case $m = m'$) shows that the situation we have just described cannot occur in the non-affine case.

Proposition 1.29. *Let $w \in W$, m be a w-essential wall and P be the irreducible component of $\mathrm{Pc}(w)$ that contains r_m.*

If P is not of affine type, then for each w-essential wall m' such that $r_{m'} \in P$, there exists an $l_0 \in \mathbb{N}$ such that for all $l \in \mathbb{Z}$ with $|l| \geq l_0$, the wall m' lies between $w^{-l}m$ and $w^l m$.

Proof. First notice that if m is a w-essential wall, then the reflection r_m belongs to $\mathrm{Pc}(w)$ by Lemma 1.25, so that P is well defined. Moreover, we have $r_{w^l m} = w^l r_m w^{-l} \in P$ for all $l \in \mathbb{Z}$.

Let $k > 0$ be such that $w^k \in W_0$ and let $\mathrm{Ess}(w) = \mathrm{Ess}(w^k) = M_1 \cup \cdots \cup M_t$ be the partition of $\mathrm{Ess}(w)$ into $\langle w^k\rangle$-orbits. Upon reordering the M_i, we may assume that $m' \in M_1$. Let also $I \subseteq \{1, \dots, t\}$ be the set of those i such that $w^l m \in M_i$ for some l. In other words the $\langle w\rangle$-orbit of m coincides with $\bigcup_{i\in I} M_i$.

For all j, set $P_j = \mathrm{Pc}(\{r_\mu \mid \mu \in M_j\})$. By Proposition 1.28, each P_j is an irreducible direct component of $\mathrm{Pc}(w)$. By hypothesis, this implies that $P = P_1 = P_i$ for all $i \in I$.

Suppose now that for infinitely many values of l, the wall $w^l m$ has a non-empty intersection with m'. We have to deduce that P is of affine type.

Recall from Lemma 1.24 that the elements of M_j are pairwise parallel for all j. Therefore, our assumption implies that for some $i \in I$, the wall m' meets infinitely many walls in M_i. By Proposition 1.28, this implies that either $P = P_1 = P_i$ is of affine type, or $\mathrm{Pc}(P_1 \cup P_i) \cong P_1 \times P_i$. The second case is impossible since $P_1 = P_i$. $\square$

On parabolic closures of a pair of reflections The following result, of independent interest, is a corollary to Proposition 1.28.

Corollary 1.30. *For each $w \in W$ with infinite irreducible parabolic closure $\mathrm{Pc}(w)$, there is a constant C such that the following holds. For all $m, m' \in \mathrm{Ess}(w)$ with $d(m, m') > C$, we have $\mathrm{Pc}(w) = \mathrm{Pc}(r_m, r_{m'})$.*

We shall use the following.

Lemma 1.31. *Let $\alpha, \beta, \gamma \in \Phi$ such that $\alpha \subsetneq \beta \subsetneq \gamma$. Then $r_\beta \in \mathrm{Pc}(\{r_\alpha, r_\gamma\})$.*

Proof. See [Cap06, Lemma 17]. □

Proof of Corollary 1.30. Retain the notation of Proposition 1.28. Since $P = \mathrm{Pc}(w)$ is irreducible, we have $P = P_i$ for all $i \in \{1, \ldots, t\}$ by Proposition 1.28. Recall that M_i is the $\langle w^k \rangle$-orbit of some w-essential wall m. For all $n \in \mathbb{Z}$, we set $m_n = w^{kn}m$. By Lemma 1.24 the elements of M_i are pairwise parallel and hence for all $i < j < n$, it follows that m_j separates m_i from m_n. For all $n \geq 0$ let now $Q_n = \mathrm{Pc}(\{r_{m_n}, r_{m_{-n}}\})$. By Lemma 1.31 we have $Q_n \leq Q_{n+1} \leq P$ for all $n \geq 0$. In particular $\bigcup_{n \geq 0} Q_n$ is a parabolic subgroup, which must thus coincide with P. It follows that $Q_n = P$ for some n. Since this argument holds for all $i \in \{1, \ldots, t\}$, the desired result follows. □

Corollary 1.32. *Any irreducible non-spherical parabolic subgroup P of W is the parabolic closure of a pair of reflections.*

Proof. Let $w \in P$ such that $P = \mathrm{Pc}(w)$. Such an w always exists by [CF10, Corollary 4.3]. (Note that this can also be deduced from Corollary 1.35 below together with [AB08, Proposition 2.43].) The conclusion now follows from Corollary 1.30. □

The parabolic closure of a product of two elements in a Coxeter group
We are now able to present the main result of this section, of which Theorem A from the introduction will be an easy corollary.

Before we state it, we prove one more technical lemma about CAT(0) spaces. Recall that W acts on the CAT(0) space X.

Lemma 1.33. *Let $w \in W$ be hyperbolic and suppose it decomposes as a product $w = w_1 w_2 \ldots w_t$ of pairwise commuting hyperbolic elements of W. Let m be a w-essential wall. Then m is also w_i-essential for some $i \in \{1, \ldots, t\}$.*

Proof. Write $w_0 := w$. Then, since the w_i are pairwise commuting for $i = 0, \ldots, t$, each w_i stabilises $\mathrm{Min}(w_j)$ for all j. Thus $M := \bigcap_{j=1}^{t} \mathrm{Min}(w_j)$ and $\mathrm{Min}(w)$ are both non-empty by Lemma 1.14, and are stabilised by each w_i, $i = 0, \ldots, t$. Therefore, if $x \in M \cap \mathrm{Min}(w)$, there is a piecewise geodesic path $x, w_1x, w_1w_2x, \ldots, w_1 \ldots w_tx = wx$ inside $M \cap \mathrm{Min}(w)$, where each geodesic segment is part of a w_i-axis for some $i \in \{1, \ldots, t\}$. Since any wall intersecting the geodesic segment $[x, wx]$ must intersect one of those axis, the conclusion follows from Lemma 1.23. □

Theorem 1.34. *For all $g, h \in W_0$, there exists a constant $K = K(g, h) \in \mathbb{N}$ such that for all $m, n \in \mathbb{Z}$ with $\min\{|m|, |n|, |m/n| + |n/m|\} \geq K$, we have $\mathrm{Pc}(g) \cup \mathrm{Pc}(h) \subseteq \mathrm{Pc}(g^m h^n)$.*

Proof. Fix $g, h \in W_0$. Let $\mathrm{Ess}(g) = M_1 \cup \cdots \cup M_k$ (resp. $\mathrm{Ess}(h) = N_1 \cup \cdots \cup N_l$) be the partition of $\mathrm{Ess}(g)$ into $\langle g \rangle$-orbits (resp. $\mathrm{Ess}(h)$ into $\langle h \rangle$-orbits). For all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$, set $P_i = \mathrm{Pc}(\{r_m \mid m \in M_i\})$ and $Q_j = \mathrm{Pc}(\{r_m \mid m \in N_j\})$.

By Lemma 1.25, we have $\mathrm{Pc}(g) = \langle \{r_m \mid m \in M_i,\ i = 1, \dots, k\} \rangle$, and Proposition 1.28 ensures that P_i is an irreducible direct component of $\mathrm{Pc}(g)$ for all i. Thus there is a subset $I \subseteq \{1, \dots, k\}$ such that $\mathrm{Pc}(g) = \prod_{i \in I} P_i$. Similarly, there is a subset $J \subseteq \{1, \dots, l\}$ such that $\mathrm{Pc}(h) = \prod_{j \in J} Q_j$.

For all $i \in I$ and $j \in J$, we finally let g_i and h_j denote the respective projections of g and h onto P_i and Q_j, so that $P_i = \mathrm{Pc}(g_i)$ and $Q_j = \mathrm{Pc}(h_j)$.

We define a collection $\mathrm{E}(g, h)$ of subsets of W as follows: a set $Z \subseteq W$ belongs to $\mathrm{E}(g, h)$ if and only if there exists a constant $K = K(g, h, Z) \in \mathbb{N}$ such that for all $m, n \in \mathbb{Z}$ with $\min\{|m|, |n|, |m/n| + |n/m|\} \geq K$ we have $Z \subseteq \mathrm{Pc}(g^m h^n)$.

Our goal is to prove that $\mathrm{Pc}(g)$ and $\mathrm{Pc}(h)$ both belong to $E(g, h)$. To this end, it suffices to show that P_i and Q_j belong to $\mathrm{E}(g, h)$ for all $i \in I$ and $j \in J$. This will be achieved in Claim 6 below.

Claim 1. *$M_s \subseteq \mathrm{Ess}(g_i)$ for all $s \in \{1, \dots, k\}$ and $i \in I$ such that $P_s = P_i$. Similarly, $N_s \subseteq \mathrm{Ess}(h_j)$ for all $s \in \{1, \dots, l\}$ and $j \in J$ such that $Q_s = Q_j$.*

Indeed, let $m \in M_s$ for some $s \in \{1, \dots, k\}$. Then $r_m \in P_s = P_i$. Moreover, as m is g-essential, it must be $g_{i'}$-essential for some $i' \in I$ by Lemma 1.33. But then $r_m \in \mathrm{Pc}(g_{i'}) = P_{i'}$ and so $i' = i$. The proof of the second statement is similar.

Claim 2. *If $i \in I$ is such that $[P_i, Q_j] = 1$ for all $j \in J$, then P_i belongs to $\mathrm{E}(g, h)$.*

Similarly, if $j \in J$ is such that $[P_i, Q_j] = 1$ for all $i \in I$, then Q_j belongs to $\mathrm{E}(g, h)$.

Indeed, suppose $[P_i, Q_j] = 1$ for some $i \in I$ and for all $j \in J$. Then P_i commutes with $\mathrm{Pc}(h)$. Thus h fixes every wall of M_i. In particular, any wall $\mu \in M_i$ is $g^m h^n$-essential for all $m, n \in \mathbb{Z}^*$ since $g^m h^n = g_i^m w$ for some $w \in W$ fixing μ and commuting with g_i. Therefore $P_i \subseteq \mathrm{Pc}(g^m h^n)$ for all $m, n \in \mathbb{Z}^*$ and so P_i belongs to $\mathrm{E}(g, h)$. The second statement is proven in the same way.

Claim 3. *Let $i \in I$ and $j \in J$ be such that $P_i = Q_j$. Then, for all $m, n \in \mathbb{Z}$, every $g_i^m h_j^n$-essential root is also $g^m h^n$-essential.*

Indeed, take $\alpha \in \Phi$ and $k > 0$ such that $(g_i^m h_j^n)^k \alpha \subsetneq \alpha$. Notice that $\mathrm{Pc}(g_i^m h_j^n) \subseteq P_i = Q_j$, and hence $r_\alpha \in P_i = Q_j$ by Lemma 1.25. Moreover, setting $g' := \prod_{t \neq i} g_t^m$ and $h' := \prod_{t \neq j} h_t^n$, we have $g' \alpha = \alpha = h' \alpha$ since g' and h' centralise $P_i = Q_j$. Therefore $(g^m h^n)^k \alpha = (g_i^m h_j^n)^k (g' h')^k \alpha = (g_i^m h_j^n)^k \alpha \subsetneq \alpha$ so that α is also $g^m h^n$-essential.

Claim 4. *Let $i \in I$ and $j \in J$ be such that $P_i = Q_j$. If P_i is of affine type, then $P_i = Q_j$ belongs to $\mathrm{E}(g, h)$.*

Since $P_i = Q_j$ is of irreducible affine type, we have $\mathrm{Pc}(w) = P_i$ for all $w \in P_i$ of infinite order. Thus, in order to prove the claim, it suffices to show that there exists some constant K such that $g_i^m h_j^n$ is of infinite order for all $m, n \in \mathbb{Z}$ with $\min\{|m|, |n|, |m/n|+|n/m|\} \geq K$. Indeed, we will then get that $\mathrm{Pc}(g_i^m h_j^n) = P_i$ is of essential type and so $P_i = \mathrm{Pc}(g_i^m h_j^n) \leq \mathrm{Pc}(g^m h^n)$ by Claim 3 and Lemma 1.25 (2).

Recalling that P_i is of affine type, we can argue in the geometric realisation of a Coxeter complex of affine type, which is a Euclidean space by Proposition 1.7. We deduce that if g_i and h_j have non-parallel translation axes, then $g_i^m h_j^n$ is of infinite order for all nonzero m, n. On the other hand, if g_i and h_j have some parallel translation axes, we consider a Euclidean hyperplane H orthogonal to these and let ℓ_i and ℓ_j denote the respective translation lengths of g_i and h_j. Then, upon replacing g_i by its inverse (which does not affect the conclusion since $E(g, h) = E(g^{-1}, h)$), we have $d(g_i^m h_j^n H, H) = |m\ell_i - n\ell_j|$. Since $g_i^m h_j^n$ is of infinite order as soon as this distance is nonzero, the claim now follows by setting $K = \ell_i/\ell_j + \ell_j/\ell_i + 1$.

Claim 5. *Let $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$ be such that $M_i \cap N_j$ is infinite. Then $P_i = Q_j$ and these belong to* $\mathrm{E}(g, h)$.

Indeed, remember that the walls in M_i are pairwise parallel by Lemma 1.24. Since $M_i \cap N_j \subseteq \mathrm{Ess}(g_{i'}) \cap \mathrm{Ess}(h_{j'})$ for some $i' \in I$ such that $P_i = P_{i'}$ and some $j' \in J$ such that $Q_j = Q_{j'}$ by Claim 1, Corollary 1.30 then yields $P_i = Q_j$.

Let now C denote the minimal distance between two parallel walls in X and set $K := \frac{|g|+|h|}{C} + 1$. Let $m, n \in \mathbb{Z}$ be such that $\min\{|m|, |n|, |m/n| + |n/m|\} \geq K$. We now show that $P_i \leq \mathrm{Pc}(g^m h^n)$. By Lemma 1.25 and Corollary 1.30, it is sufficient to check that infinitely many walls in $M_i \cap N_j$ are $g^m h^n$-essential.

Note first that for any wall $\mu \in M_i \cap N_j$, we have $g^{\epsilon m}\mu \in M_i$ and $h^{\epsilon n}\mu \in N_j$ for $\epsilon \in \{+, -\}$. Thus, since $M_i \cap N_j$ is infinite, there exist infinitely many such $\mu \in M_i \cap N_j$ with the property that $g^{\epsilon m}\mu$ lies between μ and some $\mu_\epsilon \in M_i \cap N_j$ and $h^{\epsilon n}\mu$ lies between μ and some $\mu'_\epsilon \in M_i \cap N_j$ for $\epsilon \in \{+, -\}$. We now show that any such μ is $g^m h^n$-essential, as desired. Consider thus such a μ.

Let D be a g-axis and D' be an h-axis. Since $M_i \cap N_j \subseteq \mathrm{Ess}(g) \cap \mathrm{Ess}(h)$, Lemma 1.23 implies that each of the walls μ, μ_ϵ and μ'_ϵ for $\epsilon \in \{+, -\}$ is transverse to both D and D'. In particular, the choice of μ implies that $g^{\epsilon m}\mu$ and $h^{\epsilon n}\mu$ for $\epsilon \in \{+, -\}$ are also transverse to both D and D'.

Let $\alpha \in \Phi$ be such that $\partial\alpha = \mu$ and $g^m\alpha \subsetneq \alpha$. If $h^n\alpha \subsetneq \alpha$ then clearly $g^m h^n\alpha \subsetneq \alpha$, as desired. Suppose now that $h^n\alpha \supsetneq \alpha$.

Note that the walls in $\langle g\rangle\mu \cup \langle h\rangle\mu$ are pairwise parallel since this is the case for the walls in $W_0 \cdot \mu$ by Lemma 1.24 and since $g, h \in W_0$.

Assume now that $|n| > |m|$, the other case being similar. In particular, $|n/m| > |g|/C$. Then $\mathrm{d}(\mu, g^{-m}\mu) \leq |m| \cdot |g| < |n| \cdot C \leq \mathrm{d}(\mu, h^n\mu)$ and so the wall $g^{-m}\mu$ lies between μ and $h^n\mu$. Thus $\alpha \subsetneq g^{-m}\alpha \subsetneq h^n\alpha$ and so $g^m h^n\alpha \supsetneq \alpha$, as desired.

Claim 6. *For all $i \in I$ and $j \in J$, the sets P_i and Q_j both belong to* $\mathrm{E}(g, h)$.

We only deal with P_i; the argument for Q_j is similar.

Let D denote a g-axis, and D' an h-axis in X. By Claim 5 we may assume that $M_i \cap \mathrm{Ess}(h)$ is finite. Moreover, by Claim 3 we may assume there exists a $j \in J$ such that $[P_i, Q_j] \neq 1$.

If $N_j \cap \mathrm{Ess}(g)$ is infinite, then $N_j \cap M_{i'}$ is infinite for some $i' \in \{1, \dots, k\}$ and thus Claim 5 yields that $Q_j = P_{i'} \in \mathrm{E}(g,h)$. In particular, $[Q_j, P_s] = 1$ as soon as $P_s \neq P_{i'}$. This implies $P_i = P_{i'} \in \mathrm{E}(g,h)$, as desired. We now assume that $N_j \cap \mathrm{Ess}(g)$ is finite.

Thus by Lemma 1.23, only finitely many walls in M_i intersect D' and only finitely walls in N_j intersect D.

Take $m_1 \in M_i$ and $m_2 \in N_j$. By Claim 1 and Corollary 1.30, there exists some $k_0 \in \mathbb{N}$ such that if one sets $M := \{g^{sk_0} m_1 \mid s \in \mathbb{Z}\} \subseteq M_i$ and $N := \{h^{tk_0} m_2 \mid t \in \mathbb{Z}\} \subseteq N_j$, then any two reflections associated to distinct walls of M (respectively, N) generate P_i (respectively, Q_j) as parabolic subgroups. Also, we may assume that no wall in M intersects D' and that no wall in N intersects D.

If every wall of M intersects every wall of N, then since $[P_i, Q_j] \neq 1$, Lemma 1.26 yields that $P_i = Q_j$ is of affine type and Claim 4 allows us to conclude. Up to making a different choice for m_1 and m_2 inside M and N respectively, we may thus assume that m_1 is parallel to m_2. For the same reason, we may also choose $m_1' \in M$ and $m_2' \in N$ such that D' lies between m_1 and m_1', D lies between m_2 and m_2', and such that $m_1 \cap m_2' = m_2 \cap m_1' = m_1' \cap m_2' = \varnothing$.

Let now $s_0, t_0 \in \mathbb{Z}$ be such that $g^{s_0 k_0} m_1 = m_1'$ and $h^{t_0 k_0} m_2 = m_2'$. Up to interchanging m_1 and m_1' (respectively, m_2 and m_2'), we may assume that $s_0 > 0$ and $t_0 > 0$.

Let $\alpha, \beta \in \Phi$ be such that $\partial\alpha = m_1$, $\partial\beta = m_2$ and such that D' is contained in $\alpha \cap -g^{s_0 k_0}\alpha$ and D is contained in $\beta \cap -h^{t_0 k_0}\beta$. For each $s, t \in \mathbb{Z}$, set $\alpha_s := g^{sk_0}\alpha$ and $\beta_t = h^{tk_0}\beta$ (see Figure 1.1). Since for two roots $\gamma, \delta \in \Phi$ with $\partial\gamma$ parallel to $\partial\delta$, one of the possibilities $\gamma \subseteq \delta$ or $\gamma \subseteq -\delta$ or $-\gamma \subseteq \delta$ or $-\gamma \subseteq -\delta$ must hold, this implies that

$$\alpha_{s_0} \subseteq -\beta_{t_0}, \quad -\alpha \subseteq \beta \quad \text{and} \quad \beta_{t_0} \subseteq \alpha.$$

Set $K := (s_0 + t_0 + 1)k_0$ and let $m, n \in \mathbb{Z}$ be such that $|m|, |n| > K$. We now prove that $P_i \leq \mathrm{Pc}(g^m h^n)$. By Lemma 1.25, it is sufficient to show that either α_{-1} and α or α_{s_0} and α_{s_0+1} are $g^m h^n$-essential. We distinguish several cases depending on the respective signs of m, n.

1. If $m, n > 0$, then

$$g^m h^n \alpha_{s_0+1} \subseteq g^m h^n \alpha_{s_0} \subseteq g^m h^n \beta \subsetneq g^m \beta_{t_0} \subseteq g^m \alpha \subsetneq \alpha_{s_0+1} \subseteq \alpha_{s_0}$$

   so that α_{s_0} and α_{s_0+1} are $g^m h^n$-essential.

2. If $m, n < 0$, then

$$g^m h^n \alpha_{-1} \supseteq g^m h^n \alpha \supseteq g^m h^n \beta_{t_0} \supsetneq g^m \beta \supseteq g^m \alpha_{s_0} \supsetneq \alpha_{-1} \supseteq \alpha$$

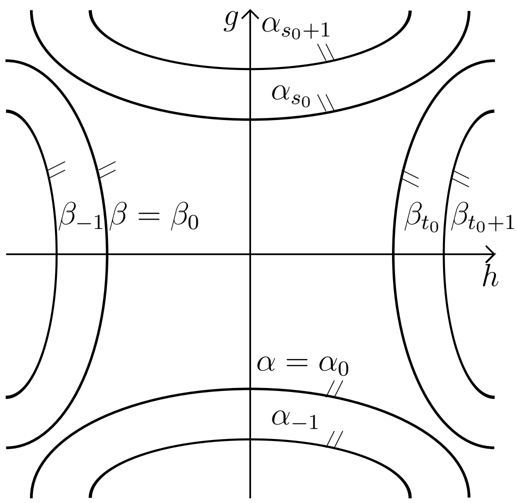

Figure 1.1: Claim 6.

so that α_{-1} and α are $g^m h^n$-essential.

3. If $m > 0$ and $n < 0$, then
$$g^m h^n \alpha_{s_0+1} \subseteq g^m h^n \alpha_{s_0} \subseteq g^m h^n(-\beta_{t_0}) \subsetneq g^m(-\beta) \subseteq g^m \alpha \subsetneq \alpha_{s_0+1} \subseteq \alpha_{s_0}$$
so that α_{s_0} and α_{s_0+1} are $g^m h^n$-essential.

4. If $m < 0$ and $n > 0$, then
$$g^m h^n \alpha_{-1} \supseteq g^m h^n \alpha \supseteq g^m h^n(-\beta) \supsetneq g^m(-\beta_{t_0}) \supseteq g^m \alpha_{s_0} \supsetneq \alpha_{-1} \supseteq \alpha$$
so that α_{-1} and α are $g^m h^n$-essential.

This concludes the proof of the theorem. $\square$

Here is a restatement of Theorem A.

Corollary 1.35. *Let H be a subgroup of W. Then there exists $h \in H \cap W_0$ such that* $[\mathrm{Pc}(H) : \mathrm{Pc}(h)] < \infty$.

Proof. Take $h \in H \cap W_0$ such that $\mathrm{Pc}(h)$ is maximal. Then $\mathrm{Pc}(h) = \mathrm{Pc}(H \cap W_0)$, for otherwise there would exist $g \in H \cap W_0$ such that $\mathrm{Pc}(g) \not\subseteq \mathrm{Pc}(h)$, and hence Theorem 1.34 would yield integers m, n such that $\mathrm{Pc}(h) \subsetneq \mathrm{Pc}(g^m h^n)$, contradicting the choice of h. The result now follows from Lemma 1.22 since $[H : H \cap W_0] < \infty$. $\square$

Remark 1.36. Note that the conclusion of Corollary 1.35 cannot be improved: indeed, one cannot expect that there is some $h \in H$ such that $\mathrm{Pc}(H) = \mathrm{Pc}(h)$ in general. Consider for example the Coxeter group $W = \langle s \rangle \times \langle t \rangle \times \langle u \rangle$, which is a direct product of three copies of $\mathbb{Z}/2\mathbb{Z}$. Then the parabolic closure of the subgroup $H = \langle st, tu \rangle$ of W is the whole of W, but there is no $h \in H$ such that $\mathrm{Pc}(h) = W$.

On walls at bounded distance from a residue We finish this section with a couple of observations on Coxeter groups which we shall need in our study of open subgroups of Kac–Moody groups in Section 8.1.

Given a subset $J \subseteq S$, we set $\Phi_J = \{\alpha \in \Phi \mid \exists v \in W_J,\ s \in J : \alpha = v\alpha_s\}$.

Lemma 1.37. *Let $L \subseteq S$ be essential. Then for each root $\alpha \in \Phi_L$, there exists $w \in W_L$ such that $w.\alpha \subsetneq \alpha$. In particular α is w-essential.*

Proof. Let $\alpha \in \Phi_L$. By [Hée93, Proposition 8.1, p309], there exists a root $\beta \in \Phi_L$ such that $\alpha \cap \beta = \varnothing$. We can then take $w = r_\alpha r_\beta$ or its inverse. $\square$

Lemma 1.38. *Let $L \subseteq S$ be essential, and let R be the standard L-residue of the Coxeter complex Σ of W.*

Then for each wall m of Σ, the following assertions are equivalent:

(1) m is perpendicular to every wall of R,

(2) $[r_m, W_L] = 1$,

(3) There exists $n > 0$ such that R is contained in an n-neighbourhood of m.

Proof. We first show that (3)⇒(2). By Lemma 1.37, if m' is a wall of R (that is, a wall intersecting R), then there exists $w \in W_L$ such that one of the two half-spaces associated to m' is w-essential. It follows that m and m' cannot be parallel since R is at a bounded distance from m. Hence m intersects every wall of R, but does not intersect R. Back to an arbitrary wall m' of R, consider a wall m'' of R that is parallel to m' and such that the reflection group generated by the two reflections $r_{m'}$ and $r_{m''}$ is infinite dihedral. Such a wall m'' exists by Lemma 1.37. Then r_m centralises these reflections by Lemma 1.21 and [CR09, Lemma 12]. As m' was arbitrary, this means that r_m centralises W_L.

The equivalence of (1) and (2) is trivial.

Finally, to show (1)⇒(3), notice that if C is a chamber of R and t a reflection associated to a wall of R, then the distance from C to m equals the distance from $t \cdot C$ to m. Indeed, if α is the root associated to m not containing R and D is the projection of C onto α, then $t \cdot D$ is the projection of $t \cdot C$ onto α. As W_L is transitive on R, (3) follows. $\square$

Chapter 2

Buildings and groups with a BN-pair

2.1 Buildings

The general reference for this section is [AB08, Chapters 4–5].

Coxeter complexes, as we saw, consist of a system of *chambers* separated by *walls*. One then calls them *apartments*. Putting together these apartments, one obtains a new, bigger complex which is called a *building*. Here is a precise axiomatic definition.

Definition 2.1. A **building** Δ is a simplicial complex, which can be expressed as the reunion of subcomplexes Σ called **apartments**, satisfying the following axioms:

(B0) Each apartment Σ is a Coxeter complex.

(B1) Any two simplices of Δ are contained in a common apartment.

(B2) Given two apartments Σ, Σ', there is an isomorphism $\phi_{\Sigma,\Sigma'} \colon \Sigma \to \Sigma'$ fixing $\Sigma \cap \Sigma'$ pointwise.

By axioms (B0) and (B2), we deduce that all apartments of a building Δ are isomorphic to a same Coxeter complex $\Sigma(W, S)$. The corresponding Coxeter system (W, S) is called the **type** of Δ. The set of all apartments of Δ is called its **apartment system**. As the reunion of apartment systems for Δ is again a possible apartment system for Δ, there is a unique maximal such system called the **complete apartment system** of Δ. The **rank** (respectively, **dimension**) of Δ is the rank (respectively, dimension) of the Coxeter complex $\Sigma(W, S)$. Moreover, Δ is **irreducible** (respectively, **spherical**, **affine**) if the corresponding property is true for its type.

Example 2.2. The buildings of type D_∞ correspond to the (simplicial) trees without endpoint. Any simplicial line in such a tree is an apartment, isomorphic to the Coxeter complex of type D_∞ (see Example 1.8).

As for Coxeter complexes, the maximal simplices of Δ are called **chambers**. Choosing a **fundamental chamber** C_0 of Δ and a **fundamental apartment** Σ_0 that contains this chamber, and fixing an isomorphism $\Sigma_0 \approx \Sigma(W,S)$, one may extend the induced type function $\lambda_{\Sigma_0}\colon \Sigma_0 \to S$ (see §1.1.4) to a **type function** $\lambda\colon \Delta \to S$, which associates to each simplex of Δ its **type**, by pre-composing λ_{Σ_0} with the isomorphisms ϕ_{Σ,Σ_0} of axiom (B2). This axiom ensures that this is well-defined. For each apartment Σ of Δ, we fix an isomorphism $\Sigma \approx \Sigma(W,S)$ preserving the type. Using axiom (B1), one may then define as in §1.1.4 and §1.1.5 the notions of **panel**, **s-adjacency** ($s \in S$), **J-gallery** ($J \subseteq S$) and **chamber distance**; again, axiom (B2) ensures that it is well-defined, that is, it does not depend on a choice of apartment.

The building Δ is called **thick** if every panel of Δ is contained in at least three chambers; it is called **thin** if every panel is contained in precisely two chambers, in which case Δ consists of a single apartment, that is, Δ is a Coxeter complex.

A **spherical simplex** of Δ is a simplex of Δ of spherical type. We will say that a group G acts on Δ by **type-preserving simplicial isometries** if this action preserves the type function as well as the simplicial structure of Δ. The type-preserving map $\rho_{\Sigma_0,C_0}\colon \Delta \to \Sigma_0$ whose restriction to each apartment Σ containing C_0 is the isomorphism ϕ_{Σ,Σ_0} is called the **retraction of Δ on Σ_0 centered at C_0**.

Lemma 2.3. *The retraction $\rho = \rho_{\Sigma_0,C_0}$ enjoys the following properties:*

(1) For any face $A \leq C_0$, $\rho^{-1}(A) = \{A\}$.

(2) ρ preserves distances from C_0, that is, $\mathrm{d}(C_0,\rho(D)) = \mathrm{d}(C_0,D)$ *for any chamber $D \in \Delta$.*

As in §1.1.5, we define for a subset J of S and a chamber C of Δ the **J-residue** of Δ containing C, denoted $R_J(C)$, as the subcomplex of Δ consisting of the chambers of Δ connected to C by a J-gallery. Equivalently, the chambers of $R_J(C)$ are those that admit as a face the unique simplex of type $S \setminus J$ that is a face of C. In particular, the map associating to a residue the intersection of its chambers yields a bijective (dual) correspondence between residues and simplices of Δ, and we will freely identify both notions when no confusion is possible. A residue of the form $R_J(C_0)$ is called **standard**. Residues of type J are buildings in their own right, of type (W_J, J). When convenient, we will identify the subcomplex $R_J(C)$ with its set $\mathrm{Ch}(R_J(C))$ of chambers (which, together with the s-adjacency relations, completely determines $R_J(C)$).

One of the main features of buildings is that they admit projections on residues. Given a chamber D and a residue R of Δ, there exists a unique chamber D' in

R at minimal chamber distance from D. It is denoted $\mathrm{proj}_R(D)$ and called the **projection** of D on R. It possesses the following **gate property**: $\mathrm{d}(D, E) = \mathrm{d}(D, D') + \mathrm{d}(D', E)$ for every chamber E of R. Given two residues R_1, R_2 of Δ, one defines more generally the **projection** of R_2 on R_1, denoted $\mathrm{proj}_{R_1}(R_2)$, as the residue in R_1 with chamber set the chambers $\mathrm{proj}_{R_1}(C)$, where C runs through the set of chambers of R_2.

2.2 Davis realisation of a building

As we saw in Section 1.3, Coxeter complexes admit nice CAT(0) realisations, allowing to bring the tools from CAT(0) theory into the picture. It is fortunate that this construction can in fact be extended to arbitrary buildings, a fact which we now describe. The general reference for this section is [Dav98].

Let Δ be a building of type (W, S). Let $\Delta_{(1)}$ be its flag complex, and let $\Delta^s_{(1)}$ be the subcomplex of $\Delta_{(1)}$ with vertex set the set of spherical simplices of Δ. We then define, exactly as for Davis complexes, the **Davis realisation** $|\Delta|_{\mathrm{CAT}(0)}$ of the building Δ as the geometric realisation of the simplicial complex $\Delta^s_{(1)}$, together with a suitably chosen locally Euclidean metric. In particular, the Davis realisation of a Coxeter complex is its Davis complex.

Proposition 2.4. *Let Δ be a building. Then $X = |\Delta|_{\mathrm{CAT}(0)}$ is a complete, locally Euclidean* CAT(0) *cell complex. Moreover, any simplicial type-preserving action of a group G on Δ induces a type-preserving cellular isometric action of G on X.*

The Davis complexes of the apartments of Δ embed as closed convex subsets of $|\Delta|_{\mathrm{CAT}(0)}$, and one can thus keep the intuitions of Section 1.3 to visualise $|\Delta|_{\mathrm{CAT}(0)}$. In the spirit of Lemma 1.18, we record the following result from [CH09, Theorem 5].

Lemma 2.5. *Let Δ be a building with Davis realisation X. Then any geodesic line in X is contained in the Davis realisation of an apartment of Δ.*

Given a group G acting on Δ by type-preserving simplicial isometries, we recall from Section 1.3 that the induced action of G on $X = |\Delta|_{\mathrm{CAT}(0)}$ can be identified with the action of G on the set of spherical residues of Δ, once we have associated to each point $x \in X$ its support. In particular, G fixes a point x of X if and only it stabilises a spherical residue of Δ (namely, the support of x). Note also that Lemma 1.17 still applies in this situation, so that every element of G is semi-simple for the G-action on X.

We conclude this section by recording the following result from [CL10, Theorem 1.1], which we will use while proving the main result of Section 3.3.

Lemma 2.6. *Let X be a complete* CAT(0) *space of finite geometric dimension and $\{X_\alpha\}_{\alpha \in A}$ be a filtering family of closed convex non-empty subspaces. Then either* *the intersection* $\bigcap_{\alpha\in A} X_\alpha$ *is non-empty, or the intersection of the visual boundaries* $\bigcap_{\alpha\in A} \partial X_\alpha$ *is a non-empty subset of* ∂X.

We recall that a family of subsets $\mathcal{F}$ of a given set is called **filtering** if for all $E, F \in \mathcal{F}$ there exists $D \in \mathcal{F}$ such that $D \subseteq E \cap F$. We point out that Davis realisations of buildings of finite rank (and closed convex subcomplexes) are examples of complete CAT(0) spaces of finite geometric dimension (see [Kle99]), since these are finite-dimensional CAT(0) cell complexes.

2.3 Combinatorial bordification of a building

In this section, we introduce some terminology and facts from [CL11], which we will use in Section 3.3.

2.3.1 Combinatorial bordification and sectors. — Let Δ be a building with Davis realisation X, and let $G = \mathrm{Aut}_0(\Delta)$ be its group of type-preserving simplicial isometries. Let $\mathrm{Res}_{\mathrm{sph}}(\Delta)$ denote the set of spherical residues of Δ. Note that, as mentioned in Section 2.2, the G-actions on X and $\mathrm{Res}_{\mathrm{sph}}(\Delta)$ coincide. To facilitate the intuition, residues are better considered in what follows as simplices (see Section 2.1).

The set $\mathrm{Res}_{\mathrm{sph}}(\Delta)$ can be turned into a metric space using the so-called **root-distance** ([CL11, Section 1.2]), whose restriction to the set of chambers coincides with the chamber distance. By looking at the projections of any given spherical residue R onto all $R' \in \mathrm{Res}_{\mathrm{sph}}(\Delta)$, one gets a map

$$\pi_{\mathrm{Res}}\colon\ \mathrm{Res}_{\mathrm{sph}}(\Delta) \to \prod_{R'\in\mathrm{Res}_{\mathrm{sph}}(\Delta)} \mathrm{St}(R') : R \mapsto (R' \mapsto \mathrm{proj}_{R'}(R)),$$

where $\mathrm{St}(R')$ denotes the **star** of R', that is, the set of all residues having R' as a face (or dually, the set of all residues contained in R', viewed as a set of chambers). Endow the above product with the product topology, where each star is a discrete set of residues. The **combinatorial bordification** of Δ, denoted $\mathscr{C}_{\mathrm{sph}}(\Delta)$, is then defined as the closure of the image of π_{Res}:

$$\mathscr{C}_{\mathrm{sph}}(\Delta) = \overline{\pi_{\mathrm{Res}}(\mathrm{Res}_{\mathrm{sph}}(\Delta))}$$

(see [CL11, Section 2.1]). By abuse of notation, we will also write $\mathrm{Res}_{\mathrm{sph}}(X)$ for $\mathrm{Res}_{\mathrm{sph}}(\Delta)$ and $\mathscr{C}_{\mathrm{sph}}(X)$ for $\mathscr{C}_{\mathrm{sph}}(\Delta)$.

Given two spherical residues $R_1, R_2 \in \mathrm{Res}_{\mathrm{sph}}(\Delta)$, we define their **convex hull** $\mathrm{Conv}(R_1, R_2)$ in $\mathrm{Res}_{\mathrm{sph}}(\Delta)$ as the intersection of all half-spaces of Σ containing $R_1 \cup R_2$, where Σ is any apartment containing $R_1 \cup R_2$. For any spherical residue $x \in \mathrm{Res}_{\mathrm{sph}}(\Delta)$ and any sequence $(R_n)_{n\in\mathbb{N}}$ of spherical residues converging to some

$\xi \in \mathscr{C}_{\mathrm{sph}}(\Delta)$, define the **combinatorial sector** $Q(x, \xi)$ pointing towards ξ and based at x as

$$Q(x, \xi) := \bigcup_{k \geq 0} \bigcap_{n \geq k} \mathrm{Conv}(x, R_n).$$

This definition turns out to be indeed independent of the sequence (R_n) converging to ξ (see [CL11, Section 2.3]).

Lemma 2.7. *Let ξ be any point in $\mathscr{C}_{\mathrm{sph}}(\Delta)$, and let $x, y \in \mathrm{Res}_{\mathrm{sph}}(\Delta)$. Then:*

(1) The combinatorial sector $Q(x, \xi)$ is entirely contained in some apartment of Δ.

(2) There exists $z \in \mathrm{Res}_{\mathrm{sph}}(\Delta)$ such that $Q(z, \xi) \subset Q(x, \xi) \cap Q(y, \xi)$.

(3) Any element $g \in G$ fixing x and ξ fixes the sector $Q(x, \xi)$ pointwise.

Proof. The first statement is contained in [CL11, Section 2.3], the second statement is [CL11, Proposition 2.30] and the third is [CL11, Lemma 4.4]. □

2.3.2 Transversal buildings. — It follows from [CL11, Section 5.1] that one can associate to every point ξ in the visual boundary ∂X of X a **transversal building** X^ξ of dimension strictly smaller than $\dim(X) := \dim(\Delta^s_{(1)})$ (see Section 2.2), on which the stabiliser G_ξ of ξ acts by type-preserving simplicial isometries. We briefly review this construction.

In what follows, we identify an apartment of Δ with its Davis realisation in X, and similarly for chambers, walls, half-spaces, and so on. Given $\xi \in \partial X$, let $\mathcal{A}_\xi$ denote the set of all apartments A such that $\xi \in \partial A$. Note that this set is nonempty by Lemma 2.5. Let also $\frac{1}{2}\mathcal{A}_\xi$ denote the (possibly empty) set of all half-apartments α (that is, half-spaces of some apartment) such that the visual boundary of the wall $\partial\alpha$ contains ξ. In particular, every $\alpha \in \frac{1}{2}\mathcal{A}_\xi$ is a half-apartment of some apartment of $\mathcal{A}_\xi$.

Given $R \in \mathrm{Res}_{\mathrm{sph}}(X)$, let R_ξ denote the intersection of all $\alpha \in \frac{1}{2}\mathcal{A}_\xi$ such that $R \subset \alpha$. Thus, in the case of chambers, the map $C \mapsto C_\xi$ identifies two adjacent chambers of X unless they are separated by some wall $\partial\alpha$ with $\alpha \in \frac{1}{2}\mathcal{A}_\xi$. Let $\mathcal{C}_\xi$ be the set of all C_ξ. We call two elements of $\mathcal{C}_\xi$ **adjacent** if they are the images of adjacent chambers of X. If $\frac{1}{2}\mathcal{A}_\xi$ is empty, there is only one chamber C_ξ (and one spherical residue R_ξ), which can be identified to the whole building X^ξ.

Let W be the type of Δ. Choose an apartment $A \in A_\xi$ and view W as a reflection group acting on A. The reflections associated to half-apartments α of A which belong to $\frac{1}{2}\mathcal{A}_\xi$ generate a subgroup of W which we denote by W_ξ. If $\frac{1}{2}\mathcal{A}_\xi$ is empty, W_ξ is equal to $\{1\}$. Note that by Lemma 1.20, the group W_ξ is a Coxeter group; moreover, the set $\{C_\xi \mid C \in \mathrm{Ch}(A)\}$ endowed with the above adjacency relation is W_ξ-equivariantly isomorphic to the graph of chambers of the Coxeter complex of W_ξ.

Here is a summary of the results from [CL11, Section 5.1].

Lemma 2.8. *With the notations above:*

(1) The Coxeter group W_ξ depends only on ξ but not on the choice of the apartment A.

(2) The set $\mathcal{C}_\xi = \{C_\xi \mid C \in \mathrm{Ch}(X)\}$ is the set of chambers of a building X^ξ of type W_ξ, with apartment system $\mathcal{A}_\xi$.

(3) The map $R \mapsto R_\xi$ is a G_ξ-equivariant map from $\mathrm{Res}_{\mathrm{sph}}(X)$ onto $\mathrm{Res}_{\mathrm{sph}}(X^\xi)$ which does not increase the root distance.

(4) We have $\dim(X^\xi) < \dim(X)$.

Example 2.9. Let Δ be the thin building of type $\tilde{A}_2$, namely the Coxeter complex $\Sigma = \Sigma(W, S)$ of the Coxeter group W of type $\tilde{A}_2$. As we saw in Example 1.9, Σ is the simplicial complex induced by the triangulation of the Euclidean plane by congruent equilateral triangles. Let X be the Davis realisation of Σ and let $\xi \in \partial X$. Then ξ is an equivalence class of parallel rays, and two cases can occur: either these rays are all transverse to any wall in X, in which case $\frac{1}{2}\mathcal{A}_\xi$ is empty and $W_\xi = \{1\}$, or there exists a ray which is contained in a wall m. In the latter case, X^ξ is a simplicial line and hence a building of type D_∞. Its associated Coxeter group is $W_\xi = \langle r_m, r_{m'} \rangle \leq W$ for some wall m' in the parallelism class $[m]$ of m, and which is "adjacent" to m. The pre-image of a (closed) chamber under the map $R \mapsto R_\xi$ is the convex hull in X of a pair of (adjacent) parallel walls in $[m]$.

To conclude this section, we give a proposition ([CL11, Theorem 5.5]) which allows to compare the respective combinatorial bordifications of X and X^ξ.

Proposition 2.10. *Let X be the Davis realisation of a building. For each $\xi \in \partial X$, there is a canonical continuous injective $\mathrm{Aut}(X)_\xi$-equivariant map $r_\xi \colon \mathscr{C}_{\mathrm{sph}}(X^\xi) \to \mathscr{C}_{\mathrm{sph}}(X)$. Furthermore, identifying $\mathscr{C}_{\mathrm{sph}}(X^\xi)$ with its image, one has the following decomposition:*

$$\mathscr{C}_{\mathrm{sph}}(X) = \mathrm{Res}_{\mathrm{sph}}(X) \cup \Big(\bigcup_{\xi \in \partial X} \mathscr{C}_{\mathrm{sph}}(X^\xi) \Big).$$

2.4 BN-pairs

As we saw in Section 1.1, there is a nice interplay between the group structure of a Coxeter group and the geometry of its associated Coxeter complex. As it turns out, this can be extended to groups acting "nicely" on thick[1] buildings.

More precisely, to any "nice" action of a group G on a thick building Δ of type (W, S), one can associate a group theoretic data to G – a *BN-pair* – such that Δ

[1] It is possible to develop the theory without this thickness assumption; however, it leads to some simplifications and is sufficient for the applications to Kac–Moody theory.

(and the G-action on Δ) can be entirely reconstructed from this data, exactly in the same way as $\Sigma(W,S)$ can be constructed from (W,S). To see what a "nice" action should mean, remember that W acts on $\Sigma(W,S)$ by type-preserving simplicial isometries and is transitive on the set of chambers of $\Sigma(W,S)$. Considering the fact that, given an apartment Σ of Δ, we want an identification of the W-action on $\Sigma(W,S)$ with the $\mathrm{Stab}_G(\Sigma)$-action on $\Sigma \approx \Sigma(W,S)$, the appropriate notion for a "nice" action – besides being type-preserving and simplicial – should at least include that any apartment stabiliser $\mathrm{Stab}_G(\Sigma)$ acts transitively on $\mathrm{Ch}\,\Sigma$. The additional requirement will then be to ask that G also acts transitively on the set of apartments of Δ. The general reference for this section is [AB08, Chapter 6].

2.4.1 Strongly transitive actions on buildings. — Let Δ be a thick building of type (W,S) with apartment system $\mathcal{A}$, and let G be a group acting by type-preserving simplicial isometries on Δ. Up to replacing $\mathcal{A}$ by the complete apartment system of Δ, we may assume that G stabilises $\mathcal{A}$. The G-action on Δ is said to be **strongly transitive** if G acts transitively on the set $\mathcal{A}$ of apartments of Δ and if the stabiliser in G of any apartment Σ is transitive on the set of chambers of Σ. Equivalently, G is transitive on the set of pairs (Σ, C) consisting of an apartment Σ and of a chamber C in that apartment. Fix such a "fundamental" pair (Σ_0, C_0). Thus W can be identified with the group of type-preserving simplicial isometries of $\Sigma_0 \approx \Sigma(W,S)$, with S the set of reflections through the walls of C_0.

Define the following subgroups of G:

$$B := \mathrm{Stab}_G(C_0), \quad N := \mathrm{Stab}_G(\Sigma_0), \quad T := B \cap N.$$

Since the G-action is type-preserving, B in fact fixes C_0 pointwise and T coincides with the pointwise fixator in G of Σ_0. Thus, by chamber transitivity, the map $N \to W$ induced by the N-action on Σ_0 is surjective and has kernel $T \lhd N$. In particular, $W \cong N/T$.

We now aim at reconstructing Δ uniquely in terms of G, B, N and S. First, since G acts transitively on $\mathrm{Ch}(\Delta)$, one has a bijection

$$G/B \overset{\approx}{\to} \mathrm{Ch}(\Delta) : gB \mapsto gC_0.$$

The experience of Coxeter groups and complexes then suggests to attach to each face A of C_0 its stabiliser in G, and then use the G-action to get a group-theoretic definition of all the simplices of Δ. We implement this strategy by describing for each face $A \leq C_0$ of type $\lambda(A) = S \setminus J$ its stabiliser P_J in G in terms of B and N. Let BW_JB denote the reunion in G of all double cosets BwB with $w \in W_J$, with the slight abuse of identifying $W \approx N/T$ with a subset of G; since $T \subset B$, the double coset BwB is well-defined. Clearly $BW_JB \subseteq P_J$. We claim that the reverse inclusion holds as well. Indeed, let $g \in P_J$ and choose an apartment Σ containing C_0 and gC_0. By strong transitivity, there exists a $b \in B$ such that $b\Sigma = \Sigma_0$. Note that $b|_\Sigma\colon \Sigma \to \Sigma_0$ is nothing else but the isomorphism ϕ_{Σ,Σ_0} from Definition 2.1. Now, by transitivity of $W \approx N/T$ on $\mathrm{Ch}(\Sigma_0)$, there exists some $w \in W_J$ such that $bgC_0 = wC_0$. Hence $w^{-1}bg \in B$ so that $g \in BwB$, as desired. Thus $P_J = BW_JB$ and we have a poset isomorphism

$$(\Delta, \leq) \overset{\cong}{\to} \{gBW_JB \mid g \in G,\ J \subseteq S\}^{\mathrm{op}} : gA \mapsto gBW_JB,$$

where A is the face of C_0 of type $S \setminus J$. Indeed, the only non-trivial part in order to check that the map above is indeed an isomorphism is injectivity, which amounts to show that two double cosets BwB and $Bw'B$ are distinct unless $w = w'$. This follows from the following so-called **Bruhat decomposition** of G:

Proposition 2.11. *$G = \coprod_{w\in W} BwB$.*

Note that the stabiliser calculation above with A the empty simplex already yields that $G = BWB$ (and hence in particular that B and N generate G). Moreover, if $\overline{\rho}\colon G \to W$ is the map induced by the retraction $\rho = \rho_{\Sigma_0,C_0}$ via the formula $\rho(gC_0) = \overline{\rho}(g)C_0$, the same kind of arguments as above yield that $BwB = \overline{\rho}^{\,-1}(w)$ for all $w \in W$, hence the Bruhat decomposition.

Finally, we give two more properties of double cosets that will be crucial in defining BN-pairs.

Lemma 2.12. *Let $w \in W$ and $s \in S$. Then:*

(1) $BwB \cdot BsB \subseteq BwB \cup BwsB$.

(2) $sBs^{-1} \nsubseteq B$.

Note that given $g \in BwB$ and $h \in BsB \subset P_s$ one has $ghP_s = gP_s$ so that $ghC_0 \sim_s gC_0$. Thus $\rho(ghC_0) \sim_s \rho(gC_0)$ and hence $\overline{\rho}(gh)$ is either $\overline{\rho}(g) = w$ or $\overline{\rho}(g)s = ws$. Therefore $gh \in \overline{\rho}^{\,-1}(w) \cup \overline{\rho}^{\,-1}(ws) = BwB \cup BwsB$, proving the first statement. The second statement is an easy translation of the thickness of Δ.

2.4.2 BN-pairs and associated buildings. — We can now go backward, starting from an abstract group G with a pair of subgroups B and N satisfying suitable axioms, to get back all the results and constructions of the previous paragraph.

Definition 2.13. A pair of subgroups B and N of a group G is a **BN-pair** for G if B and N generate G, the intersection $T = B \cap N$ is normal in N, and the quotient $W = N/T$ admits a set of generators S such that the following two conditions hold for all $s \in S$ and $w \in W$:

(BN1) *$BwB \cdot BsB \subseteq BwB \cup BwsB$.*

(BN2) *$sBs^{-1} \nsubseteq B$.*

If such a set S exists, it turns out to be uniquely determined by B and N. The group W is called the **Weyl group** associated to the BN-pair. The quadruple (G, B, N, S) is also called a **Tits system**.

Proposition 2.14. *Let (G, B, N, S) be a Tits system. Then:*

(1) (W, S) is a Coxeter group.

(2) The double cosets BW_JB for $J \subseteq S$ are subgroups of G, called the **standard parabolic subgroups** *of G. They are precisely the subgroups of G that contain B.*

(3) G admits a Bruhat decomposition $G = \coprod_{w \in W} BwB$.

We record for future reference the following refinement of axiom (BN1):

Lemma 2.15. *Let (G, B, N, S) be a Tits system with Weyl group W. Then $BsB \cdot BwB = BswB$ for all $w \in W$ and $s \in S$ such that $\ell(sw) > \ell(w)$.*

Finally, given a Tits system (G, B, N, S) with Weyl group W, we let $\Delta(G, B)$ denote the poset of cosets in G of the form gBW_JB for $g \in G$ and $J \subseteq S$, ordered by the opposite of the inclusion relation. What we saw in the previous paragraph then is that if the BN-pair (B, N) for G arises from a strongly transitive action of G on a thick building Δ, then Δ can be reconstructed as $\Delta(G, B)$.

Note that $\Delta(G, B)$ indeed only depends on B by the second statement of Proposition 2.14. We need N, however, if we want to describe the apartment system $\mathcal{A}$ in terms of G. Namely, the **fundamental apartment** $\Sigma_0 \subseteq \Delta(G, B)$ corresponds to the set of cosets wBW_JB for $J \subseteq S$ and $w \in W$, with $W = N/T$; the remaining apartments in $\mathcal{A}$ are then obtained from Σ_0 by the G-action. The **fundamental chamber** of $\Delta(G, B)$ corresponds to the trivial coset B. Let also the **type** of a simplex gBW_JB of $\Delta(G, B)$ be $S \setminus J$.

Proposition 2.16. *$\Delta(G, B)$ is a thick building with apartment system $\mathcal{A}$. The G-action by left translation on $\Delta(G, B)$ is strongly transitive and type-preserving.*

We call $\Delta(G, B)$ the **building associated to the BN-pair** (B, N). The BN-pair is said to be **spherical** (respectively, **irreducible**, of **rank** n, and so on) if the same is true for the associated Coxeter system (W, S). The conjugates in G of the standard parabolic subgroups are called the **parabolic subgroups** of G; these coincide with the stabilisers of simplices of $\Delta(G, B)$, or else to the subgroups of G containing a conjugate of B.

2.4.3 Twin BN-pairs and twin buildings. — As we will see in the second part of this thesis, Kac–Moody groups give a nice class of examples of groups endowed with a BN-pair. In fact, such a Kac–Moody group G is even endowed with a "dual" BN-pair structure, called a **twin BN-pair** or else **twin Tits system**, consisting of a pair of BN-pairs (B_+, N) and (B_-, N) for G with same Weyl group W and satisfying some compatibility axioms. Geometrically, the constructions of the previous paragraph then translate into an action of G on a **twin building**, that is, on a pair of buildings $(\Delta_+ = \Delta(G, B_+), \Delta_- = \Delta(G, B_-))$ of type W, also satisfying some compatibility axioms. Informally, this compatibility between Δ_+ and Δ_- may be characterised by a so-called **opposition relation** between the chambers of Δ_+ and those of Δ_-, whose properties mimic the properties of the opposition relation in spherical buildings, where two chambers of a spherical building (thus contained in a finite apartment) are said to be **opposite** if they are at maximal distance from one another. One may then generalise the usual notions related to buildings and define twin apartments, twin roots, strongly transitive actions and so on. Since in this thesis we will only need to consider the separate actions of G on Δ_+ and Δ_- (and in fact, most of the time, only on Δ_+), we will not try to make these notions precise. We give however the axiomatic definition of twin BN-pairs for future reference.

Definition 2.17. Let B_+, B_-, and N be subgroups of a group G such that $B_+ \cap N = B_- \cap N := T$. Assume that $T \triangleleft N$, and set $W := N/T$. The triple (B_+, B_-, N) is called a **twin BN-pair** with Weyl group W if W admits a set S of generators such that the following conditions hold for all $w \in W$ and $s \in S$ and each $\epsilon \in \{+, -\}$:

(TBN0) (G, B_ϵ, N, S) is a Tits system.

(TBN1) If $\ell(sw) < \ell(w)$, then $B_\epsilon s B_\epsilon w B_{-\epsilon} = B_\epsilon s w B_{-\epsilon}$.

(TBN2) $B_+ s \cap B_- = \emptyset$.

In this situation one also says that the quintuple (G, B_+, B_-, N, S) is a **twin Tits system**.

There is also a related notion, namely the notion of a **refined Tits system**, which consists of a sextuple (G, N, U_+, U_-, T, S) where N, U_+, U_- and T are subgroups of the group G, T is normal in N and normalises U_+ and U_- and S is a generating set for $W := N/T$ satisfying a list of axioms which in particular ensure that $(G, B_+ := TU_+, N, S)$ is a Tits system (see [KP85] or else [Rém02, 1.2.4] for a precise definition). Refined and twin Tits systems enjoy similar properties, such as refined Bruhat decompositions, called **Birkhoff decompositions**; however, in the above refined Tits system, $(G, B_- := TU_-, N, S)$ need not be a Tits system in general.

One can also consider a **symmetric refined Tits system**, that is, a refined Tits system (G, N, U_+, U_-, T, S) such that (G, N, U_-, U_+, T, S) is also a refined Tits system. In that case, $(G, B_+ := TU_+, B_- := TU_-, N, S)$ is a twin BN-pair (see [Rém02, 1.6.1]); hence, symmetric refined Tits systems may be viewed as a refined version of twin Tits systems.

Example 2.18. Let k be a field and let $R = k[t, t^{-1}]$ be the corresponding ring of Laurent polynomials. Let $G = \mathrm{SL}_3(R)$. Let N be the set of monomial matrices in G, that is, matrices with exactly one nonzero entry in each row and each column. Let also B_+ (respectively, U_+) denote the set of matrices in $\mathrm{SL}_3(k[t])$ that are upper triangular (respectively, and unipotent) mod t, and B_- (respectively, U_-) the set of matrices in $\mathrm{SL}_3(k[t^{-1}])$ that are lower triangular (respectively, and unipotent) mod t^{-1}. Then (B_+, B_-, N) is a twin BN-pair for G. The intersection $T = B_+ \cap N = B_- \cap N$ is the set of diagonal matrices of $\mathrm{SL}_3(k)$ and the quotient $W = N/T$ can be seen to be isomorphic to the Coxeter group of type $\tilde{A}_2$. Moreover, if S is the Coxeter generating set of W determined by the twin BN-pair, the sextuple (G, N, U_+, U_-, T, S) is a (symmetric) refined Tits system.

Let now $\widehat{G} := \mathrm{SL}_3(k(\!(t)\!))$, of which G is a (dense) subgroup, and let $\widehat{U}_+$ denote the set of matrices in $\mathrm{SL}_3(k[[t]])$ that are upper triangular and unipotent mod t. Then $(\widehat{G}, N, \widehat{U}_+, U_-, T, S)$ is a refined Tits system as well, but this time it is not symmetric.

2.5 Root group data

We conclude this chapter by briefly reviewing a structure on a group G, known as a *twin root datum*, which will appear in the context of Kac–Moody groups, and which allows to construct a symmetric refined Tits system for G. The general reference for this section is [Rém02, Chapter 1].

Recall from §1.1.5 the definition of roots (or half-spaces) of a given Coxeter system (W, S) with associated Coxeter complex $\Sigma = \Sigma(W, S)$. Write as before $\Phi = \Phi(\Sigma)$ for the set of these roots. Let Φ_+ denote the set of **positive roots** of Σ, that is, the set of roots in Φ containing the fundamental chamber $C_0 = \{1_W\}$ of Σ. Let also $\Phi_- := \Phi \setminus \Phi_+$ denote the set of **negative roots**. Remember that $\Phi = W \cdot \Pi$, where $\Pi = \{\alpha_s \mid s \in S\}$ is the set of **simple roots** of Σ, namely the positive roots associated to the walls of C_0. Finally, given $\alpha \in \Phi$, recall that we denoted by $-\alpha$ the unique root **opposite** to α and that $r_\alpha = r_{-\alpha} \in W$ denoted the associated reflection.

A pair of roots $\{\alpha, \beta\} \subset \Phi$ is called **prenilpotent** if there exist $w, w' \in W$ such that $\{w(\alpha), w(\beta)\} \subset \Phi_+$ and $\{w'(\alpha), w'(\beta)\} \subset \Phi_-$. In that case, we set

$$[\alpha, \beta] := \{\gamma \in \Phi \mid \gamma \supset \alpha \cap \beta \text{ and } -\gamma \supset (-\alpha) \cap (-\beta)\}$$

and $]\alpha, \beta[:= [\alpha, \beta] \setminus \{\alpha, \beta\}$.

Definition 2.19. A **twin root datum** of type (W, S) is a system $(G, (U_\alpha)_{\alpha\in\Phi})$ consisting of a group G together with a family of subgroups U_α indexed by the root system $\Phi = \Phi(\Sigma(W, S))$, which satisfy the following axioms, where $H := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$, $U_+ := \langle U_\alpha \mid \alpha \in \Phi_+\rangle$ and $U_- := \langle U_\alpha \mid \alpha \in \Phi_-\rangle$:

(TRD0) For each $\alpha \in \Phi$, we have $U_\alpha \neq \{1\}$.

(TRD1) For each prenilpotent pair $\{\alpha, \beta\} \subset \Phi$, the commutator group $[U_\alpha, U_\beta]$ is contained in the group $U_{]\alpha,\beta[} := \langle U_\gamma \mid \gamma \in]\alpha, \beta[\rangle$.

(TRD2) For each $\alpha \in \Pi$ and each $u \in U_\alpha \setminus \{1\}$, there exist elements $u', u'' \in U_{-\alpha}$ such that the product $\mu(u) := u'uu''$ conjugates U_β onto $U_{r_\alpha(\beta)}$ for each $\beta \in \Phi$. Moreover, $\mu(u)H = \mu(v)H$ for all $u, v \in U_\alpha \setminus \{1\}$.

(TRD3) For each $\alpha \in \Pi$, the group $U_{-\alpha}$ is not contained in U_+ and the group U_α is not contained in U_-.

(TRD4) $G = H\langle U_\alpha \mid \alpha \in \Phi\rangle$.

One may show that the product $u'uu''$ in (TRD2) is uniquely determined by the element u, as suggested by the notation $\mu(u)$. The subgroups U_α are called the **root subgroups** of G.

Proposition 2.20. *Let G be a group endowed with a twin root datum $(G, (U_\alpha)_{\alpha\in\Phi})$ of type (W, S) and let the subgroups H, U_+ and U_- be as in Definition 2.19. Let N be the subgroup of G generated by H together with all $\mu(u)$ such that $u \in U_\alpha \setminus \{1\}$, $\alpha \in \Pi$, where $\mu(u)$ is as in (TRD2). Then (G, N, U_+, U_-, H, S) is a symmetric refined Tits system.*

Chapter 3

Fixed point properties for groups acting on buildings

The results presented in this chapter are part of the original work of this thesis; they are contained in the paper [Mar12b].

3.1 The Bruhat–Tits fixed point theorem

Recall from Section 1.2 the Bruhat–Tits fixed point theorem, which states that whenever a group G acts by isometries on a complete CAT(0) space X, if G admits a bounded orbit, then it has a fixed point. Since conversely, if G fixes a point $x \in X$ all its orbits are contained in spheres centered at x, we can reformulate this theorem as follows.

Proposition 3.1. *Let G be a group acting by isometries on a complete* CAT(0) *space X. Then G has a fixed point in X if and only if its orbits in X are bounded.*

We now state an easy application of this fact.

We say that the group G is a **bounded product** of finitely many subgroups $U_1, \dots, U_n$, or that it is **boundedly generated** by these subgroups, which we write $G = U_1 \dots U_n$, if each element $g \in G$ can be written as $g = u_1 \dots u_n$ for some $u_i \in U_i$, $1 \leq i \leq n$.

Lemma 3.2. *G fixes a point in X as soon as one of the following holds:*

(1) G is a bounded product of subgroups each fixing a point in X.

(2) There exists a finite-index subgroup of G fixing a point in X.

Proof. Suppose that $G = U_1 \dots U_n$ for some subgroups $U_i \leq G$ with bounded orbits in X. Then each U_i maps a bounded set onto a bounded set. A straightforward induction now proves the first case. To prove the second case, let $H < G$ be a finite index subgroup of G with bounded orbits in X. Write $G = \coprod_{i=1}^{n} g_i H$ for some $g_i \in G$. Then each $x \in X$ is mapped by G onto the finite union $\bigcup_{i=1}^{n} g_i(Hx)$ of bounded subsets, which is again bounded. □

3.2 Locally compact groups

In this section, we briefly review the structure theory of locally compact groups. We also establish some properties of connected locally compact groups, including the fact that connected Lie groups are boundedly generated by one-parameter subgroups.

3.2.1 Definitions and structure theory. — Recall that a topological group G is **locally compact** if it is Hausdorff (equivalently, all singletons are closed) and if every element of G possesses a compact neighbourhood. A classical theorem of von Neumann asserts that such a locally compact group G possesses a **(left) Haar measure** μ, that is, a nontrivial, σ-additive, regular, and left invariant Borel measure. A subset of G is then **Haar measurable** (or simply **measurable**) if it is measurable by the completion of μ (again denoted by μ), that is, if it is the union of a Borelian set and of a set contained in a Borelian of measure zero. Remark that, by a theorem of Weil, locally compact groups are in fact *characterised* amongst topological groups as those possessing a left Haar measure.

The structure theory of locally compact groups has a long history, which we will not try to account for. Instead, we just mention a few key results (in addition to the theorems of von Neumann and Weil already mentioned) that will be needed later, at least for motivational purposes.

The connected case has been satisfyingly dealt with: indeed, the solution to Hilbert's fifth problem reduces the question to Lie theory and to the study of compact groups. More precisely, one has the following result (see [MZ55, Theorem 4.6]).

Lemma 3.3. *Let G be a connected locally compact group. Then there is a compact normal subgroup N of G such that G/N is a connected Lie group.*

Lie groups are then described by investigating separately the soluble groups and the simple factors, which have been classified since the time of É. Cartan; the structure theory of compact groups is also well developed (see [HM98]).

The totally disconnected case, by contrast, remains mysterious to a large extent. Remark that any abstract group equipped with the discrete topology is locally compact. Hence one cannot expect to get any meaningful classification result without some non-discreteness assumption. In their paper [CM11b], P-E. Caprace and N. Monod consider the class of *compactly generated* totally disconnected locally compact groups, and reduce the study of these groups to the (topologically) simple case. In turn, the known list of topologically simple, compactly generated and totally disconnected locally compact groups is very short; as we will see in the second part of this thesis, complete Kac–Moody groups over finite fields provide an important class of such groups, which are in addition non-linear.

3.2.2 Almost connected locally compact groups. — A topological group G is **almost connected** if its group of components G/G^0 is compact, where G^0 denotes the connected component of the identity in G. Here is a technical result that will be needed in Section 3.3.

Lemma 3.4. *Let G be an almost connected locally compact group. Then every measurable subgroup H of G of positive measure has finite index in G.*

Proof. Since G is compactly generated, it is in particular σ-compact. So there exists a compact subset $K \subseteq G$ such that $H \cap K$ has positive (finite) measure. Then by [HR79, Corollary 20.17], there is an open neighbourhood U of the identity such that $U \subseteq (H \cap K)(H \cap K)^{-1} \subseteq H$, so that H is open. Hence H contains the connected component G^0 of G. Since moreover G/G^0 is compact and the natural projection $\pi\colon G \to G/G^0$ is open, $\pi(H)$ has finite index in G/G^0, whence the lemma. □

The following result is probably well known; since we could not find it explicitly stated in the published literature, we include it here with a complete proof.

Theorem 3.5. *Let G be a connected Lie group. Then G is a bounded product of one-parameter subgroups.*

Proof. Note first that G decomposes as a product of a maximal connected compact subgroup and of finitely many one-parameter subgroups (see for example [Bor95]). We may thus assume that G is compact connected. Let $x_1, \dots, x_n$ be a basis of the Lie algebra $\mathrm{Lie}(G)$, and for each $i \in \{1, \dots, n\}$, let $U_i = \overline{\exp(\mathbb{R}x_i)}$ be the closure in G of the one-parameter subgroup associated to x_i. Since G is compact, each U_i is compact and hence so is the bounded product $A = U_1 \dots U_n$. In particular, A is closed. Since G is connected, it is generated by A, so that $G = \bigcup_{n \geq 1} A^n$. Now, by Baire's theorem, there exists an $n \in \mathbb{N}$ such that A^n has non-empty interior, and so A^{2n} contains an open neighbourhood U of the identity in G. Since G is compact, there is a finite subset $F \subset G$ such that $G = FU$. Then $F \subset A^k$ for some $k \in \mathbb{N}$ and so $G = A^{2n+k}$. Finally, note that each U_i is a connected compact abelian Lie group, hence a torus (see for example [HM98, Proposition 2.42 (ii)]). Since clearly each torus is a bounded product of one-parameter subgroups, the conclusion follows. □

A group G is said to have **finite abelian width** if it is boundedly generated by abelian subgroups.

Remark 3.6. Note that the argument above in fact yields the following: *A compact group that is generated by finitely many abelian subgroups has finite abelian width.*

Remark 3.7. It follows from Theorem 3.5 that any almost connected Lie group G has finite abelian width. Indeed, since the connected component of the identity of a Lie group is open, G is virtually connected. The claim then follows since G^0 has finite abelian width by Theorem 3.5.

3.3 Property (FB)

We are now ready to present a proof of Theorem B from the introduction. We also establish several other related results of independent interest (see in particular Theorem 3.12, Corollary 3.14, and Theorem 3.20 below).

3.3.1 Property (FB)**.** — Recall that a group H is said to satisfy **property** (FA) if every action, without inversion, of H on a simplicial tree has a fixed point. This property was first introduced by J-P. Serre ([Ser77, Chapter I §6]) and is for example satisfied by the group $\mathrm{SL}_3(\mathbb{Z})$, or else by all finitely generated torsion groups.

We wish to investigate some "higher rank" analogue of property (FA), by replacing trees with arbitrary (finite rank) buildings. While Serre's main focus was on discrete groups, we are interested here in topological groups, and more precisely in locally compact groups (satisfying some suitable non-discreteness assumption).

Let thus G denote a locally compact group. By an **action** of G on a building Δ, we mean a type-preserving simplicial isometric action on Δ. Note that this is indeed the direct higher rank analogue of a simplicial action without inversion of G on a tree. We then wish to know under which conditions G will have the property that *whenever G acts on a finite rank building, it stabilises a spherical residue.*

The reader might wonder where the "fixed point" from property (FA) disappeared in this statement: this is in fact just a matter of point of view. Indeed, as we saw in Section 2.2, G stabilises a spherical residue of Δ if and only if it fixes a point in the Davis realisation X of Δ. In the sequel, when we speak about a "fixed point" for the G-action on Δ, we always mean a fixed point for the induced G-action on X. Note that trees are already CAT(0) and can thus be identified with their Davis realisation.

We now give two examples indicating what we can expect for this higher rank analogue of property (FA).

Example 3.8. Let G be a group, and assume that G is the countable union of a strictly increasing sequence $G_1 \subsetneq G_2 \subsetneq \dots$ of proper subgroups. One can then construct a tree X together with a G-action on that tree as follows (see [Ser77, Theorem 15]). Take as vertex set for X the disjoint union of the sets of cosets

G/G_n, $n \in \mathbb{N}^*$. Two vertices are then joined by an edge if and only if they belong to two consecutive sets G/G_n and G/G_{n+1} and correspond under the canonical map $G/G_n \to G/G_{n+1}$. It is easy to verify that the graph X obtained in this way is indeed a tree, and that the G-action on X by left multiplication is isometric and without inversion. Moreover, the stabilisers in G of the vertices of X are the conjugates of the subgroups G_n, which are proper. In particular, the G-action on the tree X does not admit any fixed point. Note that we may assume this tree to be a building, since we can glue rays at each endpoint of the tree without affecting the G-action (see Example 2.2).

We now construct such a sequence of proper subgroups for G in case $G = (\mathbb{R}, +)$. Let $\mathscr{B}$ be a basis of $\mathbb{R}$ over $\mathbb{Q}$ and let $\{x_i \mid i \in \mathbb{N}\} \subset \mathscr{B}$ be a countable family of (pairwise distinct) basis elements. For each $n \in \mathbb{N}$, let V_n denote the $\mathbb{Q}$-sub-vector space of $\mathbb{R}$ with basis $\mathscr{B} \setminus \{x_i \mid i \geq n\}$. Then the additive groups of the V_n yield the desired chain. Remark that for a nonzero x in V_0, we may project this chain into $\mathbb{R}/x\mathbb{Z}$, yielding an example of a compact group acting without fixed point on a tree.

This first example seems to be rather bad news since we definitely want to include such basic connected locally compact groups as $\mathbb{R}$ or the circle group in the class of groups satisfying our (yet to be defined) higher rank fixed point property.

The "problem" with any such action of $\mathbb{R}$ on a tree, arising from Serre's construction, is that it is "pathological", in the sense that the point stabilisers are not Lebesgue measurable subsets of $\mathbb{R}$. Indeed, such stabilisers were conjugate to the proper subgroups $G_n \leq \mathbb{R}$ of which $\mathbb{R}$ is the increasing union. If all the subgroups G_n were measurable, then by σ-additivity one of them would have positive measure, and hence finite index in $\mathbb{R}$ by Lemma 3.4, a contradiction. Note in particular that the existence of such a chain $G_1 \subsetneq G_2 \subsetneq \dots$ for $\mathbb{R}$ is thus equivalent to the existence of a non-Lebesgue measurable subset of $\mathbb{R}$, hence to the axiom of choice.

The most natural solution to avoid this problem then seems to restrict our attention to **measurable** actions of locally compact groups G on buildings, that is, to G-actions whose stabilisers of spherical residues are Haar measurable subsets of G. Equivalently, G acts **measurably** on a building Δ if the point stabilisers for the induced action of G on $X = |\Delta|_{\mathrm{CAT}(0)}$ are Haar measurable.

Definition 3.9. A locally compact group G has **property** (FB) if it satisfies the following property:

(FB) *Every measurable action of G on a finite rank building stabilises a spherical residue.*

Recalling that the letters F and A from property (FA) stand for "Fixe" and "Arbre", this terminology is of course motivated (besides by a – discutable – wish for internationalisation) by the fact it describes a fixed point property on buildings.

Note that continuous actions are examples of measurable actions. In particular every action of a discrete group is measurable. Thus in the special case of discrete groups, property (FB) is a direct analogue of property (FA), where trees are replaced by arbitrary finite rank buildings.

Now that the fixed point property we want to consider has been precised, we give a second example indicating what we can expect from it.

Example 3.10. Let G be a locally compact group and assume it is equipped with a non-spherical BN-pair (B, N) such that B is open. Then G acts measurably and strongly transitively on the associated non-spherical building, and hence does not satisfy property (FB) (see Section 2.4). As we will see in the second part of this thesis (see Section 6.1), examples of such groups are provided by the class of complete non-compact Kac–Moody groups over finite fields. Note that these groups, besides being non-compact, are also totally disconnected.

This example suggests that it would be better to avoid non-compact totally disconnected locally compact groups. Note that if a (closed) quotient of a locally compact group G does not have property (FB), then neither does G itself. We will thus restrict our attention to the class of *almost connected locally compact groups*, being the precise counterpart of the class of non-compact totally disconnected ones.

Remark 3.11. In the context of almost connected locally compact groups, the notion of Haar measurability generalises both Borel and universal measurability. Indeed, since such a group G is compactly generated, it is in particular σ-compact. Hence its Haar measure μ is σ-finite. Thus one can construct from μ a complete probability measure on G whose measurable sets coincide with the Haar measurable sets. For this reason, Haar measurability is the only notion of measurability we will be considering here.

We now state our results concerning property (FB).

Theorem 3.12. *Let G be an almost connected locally compact group. If G has finite abelian width, then it has property* (FB).

Since almost connected Lie groups have finite abelian width by Remark 3.7, this implies in particular the following.

Corollary 3.13. *Every almost connected Lie group has property* (FB).

In another direction, compact groups of finite abelian width (and conjecturally all compact groups) also have property (FB).

Corollary 3.14. *Every compact group of finite abelian width has property* (FB). *In particular, compact p-adic analytic groups and profinite groups of polynomial subgroup growth have property* (FB).

We prove these results in the following paragraphs.

3.3.2 Lie groups and compact groups of finite abelian width. — The key result needed for the proof of Theorem 3.12 is the following.

Theorem 3.15. *Let G be either a compact abelian group or the group* $(\mathbb{R}, +)$. *Then G has property* (FB).

Proof. Let Δ be a finite rank building on which G acts measurably. We prove that G stabilises some spherical residue of Δ by induction on the dimension of the Davis realisation X of Δ, that is, on the dimension of the simplicial complex $\Delta^s_{(1)}$ (see Section 2.2). If X has zero dimension, then Δ is spherical and there is nothing to prove. Assume now that X has positive dimension. We first need to know that each element of G fixes some point of X.

<u>Claim 1</u>: *The action of G on X is locally elliptic.*

This follows from [CM11a, Theorem 2.5] in case G is compact, and from [CM11a, Proof of Theorem 2.5, Claim 7] in case $G = \mathbb{R}$ since $\mathbb{R}$ is divisible abelian.

For a subset $F \subset G$, let X^F denote the set of F-fixed points in X. Note that each $X^F = \bigcap_{g \in F} \mathrm{Min}(g)$ is a closed convex subset of X by Lemma 1.13.

<u>Claim 2</u>: *For each finite subset $F \subset G$, the set X^F is non-empty.*

This follows from a straightforward induction using Lemma 1.14.

Since the subsets X^F of X for finite $F \subset G$ form a filtering family of non-empty closed convex subsets of X by Claim 2, it follows from Lemma 2.6 that either $\bigcap X^F$ is non-empty, where the intersection runs over all finite $F \subset G$, in which case the induction step stands proven, or the corresponding intersection of the visual boundaries $\bigcap \partial X^F$ is a non-empty subset of ∂X. We may thus assume that the group G fixes some $\xi \in \partial X$. We now prove that G already fixes a point in X. Let X^ξ denote the transversal building to X associated to ξ (see §2.3.2). Thus, G acts on X^ξ.

<u>Claim 3</u>: *Let H be a subgroup of G. Suppose that H fixes some point $\zeta \in \mathscr{C}_{\mathrm{sph}}(X)$. Let $x \in \mathrm{Res}_{\mathrm{sph}}(X)$. Then every element of H fixes pointwise a subsector of $Q(x, \zeta)$.*

Indeed, let $h \in H$. Then Claim 1 yields a spherical residue $x_h \in \mathrm{Res}_{\mathrm{sph}}(X)$ which is fixed by h. It follows from Lemma 2.7 (3) that h fixes the combinatorial sector $Q(x_h, \zeta)$ pointwise. Since by Lemma 2.7 (2) there is some $z_h \in \mathrm{Res}_{\mathrm{sph}}(X)$ such that $Q(z_h, \zeta) \subset Q(x, \zeta) \cap Q(x_h, \zeta)$, the conclusion follows.

<u>Claim 4</u>: *The action of G on X^ξ is measurable.*

Indeed, we have to check that the stabiliser H in G of a spherical residue of X^ξ is measurable. Since $\mathrm{Res}_{\mathrm{sph}}(X^\xi) \subseteq \mathscr{C}_{\mathrm{sph}}(X^\xi)$ can be identified with a subset of $\mathscr{C}_{\mathrm{sph}}(X)$ by Proposition 2.10, we may assume that H is the stabiliser in G of a point $\zeta \in \mathscr{C}_{\mathrm{sph}}(X)$. Let $x \in \mathrm{Res}_{\mathrm{sph}}(X)$, and for each spherical residue $y \in Q(x, \zeta)$, let H_y denote the pointwise fixator in G of the combinatorial sector $Q(y, \zeta)$. Note that in fact $H_y \leq H$ for all y. It follows from Claim 3 that H is the union of all such H_y. Since $Q(x, \zeta)$ lies in some apartment by Lemma 2.7 (1) and since apartments possess only countably many spherical residues, this union is countable. Thus, it is sufficient to check that the pointwise fixator in G of a combinatorial sector $Q(y, \zeta)$ is measurable. Since this fixator is the (again countable) intersection of the stabilisers of the spherical residues in $Q(y, \zeta)$, the claim follows since such stabilisers are measurable by hypothesis.

Because $\dim X^\xi < \dim X$ by Lemma 2.8 (4), it follows from Claim 4 and from the induction hypothesis that G stabilises some spherical residue $\zeta \in \mathrm{Res}_{\mathrm{sph}}(X^\xi) \subseteq \mathscr{C}_{\mathrm{sph}}(X^\xi)$. Again, we may identify $\mathscr{C}_{\mathrm{sph}}(X^\xi)$ with a subset of $\mathscr{C}_{\mathrm{sph}}(X)$, so that G stabilises some point in $\mathscr{C}_{\mathrm{sph}}(X)$, again denoted by ζ. Then, as before, Claim 3 implies that G is covered by countably many stabilisers of points of X. Since these are measurable, one of them, say G_x for some $x \in X$, must have positive measure by σ-additivity. Then G_x has finite index in G by Lemma 3.4. We can now complete the induction step using Lemma 3.2. □

Corollary 3.16. *Let G be an almost connected locally compact group acting measurably on a finite rank building Δ with Davis realisation X. Assume that G fixes a point in the combinatorial bordification of Δ. Then G already fixes a point in X.*

Proof. This is what we have just established using Lemma 3.4 to conclude the proof of Theorem 3.15. □

Proof of Corollary 3.13. Let G be an almost connected Lie group. Then G is virtually connected since G^0 is open. By Lemma 3.2 we may thus assume that G is connected. Then by Theorem 3.5, it is a bounded product of one-parameter subgroups. The conclusion then follows from Theorem 3.15, together with Lemma 3.2. □

Proof of Corollary 3.14. The first statement is an immediate consequence of Theorem 3.15 together with Lemma 3.2. Then, since compact p-adic analytic groups are finitely generated by [DdSMS99, Corollary 8.34], they have property (FB) by Remark 3.6. Finally, profinite groups of polynomial subgroup growth also have finite abelian width by the main result of [Pyb02]. □

3.3.3 Property (FB+). — Since connected locally compact groups decompose into a compact and a Lie group part by Lemma 3.3, one can now "lift" the results established so far to give a proof of Theorem 3.12. To make this precise, we need one more technical remark.

Remark 3.17. Let G be either a compact group of finite abelian width or an almost connected Lie group. Then in fact G satisfies a slightly stronger property than property (FB), which is the following **property** (FB+):

(FB+) *For each finite rank building Δ with Davis realisation X, for each subgroup $Q \leq \mathrm{Aut}_0(X)$ having a global fixed point in X, any measurable action of G on the* CAT(0) *subcomplex* $\mathrm{Min}(Q) \leq X$ *by type-preserving cellular isometries has a global fixed point.*

Note that in case Q is trivial this is just property (FB). To see that G indeed satisfies this property (FB+) when G is either compact abelian or the group $(\mathbb{R}, +)$, remark first that Claim 1 and Claim 2 from the proof of Theorem 3.15 remain valid in this context, as well as Lemma 2.6. Moreover, setting $Y := \mathrm{Min}(Q) \subseteq X$, we can identify the visual boundary ∂Y with a subset of ∂X. Then Lemma 2.6 either yields the desired conclusion or yields a fixed point $\xi \in \partial Y \subseteq \partial X$ for the G-action. Then G acts on the closed convex subset $Y_1 := \mathrm{Min}(Q) \subseteq X^\xi$ of the transversal building X^ξ for the induced action of Q on X^ξ. Moreover, by the third statement of Lemma 2.7, each combinatorial sector $Q(x, \zeta)$ for some $x \in Y$ and some $\zeta \in Y_1 \subset \mathscr{C}_{\mathrm{sph}}(X)$ is entirely contained in Y. Then the proofs of Claim 3 and Claim 4 go through without any change and we may apply the induction hypothesis to find a global fixed point for G on Y.

Finally, the case where G is a compact group of finite abelian width or an almost connected Lie group follows from Lemma 3.2.

We summarise this in the following lemma.

Lemma 3.18. *Compact groups of finite abelian width and almost connected Lie groups have property* (FB+)*.*

The interest of this slightly more general fixed point property is that it allows to construct new examples of groups with property (FB) starting from known examples.

Lemma 3.19. *Let G be a locally compact group and let $N \lhd G$ be a closed normal subgroup of G such that G/N has property* (FB+)*. If N has property* (FB)*, then so has G.*

Proof. Suppose that G acts measurably on a building Δ with Davis realisation X. By hypothesis, $Y := \mathrm{Min}(N) \subseteq X$ is non-empty and stabilised by G. Moreover, the G-action on Y coincides with the induced G/N-action on Y, which is still measurable. Thus G/N fixes a point in Y by hypothesis. The conclusion follows. □

Proof of Theorem 3.12. Let G be an almost connected locally compact group of finite abelian width. Thus G is a bounded product of abelian subgroups, which we may assume to be closed, hence also locally compact. Therefore, by Lemma 3.2 (1), it is sufficient to prove the theorem when G is abelian, which we assume henceforth.

Let G^0 denote the connected component of G. By Lemma 3.3 we know that there exists some compact normal subgroup N of G^0 such that G^0/N is a connected

Lie group. Since by assumption G^0 is abelian, we know by Lemma 3.18 that N and G^0/N satisfy property (FB+), and hence that G^0 has property (FB) by Lemma 3.19. Finally, since G/G^0 is compact abelian, the same argument yields that G has property (FB), as desired. □

3.3.4 Further considerations and conjectures. — We remark that the "finite abelian width" hypothesis we made in the statement of Theorem 3.12 is not necessary in case the building that is acted on by the almost connected locally compact group G is a tree.

Theorem 3.20. *Let G be an almost connected locally compact group. Suppose that G acts measurably by type-preserving simplicial isometries on a tree. Then G has a global fixed point.*

The proof of this theorem will be given below.

In fact, we conjecture that this hypothesis is also unnecessary in general.

Conjecture 3.21. *Every almost connected locally compact group has property* (FB).

More generally, we make the following conjecture, which would imply Conjecture 3.21.

Conjecture 3.22. *Let G be a group acting by type-preserving simplicial isometries on a building Δ. If the G-action on the Davis realisation X of Δ is locally elliptic, then G has a global fixed point in $X \cup \partial X$, where ∂X denotes the visual boundary of X.*

Conjecture 3.22 is in fact also equivalent to asserting that, under the same hypotheses, G fixes a point in the combinatorial bordification of Δ.

We now explain the claimed connections between these conjectures.

Proof that Conjecture 3.22 implies Conjecture 3.21. Let G be an almost connected locally compact group acting measurably by type-preserving simplicial isometries on a building Δ with Davis realisation X. Assume that Conjecture 3.22 holds. We have to prove that G fixes a point in X.

Claim 1: *Let H be a group acting locally elliptically on X. Then H fixes a point in the combinatorial bordification of X.*

This follows from a straightforward induction on $\dim X$ using Conjecture 3.22 and Proposition 2.10.

Claim 2: *Let H be an almost connected locally compact group acting measurably and locally elliptically on X. Then H already fixes a point in X.*

Since H fixes a point in the combinatorial bordification of X by Claim 1, the claim follows from Corollary 3.16.

Let N be a compact normal subgroup of the connected component G^0 of G such that G^0/N is a connected Lie group (see Lemma 3.3).

By [CM11a, Theorem 2.5], we know that N acts locally elliptically on X, and hence fixes a point in X by Claim 2. Consider now the induced action of the connected Lie group G^0/N on the fixed point set $\mathrm{Min}(N)$ of N in X. Since G^0/N has property (FB+) by Lemma 3.18, it fixes a point in $\mathrm{Min}(N)$. This shows that G^0 fixes a point of X.

In turn, one can consider the action of the compact group G/G^0 on $\mathrm{Min}(G^0) \subseteq X$. This is a locally elliptic action by [CM11a, Theorem 2.5], and hence the G-action on X is locally elliptic. Thus Claim 2 yields the desired fixed point for the G-action on X. □

Remark 3.23. Note that if, given a specific building Δ, we want to show that any almost connected locally compact group G acting measurably by type-preserving simplicial isometries on Δ has a global fixed point, it is sufficient to check that Conjecture 3.22 holds for group actions on Δ, as well as on the "iterated transversal buildings" of Δ (see the proof of Claim 1 above). Here, we mean by "iterated transversal buildings" of Δ either the transversal buildings to Δ, or the transversal buildings to these transversal buildings, and so on.

In particular, if Conjecture 3.22 holds for group actions on trees, then so does Theorem 3.20. More generally, remark that if Δ is of type (W, S), then the type of an "iterated transversal building" of Δ is a subgroup of W (see Lemma 2.8). So, for example, if Conjecture 3.22 holds for group actions on affine (respectively, right-angled) buildings, then Conjecture 3.21 also holds for the same class of buildings.

Proof of Theorem 3.20. As mentioned in Remark 3.23 above, this follows from the fact that Conjecture 3.22 holds when the building is a tree (see for example [Ser77, 6.5 Exercise 2]). □

Remark 3.24. Note that, by the same argument as above, Conjecture 3.21 now reduces to proving that compact groups have property (FB+). Since every compact group is a product of its identity component with a totally disconnected closed subgroup ([HM98, Theorem 9.41]), Lemma 3.2 in turn reduces the problem to compact connected and profinite groups. We now mention some consequences of Lemma 3.18 in each of these two cases.

If G is compact connected, then there exists a family $\{S_j \mid j \in J\}$ of simple simply connected compact Lie groups such that G is a quotient of $\prod_{j\in J} S_j \times Z_0(G)$ by a closed central subgroup, where $Z_0(G)$ denotes the identity component of the center of G ([HM98, Theorem 9.24]). Thus, in that case, Lemma 3.2 reduces the problem to compact connected groups of the form $\prod_{j\in J} S_j$. Note that the simple simply connected compact Lie groups are classified and belong to countably many isomorphism classes. Index these classes by $\mathbb{N}$. Then one can write G as a countable product $G = \prod_{i\in\mathbb{N}} T_i$, where T_i is the product of all S_j belonging to the i-th isomorphism class. Moreover, each T_i has property (FB+). Indeed, let S be a representative for the i-th class, so that T_i is isomorphic to a product $\prod_{j\in I} S$ of copies of S. By Theorem 3.5, the group S is boundedly generated by compact abelian subgroups $A_1, \ldots, A_n$. Hence T_i is boundedly generated by the compact abelian subgroups $\prod_{j\in I} A_1, \ldots, \prod_{j\in I} A_n$, whence the claim. Thus, if only finitely many isomorphism classes appear in the product decomposition of G, then G has property (FB+).

If G is a profinite group, then G is a closed subgroup of a product $\prod_{j\in J} S_j$ of finite groups. Suppose that $G = \prod_{j\in J} S_j$. Then, as in the case of compact connected groups, we may express G as a countable product $G = \prod_{i\in\mathbb{N}} T_i$, where T_i is this time the product of all S_j of order i. Again, the same argument shows that each T_i has property (FB+), since clearly a group of order i is boundedly generated by i abelian subgroups. Finally, as noted in Corollary 3.14, profinite groups of polynomial subgroup growth also have finite abelian width and thus property (FB+).

Part II

Kac–Moody groups

Chapter 4

Kac–Moody algebras

The original results of this chapter are presented in Sections 4.3 and 4.4.

4.1 Definition and basic properties

We begin by giving the basics on Kac–Moody algebras, following [Kac90, Chapters 1–5 and 9] (see also [Kum02, Chapter 1]).

4.1.1 Basic definitions. — Let $A = (a_{ij})_{1\leq i,j\leq n}$ be an $n \times n$ complex matrix of rank l.

The matrix A is called a **generalised Cartan matrix** if it satisfies the following conditions:

(C1) $a_{ii} = 2$ for $i = 1, \dots, n$;

(C2) a_{ij} are nonpositive integers for $i \neq j$;

(C3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

A **realisation** of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ are indexed subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, satisfying the following three conditions:

(1) both sets Π and $\Pi^\vee$ are linearly independent;

(2) $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ for $i, j = 1, \dots, n$;

(3) $\dim \mathfrak{h} = 2n - l$.

Such a realisation of A always exists and is unique (up to isomorphism).

The matrix A is called **decomposable** if, after reordering the indices (that is, after a permutation of its rows and the same permutation of its columns), it decomposes into a nontrivial direct sum $A = \left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right)$. Note that in this case one can obtain a realisation for A as the direct sum

$$(\mathfrak{h}_1 \oplus \mathfrak{h}_2, \Pi_1 \times \{0\} \cup \{0\} \times \Pi_2, \Pi_1^\vee \times \{0\} \cup \{0\} \times \Pi_2^\vee)$$

of realisations $(\mathfrak{h}_i, \Pi_i, \Pi_i^\vee)$ of A_i, $i = 1, 2$. If A is not decomposable, it is **indecomposable**.

The set Π is called the **root basis**, $\Pi^\vee$ the **coroot basis**, and elements from Π (respectively, $\Pi^\vee$) are called **simple roots** (respectively, **simple coroots**). Set

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i, \quad Q_+ = \sum_{i=1}^{n} \mathbb{N}\alpha_i.$$

The lattice Q is called the **root lattice**. For $\alpha = \sum_i k_i\alpha_i \in Q$, the number $\mathrm{ht}(\alpha) := \sum_i k_i$ is called the **height** of α. One introduces a partial ordering $\geq$ on $\mathfrak{h}^*$ by setting $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$.

Definition 4.1. Let $A = (a_{ij})_{1\leq i,j\leq n}$ be an $n \times n$ complex matrix and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation of A. We then define $\tilde{\mathfrak{g}}(A)$ to be the complex Lie algebra with generators e_i, f_i $(1 \leq i \leq n)$ and $\mathfrak{h}$, and the following defining relations:

$$\begin{cases} [e_i, f_j] &= -\delta_{ij}\alpha_i^\vee & (i, j = 1, \dots, n), \\ [h, h'] &= 0 & (h, h' \in \mathfrak{h}), \\ [h, e_i] &= \langle \alpha_i, h \rangle e_i, & \\ [h, f_i] &= -\langle \alpha_i, h \rangle f_i & (i = 1, \dots, n;\ h \in \mathfrak{h}). \end{cases} \tag{4.1}$$

We denote by $\tilde{\mathfrak{n}}_+$ (respectively, $\tilde{\mathfrak{n}}_-$) the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by $e_1, \dots, e_n$ (respectively, $f_1, \dots, f_n$).

Proposition 4.2. *With the previous notations:*

(1) $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$ *(direct sum of vector spaces).*

(2) $\tilde{\mathfrak{n}}_+$ *(respectively* $\tilde{\mathfrak{n}}_-$*) is freely generated by* $e_1, \dots, e_n$ *(respectively,* $f_1, \dots, f_n$*).*

(3) The map $e_i \mapsto -f_i$, $f_i \mapsto -e_i$ $(i = 1, \dots, n)$, $h \mapsto -h$ $(h \in \mathfrak{h})$*, can be uniquely extended to an involution* $\tilde{\omega}$ *of the Lie algebra* $\tilde{\mathfrak{g}}(A)$*.*

(4) With respect to $\mathfrak{h}$ *one has the root space decomposition:*

$$\tilde{\mathfrak{g}}(A) = \Big(\bigoplus_{\substack{\alpha\in Q_+ \\ \alpha\neq 0}} \tilde{\mathfrak{g}}_{-\alpha}\Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\substack{\alpha\in Q_+ \\ \alpha\neq 0}} \tilde{\mathfrak{g}}_{\alpha}\Big),$$

where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}}(A) \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$*. Moreover,* $\dim \tilde{\mathfrak{g}}_\alpha < \infty$ *and* $\tilde{\mathfrak{g}}_\alpha \subset \tilde{\mathfrak{n}}_\pm$ *for* $\pm\alpha \in Q_+$, $\alpha \neq 0$*.*

(5) *Among the ideals of* $\tilde{\mathfrak{g}}(A)$ *intersecting* $\mathfrak{h}$ *trivially, there exists a unique maximal ideal* $\mathfrak{i}$. *Furthermore,* $\mathfrak{i} = (\mathfrak{i} \cap \tilde{\mathfrak{n}}_-) \oplus (\mathfrak{i} \cap \tilde{\mathfrak{n}}_+)$ *(direct sum of ideals), and* $\mathfrak{i}$ *contains the elements of the form*

$$(\operatorname{ad} f_i)^{1-a_{ij}} f_j \quad \textit{and} \quad (\operatorname{ad} e_i)^{1-a_{ij}} e_j \qquad \textit{for } 1 \le i \ne j \le n.$$

Definition 4.3. Let $A = (a_{ij})_{1\le i,j\le n}$ be an $n \times n$ generalised Cartan matrix and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation of A. The **Kac–Moody algebra** with generalised Cartan matrix A is the complex Lie algebra $\mathfrak{g}(A)$ obtained from $\tilde{\mathfrak{g}}(A)$ by adding the following **Serre relations**:

$$\begin{cases} (\operatorname{ad} e_i)^{1-a_{ij}} e_j &= 0, \\ (\operatorname{ad} f_i)^{1-a_{ij}} f_j &= 0 \qquad (1 \le i \ne j \le n). \end{cases} \tag{4.2}$$

We keep the same notation for the images of $e_i, f_i, \mathfrak{h}$ in $\mathfrak{g}(A)$. The subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$ is called its **Cartan subalgebra**. The elements e_i, f_i $(i = 1, \ldots, n)$ are called the **Chevalley generators** of $\mathfrak{g}(A)$. They generate the **derived subalgebra** $\mathfrak{g}_A := [\mathfrak{g}(A), \mathfrak{g}(A)]$.

Remark 4.4. The definition of Kac–Moody algebras given above is slightly different from the one given in [Kac90, Chapter 1] in two respects. The main difference is that Kac defines $\mathfrak{g}(A)$ to be the quotient of $\tilde{\mathfrak{g}}(A)$ by the ideal $\mathfrak{i}$ of Proposition 4.2 (5) (which contains the Serre relations). However, as Kac notices (see [Kac90, Proposition 5.12]), picking one or the other definition of $\mathfrak{g}(A)$ does not make a difference for most of the results, and in particular for all the results we are about to collect in this chapter. The second difference with Kac's definition is purely cosmetic: the minus sign in the relation $[e_i, f_j] = -\delta_{ij}\alpha_i^\vee$ of (4.1) is completely arbitrary and does not appear in [Kac90]. This is the *Tits convention*, which allows to remove a minus sign elsewhere (we will mention when this happens).

For the rest of this section, we fix a generalised Cartan matrix $A = (a_{ij})_{1\le i,j\le n}$ and a realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A.

It follows from Proposition 4.2 that we have the following **root space decomposition** with respect to $\mathfrak{h}$:

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ is the **root space** attached to α. Note that $\mathfrak{g}_0 = \mathfrak{h}$. This gives a Q-**gradation** of $\mathfrak{g}(A)$, that is, a Q-gradation of $\mathfrak{g}(A)$ as vector space such that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in Q$. An element $x \in \mathfrak{g}_\alpha$ for some $\alpha \in Q$ will be called **homogeneous**. Regrouping the elements of Q by height, one also gets a $\mathbb{Z}$-gradation

$$\mathfrak{g}(A) = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n \quad \text{where} \quad \mathfrak{g}_n := \bigoplus_{\substack{\alpha \in Q \\ \operatorname{ht}(\alpha) = n}} \mathfrak{g}_\alpha.$$

An element $\alpha \in Q$ is called a **root** if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$. A root $\alpha > 0$ (respectively, $\alpha < 0$) is called **positive** (respectively, **negative**). It follows from Proposition 4.2 (4) that every root is either positive or negative. Denote by Δ, Δ_+ and Δ_- the sets of all roots, positive and negative roots, respectively, so that

$$\Delta = \Delta_+ \cup \Delta_-.$$

Let $\mathfrak{n}_+$ (respectively, $\mathfrak{n}_-$) denote the subalgebra of $\mathfrak{g}(A)$ generated by $e_1, \dots, e_n$ (respectively, $f_1, \dots, f_n$). By Proposition 4.2 (1), we have **triangular decompositions**

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

and

$$\mathfrak{g}_A = \mathfrak{n}_- \oplus \sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee \oplus \mathfrak{n}_+ \qquad \text{(direct sums of vector spaces)}.$$

Note that $\mathfrak{g}_\alpha \subset \mathfrak{n}_+$ if $\alpha > 0$ and $\mathfrak{g}_\alpha \subset \mathfrak{n}_-$ if $\alpha < 0$. More precisely, for $\alpha > 0$ (respectively, $\alpha < 0$), $\mathfrak{g}_\alpha$ is the linear span of the elements of the form $[e_{i_1}, \dots, e_{i_s}]$ (respectively, $[f_{i_1}, \dots, f_{i_s}]$), such that $\alpha_{i_1} + \dots + \alpha_{i_s} = \alpha$ (respectively, $= -\alpha$), where

$$[x_1, x_2, \dots, x_s] := \mathrm{ad}(x_1)\,\mathrm{ad}(x_2) \dots \mathrm{ad}(x_{s-1})(x_s).$$

In particular, one has the following.

Lemma 4.5. *With the previous notations:*

(1) $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$ *and* $\mathfrak{g}_{s\alpha_i} = \{0\}$ *if* $|s| > 1$.

(2) If $\beta \in \Delta_+ \setminus \{\alpha_i\}$, *then* $(\beta + \mathbb{Z}\alpha_i) \cap \Delta \subset \Delta_+$.

The involution $\tilde{\omega}$ from Proposition 4.2 (3) induces an involutive automorphism ω of $\mathfrak{g}(A)$ called the **Chevalley involution** of $\mathfrak{g}(A)$. As $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ have the same dimension. In particular,

$$\Delta_- = -\Delta_+.$$

4.1.2 The Weyl group of a Kac–Moody algebra. — For a given Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V, we call an element $x \in \mathfrak{g}$ **locally finite** on V if for any $v \in V$ there exists a finite-dimensional subspace $W \leq V$ containing v such that $xW \subseteq W$. If in addition $x|_W$ is a nilpotent transformation of W, then x is said to be **locally nilpotent** on V. Equivalently, x is locally nilpotent on V if for all $v \in V$ there is some $N \in \mathbb{N}$ such that $x^N(v) = 0$. Note that $\mathrm{ad}\, e_i$ and $\mathrm{ad}\, f_i$ are locally nilpotent on $\mathfrak{g}(A)$ (we will also say that e_i and f_i are ad-**locally nilpotent**).

A $\mathfrak{g}(A)$-module V is called **$\mathfrak{h}$-diagonalisable** if $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda := \{v \in V \mid h(v) = \langle \lambda, h \rangle v \ \forall h \in \mathfrak{h}\}$. As usual, V_λ is called a **weight space** and

$\lambda \in \mathfrak{h}^*$ is called a **weight** if $V_\lambda \neq \{0\}$. An $\mathfrak{h}$-diagonalisable $\mathfrak{g}(A)$-module V is called **integrable** if e_i, f_i $(1 \leq i \leq n)$ are locally nilpotent on V. Viewing V as a $\mathfrak{g}_{(i)}$-module, where $\mathfrak{g}_{(i)} := \mathbb{C}e_i + \mathbb{C}\alpha_i^\vee + \mathbb{C}f_i \cong \mathfrak{sl}_2$, one may use $\mathfrak{sl}_2$-module theory to prove the following.

Lemma 4.6. *Let V be an integrable $\mathfrak{g}(A)$-module. Then:*

(1) If λ is a weight then $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$ for all $i = 1, \ldots, n$.

(2) If λ is a weight and $\lambda + \alpha_i$ (respectively, $\lambda - \alpha_i$) is not a weight, then $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ (respectively, $\langle \lambda, \alpha_i^\vee \rangle \leq 0$).

(3) If λ is a weight, then $\lambda' := \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ is also a weight, and $\dim V_\lambda = \dim V_{\lambda'}$.

For $i = 1, \ldots, n$, let

$$r_i \colon \mathfrak{h}^* \to \mathfrak{h}^* : \lambda \mapsto r_i(\lambda) := \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$$

and

$$r_i^\vee \colon \mathfrak{h} \to \mathfrak{h} : x \mapsto r_i^\vee(x) := x - \langle \alpha_i, x \rangle \alpha_i^\vee$$

respectively denote the **fundamental reflections** and **dual fundamental reflections** for $\mathfrak{g}(A)$. Note that as $(\mathfrak{h}^*, \Pi^\vee, \Pi)$ is a realisation of the generalised Cartan matrix ${}^tA$, the dual fundamental reflections for $\mathfrak{g}(A)$ are just the fundamental reflections for $\mathfrak{g}({}^tA)$.

Definition 4.7. The **Weyl group** of $\mathfrak{g}(A)$ is the subgroup $W = W(A)$ of $\mathrm{GL}(\mathfrak{h}^*)$ generated by all fundamental reflections for $\mathfrak{g}(A)$. The two groups $W(A)$ and $W({}^tA) = \langle r_i^\vee \mid 1 \leq i \leq n \rangle \leq \mathrm{GL}(\mathfrak{h})$ are contragredient linear groups and will therefore be identified.

It follows from Lemma 4.6 that the set of weights of an integral $\mathfrak{g}(A)$-module V is W-invariant. Applying this to the adjoint representation and using Lemma 4.5 (2), we get the following.

Lemma 4.8. *With the notations above:*

(1) Δ is W-invariant and $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{w(\alpha)}$ *for all $w \in W$.*

(2) If $\alpha \in \Delta_+$ is such that $r_i(\alpha) < 0$, then $\alpha = \alpha_i$.

(3) $\langle w(\lambda), w(h) \rangle = \langle \lambda, h \rangle$ for all $\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$ and $w \in W$.

We now give an alternative approach to the Weyl group $W = W(A)$. For a locally finite operator a on a vector space V, we define the **exponential**

$$\exp(a) := \sum_{n\geq 0} \frac{a^n}{n!} = \mathrm{Id}_V + \frac{1}{1!}a + \frac{1}{2!}a^2 + \dots,$$

which has the usual properties. We record for future reference the following formulas (the second formula can be found in [Bou72, II §6 ex.1 p90]).

Lemma 4.9. *We have the following:*

(1) Let R be an associative algebra. Then for all $x, a \in R$ one has

$$\mathrm{ad}(x)^k a = \sum_{s=0}^{k} (-1)^s \binom{k}{s} x^{k-s} a x^s.$$

(2) Let V be an integrable $\mathfrak{g}(A)$-module. Then for any locally finite x on V, one has

$$(\exp e_i).(\exp x).(\exp -e_i) = \exp\Big(\sum_{m\in\mathbb{N}} (\mathrm{ad}(e_i)^m/m!)(x)\Big).$$

For an integrable $\mathfrak{g}(A)$-module (V, π), define the linear automorphisms

$$r_i^\pi := \exp(f_i)\exp(e_i)\exp(f_i) = \exp(e_i)\exp(f_i)\exp(e_i) \in \mathrm{GL}(V).$$

[Note: this is where the Tits convention mentioned in Remark 4.4 comes into play, whence the sign difference in the definition of r_i^π given by Kac].

Computing in $\mathfrak{sl}_2$, one can then prove the following.

Proposition 4.10. *Let (V, π) be an integrable $\mathfrak{g}(A)$-module. Then for all $\lambda \in \mathfrak{h}^*$ and $i = 1, \dots, n$:*

(1) $r_i^\pi(V_\lambda) = V_{r_i(\lambda)}$ and $(r_i^{\mathrm{ad}})|_{\mathfrak{h}} = r_i^\vee \in \mathrm{GL}(\mathfrak{h})$.

(2) $r_i^{\mathrm{ad}} \in \mathrm{Aut}(\mathfrak{g}(A))$, that is, $r_i^{\mathrm{ad}}([x, y]) = [r_i^{\mathrm{ad}}(x), r_i^{\mathrm{ad}}(y)]$ for all $x, y \in \mathfrak{g}(A)$.

(3) $(r_i^\pi)^2 v = (-1)^{\langle\lambda,\alpha_i^\vee\rangle} v$ for all $v \in V_\lambda$.

(4) $\underbrace{r_i^\pi r_j^\pi r_i^\pi \dots}_{m_{ij}\ factors} = \underbrace{r_j^\pi r_i^\pi r_j^\pi \dots}_{m_{ij}\ factors}$ for all i, j such that $r_i r_j$ has finite order m_{ij} in W.

It follows that the W-action on $\mathfrak{h}$ lifts to an action on $\mathfrak{g}(A)$ of the group

$$W^* := \langle r_i^{\mathrm{ad}} \mid 1 \leq i \leq n\rangle \leq \mathrm{Aut}(\mathfrak{g}(A)).$$

Corollary 4.11. *With the notations above:*

(1) *There exists a unique surjective group homomorphism* $\pi\colon W^* \to W$ *such that* $\pi(r_i^{\mathrm{ad}}) = r_i$ *for all* $i = 1, \ldots, n$.

(2) *For all* $w^* \in W^*$ *and* $i = 1, \ldots, n$, *the pair* $E_\alpha := \{w^* e_i, -w^* e_i\}$ *only depends on the root* $\alpha := \pi(w^*)\alpha_i \in \Delta$, *i.e., it is the same for any decomposition* $\alpha = \pi(v^*)\alpha_j$.

As expected, the Weyl group of a Kac–Moody algebra is a Coxeter group.

Proposition 4.12. *Let* $W = W(A)$ *be the Weyl group of* $\mathfrak{g}(A)$. *Then:*

(1) *For all* $i, j = 1, \ldots, n$ *and* $w \in W$: $w\alpha_i = \alpha_j \Leftrightarrow w\alpha_i^\vee = \alpha_j^\vee \Leftrightarrow wr_iw^{-1} = r_j$.

(2) (W, S) *is a Coxeter system, where* $S = \{r_i \mid 1 \leq i \leq n\}$.

(3) *For* $i \neq j$, *the order* $m_{ij} \in \mathbb{N}^* \cup \{\infty\}$ *of the product* r_ir_j *in* W *depends only on* $a_{ij}a_{ji}$ *and is given in the following table:*

$a_{ij}a_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

In particular, the Weyl groups of Kac–Moody algebras correspond precisely to the **crystallographic Coxeter groups**, namely, those whose Coxeter matrix has non-diagonal entries in $\{2, 3, 4, 6, \infty\}$. Note that the set of roots (or half-spaces) of the Coxeter complex $\Sigma(W, S)$, as defined in §1.1.5, corresponds to the set of *real* roots of Δ, namely, those in $W.\Pi$ (see also §4.1.5 below).

4.1.3 The invariant bilinear form. — Consider the rescaling $e_i \mapsto e_i$, $f_i \mapsto \epsilon_i f_i$ $(1 \leq i \leq n)$ of the Chevalley generators of the Kac–Moody algebra $\mathfrak{g}(A)$, where ϵ_i are nonzero numbers. Then $\alpha_i^\vee \mapsto \epsilon_i\alpha_i^\vee$ and we can extend this map to an isomorphism $\mathfrak{g}(A) \to \mathfrak{g}(DA)$, where $D = \mathrm{diag}(\epsilon_1, \ldots \epsilon_n)$.

The generalised Cartan matrix A is called **symmetrisable** if there exists an invertible diagonal matrix $D = \mathrm{diag}(\epsilon_1, \ldots \epsilon_n)$ and a symmetric matrix $B = (b_{ij})$ such that $A = DB$. Note that we can (and will) always assume that the diagonal entries ϵ_i of D are positive rational numbers. We also say in this case that $\mathfrak{g}(A)$ is **symmetrisable**.

Proposition 4.13. *Let* $A = (a_{ij})_{1 \leq i,j \leq n}$ *be a symmetrisable generalised Cartan matrix and fix a decomposition* $A = DB$ *as above. Let* $(\mathfrak{h}, \Pi, \Pi^\vee)$ *be a realisation of* A *and fix a complementary subspace* $\mathfrak{h}'$ *to* $\sum_{i=1}^n \mathbb{C}\alpha_i^\vee$ *in* $\mathfrak{h}$. *Then there exists a non-degenerate symmetric bilinear* $\mathbb{C}$*-valued form* $(\cdot|\cdot)\colon \mathfrak{g}(A) \times \mathfrak{g}(A) \to \mathbb{C}$ *on* $\mathfrak{g}(A)$ *such that:*

(1) $(\cdot|\cdot)|_{\mathfrak{h}}$ *is defined by* $(\alpha_i^\vee|h) = \langle \alpha_i, h\rangle \epsilon_i$ *for* $h \in \mathfrak{h}$, $i = 1, \ldots, n$, *and* $(h|h') = 0$ *for* $h, h' \in \mathfrak{h}'$. *Moreover, it is nondegenerate: there exists a linear map* $\nu\colon \mathfrak{h} \to \mathfrak{h}^*$ *defined by* $\langle \nu(h), h'\rangle = (h|h')$ *for all* $h, h' \in \mathfrak{h}$.

(2) $(\cdot|\cdot)$ *is* **invariant***, that is,* $([x,y]|z) = (x|[y,z])$ *for all* $x, y, z \in \mathfrak{g}(A)$.

(3) $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = 0$ *if* $\alpha + \beta \neq 0$.

(4) $(\cdot|\cdot)|_{\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}}$ *is nondegenerate for* $\alpha \neq 0$*, and hence* $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_{-\alpha}$ *are nondegenerately paired by* $(\cdot|\cdot)$.

(5) $[x,y] = (x|y)\nu^{-1}(\alpha)$ *for* $x \in \mathfrak{g}_\alpha$*,* $y \in \mathfrak{g}_{-\alpha}$*,* $\alpha \in \Delta$.

(6) $(\cdot|\cdot)|_{\mathfrak{h}}$ *is* W*-invariant.*

Remark that one can also define the above bilinear form for ${}^tA$. In particular, we have a symmetric nondegenerate bilinear form $(\cdot|\cdot)\colon \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ defined by

$$(\alpha|\beta) := (\nu^{-1}(\alpha)|\nu^{-1}(\beta)) \quad \text{for all } \alpha, \beta \in \Delta.$$

Note that $\nu(\alpha_i^\vee) = \epsilon_i\alpha_i$ for all $i = 1, \dots, n$. In particular, $(\alpha_i|\alpha_j) = a_{ij}/\epsilon_i = a_{ji}/\epsilon_j$.

We conclude this paragraph by mentioning the following important result of Gabber–Kac (which sheds some light on Remark 4.4 in the symmetrisable case).

Proposition 4.14. *Let* A *be a symmetrisable generalised Cartan matrix. Let* $\tilde{\mathfrak{g}}(A)$ *and* $\mathfrak{i}$ *be as in Proposition 4.2. Then* $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{i}$.

4.1.4 Types of generalised Cartan matrices. — We now briefly review the classification of indecomposable generalised Cartan matrices. For a real column vector $u = {}^t(u_1, \dots, u_n)$, we write $u > 0$ if all $u_i > 0$ and $u \geq 0$ if all $u_i \geq 0$.

Proposition 4.15. *Let* $A = (a_{ij})_{1\leq i,j\leq n}$ *be an indecomposable generalised Cartan matrix. Then exactly one of the following holds:*

(Fin) *There exists* $u > 0$ *such that* $Au > 0$*. In this case,* $\det A \neq 0$ *and* $Av \geq 0$ *implies* $v > 0$ *or* $v = 0$.

(Aff) *There exists* $u > 0$ *such that* $Au = 0$*. In this case,* $\operatorname{corank} A = 1$ *and* $Av \geq 0$ *implies* $Av = 0$.

(Ind) *There exists* $u > 0$ *such that* $Au < 0$*. In this case,* $Av \geq 0$*,* $v \geq 0$ *implies* $v = 0$.

Referring to cases (Fin), (Aff), or (Ind), we will say that A is of **finite**, **affine**, or **indefinite type**, respectively. The matrices of finite and affine type are completely classified by their Dynkin diagram (see [Kac90, §4.8] for a list). We recall that the **Dynkin diagram** of $A = (a_{ij})_{1\leq i,j\leq n}$, denoted $S(A)$, is a (labeled) graph on n vertices $\alpha_1, \dots, \alpha_n$ such that α_i and α_j are connected by an edge if and only if $a_{ij}a_{ji} \neq 0$ (see *loc. cit.* for a precise definition). Note that the Kac–Moody algebras associated to indecomposable matrices of finite type are precisely the simple finite-dimensional Lie algebras. Note also that the Weyl group of a matrix of finite type is finite, while the Weyl group of a matrix of affine type is an affine Coxeter group, as defined in §1.1.7.

4.1.5 Real and imaginary roots. — We conclude this section by giving an explicit description of the root system Δ of a Kac–Moody algebra $\mathfrak{g}(A)$.

A root α is called **real** if there exists $w \in W$ such that $w(\alpha)$ is a simple root. Denote by Δ^{re} and Δ^{re}_+ the sets of all real and positive real roots, respectively.

For $\alpha \in \Delta^{\mathrm{re}}$, say $\alpha = w(\alpha_i)$ for some $w \in W$ and $\alpha_i \in \Pi$, define the **dual (real) root** $\alpha^\vee \in \Delta^{\vee\,\mathrm{re}}$ by $\alpha^\vee = w(\alpha_i^\vee)$, where $\Delta^\vee \subset \mathfrak{h}$ is the set of roots of $\mathfrak{g}({}^tA)$. This is well-defined by Proposition 4.12 (1). Moreover, the reflection

$$r_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^* : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

belongs to W since $r_\alpha = w r_i w^{-1}$.

Proposition 4.16. *Let α be a real root of $\mathfrak{g}(A)$. Then:*

(1) $\dim \mathfrak{g}_\alpha = 1$ *and $k\alpha$ is a root if and only if $k \in \{\pm 1\}$.*

(2) If $\beta \in \Delta$ then there exist nonnegative integers p and q related by the equation $p - q = \langle \beta, \alpha^\vee \rangle$, such that $\beta + k\alpha \in \Delta \cup \{0\}$ if and only if $-p \leq k \leq q$, $k \in \mathbb{Z}$.

(3) If A is symmetrisable and $(\cdot|\cdot) \colon \mathfrak{g}(A) \times \mathfrak{g}(A) \to \mathbb{C}$ is the bilinear form from Proposition 4.13, then $(\alpha|\alpha) > 0$ and $\alpha^\vee = 2\nu^{-1}(\alpha)/(\alpha|\alpha)$.

(4) Provided that $\pm\alpha \notin \Pi$, there exists i such that $|\operatorname{ht}(r_i(\alpha))| < |\operatorname{ht}(\alpha)|$.

A root α which is not real is called an **imaginary root**. Denote by Δ^{im} and Δ^{im}_+ the sets of all imaginary and positive imaginary roots, respectively. Thus $\Delta = \Delta^{\mathrm{re}} \cup \Delta^{\mathrm{im}}$ and $\Delta^{\mathrm{im}} = \Delta^{\mathrm{im}}_+ \cup (-\Delta^{\mathrm{im}}_+)$.

Proposition 4.17. *With the notations above:*

(1) The set Δ^{im}_+ is W-invariant.

(2) For $\alpha \in \Delta^{\mathrm{im}}_+$ there exists a unique root β with $\langle \beta, \alpha_i^\vee \rangle \leq 0$ for all $i = 1, \ldots, n$, which is W-equivalent to α.

(3) If A is symmetrisable and $(\cdot|\cdot) \colon \mathfrak{g}(A) \times \mathfrak{g}(A) \to \mathbb{C}$ is the bilinear form from Proposition 4.13, then a root α is imaginary if and only if $(\alpha|\alpha) \leq 0$.

We now give an explicit description of the set of imaginary roots. For $\alpha = \sum_i k_i \alpha_i \in Q$ we define the **support** of α, denoted $\operatorname{supp}\alpha$, to be the subdiagram of $S(A)$ which consists of the vertices α_i such that $k_i \neq 0$, and of all the edges joining these vertices. Set

$$K := \{\alpha \in Q_+ \setminus \{0\} \mid \langle \alpha, \alpha_i^\vee \rangle \leq 0 \text{ for all } i \text{ and } \operatorname{supp}\alpha \text{ is connected}\}.$$

Proposition 4.18. *With the notations above:*

(1) The support of every root $\alpha \in \Delta$ is connected.

(2) $\Delta^{\mathrm{im}}_+ = \bigcup_{w\in W} w(K)$.

(3) If $\alpha \in \Delta^{\mathrm{im}}_+$ and r is a nonzero rational number such that $r\alpha \in Q$, then $r\alpha \in \Delta^{\mathrm{im}}$. In particular, $n\alpha \in \Delta^{\mathrm{im}}$ for all nonzero $n \in \mathbb{Z}$.

It remains to see when imaginary roots exist.

Proposition 4.19. *Let A be an indecomposable generalised Cartan matrix.*

(1) If A is of finite type, then Δ^{im} is empty.

(2) If A is of affine type, then $\Delta^{\mathrm{im}}_+ = \{n\delta \mid n \in \mathbb{N}^*\}$ where $\delta \in Q_+$, viewed as a column vector of $\mathbb{R}^n$ in the basis of simple roots, is such that $A\delta = 0$ and is of minimal height for this property.

(3) If A is of indefinite type, then there exists a positive imaginary root $\alpha = \sum_i k_i\alpha_i$ such that $k_i > 0$ and $\langle \alpha, \alpha_i^\vee \rangle < 0$ for all $i = 1, \dots, n$.

We mention for future reference the following characterisation of **isotropic** roots of a symmetrisable Kac–Moody algebra $\mathfrak{g}(A)$, that is, of the roots α such that $(\alpha|\alpha) = 0$. Note by Proposition 4.17 that such a root must in particular be imaginary.

Lemma 4.20. *Let A be symmetrisable. A root α is isotropic if and only if the unique root β of minimal height in its W-orbit provided by Proposition 4.17 (2) has affine support, that is, the subdiagram* $\operatorname{supp}\beta$ *of $S(A)$ is of affine type.*

We conclude this section by giving some vocabulary concerning sets of roots.

Definition 4.21. Given two subsets $\Psi \subseteq \Psi' \subseteq \Delta \cup \{0\}$, one says that Ψ is **closed** (respectively, an **ideal** in Ψ') if $\alpha + \beta \in \Psi$ whenever $\alpha, \beta \in \Psi$ (respectively, $\alpha \in \Psi$, $\beta \in \Psi'$) and $\alpha + \beta \in \Delta \cup \{0\}$. The set Ψ is called **prenilpotent** if there exist $w, w' \in W$ such that $w\Psi \subset \Delta_+$ and $w'\Psi \subseteq \Delta_-$(see also Section 2.5). In this case, Ψ is finite and contained in the subset $w^{-1}(\Delta^{\mathrm{re}}_+) \cap (w')^{-1}(\Delta^{\mathrm{re}}_-)$ of Δ^{re}, which is **nilpotent**, that is, both prenilpotent and closed (see e.g. [Rém02, 1.4.1]).

4.2 Integral enveloping algebra

Let $\mathfrak{g}(A)$ be a Kac–Moody algebra and let $\mathfrak{g}_A = [\mathfrak{g}(A), \mathfrak{g}(A)]$ be its derived algebra (see Definition 4.3). In this section, we present a $\mathbb{Z}$-form of the universal enveloping algebra of $\mathfrak{g}_A$, which was introduced by J. Tits ([Tit87, §4]). Besides the original paper, we also mention [Rém02, Section 7.4] and [Rou12, Section 2] as general references for what follows.

4.2.1 The $\mathbb{Z}$-form $\mathcal{U}$. — We fix a generalised Cartan matrix $A = (a_{ij})_{i,j\in I}$ and a realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ for A. For the rest of this section, we set $\mathfrak{h}' := \sum_{i\in I} \mathbb{C}\alpha_i^\vee \subseteq \mathfrak{h}$, so that $\mathfrak{g}_A = \mathfrak{n}_- \oplus \mathfrak{h}' \oplus \mathfrak{n}_+$. We also set $Q^\vee := \sum_{i\in I} \mathbb{Z}\alpha_i^\vee \subseteq \mathfrak{h}'$. We recall that the (derived) Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}_A$ was defined over $\mathbb{C}$. Let $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$.

A **$\mathbb{Z}$-form** of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ is a $\mathbb{Z}$-subalgebra $\mathcal{U}_\mathbb{Z}$ of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ such that the canonical map $\mathcal{U}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C} \to \mathcal{U}_\mathbb{C}(\mathfrak{g})$ is an isomorphism.

For an element $u \in \mathcal{U}_\mathbb{C}(\mathfrak{g})$ and an integer $n \in \mathbb{N}$, define the elements

$$u^{(n)} := \frac{u^n}{n!} \qquad \text{and} \qquad \binom{u}{n} := \frac{1}{n!} u(u-1)\dots(u-n+1)$$

of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$. Let $\mathcal{U}^+, \mathcal{U}^-$, and $\mathcal{U}^0$ be the $\mathbb{Z}$-subalgebras of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ respectively generated by the elements $e_i^{(n)}$ ($i \in I$, $n \in \mathbb{N}$), $f_i^{(n)}$ ($i \in I$, $n \in \mathbb{N}$), and $\binom{h}{n}$ ($h \in Q^\vee$, $n \in \mathbb{N}$). Finally, let $\mathcal{U}$ be the $\mathbb{Z}$-subalgebra of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ generated by $\mathcal{U}^+$, $\mathcal{U}^-$ and $\mathcal{U}^0$.

Proposition 4.22. *With the notations above:*

(1) $\mathcal{U}^+, \mathcal{U}^-$, and $\mathcal{U}^0$ are $\mathbb{Z}$-forms of the corresponding enveloping algebras $\mathcal{U}_\mathbb{C}(\mathfrak{n}_+)$, $\mathcal{U}_\mathbb{C}(\mathfrak{n}_-)$, and $\mathcal{U}_\mathbb{C}(\mathfrak{h}')$, respectively.

(2) $\mathcal{U}$ is a $\mathbb{Z}$-form of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$.

(3) There is a unique decomposition $\mathcal{U} = \mathcal{U}^+\mathcal{U}^-\mathcal{U}^0$.

We set $\mathfrak{g}_\mathbb{Z} := \mathfrak{g} \cap \mathcal{U}$, $\mathfrak{n}_\mathbb{Z}^+ := \mathfrak{n}_+ \cap \mathcal{U}^+$, $\mathfrak{n}_\mathbb{Z}^- := \mathfrak{n}_- \cap \mathcal{U}^-$, and $\mathfrak{h}_\mathbb{Z} := \mathfrak{h}' \cap \mathcal{U}^0$. We then have a triangular decomposition

$$\mathfrak{g}_\mathbb{Z} = \mathfrak{n}_\mathbb{Z}^- \oplus \mathfrak{h}_\mathbb{Z} \oplus \mathfrak{n}_\mathbb{Z}^+.$$

For a ring k, we also write $\mathcal{U}_k, \mathcal{U}_k^\pm, \mathcal{U}_k^0, \mathfrak{g}_k, \mathfrak{n}_k^\pm, \mathfrak{h}_k$ for the corresponding tensor products $\mathcal{U} \otimes k, \mathcal{U}^\pm \otimes k, \mathcal{U}^0 \otimes k, \mathfrak{g}_\mathbb{Z} \otimes k, \mathfrak{n}_\mathbb{Z}^\pm \otimes k, \mathfrak{h}_\mathbb{Z} \otimes k$ over $\mathbb{Z}$. We then have root space decompositions

$$\mathfrak{g}_k = \mathfrak{h}_k \oplus (\bigoplus_{\alpha\in\Delta} \mathfrak{g}_{\alpha k}),$$

where $\mathfrak{g}_{\alpha k} := (\mathfrak{g}_\alpha \cap \mathfrak{g}_\mathbb{Z}) \otimes k$.

4.2.2 Gradation and filtration on $\mathcal{U}$. — The Q-gradation on $\mathfrak{g}$ induces a Q-gradation of the associative algebra $\mathcal{U}_\mathbb{C}(\mathfrak{g})$; the corresponding weight space for $\alpha \in Q$ is denoted $\mathcal{U}_\mathbb{C}(\mathfrak{g})_\alpha$. Set also $\mathcal{U}_\alpha := \mathcal{U}_\mathbb{C}(\mathfrak{g})_\alpha \cap \mathcal{U}$.

Note that $\mathcal{U}$ is a sub-$\mathcal{U}$-module for the adjoint representation ad of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ on itself. Indeed, $(\mathrm{ad}(e_i^{(n)}))(u) = \frac{(\mathrm{ad}(e_i))^n}{n!}(u) = \sum_{s=0}^n (-1)^s e_i^{(n-s)} u e_i^{(s)}$ by Lemma 4.9, and for $h \in \mathfrak{h}'$, ad $\binom{h}{n}$ acts on $\mathcal{U}_\mathbb{C}(\mathfrak{g})_\alpha$ by multiplication by $\binom{\alpha(h)}{n} \in \mathbb{Z}$.

The following lemma is [Rou12, Proposition 2.2 and Corollary 2.3].

Lemma 4.23. *Let k be a field of characteristic p. Then:*

(1) The Lie algebra $\mathfrak{g}_\mathbb{Z}$ (respectively, $\mathfrak{n}^+_\mathbb{Z}$, $\mathfrak{n}^-_\mathbb{Z}$) is generated by $\mathfrak{h}_\mathbb{Z}$ and the elements e_i, f_i ($i \in I$) as a $\mathcal{U}$-module (respectively, $\mathcal{U}^+$-module, $\mathcal{U}^-$-module).

(2) If $p > -a_{ij}$ for all $i, j \in I$ or if $p = 0$, then the Lie algebra $\mathfrak{g}_k$ (respectively, $\mathfrak{n}^+_k$, $\mathfrak{n}^-_k$) is generated by $\mathfrak{h}_k$ and the elements e_i, f_i ($i \in I$).

The canonical filtration of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ (induced by the filtration on $\mathbb{C} \oplus (\bigoplus_{n\in\mathbb{N}^*} \mathfrak{g}^{\otimes n})$) also induces a filtration on $\mathcal{U}$. In particular, the elements of $\mathcal{U}$ of filtration at most 1 are those in $\mathbb{Z} \oplus \mathfrak{g}_\mathbb{Z}$.

4.2.3 Completion of $\mathcal{U}$. — Recall that the **formal topology** on a graded vector space $V = \bigoplus_{\alpha\in M} V_\alpha$, where M is an abelian group, is the topology on V with fundamental system of neighbourhoods of zero the subsets $V^F := \bigoplus_{\alpha\in M\setminus F} V_\alpha$ for F a finite subset of M. The **(formal) completion** $\widehat{V}$ of V, that is, the completion of V with respect to the formal topology, is then just $\widehat{V} = \prod_{\alpha\in M} V_\alpha$. For $M = \mathbb{Z}$, we also define the **positive completion** $\widehat{V}^p$ of $V = V^- \oplus V^0 \oplus V^+$ as $\widehat{V}^p := V^- \oplus V^0 \oplus \widehat{V^+}$, where $V^\pm := \bigoplus_{\pm n\in\mathbb{N}^*} V_n$.

Consider the Q_+-gradation $\mathcal{U}^+ = \bigoplus_{\alpha\in Q_+} \mathcal{U}^+_\alpha$ of $\mathcal{U}^+$. Regrouping by height, one gets a (total) $\mathbb{N}$-gradation $\mathcal{U}^+ = \bigoplus_{n\in\mathbb{N}} \mathcal{U}^+_n$, where each $\mathcal{U}^+_n$ is a direct sum of finitely many $\mathcal{U}^+_\alpha$. An element of $\mathcal{U}^+_n$ is said to be of **degree** n. The completion

$$\widehat{\mathcal{U}}^+ = \prod_{\alpha\in Q_+} \mathcal{U}^+_\alpha$$

with respect to this gradation is a $\mathbb{Z}$-algebra for the natural extension of the multiplication of $\mathcal{U}^+$. For a ring k, one may also define the algebra $\widehat{\mathcal{U}}^+_k = \prod_{\alpha\in Q_+} \mathcal{U}^+_\alpha \otimes k$ (in general different from $\widehat{\mathcal{U}}^+ \otimes k$).

Consider also the positive completion $\widehat{\mathcal{U}}^p$ of $\mathcal{U}$ relatively to the total degree in $\mathbb{Z}$; it is a $\mathbb{Z}$-algebra containing $\widehat{\mathcal{U}}^+$. Similarly, the k-algebra $\widehat{\mathcal{U}}^p_k$ contains $\widehat{\mathcal{U}}^+_k$, as well as the Lie algebra $\widehat{\mathfrak{g}}^p_k = (\oplus_{\alpha<0} \mathfrak{g}_{\alpha k}) \oplus \mathfrak{h}_k \oplus (\prod_{\alpha>0} \mathfrak{g}_{\alpha k})$.

The adjoint action ad of $\mathcal{U}$ on itself respects the gradation, and can thus be extended to an action of $\widehat{\mathcal{U}}^p$ (or $\widehat{\mathcal{U}}^p_k$) on itself. The Lie algebra $\widehat{\mathfrak{g}}^p_k$ is then a sub-ad$(\widehat{\mathcal{U}}^p_k)$-module.

Finally, for a root $\alpha \in \Delta$, consider the $\mathbb{Z}$-subalgebra $\mathcal{U}^\alpha$ of $\mathcal{U}$ defined by

$$\mathcal{U}^\alpha := \mathcal{U}_\mathbb{C}(\oplus_{n\geq 1}\mathfrak{g}_{n\alpha}) \cap (\oplus_{n\geq 0}\mathcal{U}_{n\alpha}) = \mathcal{U}_\mathbb{C}(\oplus_{n\geq 1}\mathfrak{g}_{n\alpha}) \cap \mathcal{U}.$$

If $\Psi \subseteq \Delta \cup \{0\}$ is a closed set, $\mathfrak{g}_\Psi := \oplus_{\alpha\in\Psi}\mathfrak{g}_\alpha$ is a Lie algebra; we then define the $\mathbb{Z}$-subalgebra

$$\mathcal{U}(\Psi)$$

of $\mathcal{U}$, which is generated by all $\mathcal{U}^\alpha$ for $\alpha \in \Psi$. Then $\mathcal{U}_\mathbb{C}(\Psi) := \mathcal{U}(\Psi) \otimes_\mathbb{Z} \mathbb{C}$ is the enveloping algebra of $\mathfrak{g}_\Psi$ and $\mathcal{U}(\Psi) = \mathcal{U}_\mathbb{C}(\Psi) \cap \mathcal{U}$. If in addition $\Psi \subseteq \Delta_+$ then the algebra $\mathcal{U}(\Psi)$ is graded by Q_+; its completion $\widehat{\mathcal{U}}(\Psi) = \prod_{\alpha\in Q_+} \mathcal{U}(\Psi)_\alpha$ is a $\mathbb{Z}$-subalgebra of $\widehat{\mathcal{U}}^+$. As before, one can also consider for any ring k the k-algebras $\mathcal{U}_k^\alpha := \mathcal{U}^\alpha \otimes_{\mathbb{Z}} k$, $\mathcal{U}_k(\Psi) := \mathcal{U}(\Psi) \otimes_{\mathbb{Z}} k$ and

$$\widehat{\mathcal{U}}_k(\Psi) = \prod_{\alpha\in Q_+} \mathcal{U}(\Psi)_\alpha \otimes_{\mathbb{Z}} k$$

(in general different from $\widehat{\mathcal{U}}(\Psi) \otimes_{\mathbb{Z}} k$).

Note that $\widehat{\mathcal{U}}^+ = \widehat{\mathcal{U}}(\Delta_+)$. Replacing Δ_+ by $w(\Delta_+)$ for some $w \in W = W(A)$, it of course also makes sense to define the completion $\widehat{\mathcal{U}}(w(\Delta_+))$ of $\mathcal{U}(w(\Delta_+))$ with respect to the corresponding $\mathbb{N}$-gradation.

4.2.4 The bialgebra structure on $\mathcal{U}$. — The algebra $\mathcal{U}$ (or $\mathcal{U}^+$, $\mathcal{U}^-$, $\mathcal{U}^0$) also admits a structure of cocommutative and co-invertible $\mathbb{Z}$-bialgebra. Its comultiplication ∇, its co-unit ϵ and its co-inversion τ respect the gradation and the filtration of $\mathcal{U}$. They are given on the generators by the following formulas: for $h \in \mathfrak{h}'$ and $n \in \mathbb{N}$,

$$\nabla\binom{h}{n} = \sum_{k+l=n} \binom{h}{k} \otimes \binom{h}{l}, \quad \epsilon\binom{h}{n} = 0 \text{ for } n > 0$$

$$\text{and} \quad \tau\binom{h}{n} = \binom{-h}{n} = (-1)^n\binom{h+n-1}{n} = (-1)^n \sum_{k+l=n} \binom{n-1}{k}\binom{h}{l}.$$

For $i \in I$ and $n \in \mathbb{N}$,

$$\nabla e_i^{(n)} = \sum_{k+l=n} e_i^{(k)} \otimes e_i^{(l)}, \quad \epsilon e_i^{(n)} = 0 \text{ for } n > 0 \quad \text{and} \quad \tau e_i^{(n)} = (-1)^n e_i^{(n)};$$

$$\nabla f_i^{(n)} = \sum_{k+l=n} f_i^{(k)} \otimes f_i^{(l)}, \quad \epsilon f_i^{(n)} = 0 \text{ for } n > 0 \quad \text{and} \quad \tau f_i^{(n)} = (-1)^n f_i^{(n)}.$$

4.2.5 The W^*-action on $\mathcal{U}$. — Recall from Proposition 4.10 that the action on $\mathfrak{h}'$ of the Weyl group W of $\mathfrak{g}$ lifts to an action of W^* on $\mathfrak{g}$. We will also write s_i^* for the automorphism $r_i^{\mathrm{ad}} = \exp(f_i)\exp(e_i)\exp(f_i) = \exp(e_i)\exp(f_i)\exp(e_i)$ of W^* ($i \in I$). Note that these automorphisms extend to $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$. Moreover, since $\mathcal{U}$ is an $\mathrm{ad}(\mathcal{U})$-module, each s_i^* preserves $\mathcal{U}$, and hence one also has a W^*-action on $\mathcal{U}$.

For a real root $\alpha \in \Delta^{\mathrm{re}}$, let $E_\alpha = \{\pm e_\alpha\}$ be the pair of elements of $\mathfrak{g}_{\mathbb{Z}}$ provided by Corollary 4.11 (2). Thus $w^* e_\alpha = \pm e_{w\alpha}$ for any $w^* \in W^*$ such that $\pi(w^*) = w$, in the notations of Corollary 4.11. Writing $f_\alpha := e_{-\alpha}$ for $\alpha \in \Delta_+^{\mathrm{re}}$, we may also assume that $e_{\alpha_i} = e_i$, $f_{\alpha_i} = f_i$ and $[e_\alpha, f_\alpha] = -\alpha^\vee$.

Lemma 4.24. *Let α be a real root. Then:*

(1) $\mathfrak{g}_{\alpha\mathbb{Z}} = \mathbb{Z}e_\alpha$.

(2) $\mathcal{U}_{\mathbb{C}}(\mathfrak{g}_\alpha) \cap \mathcal{U}_{n\alpha} = \mathbb{Z}e_\alpha^{(n)}$ *for all* $n \in \mathbb{N}$.

4.3 Free Lie algebras in Kac–Moody algebras

In this section, we obtain a characterisation of the type of a Kac–Moody algebra $\mathfrak{g}(A)$ in terms of the presence in $\mathfrak{g}(A)$ of free Lie subalgebras. Since this characterisation is essentially contained in [Kac90] in the symmetrisable case, our contribution is to extend it to the general case.

Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a generalised Cartan matrix, and fix a realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A. We claim that

$$\bigoplus_{\beta\in\Delta_+^{\mathrm{im}}} \mathfrak{g}_\beta = \bigcap_{w^*\in W^*} w^*(\mathfrak{n}_+).$$

Indeed, using Proposition 4.10 (1), the inclusion from left to right follows from Proposition 4.17 (1) and the other inclusion follows from Proposition 4.16 (4). It follows that $\mathfrak{n}_+^{\mathrm{im}} := \bigoplus_{\beta\in\Delta_+^{\mathrm{im}}} \mathfrak{g}_\beta$ is a Lie subalgebra of $\mathfrak{n}_+$, which we call the **positive imaginary subalgebra** of $\mathfrak{g}(A)$.

The following proposition is [Kac90, Corollary 9.12].

Proposition 4.25. *Let* A *be symmetrisable and let* $\alpha \in \Delta$ *be such that* $(\alpha|\alpha) \neq 0$. *Then* $\oplus_{k\geq 1}\mathfrak{g}_{k\alpha}$ *is a free Lie algebra on a basis of the space* $\oplus_{k\geq 1}\mathfrak{g}^0_{k\alpha}$, *where*

$$\mathfrak{g}^0_{k\alpha} = \{x \in \mathfrak{g}_{k\alpha} \mid (x|y) = 0 \textit{ for all } y \in \langle \bigoplus_{s=1}^{k-1} \mathfrak{g}_{-s\alpha}\rangle \},$$

where $\langle \bigoplus_{s=1}^{k-1} \mathfrak{g}_{-s\alpha}\rangle$ *denotes the subalgebra of* $\mathfrak{g}(A)$ *generated by* $\bigoplus_{s=1}^{k-1} \mathfrak{g}_{-s\alpha}$.

Corollary 4.26. *Let* A *be indecomposable and symmetrisable. Then exactly one of the following holds.*

(1) A *is of spherical type and* $\mathfrak{n}_+^{\mathrm{im}} = \{0\}$.

(2) A *is of affine type and* $\mathfrak{n}_+^{\mathrm{im}}$ *is abelian and infinite-dimensional.*

(3) A *is of indefinite type and* $\mathfrak{n}_+^{\mathrm{im}}$ *contains a nonabelian free Lie subalgebra.*

Proof. If A is of spherical type, this follows from Proposition 4.19 (1). If A is of affine type, this follows from [Kac90, Proposition 8.4]. Finally, if A is of indefinite type, then by Proposition 4.19 (3) there exists a positive imaginary root $\alpha = \sum_{i=1}^n k_i\alpha_i$ such that $k_i > 0$ and $\langle \alpha, \alpha_i^\vee\rangle < 0$ for all $i = 1, \dots, n$. It follows from Lemma 4.20 that $(\alpha|\alpha) \neq 0$, whence the result by Proposition 4.25. □

We aim at proving Corollary 4.26 in the general case, that is, without assuming that A is symmetrisable. We first recall an easy characterisation of symmetrisable generalised Cartan matrices.

Lemma 4.27. *A is symmetrisable if and only if $a_{i_1i_2}a_{i_2i_3}\dots a_{i_ki_1} = a_{i_2i_1}a_{i_3i_2}\dots a_{i_1i_k}$ for all $i_1,\dots,i_k$ and $k \geq 3$. In particular, if the Dynkin diagram $S(A)$ of A contains no loop (that is, $D(A)$ is a disjoint union of trees), then A is symmetrisable.*

Proof. See [Kac90, Exercise 2.1]. □

We next give a criterion yielding free Lie subalgebras of $\mathfrak{n}_+^{\mathrm{im}}$ of the form $\bigoplus_{k\geq 1}\mathfrak{g}_{k\gamma}$ for some $\gamma \in \Delta_+^{\mathrm{im}}$, in the general case.

Lemma 4.28. *Let A be an arbitrary generalised Cartan matrix. Assume that there exist real roots $\alpha,\beta \in \Delta^{\mathrm{re}}$ with $\alpha-\beta \notin \Delta$ and such that if $m := \langle \beta,\alpha^\vee\rangle$ and $n := \langle\alpha,\beta^\vee\rangle$, then $m,n \leq -2$ and $mn > 4$. Let $\mathfrak{g}_{\alpha,\beta}$ be the Lie subalgebra of $\mathfrak{g} = \mathfrak{g}(A)$ generated by $\mathfrak{g}_{\pm\alpha}\oplus\mathfrak{g}_{\pm\beta}$. Then the Lie subalgebra $\oplus_{k\geq 1}\mathfrak{g}_{k(\alpha+\beta)}$ of $\mathfrak{g}_{\alpha,\beta}$ is free non-abelian.*

Proof. Let $B = (b_{ij})$ be the 2×2 (hence symmetrisable) generalised Cartan matrix defined by $B = \left(\begin{smallmatrix} 2 & m \\ n & 2\end{smallmatrix}\right)$, and let $\tilde{\mathfrak{g}}(B)$ be the Lie algebra generated by the symbols $\tilde{e}_i, \tilde{f}_i, \tilde{\alpha}_i^\vee$, $i = 1,2$, subject to the relations

$$[\tilde{\alpha}_1^\vee,\tilde{\alpha}_2^\vee] = 0, \quad [\tilde{e}_i,\tilde{f}_j] = -\delta_{ij}\tilde{\alpha}_i^\vee, \quad [\tilde{\alpha}_i^\vee,\tilde{e}_j] = b_{ij}\tilde{e}_j \quad \text{and} \quad [\tilde{\alpha}_i^\vee,\tilde{f}_j] = -b_{ij}\tilde{f}_j.$$

Thus $\mathfrak{g}(B) = \tilde{\mathfrak{g}}(B)/\mathfrak{i}$, where $\mathfrak{i}$ is the unique maximal ideal of $\tilde{\mathfrak{g}}(B)$ intersecting $\mathbb{C}\tilde{\alpha}_1^\vee + \mathbb{C}\tilde{\alpha}_2^\vee$ trivially, or else the ideal of $\tilde{\mathfrak{g}}(B)$ defined by the Serre relations $(\operatorname{ad}\tilde{e}_i)^{1-b_{ij}}\tilde{e}_j = (\operatorname{ad}\tilde{f}_i)^{1-b_{ij}}\tilde{f}_j = 0$ (see Proposition 4.14).

Fix nonzero vectors $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ (respectively, $x_{\pm\beta}\in\mathfrak{g}_{\pm\beta}$) such that $[x_\alpha,x_{-\alpha}] = -\alpha^\vee$ (respectively, $[x_\beta,x_{-\beta}] = -\beta^\vee$). We claim that there is a surjective Lie algebra homomorphism $\pi\colon \tilde{\mathfrak{g}}(B)\to\mathfrak{g}_{\alpha,\beta}$ defined on the generators by

$$\pi(\tilde{e}_1) = x_\alpha,\ \pi(\tilde{e}_2) = x_\beta,\ \pi(\tilde{f}_1) = x_{-\alpha},\ \pi(\tilde{f}_2) = x_{-\beta},\ \pi(\tilde{\alpha}_1^\vee) = \alpha^\vee,\ \pi(\tilde{\alpha}_2^\vee) = \beta^\vee.$$

Indeed, using the fact that $\alpha-\beta\notin\Delta$, it is a routine exercise to check that the defining relations of $\tilde{\mathfrak{g}}(B)$ are satisfied in $\mathfrak{g}_{\alpha,\beta}$.

Let $\tilde{\alpha}_1,\tilde{\alpha}_2$ denote the simple roots associated to B. Let also $(\cdot|\cdot)_B$ denote the bilinear form on $\mathbb{C}\tilde{\alpha}_1+\mathbb{C}\tilde{\alpha}_2$ associated to the symmetrisation

$$B = \begin{pmatrix} -1/n & 0 \\ 0 & -1/m\end{pmatrix}\begin{pmatrix} -2n & -mn \\ -mn & -2m\end{pmatrix}$$

of B. Then

$$(\tilde{\alpha}_1+\tilde{\alpha}_2|\tilde{\alpha}_1+\tilde{\alpha}_2)_B = -2(m+n+mn) < 0.$$

Thus $\tilde{\alpha}_1+\tilde{\alpha}_2$ is an imaginary root of $\mathfrak{g}(B)$ by Proposition 4.17, and the Lie subalgebra $\oplus_{k\geq1}\mathfrak{g}(B)_{k(\tilde{\alpha}_1+\tilde{\alpha}_2)}$ of $\mathfrak{g}(B)$ is free non-abelian by Proposition 4.25. Lifting this subalgebra in $\tilde{\mathfrak{g}}(B)$ and applying π, we conclude that the Lie subalgebra $\oplus_{k\geq1}\mathfrak{g}_{k(\alpha+\beta)}$ of $\mathfrak{g}$ is free non-abelian since $\ker\pi\subseteq\mathfrak{i}$. □

Finally, we give an existence result in the non-symmetrisable case.

Lemma 4.29. *Assume that $A = (a_{ij})$ is not symmetrisable. Then there exist real roots $\alpha, \beta \in \Delta_+^{\mathrm{re}}$ with $\alpha - \beta \notin \Delta$ and such that $\langle \alpha, \beta^\vee \rangle \leq -3$ and $\langle \beta, \alpha^\vee \rangle \leq -2$.*

Proof. By Lemma 4.27, we may assume up to re-indexing the set of simple roots of A that $a_{12}a_{23}\dots a_{n-1,n}a_{n,1} \neq 0$ for some $n \geq 3$, with $a_{12} \leq -2$. Let $\alpha_1, \dots, \alpha_n$ be the corresponding simple roots.

Assume first that $n > 3$. For each $i = 1, \dots, n-1$, we define a real root β_i of $\mathfrak{g} = \mathfrak{g}(A)$ as follows:

$$\beta_i = \begin{cases} r_{\alpha_{n-1}}(\alpha_n) = \alpha_n - a_{n-1,n}\alpha_{n-1} & \text{if } i = n-1, \\ \alpha_i & \text{otherwise.} \end{cases}$$

We claim that the $(n-1) \times (n-1)$ matrix with (i,j)-th entry $b_{ij} = \langle \beta_j, \beta_i^\vee \rangle$ is a generalised Cartan matrix such that $b_{12}b_{23}\dots b_{n-2,n-1}b_{n-1,1} \neq 0$ and $b_{12} = a_{12} \leq -2$.

Indeed, for $i = 1, \dots, n-2$ we get

$$b_{i,n-1} = \langle \alpha_n - a_{n-1,n}\alpha_{n-1}, \alpha_i^\vee \rangle = a_{i,n} - a_{n-1,n}a_{i,n-1}$$

and

$$b_{n-1,i} = \langle \alpha_i, \alpha_n^\vee - a_{n,n-1}\alpha_{n-1}^\vee \rangle = a_{n,i} - a_{n,n-1}a_{n-1,i}.$$

Hence $b_{i,n-1}$ and $b_{n-1,i}$ are nonpositive for each $i = 1, \dots, n-2$ and $b_{i,n-1} = 0 \Leftrightarrow b_{n-1,i} = 0$. Moreover, setting $i = 1$ (resp. $i = n-2$) we get that $b_{n-1,1} < 0$ (resp. $b_{n-2,n-1} < 0$), whence the claim.

Thus, repeating this process inductively, one eventually gets real roots $\gamma_1, \gamma_2, \gamma_3$ of $\mathfrak{g}$, where $\gamma_1 = \alpha_1$, $\gamma_2 = \alpha_2$ and $\gamma_3 = r_{\alpha_3}\dots r_{\alpha_{n-1}}(\alpha_n) \in \oplus_{i=3}^n \mathbb{N}\alpha_i$, such that the 3×3 matrix with (i,j)-th entry $c_{ij} = \langle \gamma_j, \gamma_i^\vee \rangle$ is a generalised Cartan matrix and moreover $c_{12}c_{23}c_{31} \neq 0$ and $c_{12} \leq -2$.

Set $\alpha = r_{\gamma_2}(\gamma_3) = \gamma_3 - c_{23}\gamma_2$ and $\beta = \gamma_1$. Then clearly α and β are positive real roots of $\mathfrak{g}$ such that $\alpha - \beta = -\gamma_1 - c_{23}\gamma_2 + \gamma_3 \notin \Delta$.

Moreover, since $\alpha^\vee = \gamma_3^\vee - c_{32}\gamma_2^\vee$ and $\beta^\vee = \gamma_1^\vee$, we get

$$\langle \alpha, \beta^\vee \rangle = c_{13} - c_{23}c_{12} \leq -3$$

and

$$\langle \beta, \alpha^\vee \rangle = c_{31} - c_{32}c_{21} \leq -2,$$

since $c_{ij} \leq -1$ for $i \neq j$ and $c_{12} \leq -2$. This concludes the proof of the lemma. □

Theorem 4.30. *Let A be an arbitrary indecomposable generalised Cartan matrix. Then exactly one of the following holds.*

(1) A is of spherical type and $\mathfrak{n}_+^{\mathrm{im}} = \{0\}$.

(2) A is of affine type and $\mathfrak{n}_+^{\mathrm{im}}$ *is abelian and infinite-dimensional.*

(3) A is of indefinite type and $\mathfrak{n}_+^{\mathrm{im}}$ *contains a nonabelian free Lie subalgebra.*

Proof. This readily follows from Corollary 4.26 together with Lemmas 4.28 and 4.29 since generalised Cartan matrices of affine type are symmetrisable (see [Kac90, Lemma 4.6]). □

We conclude this section by giving a geometric version of this result. Fix as before a generalised Cartan matrix A and a realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A with set of simple roots $\Pi = \{\alpha_i \mid i \in I\}$. Let W be the associated Weyl group and let Σ be its Coxeter complex. Recall that the set of half-spaces of Σ corresponds to the set of real roots Δ^{re} of $\mathfrak{g}(A)$: $\Delta^{\mathrm{re}} = W.\Pi = \Phi(\Sigma)$, where we identify the two notions of simple roots. Let C_0 denote the fundamental chamber of Σ.

We have the following dictionary.

Proposition 4.31. *Let* α, β *be two real roots of* $\mathfrak{g}(A)$*, either viewed as half-spaces in* Σ *or as elements of* $\mathfrak{h}^*$*. Then:*

(1) α *contains* C_0 *if and only if* $\alpha \in \Delta_+^{\mathrm{re}}$.

(2) $\partial\alpha$ *and* $\partial\beta$ *are parallel and distinct if and only if* $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \geq 4$.

(3) $\alpha \subsetneq -\beta$ *or* $-\alpha \subsetneq \beta$ *if and only if* $m := \langle \alpha, \beta^\vee \rangle$ *and* $n := \langle \beta, \alpha^\vee \rangle$ *are such that* $m, n < 0$ *and* $mn \geq 4$.

(4) If $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 4$*, then* $\mathrm{Pc}(r_\alpha, r_\beta)$ *is an affine Coxeter group.*

Proof. If α is a simple root then (1) follows from the identification of the two notions of simple roots. Moreover, given a root γ in Σ containing C_0, the root $r_i(\gamma)$ does not contain C_0 anymore if and only $\partial\gamma$ separates C_0 from $r_i(C_0)$, that is, if and only if $\gamma = \alpha_i$. Thus (1) follows from an induction on $\ell(w)$ where $\alpha = w\alpha_i$ for some i, using Lemma 4.8 (2).

Since the walls $\partial\alpha$ and $\partial\beta$ are parallel and distinct if and only if the product $r_\alpha r_\beta$ has infinite order in W, statement (2) follows from Proposition 4.12 (3) (the roots considered in this proposition are actually simple roots, but the proof is exactly the same for arbitrary real roots, see [Kac90, proof of Proposition 3.13]).

To prove (3), we may assume using the W-action that $\beta = \alpha_j$ is a simple root. Then $\alpha \subsetneq -\beta$ if and only if $\partial\alpha$ and $\partial\beta$ are parallel and distinct, α does not contain C_0 and $r_\alpha(\beta)$ does not contain C_0 either (use the fact that for two distinct parallel walls $\partial\gamma$ and $\partial\delta$, exactly one of the possibilities $\gamma \subseteq \delta$ or $\gamma \subseteq -\delta$ or $-\gamma \subseteq \delta$ or $-\gamma \subseteq -\delta$ holds). Since α and $r_\alpha(\beta) = \alpha_j - m\alpha$ do not contain C_0 if and only if they are negative by (1), which only occurs for $m < 0$, statement (3) then follows from (2).

Finally, we prove (4). Note that setting $\beta_1 := r_\beta(\alpha)$, one has

$$\langle \alpha, \beta_1^\vee \rangle = \langle \beta_1, \alpha^\vee \rangle = 2 - \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = -2.$$

Thus, since $\mathrm{Pc}(r_\alpha, r_\beta) = \mathrm{Pc}(r_\alpha, r_{\beta_1})$ by Lemma 1.31, we may assume up to replacing β by $r_\beta(\alpha)$ that $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -2$. Note that since $\gamma := r_\alpha(\beta) = \beta + 2\alpha$, Proposition 4.16 (2) implies that $\alpha + \beta \in \Delta$. Similarly, since $r_\beta(\gamma) = \gamma + 2\beta$, one also knows that $\gamma + \beta = 2(\alpha + \beta) \in \Delta$. Then Proposition 4.16 (1) implies that $\alpha + \beta \in \Delta^{\mathrm{im}}$. Up to replacing α and β by their opposite root, we may assume that $\alpha + \beta \in \Delta_+^{\mathrm{im}}$. Using the W-action, we may then moreover assume by Proposition 4.17 (2) that $\langle \alpha + \beta, \alpha_i^\vee \rangle \leq 0$ for all $i \in I$.

Note that for a real root $\gamma \in \Delta^{\mathrm{re}}$, one has $\mathrm{supp}(\gamma) = \mathrm{supp}(\gamma^\vee)$. Indeed, this follows from the fact that $\mathrm{supp}(\gamma) \subseteq J$ if and only if there exists a decomposition $\gamma = w\alpha_i$ with $i \in J$ and $w \in W_J := \langle r_j \mid j \in J \rangle$, and similarly for $\mathrm{supp}(\gamma^\vee)$. (This last statement is obtained by induction on $|\,\mathrm{ht}(\gamma)|$ using Proposition 4.16 (4)).

Let $P := \mathrm{supp}(\alpha)$, $Q := \mathrm{supp}(\beta)$, and $J := P \cup Q \subseteq I$. Write $\alpha^\vee = \sum_{k \in P} \lambda_k \alpha_k^\vee$, where the integers λ_k are nonzero and of the same sign. Then

$$0 = \langle \alpha + \beta, \alpha^\vee \rangle = \sum_{k \in P} \lambda_k \langle \alpha + \beta, \alpha_k^\vee \rangle$$

and hence $\langle \alpha + \beta, \alpha_k^\vee \rangle = 0$ for all $k \in P$. Using the same argument for $\beta^\vee$, we thus have that $\langle \alpha + \beta, \alpha_k^\vee \rangle = 0$ for all $k \in J$. Let A_J be the submatrix of A whose rows and columns are indexed by J. Note that A_J is indecomposable by Proposition 4.18 (1). Viewing the positive root $\delta := \alpha + \beta$ as a column vector in the basis $\{\alpha_i \mid i \in J\}$, we then have $A_J \delta = 0$ and $\delta > 0$, and thus A_J is of affine type by Proposition 4.15.

Let W_J be the associated affine Coxeter group, that is, the subgroup of W generated by the reflections r_j with $j \in J$. Since $r_\alpha, r_\beta \in W_J$, it follows that $\mathrm{Pc}(r_\alpha, r_\beta) \leq W_J$. Since $\mathrm{Pc}(r_\alpha, r_\beta)$ is infinite by (2) and irreducible by Lemma 1.20, it is also affine, as desired. □

Theorem 4.32. *Let $\alpha \subsetneq -\beta$ be two nested roots in Σ. Assume that $\mathrm{Pc}(r_\alpha, r_\beta)$ is not affine. Then there is some $\delta \in \{\alpha + r_\beta(\alpha), \beta + r_\alpha(\beta)\}$ such that the Lie algebra $\oplus_{k \geq 1} \mathfrak{g}_{k\delta}$ is free non-abelian.*

Proof. Setting $m := \langle \alpha, \beta^\vee \rangle$ and $n := \langle \beta, \alpha^\vee \rangle$, we know by Proposition 4.31 (3) and (4) that $m, n < 0$ and $mn > 4$. In particular, $(m, n) \neq (-1, -1)$. We deal with the case $m \leq -2$, the case $n \leq -2$ being similar. Choose $\delta = \alpha + r_\beta(\alpha)$. Set $\gamma := r_\beta(\alpha) = \alpha - m\beta$. Then

$$\langle \gamma, \alpha^\vee \rangle = \langle \alpha, \gamma^\vee \rangle = 2 - mn < -2$$

and moreover $\alpha - \gamma = m\beta \notin \Delta$ by Proposition 4.16 (1). We are thus in a position to apply Lemma 4.28, yielding the claim. □

4.4 Kac–Moody algebras in positive characteristic are restricted

Let $\mathfrak{g} = \mathfrak{g}_A$ be a (derived) Kac–Moody algebra and let k be a field of positive characteristic p. We consider as in Section 4.2 the $\mathbb{Z}$-form $\mathcal{U}$ of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ and the Lie algebra $\mathfrak{g}_k = \mathfrak{g}_{\mathbb{Z}} \otimes k$ over k. In this section, we establish that the Lie algebra $\mathfrak{g}_k$ is restricted, and we give explicit recursive formulas for computing its p-operation.

We begin by recalling the notion of a *restricted Lie algebra*, which was first introduced by N. Jacobson [Jac37] in 1937 (see also [Jac79, V §7 Definition 4]). Informally, a Lie algebra over k is restricted if it is equipped with a *p-power* operator possessing formal properties derived from those of the corresponding operator $x \mapsto x^p$ of an associative algebra.

Definition 4.33. Let $\mathfrak{L}$ be a Lie algebra over a field k of characteristic $p > 0$. Then $\mathfrak{L}$ is a **restricted Lie algebra** if there exists a **p-operation** on $\mathfrak{L}$, that is, a map $\mathfrak{L} \to \mathfrak{L} : X \mapsto X^{[p]}$ such that

(R1) $\mathrm{ad}(X^{[p]}) = \mathrm{ad}(X)^p$ for all $X \in \mathfrak{L}$,

(R2) $(tX)^{[p]} = t^p X^{[p]}$ for all $t \in k$ and $X \in \mathfrak{L}$,

(R3) $(X+Y)^{[p]} = X^{[p]} + Y^{[p]} + \sum_{i=1}^{p-1} s_i(X,Y)/i$ for all $X, Y \in \mathfrak{L}$, where $s_i(X,Y)$ is the coefficient of t^{i-1} in the formal expression $\mathrm{ad}(tX+Y)^{p-1}(X)$.

The **restricted enveloping algebra** of $\mathfrak{L}$ is the quotient of its universal enveloping algebra by the two-sided ideal generated by elements of the form $X^p - X^{[p]}$.

The study of (finite-dimensional) simple Lie algebras in positive characteristic goes back to the 1930s. This study culminated in a complete classification of finite-dimensional restricted simple Lie algebras over an algebraically closed field k of characteristic $p > 7$ (see [BW88]), which was later extended to general (non-necessarily restricted) Lie algebras by H. Strade (see [Str04] and its sequels). The outcome of this classification is that all such restricted Lie algebras are of so-called classical or Cartan type, and one has to include in addition the algebras of so-called Melikian type if we drop the "restricted" assumption. The (restricted) Lie algebras of classical type are those of the form $\mathfrak{g}_k$ where $\mathfrak{g}$ is a Kac–Moody algebra of finite type. We now give a proof that $\mathfrak{g}_k$ is restricted for arbitrary Kac–Moody algebras.

We begin with two computational lemmas, which are valid in arbitrary associative $\mathbb{Z}$-algebras. For an algebra A with Lie bracket $[\cdot,\cdot]$, recall the notation

$$[x_1, x_2, \ldots, x_n] := \mathrm{ad}(x_1)\,\mathrm{ad}(x_2) \ldots \mathrm{ad}(x_{n-1})(x_n)$$

for all $x_1, \ldots, x_n \in A$, where $\mathrm{ad}(x)(y) := [x,y]$ is the adjoint representation. As usual, we will denote the omission of an element x by $\widehat{x}$; for instance $[x_1, \widehat{x_2}, x_3] :=$

$[x_1, x_3]$. Moreover, in a sum of the form

$$\sum_{1\leq r_1<\cdots<r_t\leq n} [x_1, \dots, \widehat{x_{r_j}}, \dots, x_n],$$

the free index j is understood as varying from 1 to t.

Lemma 4.34. *Let A be an associative $\mathbb{Z}$-algebra, and let x_i $(i \in \mathbb{N}^*)$ and y be elements of A. Then for all $i \geq 0$,*

$$x_1 \dots x_i y = \sum_{t=0}^{i} \sum_{1\leq r_1<\cdots<r_t\leq i} [x_1, \dots, \widehat{x_{r_j}}, \dots, x_i, y] x_{r_1} \dots x_{r_t},$$

where the terms for $i = 0$ and $t = 0$ should be respectively interpreted as y and $[x_1, \dots, x_i, y]$.

Proof. We prove the claim by induction on $i \geq 0$. If $i = 0$, there is nothing to prove. Assume now that $i \geq 1$. Then

$$\begin{aligned}
&\sum_{t=0}^{i} \sum_{1\leq r_1<\cdots<r_t\leq i} [x_1, \dots, \widehat{x_{r_j}}, \dots, x_i, y] x_{r_1} \dots x_{r_t} \\
&\quad = [x_1, \dots, x_i, y] + \sum_{t=1}^{i} \sum_{1\leq r_1<\cdots<r_t\leq i-1} [x_1, \dots, \widehat{x_{r_j}}, \dots, x_i, y] x_{r_1} \dots x_{r_t} \\
&\quad + \sum_{t=1}^{i} \sum_{1\leq r_1<\cdots<r_{t-1}\leq i-1} [x_1, ..., \widehat{x_{r_j}}, ..., x_{i-1}, y] x_{r_1} \dots x_{r_{t-1}} x_i \\
&\quad = \sum_{t=0}^{i-1} \sum_{1\leq r_1<\cdots<r_t\leq i-1} [x_1, \dots, \widehat{x_{r_j}}, \dots, x_i, y] x_{r_1} \dots x_{r_t} \\
&\quad + \sum_{t=0}^{i-1} \sum_{1\leq r_1<\cdots<r_t\leq i-1} [x_1, ..., \widehat{x_{r_j}}, ..., x_{i-1}, y] x_{r_1} \dots x_{r_t} x_i \\
&\quad = x_1 \dots x_{i-1} [x_i, y] + x_1 \dots x_{i-1} y x_i \\
&\quad = x_1 \dots x_i y,
\end{aligned}$$

as desired. □

Lemma 4.35. *Let A be an associative $\mathbb{Z}$-algebra, and let $(b_i)_{i\geq 1}$ be a sequence of elements of A. Then for all $n \geq 1$,*

$$\sum_{\sigma\in\mathrm{Sym}(n+1)} b_{\sigma(1)} \dots b_{\sigma(n+1)} = \sum_{k=0}^{n} \tbinom{n+1}{k+1} \sum_{\sigma\in\mathrm{Sym}(n)} [b_{\sigma(1)}, \dots, b_{\sigma(k)}, b_{n+1}] b_{\sigma(k+1)} \dots b_{\sigma(n)}.$$

Proof. Applying Lemma 4.34 $n+1$ times, we get

$$\begin{aligned}
\sum_{\sigma\in\mathrm{Sym}(n+1)} b_{\sigma(1)}\dots b_{\sigma(n+1)} &= \sum_{\sigma\in\mathrm{Sym}(n)}\sum_{i=0}^{n} b_{\sigma(1)}\dots b_{\sigma(i)}b_{n+1}b_{\sigma(i+1)}\dots b_{\sigma(n)}\\
&= \sum_{\sigma\in\mathrm{Sym}(n)}\sum_{i=0}^{n}\sum_{t=0}^{i}\sum_{1\leq r_1<\dots<r_t\leq i}[b_{\sigma(1)},\dots,\widehat{b_{\sigma(r_j)}},\dots,b_{\sigma(i)},b_{n+1}]b_{\sigma(r_1)}\dots b_{\sigma(r_t)}b_{\sigma(i+1)}\dots b_{\sigma(n)}\\
&= \sum_{i=0}^{n}\sum_{t=0}^{i}\sum_{1\leq r_1<\dots<r_t\leq i}\sum_{\sigma\in\mathrm{Sym}(n)}[b_{\sigma(1)},\dots,b_{\sigma(i-t)},b_{n+1}]b_{\sigma(i-t+1)}\dots b_{\sigma(n)}\\
&= \sum_{i=0}^{n}\sum_{t=0}^{i}\binom{i}{t}\sum_{\sigma\in\mathrm{Sym}(n)}[b_{\sigma(1)},\dots,b_{\sigma(i-t)},b_{n+1}]b_{\sigma(i-t+1)}\dots b_{\sigma(n)}.
\end{aligned}$$

Setting $k=i-t$ and using the identity $\sum_{t=0}^{n-k}\binom{t+k}{t}=\binom{n+1}{k+1}$, we deduce that

$$\begin{aligned}
\sum_{\sigma\in\mathrm{Sym}(n+1)} b_{\sigma(1)}\dots b_{\sigma(n+1)} &= \sum_{k=0}^{n}\sum_{t=0}^{n-k}\binom{t+k}{t}\sum_{\sigma\in\mathrm{Sym}(n)}[b_{\sigma(1)},\dots,b_{\sigma(k)},b_{n+1}]b_{\sigma(k+1)}\dots b_{\sigma(n)}\\
&= \sum_{k=0}^{n}\binom{n+1}{k+1}\sum_{\sigma\in\mathrm{Sym}(n)}[b_{\sigma(1)},\dots,b_{\sigma(k)},b_{n+1}]b_{\sigma(k+1)}\dots b_{\sigma(n)},
\end{aligned}$$

as desired. □

Corollary 4.36. *Let A be an associative $\mathbb{Z}$-algebra. Let p be a prime number and let $b_1,\dots,b_p\in A$. Then*

$$\sum_{\sigma\in\mathrm{Sym}(p)} b_{\sigma(1)}\dots b_{\sigma(p)} \equiv \sum_{\sigma\in\mathrm{Sym}(p-1)}[b_{\sigma(1)},\dots,b_{\sigma(p-1)},b_p] \mod pA.$$

Proof. This follows from Lemma 4.35 with $n+1=p$. □

We next apply the above lemmas to our (derived) Kac–Moody algebra $\mathfrak{g}=\mathfrak{g}_A$, where as usual, we have fixed a realisation of $A=(a_{ij})_{i,j\in I}$, and defined all notions and notations as in the previous sections of this chapter.

Lemma 4.37. *Let $\beta\in\Delta_+$ and let $x\in\mathfrak{g}_{\beta\mathbb{Z}}$. Let $y=(\mathrm{ad}(e_i)^k/k!)(x)\in\mathfrak{g}_{\gamma\mathbb{Z}}$ for some $i\in I$ and $k\in\mathbb{N}^*$, where $\gamma=\beta+k\alpha_i$. If $x^p\in\mathfrak{g}_{\mathbb{Z}}+p\mathcal{U}$, then $y^p\in\mathfrak{g}_{\mathbb{Z}}+p\mathcal{U}$ as well.*

Proof. Note first that we may assume β and α_i to be linearly independant for otherwise $\beta=\alpha_i$ and thus $y=0$. We work inside the algebra $\widehat{\mathcal{U}}^+_{\mathbb{C}}=\prod_{\alpha\in Q_+}\mathcal{U}^+_{\alpha\mathbb{C}}$, and for $z\in\mathfrak{n}^+$, we write $\exp z=\sum_{n\in\mathbb{N}}z^{(n)}$, where as before $z^{(n)}:=z^n/n!$.

Recall from Lemma 4.9 (2) that

$$(\exp e_i).(\exp x).(\exp -e_i) = \exp(\sum_{m\in\mathbb{N}} (\mathrm{ad}(e_i)^m/m!)(x)).$$

Let us identify the terms of weight $p\gamma$ in both sides of this relation. On the left-hand side, we find $\sum_{r+s=pk}(-1)^s e_i^{(r)} x^{(p)} e_i^{(s)}$. On the right-hand side, this corresponds to the terms of weight $p\gamma$ in $(\sum_{m=0}^{pk}(\mathrm{ad}(e_i)^m/m!)(x))^p/p!$. Let I_k denote the set of $pk{+}1$-tuples $\overline{j} = (j_0, \ldots, j_{pk})$ of nonnegative integers such that $\sum_{t=0}^{pk} j_t = p-1$ and $\sum_{t=1}^{pk} tj_t = pk$. Then multiplying both sides of the relation by $p!$, we get

$$\sum_{r+s=pk}(-1)^s e_i^{(r)} x^p e_i^{(s)} = y^p + \sum_{\overline{j}\in I_k} S_p(\underbrace{a_0,\ldots,a_0}_{j_0}, \underbrace{a_1,\ldots,a_1}_{j_1}, \ldots, \underbrace{a_{pk},\ldots,a_{pk}}_{j_{pk}}, x), \tag{4.3}$$

where $a_m = (\mathrm{ad}(e_i)^m/m!)(x)$ and $S_p(b_1,\ldots,b_p) = \sum_{\sigma\in\mathrm{Sym}(p)} b_{\sigma(1)}\ldots b_{\sigma(p)}$. Since

$$\sum_{r+s=pk}(-1)^s e_i^{(r)} x^p e_i^{(s)} = (\mathrm{ad}(e_i)^{pk}/(pk)!)(x^p) \tag{4.4}$$

by Lemma 4.9 (1) and since $\mathrm{ad}(e_i)^m/m!$ preserves $\mathcal{U}$, the conclusion follows from Corollary 4.36. □

Corollary 4.38. *Let y be a homogeneous element of $\mathfrak{n}_{\mathbb{Z}}^+$. Then $y^p \in \mathfrak{g}_{\mathbb{Z}} + p\mathcal{U}$.*

Proof. This readily follows from Lemma 4.37 together with the fact that $\mathfrak{n}_{\mathbb{Z}}^+$ is generated as a $\mathcal{U}$-module by the elements e_i for $i \in I$ (see Lemma 4.23 (1)). □

Theorem 4.39. *Let k be a field of characteristic $p > 0$. Then*

(1) $\mathfrak{g}_k$ is a restricted Lie algebra.

(2) The p-operation on $\mathfrak{n}_k^+$ is given on homogeneous elements by $y \mapsto y^{[p]} := y^p$, and is defined on $\mathfrak{h}_k$ by setting $(h\otimes 1)^{[p]} := h\otimes 1$ for all $h \in \mathfrak{h}_{\mathbb{Z}}$.

(3) The restricted enveloping algebra of $\mathfrak{g}_k$ coincides with its universal enveloping algebra.

Proof. By [Jac79, Theorem 11 p190], it is sufficient to check the axiom (R1) on a k-basis of $\mathfrak{g}_k$ (and then extend the p-operation to the whole of $\mathfrak{g}_k$ using (R2) and (R3)). Since $\mathfrak{g}_k$ decomposes as a sum of $\mathfrak{n}_k^{\pm}$ and $\mathfrak{h}_k$, we may deal separately with each subalgebra. The p-operation on a basis of homogeneous elements of $\mathfrak{n}_k^+$ is given as in (2) using Corollary 4.38. Since $\mathrm{ad}(y)^p z = \sum_{i=0}^{p}\binom{p}{i}(-1)^i y^{p-i} z y^i = \mathrm{ad}(y^p)z$ for all $z \in \mathfrak{g}_k$, this clearly satisfies (R1). The case of $\mathfrak{n}_k^-$ is of course identical. Finally, the definition of the p-operation on $\mathfrak{h}_k$ given by (2) clearly satisfies (R1) since $\lambda^p = \lambda$ for all $\lambda \in \mathbb{F}_p$. □

We remark that this fact was already noticed, without a proof, in [Mat96, (1.4)]. The advantage of our approach is that it gives explicit (recursive) formulas to compute the p-operation (see the proof of Lemma 4.37).

Chapter 5

Minimal Kac–Moody groups

We are now ready to introduce the main objects of this second part, namely, the Kac–Moody groups. More precisely, we begin by introducing in this chapter the so-called *“minimal” Kac–Moody groups* and *Tits functors*, as defined by J. Tits ([Tit87]). We will then describe “maximal” versions of these groups in the next chapter. The general references for this chapter are [Tit87] and [Rém02, Chapters 6–9] (see also [Rou12] and references therein).

5.1 Construction of minimal Kac–Moody groups

5.1.1 Kac–Moody root data and associated tori. — Let $A = (a_{ij})_{i,j \in I}$ be a generalised Cartan matrix and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation of A, where $\Pi = \{\alpha_i \mid i \in I\}$ and $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$.

Recall that the above realisation of A is (up to isomorphism) uniquely determined by A. We now introduce other kinds of “realisations” of A, which are less restrictive, and which do not only depend on A anymore.

Definition 5.1. A **Kac–Moody root datum** is a quintuple

$$\mathcal{D} = (I, A, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I}),$$

where I is a set indexing a generalised Cartan matrix $A = (a_{ij})$, Λ is a free $\mathbb{Z}$-module of finite rank (whose $\mathbb{Z}$-dual will be denoted $\Lambda^\vee$) and the elements c_i of Λ and h_i of $\Lambda^\vee$ are such that $\langle c_j, h_i \rangle = a_{ij}$ for all $i, j \in I$.

Example 5.2. We wish to encode the realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A as a particular Kac–Moody root datum $\mathcal{D} = (I, A, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ such that $\mathfrak{h} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathfrak{h}^* = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, and such that $c_i = \alpha_i \otimes 1$ and $h_i = \alpha_i^\vee \otimes 1$ for all $i \in I$ under this identification.

Set $I = \{1, \dots, n\}$ and let l denote the rank of A. Hence Λ and $\Lambda^\vee$ should be dual free $\mathbb{Z}$-modules of rank $2n - l$, say $\Lambda = \oplus_{i=1}^{2n-l} \mathbb{Z} u_i$ and $\Lambda^\vee = \oplus_{i=1}^{2n-l} \mathbb{Z} v_i$ for some dual bases $\{u_i\}$ and $\{v_i\}$. Note that $2n-l$ is the smallest possible dimension for $\mathfrak{h}$ under the requirement that Π and $\Pi^\vee$ both be free and that $\langle\alpha_j,\alpha_i^\vee\rangle = a_{ij}$ for all $i,j\in I$: the elements $\alpha_i^\vee$ and α_j can respectively be viewed as the first n basis vectors of $\mathbb{C}^{2n-l}$ and as the n linear forms on $\mathbb{C}^{2n-l}$ given by the n first columns of the invertible matrix

$$B=\begin{pmatrix} A_1 & A_2 & 0\\ A_3 & A_4 & \mathrm{Id}_{n-l}\\ 0 & \mathrm{Id}_{n-l} & 0\end{pmatrix},$$

where we have reordered the set I so that $A=\left(\begin{smallmatrix} A_1 & A_2\\ A_3 & A_4\end{smallmatrix}\right)$ with A_1 an $l\times l$ invertible matrix.

At the level of $\mathcal{D}$, the elements c_i (respectively, h_i) should be free in Λ (respectively, $\Lambda^\vee$); when this happens, the Kac–Moody root datum $\mathcal{D}$ is called **free** (respectively, **cofree**). Another property of $\mathcal{D}$ which will be required by S. Kumar and O. Mathieu when constructing their maximal Kac–Moody group (see §6.3) is that $\mathcal{D}$ should be **cotorsion-free**, that is, $\Lambda^\vee/\sum_{i=1}^n \mathbb{Z}h_i$ should be torsion-free.

All these requirements can for example be met by setting

$$h_i := v_i \quad\text{and}\quad c_i := \sum_{s=1}^{2n-l} B_{si}u_s = \sum_{s=1}^{n} a_{si}u_i + \delta_{l+1\le i\le n}u_{n-l+i}$$

for all $i\in I$, where $\delta_{l+1\le i\le n}$ equals 1 when $l+1\le i\le n$ and equals 0 otherwise. We will denote the Kac–Moody root datum obtained this way by $\mathcal{D}_{\mathrm{Kac}}$.

Example 5.3. Given a generalised Cartan matrix $A=(a_{ij})_{i,j\in I}$, there is a unique Kac–Moody root datum associated to A such that $\Lambda^\vee$ is freely generated by the h_i, that is, $\Lambda^\vee=\bigoplus_{i\in I}\mathbb{Z}h_i$. It is denoted $\mathcal{D}^A_{\mathrm{sc}}$ and is called the **simply connected** root datum associated to A. Note that it can be obtained from $\mathcal{D}_{\mathrm{Kac}}$ by taking for $\Lambda^\vee$ the submodule of $\Lambda^\vee_{\mathrm{Kac}}$ generated by the h_i and by restricting the elements c_i of the dual module Λ_{Kac} to $\Lambda^\vee$.

Given a Kac–Moody root datum $\mathcal{D}$ as above, one can then define the **Kac–Moody algebra $\mathfrak{g}_{\mathcal{D}}$ of type $\mathcal{D}$** as the Lie algebra generated by $\mathfrak{h}_{\mathcal{D}} := \Lambda^\vee\otimes_{\mathbb{Z}}\mathbb{C}$ and the symbols $\{e_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ subject to the relations (4.1) and (4.2) from Definitions 4.1 and 4.3, where we have to replace the symbols $\mathfrak{h}$, α_i, $\alpha_i^\vee$ in these definitions by the symbols $\mathfrak{h}_{\mathcal{D}}$, c_i, h_i, respectively.

Note that $\mathfrak{g}_{\mathcal{D}_{\mathrm{Kac}}}$ is just the Kac–Moody algebra $\mathfrak{g}(A)$ (of which $(\mathfrak{h}_{\mathcal{D}_{\mathrm{Kac}}},\{c_i\},\{h_i\})$ is a realisation), while $\mathfrak{g}_{\mathcal{D}^A_{\mathrm{sc}}}$ is its derived algebra $\mathfrak{g}_A=[\mathfrak{g}(A),\mathfrak{g}(A)]=\mathfrak{n}_-\oplus\mathfrak{h}_{\mathcal{D}^A_{\mathrm{sc}}}\oplus\mathfrak{n}_+$.

Remark 5.4. A few words to reassure the reader at this point that there will be no "parallel" theory of Kac–Moody algebras of type $\mathcal{D}$, and that the effort made in going through the previous chapter was not in vain.

As we will see, the point in introducing Kac–Moody root data is that we will be able to associate to *any* such datum $\mathcal{D}$ a Tits functor $\mathfrak{G}_{\mathcal{D}}$ (and hence Kac–Moody groups *of type* $\mathcal{D}$), and it is convenient to introduce Kac–Moody groups in this general setting. Note, however, that in the original results presented in this thesis we will only consider Kac–Moody groups of simply connected type (mainly to avoid technicalities), that is, Kac–Moody groups associated to a simply connected root datum $\mathcal{D}^A_{\mathrm{sc}}$, in which case $\mathfrak{g}_{\mathcal{D}}$ is the familiar object $\mathfrak{g}_A$.

When studying a Kac–Moody algebra of type $\mathcal{D}$ for an arbitrary Kac–Moody root datum $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$, one also considers a *root lattice* $Q := \oplus_{i\in I}\mathbb{Z}\alpha_i$ freely generated by symbols α_i, and a Q-gradation of $\mathfrak{g}_{\mathcal{D}}$ obtained by setting $\deg(h) = 0$ for all $h \in \mathfrak{h}_{\mathcal{D}}$, $\deg(e_i) = \alpha_i$ and $\deg(f_i) = -\alpha_i$. In other words, one can describe the homogeneous space $\mathfrak{g}_\alpha$ of $\mathfrak{g}_{\mathcal{D}}$ associated to the (say positive) weight $\alpha \in Q$ as the vector space generated by all $[e_{i_1}, \dots, e_{i_s}]$ such that $\alpha_{i_1} + \dots + \alpha_{i_s} = \alpha$, as in §4.1.1. This allows in particular to define as before a set of *roots* $\Delta \subset Q$, and this definition only depends on A.

This time, however, the homogeneous space $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$ need not correspond to a weight space for the adjoint action of $\mathfrak{h}_{\mathcal{D}}$. Indeed, as soon as $\mathcal{D}$ is not free, the **character map**

$$c\colon\ Q \to Q(\mathcal{D}) : \alpha \mapsto c_\alpha \quad \text{defined by } \alpha_i \mapsto c_i \text{ for all } i \in I,$$

where $Q(\mathcal{D}) := \sum_{i\in I} \mathbb{Z}c_i \subseteq \Lambda$, is not injective, so that the inclusion

$$\mathfrak{g}_\alpha \subset \mathfrak{g}_{c_\alpha} := \{x \in \mathfrak{g}_{\mathcal{D}} \mid [h, x] = c_\alpha(h)x \ \forall h \in \mathfrak{h}_{\mathcal{D}}\}$$

is in general strict for $\alpha \in \Delta \cup \{0\}$. Of course, if one wants $\mathfrak{g}_\alpha$ to correspond to a root space for the adjoint action of $\mathfrak{h}_{\mathcal{D}}$, one can just modify $\mathfrak{g}_{\mathcal{D}}$ in degree zero by enlarging $\mathfrak{h}_{\mathcal{D}}$ (or equivalently, Λ and $\Lambda^\vee$), in the same way one would pass from the Kac–Moody root datum $\mathcal{D}^A_{\mathrm{sc}}$ to $\mathcal{D}_{\mathrm{Kac}}$ (see [Rém02, 7.3.2] for precise statements). When the character map is injective – that is, when $\mathcal{D}$ is free –, we can (and we will) identify the root lattice Q with a subset of Λ and the root $\alpha \in Q$ with its image c_α.

One can also define the $\mathbb{Z}$-form $\mathcal{U}_{\mathcal{D}}$ of the universal enveloping algebra of $\mathfrak{g}_{\mathcal{D}}$, which is defined exactly as in §4.2.1 by replacing $\mathfrak{h}'$ with $\mathfrak{h}_{\mathcal{D}}$. Thus, the $\mathbb{Z}$-form $\mathcal{U}$ defined in §4.2.1 is just $\mathcal{U}_{\mathcal{D}^A_{\mathrm{sc}}}$.

To sum up, the algebras $\mathfrak{g}_{\mathcal{D}}$ and $\mathcal{U}_{\mathcal{D}}$ depend on $\mathcal{D}$ (and not just on A) only in degree zero: the set of roots Δ, the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$, the subalgebras $\mathcal{U}^+_{\mathcal{D}}$ and $\mathcal{U}^-_{\mathcal{D}}$, and so on, only depend on A. The reader who is not comfortable with this claim may safely always assume that the Kac–Moody root datum that is considered is of simply connected type, since, as we mentioned above, all our original results will take place in this particular framework.

Starting with an arbitrary Kac–Moody root datum $\mathcal{D}$, the first piece in the construction of a Kac–Moody group of type $\mathcal{D}$ is the following “torus” group scheme which, for a given generalised Cartan matrix A, encodes the specificities of the root datum $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$.

Definition 5.5. Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac–Moody root datum. Its associated **split torus scheme** $\mathfrak{T}_\Lambda$ is the group scheme over $\mathbb{Z}$ (viewed as a group functor over the category of $\mathbb{Z}$-algebras) defined by $\mathfrak{T}_\Lambda(R) = \Lambda^\vee \otimes_\mathbb{Z} R^\times$ for any ring R. Alternatively, $\mathfrak{T}_\Lambda = \mathrm{Spec}(\mathbb{Z}[\Lambda])$, that is, $\mathfrak{T}_\Lambda(R) = \mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], R) \approx \mathrm{Hom}_{\mathrm{gr}}(\Lambda, R^\times)$. It is a torus, isomorphic to $(\mathfrak{Mult})^n$, where $\mathfrak{Mult}(R) := R^\times$ and n is the rank of the free $\mathbb{Z}$-module Λ. For a given ring R, the isomorphism $\Lambda^\vee \otimes_\mathbb{Z} R^\times \approx \mathrm{Hom}_{\mathrm{gr}}(\Lambda, R^\times)$ is given by

$$\Lambda^\vee \otimes_\mathbb{Z} R^\times \to \mathrm{Hom}_{\mathrm{gr}}(\Lambda, R^\times) : h \otimes r \mapsto \Big[r^h \colon \Lambda \to R^\times : \lambda \mapsto r^h(\lambda) := r^{\langle \lambda, h\rangle}\Big].$$

Example 5.6. Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac–Moody root datum. Note that $\mathfrak{T}_\Lambda(R)$ is generated by the elements r^{h_i} for $r \in R^\times$ and $i \in I$ as soon as $\Lambda^\vee$ is generated by the h_i, for example when $\mathcal{D} = \mathcal{D}^A_{\mathrm{sc}}$.

Set now $\mathcal{D} = \mathcal{D}^A_{\mathrm{sc}}$, and consider the torus $\mathfrak{T}_{\Lambda_{\mathrm{Kac}}}$ associated to $\mathcal{D}_{\mathrm{Kac}}$. Then $\mathfrak{T}_{\Lambda_{\mathrm{Kac}}}(R)$ is generated by $\{r^{v_i} \mid r \in R^\times,\ 1 \le i \le 2n - l\}$ in the notations of Example 5.2, and one may view $\mathfrak{T}_\Lambda(R)$ as the subtorus of $\mathfrak{T}_{\Lambda_{\mathrm{Kac}}}(R)$ generated by all $r^{v_i} = r^{h_i}$ for $i = 1, \dots, n$.

5.1.2 Tits' construction of Tits functors. —

Starting from a Kac–Moody root datum $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ and its associated split torus scheme $\mathfrak{T}_\Lambda$, we now wish to define a group functor $\mathfrak{G}_\mathcal{D} \colon$ $\mathbb{Z}$-alg $\to$ Gr on the category of $\mathbb{Z}$-algebras that generalises the Chevalley–Demazure group schemes to this (infinite-dimensional) Kac–Moody setting. In particular, we wish to be able to recognise in $\mathfrak{G}_\mathcal{D}$ the split torus $\mathfrak{T}_\Lambda$, as well as subgroups isomorphic to SL_2 obtained by exponentiating the Lie subalgebras $\mathbb{C}e_i + \mathbb{C}[e_i, f_i] + \mathbb{C}f_i \cong \mathfrak{sl}_2$ of $\mathfrak{g}_A$ for $i \in I$.

More precisely, we consider systems of the form $\mathcal{F} = (\mathfrak{G}, (\varphi_i)_{i\in I}, \eta)$ consisting of a group functor $\mathfrak{G} \colon$ $\mathbb{Z}$-alg $\to$ Gr, of a collection $(\varphi_i)_{i\in I}$ of morphisms of functors $\varphi_i \colon \mathrm{SL}_2 \to \mathfrak{G}$, and of a morphism of functors $\eta \colon \mathfrak{T}_\Lambda \to \mathfrak{G}$. Here is a list of axioms one would like $\mathfrak{G}$ to satisfy (see [Tit87, §2]):

(KMG1) If $\mathbb{K}$ is a field, $\mathfrak{G}(\mathbb{K})$ is generated by the images of $\varphi_i(\mathbb{K})$ and $\eta(\mathbb{K})$.

(KMG2) For every ring R, the homomorphism $\eta(R) \colon \mathfrak{T}_\Lambda(R) \to \mathfrak{G}(R)$ is injective.

(KMG3) For $i \in I$ and $r \in R^\times$, one has $\varphi_i \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} = \eta(r^{h_i})$.

(KMG4) If $\iota \colon R \to \mathbb{K}$ is an injective homomorphism of a ring R in a field $\mathbb{K}$, then $\mathfrak{G}(\iota) \colon \mathfrak{G}(R) \to \mathfrak{G}(\mathbb{K})$ is injective.

(KMG5) There is a homomorphism $\mathrm{Ad} \colon \mathfrak{G}(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g}_A)$ whose kernel is contained in $\eta(\mathfrak{T}_\Lambda(\mathbb{C}))$, such that, for $c \in \mathbb{C}$,

$$\mathrm{Ad}\left(\varphi_i\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\right) = \exp \mathrm{ad}\, ce_i, \quad \mathrm{Ad}\left(\varphi_i\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right) = \exp \mathrm{ad}(-cf_i),$$

and, for $t \in \mathfrak{T}_\Lambda(\mathbb{C})$,

$$\mathrm{Ad}(\eta(t))(e_i) = t(c_i) \cdot e_i, \quad \mathrm{Ad}(\eta(t))(f_i) = t(-c_i) \cdot f_i.$$

Starting from the Kac–Moody root datum $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$, we now give the construction, following [Tit87, §3.6], of a functor $\mathfrak{G}_\mathcal{D}\colon$ $\mathbb{Z}$-alg $\to$ Gr, which will be in some sense "universal" amongst the functors $\mathfrak{G}$ that are part of a system $\mathcal{F} = (\mathfrak{G}, (\varphi_i)_{i\in I}, \eta)$ and that satisfy the conditions (KMG1) to (KMG5) above (see Proposition 5.9 below for a precise statement).

Denote as before by Δ^{re} the set of real roots of $\mathfrak{g} = \mathfrak{g}_A$, where as usual we have fixed a realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A with set of simple roots $\Pi = \{\alpha_i \mid i \in I\}$. We also let $W = W(A)$ be the associated Weyl group. To avoid any confusion as to how W should be considered, we write $\{s_i \mid i \in I\}$ for its generating set when W is viewed as an abstract Coxeter group, and we keep the notation $\{r_i \mid i \in I\}$ for its set of fundamental reflections when W is viewed as a subgroup of $\mathrm{GL}(\mathfrak{h})$ (and of course, s_i corresponds to r_i for each $i \in I$).

Recall from Definition 4.21 that a set $\Psi \subset \Delta^{\mathrm{re}}$ of real roots is prenilpotent if there exist $w, w' \in W$ such that $w\Psi \subset \Delta^{\mathrm{re}}_+$ and $w'\Psi \subset \Delta^{\mathrm{re}}_-$, and that such a set is necessarily finite. For a prenilpotent pair $\{\alpha, \beta\}$, the interval

$$[\alpha, \beta]_\mathbb{N} := (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta^{\mathrm{re}}$$

is always contained in the interval $[\alpha, \beta]$ defined in Section 2.5. Note, however, that the inclusion might be strict (see [Rém02, 6.2.4]).

For any $\alpha \in \Delta^{\mathrm{re}}$, let $\mathfrak{U}_\alpha$ denote the affine group scheme over $\mathbb{Z}$ isomorphic to $\mathbb{G}_a$ and with Lie algebra $\mathfrak{g}_{\alpha\mathbb{Z}} = \mathbb{Z}e_\alpha$ (see Lemma 4.24). Also, for a nilpotent set of roots Ψ, let U_Ψ denote the complex algebraic group whose Lie algebra is the direct sum $\bigoplus_{\gamma\in\Psi} \mathfrak{g}_\gamma$. The following lemma is [Tit87, Proposition 1].

Lemma 5.7. *Let Ψ be a nilpotent set of roots. Then there exists a unique group scheme $\mathfrak{U}_\Psi$ containing all $\mathfrak{U}_\gamma$ for $\gamma \in \Psi$, whose value on $\mathbb{C}$ is the group U_Ψ, and such that, for any order on Ψ, the product morphism $\prod_{\gamma\in\Psi} \mathfrak{U}_\gamma \to \mathfrak{U}_\Psi$ is an isomorphism of the underlying schemes.*

We next define the **Steinberg group functor** $\mathfrak{St}_A$, which depends only on A, as the inductive limit of the functors $\mathfrak{U}_\gamma$ and $\mathcal{U}_{[\alpha,\beta]_\mathbb{N}}$, where $\gamma \in \Delta^{\mathrm{re}}$ and $\{\alpha, \beta\}$ runs over all prenilpotent pairs of roots, relative to all canonical injections $\mathfrak{U}_\gamma \to \mathfrak{U}_{[\alpha,\beta]_\mathbb{N}}$ for $\gamma \in [\alpha, \beta]_\mathbb{N}$. Informally, $\mathfrak{St}_A(R)$ is thus obtained by taking a free product of the **root groups** $\mathfrak{U}_\gamma(R)$ and by adding the relations between them which one sees in the groups $\mathcal{U}_{[\alpha,\beta]_\mathbb{N}}(R)$ for $\{\alpha, \beta\}$ prenilpotent. It turns out that the canonical homomorphisms $\mathfrak{U}_\gamma(R) \to \mathfrak{St}_A(R)$ are injective, and one may thus identify each $\mathfrak{U}_\gamma(R)$ with its image in $\mathfrak{St}_A(R)$.

The identification of $\mathfrak{U}_\alpha$ with the functor $R \mapsto \{\exp(re_\alpha)\}_{r\in R}$ allows to define an action of the group W^* on $\mathfrak{St}_A$ by

$$w^* \cdot \{\exp(re_\alpha)\}_{r\in R} := \{\exp(rw^*e_\alpha)\}_{r\in R}$$

for all $w^* \in W^*$ and all rings R (see §4.2.5). Note also that we have an action of W on the torus $\mathfrak{T}_\Lambda(R) = \mathrm{Hom}_{\mathrm{gr}}(\Lambda, R^\times)$ defined on the generators s_i, $i \in I$, by

$$s_i(t)\colon\ \Lambda \to R^\times : \lambda \mapsto t(\lambda - \langle \lambda, h_i\rangle c_i)$$

for all $t \in \mathfrak{T}(R)$.

For each $i \in I$, denote by x_i (respectively, x_{-i}) the isomorphism $\mathbb{G}_a \xrightarrow{\sim} \mathfrak{U}_{\alpha_i}$ (respectively, $\mathbb{G}_a \xrightarrow{\sim} \mathfrak{U}_{-\alpha_i}$) induced by the choice of $e_i = e_{\alpha_i}$ (respectively, $f_i = e_{-\alpha_i}$) as basis element of $\mathfrak{g}_{\alpha_i}$ (respectively, $\mathfrak{g}_{-\alpha_i}$). Given a ring R, introduce for each $r \in R^\times$ and $i \in I$ the element $\tilde{s}_i(r) := x_i(r)x_{-i}(r^{-1})x_i(r)$ of $\mathfrak{St}_A(R)$ (remembering that each $\mathfrak{U}_\gamma(R)$ is identified with its image in $\mathfrak{St}_A(R)$) and set $\tilde{s}_i := \tilde{s}_i(1)$.

Definition 5.8. With the notations above, the **Tits functor of type $\mathcal{D}$** is the group functor $\mathfrak{G}_\mathcal{D}\colon\ \mathbb{Z}\text{-alg} \to \mathrm{Gr}$, such that for each ring R, the group $\mathfrak{G}_\mathcal{D}(R)$ is the quotient of the free product $\mathfrak{St}_A(R)*\mathfrak{T}_\Lambda(R)$ of $\mathfrak{St}_A(R)$ and $\mathfrak{T}_\Lambda(R) = \mathrm{Hom}_{\mathrm{gr}}(\Lambda, R^\times)$ by the following relations, where $i \in I$, $r \in R$, $t \in \mathfrak{T}_\Lambda(R)$, r^{h_i} is as in Definition 5.5, and where we identify each element with its canonical image in $\mathfrak{St}_A(R) * \mathfrak{T}_\Lambda(R)$:

$$t \cdot x_i(r) \cdot t^{-1} = x_i(t(c_i)r), \tag{5.1}$$
$$\tilde{s}_i \cdot t \cdot \tilde{s}_i^{-1} = s_i(t), \tag{5.2}$$
$$\tilde{s}_i(r^{-1}) = \tilde{s}_i \cdot r^{h_i} \qquad \text{for } r \neq 0, \tag{5.3}$$
$$\tilde{s}_i \cdot u \cdot \tilde{s}_i^{-1} = s_i^*(u) \quad \text{for } u \in \mathfrak{U}_\gamma(R), \quad \gamma \in \Delta^{\mathrm{re}}. \tag{5.4}$$

A **(split, minimal) Kac–Moody group** of type $\mathcal{D}$ over a field $\mathbb{K}$ is then the value on $\mathbb{K}$ of the Tits functor $\mathfrak{G}_\mathcal{D}$ of type $\mathcal{D}$.

We can now make the connection with the properties (KMG1) to (KMG5) introduced at the beginning of this paragraph. For any ring R, let $\mathfrak{U}^+(R)$ (respectively, $\mathfrak{U}^-(R)$) denote the subgroup of $\mathfrak{G}_\mathcal{D}(R)$ generated by all $\mathfrak{U}_\gamma(R)$ for γ a positive (respectively, negative) root. Let also x_+ (respectively, x_-) be the homomorphism $r \mapsto \left(\begin{smallmatrix}1 & r\\ 0 & 1\end{smallmatrix}\right)$ (respectively, $r \mapsto \left(\begin{smallmatrix}1 & 0\\ r & 1\end{smallmatrix}\right)$) of $\mathbb{G}_a$ in SL_2. We now state the main result of [Tit87].

Proposition 5.9. *Let $\mathcal{F} = (\mathfrak{G}, (\varphi_i)_{i\in I}, \eta)$ be a system which satisfies the axioms (KMG1) to (KMG5) for a given Kac–Moody root datum $\mathcal{D}$. Then:*

(1) *There exists a unique homomorphism of group functors $\pi\colon\ \mathfrak{G}_\mathcal{D} \to \mathfrak{G}$ such that the composed map $\mathfrak{T}_\Lambda \to \mathfrak{G}_\mathcal{D} \xrightarrow{\pi} \mathfrak{G}$ coincides with η and that the composed map $\mathbb{G}_a \xrightarrow{x_{\pm i}} \mathfrak{U}_{\pm\alpha_i} \to \mathfrak{G}_\mathcal{D} \xrightarrow{\pi} \mathfrak{G}$ is the same as $\varphi_i \circ x_\pm$.*

(2) *If $\mathbb{K}$ is a field, then $\pi(\mathbb{K})$ is an isomorphism unless $\varphi_i(\mathrm{SL}_2(\mathbb{K}))$ is contained in $\pi(\mathfrak{U}^+(\mathbb{K}))$ or in $\pi(\mathfrak{U}^-(\mathbb{K}))$ for some i.*

As mentioned by Tits [Tit87, §3.10 Remark (b)], by using highest-weight modules (which will be defined in Section 6.2) and the integral form $\mathcal{U}_\mathcal{D}$ of the enveloping algebra of $\mathfrak{g}_\mathcal{D}$, one can construct for any Kac–Moody root datum $\mathcal{D}$ a system $\mathcal{F} = (\mathfrak{G}, (\varphi_i)_{i\in I}, \eta)$ that satisfies all properties (KMG1) to (KMG5), as well as the additional non-degeneracy condition in the second point of Proposition 5.9 above. In particular, $\mathfrak{G}_\mathcal{D}$ satisfies all these properties, and these completely characterise the restriction of $\mathfrak{G}_\mathcal{D}$ to the category of fields.

5.1.3 The adjoint representation of a Kac–Moody group. — We now briefly mention a "functorial version" of the representation Ad from property (KMG5), as well as a more "down to earth" construction of the Steinberg group functor $\mathfrak{St}_A$ associated to a generalised Cartan matrix A.

Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac–Moody root datum, and fix as usual a realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A. We let $\mathcal{U} = \mathcal{U}_\mathcal{D}$ denote the $\mathbb{Z}$-form of the enveloping algebra of $\mathfrak{g} = \mathfrak{g}_\mathcal{D}$ (see Remark 5.4). Remember from §4.2.5 that for each real root $\alpha \in \Delta^{\mathrm{re}}$, we have an element e_α of $\mathfrak{g}$ such that $\mathfrak{g}_{\alpha\mathbb{Z}} = \mathbb{Z}e_\alpha$. Denote by $x_\alpha\colon\ \mathbb{G}_a \xrightarrow{\sim} \mathfrak{U}_\alpha$ the induced isomorphism, that is,

$$x_\alpha\colon\ \mathbb{G}_a(R) \xrightarrow{\sim} \mathfrak{U}_\alpha(R) : r \mapsto \exp(re_\alpha)$$

for all rings R. As before, we identify $\mathfrak{U}_\alpha$ with its canonical image in $\mathfrak{St}_A$.

We wish to define a natural transformation $\mathrm{Ad}\colon\ \mathfrak{G}_\mathcal{D} \to \mathrm{Aut}(\mathcal{U})$, which over $\mathbb{C}$ induces the map of the same name from property (KMG5). For this, we turn (a subgroup of) $\mathrm{Aut}(\mathcal{U})$ into a group functor by letting $\mathrm{Aut}_{\mathrm{filt}}(\mathcal{U})_R$ denote, for each ring R, the set of R-automorphisms $\alpha\colon\ \mathcal{U}_R \to \mathcal{U}_R$ such that α respects the filtration of $\mathcal{U}_R$ and preserves the ideal $\mathcal{U}_R^+$.

Recall from Remark 5.4 the definition of the character map

$$c\colon\ Q \twoheadrightarrow Q(\mathcal{D}) : \alpha \mapsto c_\alpha$$

mapping α_i to c_i for each $i \in I$, where $Q(\mathcal{D}) := \sum_{i\in I} \mathbb{Z}c_i \subseteq \Lambda$. The following can be found in [Rém02, 9.5.3].

Proposition 5.10. *There exists a natural transformation* $\mathrm{Ad}\colon\ \mathfrak{G}_\mathcal{D} \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U})$, *characterised by*

$$\mathrm{Ad}_R(x_\alpha(r)) = \exp(\mathrm{ad}\, e_\alpha \otimes r) = \sum_{n\geq 0} \frac{(\mathrm{ad}\, e_\alpha)^n}{n!} \otimes r^n,$$

$$\mathrm{Ad}_R(\mathfrak{T}_\Lambda(R)) \textit{ fixes } \mathcal{U}_R^0 \quad \textit{and} \quad \mathrm{Ad}_R(t)(e_\alpha \otimes r) = t(c_\alpha)\cdot(e_\alpha \otimes r),$$

for all rings R, for all $t \in \mathfrak{T}_\Lambda(R) = \mathrm{Hom}_{\mathrm{gr}}(\Lambda, R^\times)$, for all $\alpha \in \Delta^{\mathrm{re}}$ and for all $r \in R$.

This transformation Ad is called the **adjoint representation** of $\mathfrak{G}_{\mathcal{D}}$. Note that, since it preserves the filtration of $\mathcal{U}$, it stabilises $\mathfrak{g}$, and the induced action on $\mathfrak{g}_A$ over $\mathbb{C}$ coincides with the action described in property (KMG5).

Over a field $\mathbb{K}$, the kernel of $\mathrm{Ad}_{\mathbb{K}}$ coincides with the center of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$, which is contained in $\mathfrak{T}_\Lambda(\mathbb{K})$. More precisely, one has

$$\ker \mathrm{Ad}_{\mathbb{K}} = \mathscr{Z}(\mathfrak{G}_{\mathcal{D}}(\mathbb{K})) = \{t \in \mathfrak{T}_\Lambda(\mathbb{K}) \mid t(c_i) = 1 \text{ for all } i \in I\}.$$

Moreover, the image $\mathrm{Ad}_{\mathbb{K}}(\mathfrak{G}_{\mathcal{D}}(\mathbb{K}))$ of $\mathrm{Ad}_{\mathbb{K}}$ is contained in $\mathfrak{G}_{\mathrm{ad}(\mathcal{D})}(\mathbb{K})$ with equality when $\mathbb{K}$ is algebraically closed, where $\mathrm{ad}(\mathcal{D})$ is the **adjoint root datum** associated to $\mathcal{D}$, that is, the Kac–Moody root datum $\mathrm{ad}(\mathcal{D})$ obtained from $\mathcal{D}$ by replacing Λ with $\Lambda^{\mathrm{ad}} := \sum_{i\in I}\mathbb{Z}c_i$ and h_i with its restriction to Λ^{ad} (see [Rém02, 9.6.2]). The group $\mathrm{Ad}_{\mathbb{K}}(\mathfrak{G}_{\mathcal{D}}(\mathbb{K}))$ is called the **adjoint form** of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ (see [HKM12, Remark 5.3]). It is not to be confused with the notion of an **adjoint Kac–Moody group**, that is, a Kac–Moody group $\mathfrak{G}_{\mathrm{ad}(\mathcal{D})}(\mathbb{K})$ associated to an adjoint root datum. Note that the adjoint representation of an adjoint Kac–Moody group is injective.

Finally, we mention as promised a more concrete construction of the Steinberg group functor $\mathfrak{St}_A$ (see [Rém02, 9.2.2]).

Lemma 5.11. *Let $\{\alpha,\beta\} \subset \Delta^{\mathrm{re}}$ be a prenilpotent pair of distinct roots. Put an arbitrary order on the (finite) set $]\alpha,\beta[_{\mathbb{N}} := [\alpha,\beta]_{\mathbb{N}} \backslash \{\alpha,\beta\}$. Then there exist integers $C_{ij}^{\alpha\beta} \in \mathbb{Z}$ depending only on α,β and the chosen order, such that in $\mathcal{U}_{[\alpha,\beta]_{\mathbb{N}}}(\mathbb{Z}[[t,u]])$ one has*

$$[\exp(te_\alpha), \exp(ue_\beta)] = \prod_\gamma \exp(t^i u^j C_{ij}^{\alpha\beta} e_\gamma),$$

where $\gamma = i\alpha + j\beta$ runs through $]\alpha,\beta[_{\mathbb{N}}$ in the prescribed order.

For a ring R, $\mathfrak{St}_A(R)$ is then generated by the symbols $x_\alpha(r)$, $\alpha \in \Delta^{\mathrm{re}}$, $r \in R$, subject to the relations

$$x_\alpha(r + r') = x_\alpha(r) \cdot x_\alpha(r')$$

and

$$[x_\alpha(r), x_\beta(r')] = \prod_\gamma x_\gamma(C_{ij}^{\alpha\beta} r^i r'^j)$$

for $r, r' \in R$, $\{\alpha,\beta\}$ prenilpotent and $\gamma, C_{ij}^{\alpha\beta}$ as in Lemma 5.11.

5.2 The Kac–Peterson topology

In this section, we describe a possible topology on a minimal Kac–Moody group $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ over a topological field $\mathbb{K}$, which was introduced in [PK83] and further studied in [HKM12].

Let thus $\mathbb{K}$ be a Hausdorff topological field and let $G = \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ be a minimal Kac–Moody group over $\mathbb{K}$. Let $\varphi_i \colon \mathrm{SL}_2(\mathbb{K}) \to G$, $i \in I$, denote the group homomorphisms from §5.1.2. Thus, each $X_i := \varphi_i(\mathrm{SL}_2(\mathbb{K})) = \langle U_{\alpha_i}, U_{-\alpha_i}\rangle$ is an embedded copy of $\mathrm{SL}_2(\mathbb{K})$ in G, which we endow with the natural topology on $\mathrm{SL}_2(\mathbb{K})$ coming from the field $\mathbb{K}$.

The **Kac–Peterson topology** on G is by definition the finest topology such that for all $n \in \mathbb{N}$ and all $i_1, \dots, i_n \in I$, the multiplication map

$$X_{i_1} \times \cdots \times X_{i_n} \to G : (x_1, \dots, x_n) \mapsto x_1 \dots x_n$$

is continuous, where each product $X_{i_1} \times \cdots \times X_{i_n}$ has the product topology. Note that of course, the Kac–Peterson topology may as well be defined on the adjoint form of G.

Proposition 5.12 ([HKM12, Proposition 5.15 and Remark 5.16])**.** *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and assume that $G = \mathrm{Ad}_{\mathbb{K}}(\mathfrak{G}_{\mathcal{D}^A_{\mathrm{sc}}}(\mathbb{K}))$ is the adjoint form of a Kac–Moody group of simply connected type over $\mathbb{K}$. Then the Kac–Peterson topology on G turns G into a connected Hausdorff topological group. Moreover, each X_i is closed in G, and the induced topology from G coincides with the natural topology of $\mathrm{SL}_2(\mathbb{K})$.*

5.3 Kac–Moody groups and buildings

We conclude this chapter by relating the minimal Kac–Moody groups we have just defined to the geometric notions introduced in Chapter 2.

5.3.1 The geometry of Kac–Moody groups. — We already saw that one could attach to a Kac–Moody group $\mathfrak{G}_{\mathcal{D}}$ a Coxeter system (W, S) through the Weyl group of the corresponding Kac–Moody algebra. As alluded to in Section 2.5, $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ in fact possesses a rich geometry given by the **root group datum** $\{\mathfrak{U}_\alpha(\mathbb{K}) \mid \alpha \in \Delta^{\mathrm{re}}\}$. More precisely, we have the following result (see [Rém02, 8.4.1]).

Proposition 5.13. *Let $\mathcal{D}$ be a Kac–Moody root datum and let $G = \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ be a Kac–Moody group of type $\mathcal{D}$ over the field $\mathbb{K}$. Let (W, S) be the associated Coxeter system. For each real root $\alpha \in \Delta^{\mathrm{re}}$, write $U_\alpha := \mathfrak{U}_\alpha(\mathbb{K})$. Set also $T := \mathfrak{T}_\Lambda(\mathbb{K})$, and let N be the subgroup of G generated by T and all $\tilde{s}_i$, $i \in I$ (see §5.1.2). Finally, let $U^\pm := \mathfrak{U}^\pm(\mathbb{K})$ be as in Proposition 5.9 and set $B^\pm := TU^\pm$. Then:*

(1) $(G, (U_\alpha)_{\alpha\in\Delta^{\mathrm{re}}})$ is a twin root datum.

(2) (G, N, U^+, U^-, T, S) is a symmetric refined Tits system.

(3) (G, B^+, B^-, N, S) is a twin Tits system.

In particular, G possesses BN-pairs $(B^\pm, N)$ and hence acts strongly transitively by simplicial isometries on the corresponding thick buildings $\Delta_\pm = \Delta(G, B^\pm)$.

Remark 5.14. Notice that $G = \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ is by definition generated by T and the U_α, $\alpha \in \Delta^{\mathrm{re}}$. However, when $\Lambda^\vee$ is generated by the h_i, $i \in I$ (for example, when $\mathcal{D}$ is the Kac–Moody root datum of simply connected type), the defining relation (5.3) implies that G is in fact already generated by the U_α, $\alpha \in \Delta^{\mathrm{re}}$.

Remark 5.15. If $G = \mathrm{Ad}(\mathfrak{G}_{\mathcal{D}}(\mathbb{K})) = \mathfrak{G}_{\mathcal{D}}(\mathbb{K})/Z(\mathfrak{G}_{\mathcal{D}}(\mathbb{K}))$ is the adjoint form of a Kac–Moody group $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$, then G inherits a twin root datum $(G, (U_\alpha)_{\alpha\in\Delta^{\mathrm{re}}})$ from $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ with same associated twin building, where the U_α are the canonical images in G of the corresponding subgroups of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$. The difference then is that the G-action on the twin building is effective (see [HKM12, Remark 5.3]).

5.3.2 Levi decompositions of parabolic subgroups. — Keeping with the notations of the previous paragraph, we denote by $P_J := B^+ W_J B^+$ the standard parabolic subgroup of G of type $J \subseteq S$ associated to the positive BN-pair (B^+, N). Thus, P_J is the stabiliser in G of the J-residue $R_J(C_0)$ of Δ_+ containing the fundamental chamber C_0 (see §2.4.2).

We view Δ^{re} as the set of half-spaces $\Phi = \Phi(\Sigma_0)$ of the fundamental apartment Σ_0 of the building Δ_+. Set $\Phi_J := \{\alpha \in \Phi \mid \exists v \in W_J,\ s \in J \ :\ \alpha = v\alpha_s\}$ and

$$L_J^+ := \langle U_\alpha \mid \alpha \in \Phi_J \rangle.$$

Finally, set $L_J = T \cdot L_J^+$ and denote by U_J the normal closure in B^+ of

$$\langle U_\alpha \mid \alpha \in \Phi,\ \alpha \supset R_J(C_0) \cap \Sigma_0 \rangle.$$

The following can be found in [Rém02, 6.2.2].

Proposition 5.16. *Let J be a subset of S and consider the G-action on Δ_+. Then:*

(1) U_J is the pointwise fixator of the standard J-residue $R_J(C_0)$.

(2) L_J stabilises $R_J(C_0)$ and acts transitively on it.

(3) There is a semidirect decomposition $P_J = L_J \ltimes U_J$.

The group U_J is called the **unipotent radical** of the parabolic subgroup P_J, and L_J is called its **Levi factor**.

Chapter 6

Maximal Kac–Moody groups

As mentioned in the introduction to the previous chapter, there also exist "maximal" (or "complete") versions of Kac–Moody groups. These are complete topological groups which (generically) contain a minimal Kac–Moody group as a dense subgroup. Recalling that minimal Kac–Moody groups were solely constructed from the *real* root spaces of the associated Kac–Moody algebra – as imposed by property (KMG1) –, their maximal versions will allow to also take into account the imaginary roots; in particular, one will also have "root groups" $\mathfrak{U}_{(\beta)}$ for $\beta \in \Delta^{\mathrm{im}}$.

Different completions $\widehat{\mathfrak{G}}_{\mathcal{D}}(\mathbb{K})$ of a given minimal Kac–Moody group $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ were considered in the literature, and therefore all deserve the name of "complete Kac–Moody groups". There are essentially three such constructions, from very different points of view. While this diversity accounts for the richness of the theory, it would be nice to have a unified approach to "maximal Kac–Moody groups", as in the minimal case. Fortunately, the different constructions are strongly related, and hopefully equivalent, as we will see in the last section of this chapter.

In the first section, we describe a "geometric" construction of $\widehat{\mathfrak{G}}_{\mathcal{D}}(\mathbb{K})$, which we will use over finite fields in Section 8.1. The second section treats a construction of $\widehat{\mathfrak{G}}_{\mathcal{D}}(\mathbb{K})$ using highest-weight modules over the Kac–Moody algebra. The third section contains a purely algebraic construction of $\widehat{\mathfrak{G}}_{\mathcal{D}}(\mathbb{K})$, which is very close in spirit to the functorial construction of $\mathfrak{G}_{\mathcal{D}}$. It therefore seems *a priori* more suitable to establish algebraic properties of maximal Kac–Moody groups. An illustration of this idea will be presented in Section 8.2.

6.1 The group à la Caprace–Rémy–Ronan

The references for this section are [RR06] and [CR09] (see also [Rou12, 6.1]).

Let $G = \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ be a minimal Kac–Moody group over a field $\mathbb{K}$. We keep the notations from Section 5.3. In particular, (B^+, N) is the positive BN-pair for G with associated building $\Delta_+ = \Delta(G, B^+)$. We denote by $\pi_+\colon G \to \mathrm{Aut}(\Delta_+)$ the corresponding strongly transitive action. We equip $\mathrm{Aut}(\Delta_+)$ with the topology of uniform convergence on bounded sets (which, identifying Δ with its geometric realisation, coincides with the compact-open topology in case $\mathbb{K}$ is finite).

Definition 6.1. The **complete Kac–Moody group à la Rémy–Ronan**, which was introduced in [RR06], is the completion in $\mathrm{Aut}(\Delta_+)$ of $\pi_+(G)$. It will be denoted in this thesis by $G^{rr} = \mathfrak{G}^{rr}_{\mathcal{D}}(\mathbb{K})$. We notice that, although the notation may suggest it, we do not claim that there is a group functor $\mathfrak{G}^{rr}_{\mathcal{D}}$.

We will also denote by $U^{rr+} = \mathfrak{U}^{rr+}(\mathbb{K})$ and $B^{rr+} = \mathfrak{B}^{rr+}(\mathbb{K})$ the respective completions of $\pi_+(U^+)$ and $\pi_+(B^+)$ in G^{rr}. Note that B^{rr+} is open in G^{rr}.

Notice that G^{rr} does not really contain G as a dense subgroup, but rather the quotient of G by the kernel $\bigcap_{g\in G} gB^+g^{-1}$ of the G-action on Δ_+. As it turns out, this kernel in fact coincides with the centraliser $Z'(G)$ in G of all root groups U_α ($\alpha \in \Phi$), which is contained in T (see [RR06, 1.B. Lemma 1]).

To remediate this, P-E. Caprace and B. Rémy [CR09, 1.2] consider a slight variant of this group, which we now describe. For $r \in \mathbb{N}$, let $D(r)$ denote the reunion of the (closed) chambers in Δ_+ that are at chamber distance at most r from the fundamental chamber C_0. The fixators $U^+_{D(r)}$ of $D(r)$ in U^+ form a filtration of U^+ which is exhaustive (that is, $U^+_{D(0)} = U^+$) and separated (that is, $\bigcap_{r\in\mathbb{N}} U^+_{D(r)} = \{1\}$).

Definition 6.2. The **complete Kac–Moody group à la Caprace–Rémy–Ronan** is the completion of G with respect to its (non-exhaustive) filtration by the fixators $U^+_{D(r)}$; it is equipped with the corresponding topology. It will be denoted in this thesis by $G^{crr} = \mathfrak{G}^{crr}_{\mathcal{D}}(\mathbb{K})$.

It admits G^{rr} as a quotient and contains U^{rr+}, which we identify with the completion of U^+ with respect to its filtration by the $U^+_{D(r)}$ (see also Proposition 6.3 below).

As mentioned in the introduction of this chapter, we will be mainly interested in $G^{crr} = \mathfrak{G}^{crr}(\mathbb{K})$ over *finite* fields $\mathbb{K}$. Here is a list of the main properties of G^{crr} in this case (see [CR09, Proposition 1]).

Proposition 6.3. *Assume that $\mathbb{K}$ is finite. Then:*

(1) G^{crr} is a locally compact totally disconnected second countable topological group.

(2) The canonical map $\pi_+\colon G \to \mathrm{Aut}(\Delta_+)$ has a unique extension to a continuous surjective open homomorphism $\overline{\pi}_+\colon G^{crr} \to G^{rr}$.

(3) The kernel of $\overline{\pi}_+$ is the discrete subgroup $Z'(G) < G^{crr}$, which is contained in T.

(4) We have $\mathrm{Stab}_{G^{crr}}(C_0) \cong T \ltimes U^{rr+}$.

(5) The sextuple $(G^{crr}, N, U^{rr+}, U^-, T, S)$ *is a refined Tits system.*

Note that statements (4) and (5) in this proposition, which remain valid over arbitrary fields, imply that the positive building $\Delta(G^{crr}, TU^{rr+})$ associated to G^{crr} identifies with Δ_+: this is because G^{crr} acts strongly transitively on Δ_+ with TU^{rr+} as stabiliser of the fundamental chamber (see §2.4.2).

6.2 The group à la Carbone–Garland–Rousseau

The references for this section are [CG03] and [Rou12, 6.2]. Basics on highest-weight modules over Kac–Moody algebras can be found in [Kac90, Chapter 9].

Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac–Moody root datum and let $G = \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ be a minimal Kac–Moody group over a field $\mathbb{K}$. Let also $\mathfrak{g} = \mathfrak{g}_{\mathcal{D}}$ be the associated Kac–Moody algebra and $\mathcal{U} = \mathcal{U}_{\mathcal{D}}$ be the corresponding $\mathbb{Z}$-form of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$. Set $\mathfrak{h} = \mathfrak{h}_{\mathcal{D}} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{C}$.

We recall from [Kac90, §9.2] that an $\mathfrak{h}$-diagonalisable $\mathfrak{g}$-module V is called a **highest-weight module** with **highest-weight** $\lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v_\lambda \in V$ such that

(HWM1) $\mathfrak{n}_+(v_\lambda) = 0$ and $h(v_\lambda) = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$,

(HWM2) $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})(v_\lambda) = V$.

A $\mathfrak{g}$-module V with highest-weight λ is called a **Verma module** if every $\mathfrak{g}$-module with highest-weight λ is a quotient of V. As it turns out, there exists for each $\lambda \in \mathfrak{h}^*$ a unique (up to isomorphism) Verma module, which is denoted $M(\lambda)$. Moreover, $M(\lambda)$ possesses a unique proper maximal submodule $M'(\lambda) < M(\lambda)$. The quotient $L(\lambda) := M(\lambda)/M'(\lambda)$ is then the unique irreducible highest-weight module with highest-weight λ. The sub-$\mathcal{U}$-module $L_{\mathbb{Z}}(\lambda) := \mathcal{U}(v_\lambda)$ is a $\mathbb{Z}$-form of $L(\lambda)$; one can then define $L_k(\lambda) := L_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} k$ for any ring k.

A nonzero weight $\lambda \in \mathfrak{h}^*$ is called **dominant** if $\lambda(h_i) \geq 0$ for all $i \in I$. It is moreover **regular** if $\lambda(h_i) > 0$ for all $i \in I$. For $\lambda \in \Lambda \subset \mathfrak{h}^*$ a dominant weight, the $\mathfrak{g}$-module $L(\lambda)$ is integrable. It follows from [Kac90, Proposition 3.6] that, for each $i \in I$, $L(\lambda)$ as a $\mathfrak{g}_{(i)}$-module (where $\mathfrak{g}_{(i)} = \mathbb{C}e_i + \mathbb{C}h_i + \mathbb{C}f_i \cong \mathfrak{sl}_2(\mathbb{C})$) decomposes into a direct sum of finite-dimensional irreducible $\mathfrak{h}$-invariant modules, and hence the action of $\mathfrak{g}_{(i)}$ on $L(\lambda)$ can be "integrated" to the action of the group $\mathrm{SL}_2(\mathbb{C})$. More generally, we have for each $i \in I$ an integrated action of $\varphi_i(\mathrm{SL}_2(\mathbb{K}))$ and $\mathfrak{T}_\Lambda(\mathbb{K})$ on $L_{\mathbb{K}}(\lambda)$, where φ_i is as in property (KMG1) from §5.1.2, and thus also a representation $\pi_\lambda \colon G \to \mathrm{Aut}(L_{\mathbb{K}}(\lambda))$.

Write $V^\lambda := L_{\mathbb{K}}(\lambda)$ and for each $n \in \mathbb{N}$, let $V_\lambda(n)$ be the sum of all subspaces of V^λ of weight $\lambda - \alpha$ with $\alpha \in Q_+$ and $\mathrm{ht}(\alpha) \leq n$. Thus each $V_\lambda(n)$ is a finite-dimensional sub-vector space of V^λ. Consider the filtration of G with respect to the fixators $G_{V_\lambda(n)}$ of the $V_\lambda(n)$. Note that this filtration is not separated, since $\bigcap_{n\in\mathbb{N}} G_{V_\lambda(n)} = \ker\pi_\lambda = Z(G)\cap\ker(\lambda)$, where $\lambda\colon T\to\mathbb{K}^* : t\mapsto t(\lambda)$ (see [Rou12, 6.2] for the last equality).

Definition 6.4. Let $\lambda\in\Lambda$ be a regular dominant weight. The **complete Kac–Moody group à la Carbone–Garland**, introduced in [CG03], is the Hausdorff completion of G with respect to its filtration by the $G_{V_\lambda(n)}$, $n\in\mathbb{N}$. It will be denoted in this thesis by $G^{cg\lambda}$.

We recall that the *Hausdorff* completion of G with respect to the $G_{V_\lambda(n)}$ is just the completion of the Hausdorff group $G/\bigcap_n G_{V_\lambda(n)}$ with respect to the corresponding filtration. Hence it contains G as a dense subgroup if and only if the filtration is separated. For example, G^{rr} was the Hausdorff completion of G with respect to the filtration of fixators $G_{D(r)}\subset B^+$. As for G^{rr}, if one wants $G^{cg\lambda}$ to contain G as a dense subgroup, and not just a quotient of G, one has then to slightly modify the construction.

This can be achieved by considering the action of G on the direct sum $V^{\lambda_1}\oplus\cdots\oplus V^{\lambda_r}$ where $\lambda_1,\dots,\lambda_r$ generate Λ (see [Rou12, 6.2]). One then considers $V(n) = V_{\lambda_1}(n)\oplus\cdots\oplus V_{\lambda_r}(n)$ for each $n\in\mathbb{N}$. The corresponding filtration of U^+ by the $U^+_{V(n)} = U^+\cap G_{V(n)}$ is then separated, and one denotes by U^{cgr+} the completion of U^+ for this filtration.

Definition 6.5. Let $\lambda\in\mathfrak{h}^*$ be a regular dominant weight. The **complete Kac–Moody group à la Carbone–Garland–Rousseau** is the completion of G with respect to its filtration by the $U^+_{V(n)}$, $n\in\mathbb{N}$. It will be denoted in this thesis by $G^{cgr} = \mathfrak{G}^{cgr}_{\mathcal{D}}(\mathbb{K})$.

It can then be shown that these completions admit a refined Tits system with associated positive building Δ_+, as it was the case for G^{crr} (see [CG03, Theorem 0.1]).

6.3 The group à la Kumar–Mathieu–Rousseau

The group à la Kumar

We begin by describing the group à la Kumar, which is defined over $\mathbb{K}=\mathbb{C}$. The reference for the following paragraphs is [Kum02, Chapters 4.4 and 6].

6.3.1 Pro-groups and pro-Lie algebras. — From now on and till the end of §6.3.5, we set $\mathbb{K}=\mathbb{C}$, although most of what follows remains valid over an arbitrary algebraically closed field of characteristic zero.

Definition 6.6. A **pro-algebraic group over** $\mathbb{K}$ (or just **pro-group**) is a group G together with a nonempty family $\mathcal{F}$ of normal subgroups of G such that for each $N \in \mathcal{F}$, the quotient group G/N is given the structure of an affine algebraic $\mathbb{K}$-group, and such that:

(a_1) $N_1 \cap N_2 \in \mathcal{F}$ whenever $N_1, N_2 \in \mathcal{F}$.

(a_2) Given $N_1 \in \mathcal{F}$ and $N_2 \lhd G$ containing N_1, one has $N_2 \in \mathcal{F}$ if and only if N_2/N_1 is a closed subgroup of G/N_1.

(a_3) For all $N_1, N_2 \in \mathcal{F}$ with $N_1 \subseteq N_2$, there is a morphism $\gamma_{N_2,N_1}\colon G/N_1 \to G/N_2$ of algebraic $\mathbb{K}$-groups.

(a_4) The homomorphism $\gamma\colon G \to \varprojlim G/N : g \mapsto (gN)_{N\in\mathcal{F}}$ is bijective, where

$$\varprojlim_{N\in\mathcal{F}} G/N = \{(g_N N) \in \prod_{N\in\mathcal{F}} G/N \mid N_1 \subseteq N_2 \ \Rightarrow \ g_{N_1} \equiv g_{N_2} \mod N_1\}.$$

The family $\mathcal{F}$ is called a **(complete) defining set** for G. The group G is equipped with the **pro-topology** with respect to $\mathcal{F}$, namely the inverse limit topology provided by (a_4), where each G/N is equipped with the Zariski topology.

Given two pro-groups $(G, \mathcal{F})$ and $(G', \mathcal{F}')$ over $\mathbb{K}$, a group homomorphism $\phi\colon G \to G'$ is called a **pro-group morphism** if $\phi^{-1}(N') \in \mathcal{F}$ for all $N' \in \mathcal{F}'$ and moreover $\phi_{N'}\colon G/\phi^{-1}(N') \to G'/N'$ is a morphism of algebraic groups.

A pro-group G with defining set $\mathcal{F}$ is called a **pro-unipotent group** if each G/N, $N \in \mathcal{F}$, is a unipotent algebraic group. The category of pro-unipotent groups over $\mathbb{K}$ with pro-group morphisms is denoted $\mathcal{C}_{uni}$.

Example 6.7. If G is an algebraic $\mathbb{K}$-group, then G admits a unique structure of pro-algebraic group with defining set $\mathcal{F}$ the set of closed normal subgroups N of G, where each G/N is equipped with its standard algebraic group structure (see [Spr98, 5.5.10]).

Example 6.8. Let $\{G_i\}_{i\in\mathbb{N}}$ be a family of algebraic $\mathbb{K}$-groups with surjective morphisms $\phi_i\colon G_{i+1} \twoheadrightarrow G_i$ for all $i \in \mathbb{N}$. Then $G = \varprojlim G_i$ is a pro-group with defining set $\mathcal{F}$ the set of subgroups of the form $\pi_i^{-1}(N_i)$, $i \in I$, for N_i a closed normal subgroup of G_i and $\pi_i\colon G \twoheadrightarrow G_i$ the canonical projection.

Definition 6.9. A **pro-Lie algebra over** $\mathbb{K}$ is a Lie algebra $\mathfrak{s}$ together with a nonempty family $\mathcal{F}$ of ideals of $\mathfrak{s}$ of finite codimension, such that:

(b_1) $\mathfrak{a}_1 \cap \mathfrak{a}_2 \in \mathcal{F}$ whenever $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{F}$.

(b_2) Given $\mathfrak{a}_1 \in \mathcal{F}$, any ideal $\mathfrak{a}_2 \lhd \mathfrak{s}$ containing $\mathfrak{a}_1$ belongs to $\mathcal{F}$.

(b_3) The Lie algebra homomorphism $\gamma\colon \mathfrak{s} \to \varprojlim \mathfrak{s}/\mathfrak{a} : x \mapsto (x+\mathfrak{a})_{\mathfrak{a}\in\mathcal{F}}$ is an isomorphism, where

$$\varprojlim_{\mathfrak{a}\in\mathcal{F}} \mathfrak{s}/\mathfrak{a} = \{(x_{\mathfrak{a}}+\mathfrak{a}) \in \prod_{\mathfrak{a}\in\mathcal{F}} \mathfrak{s}/\mathfrak{a} \mid \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \ \Rightarrow\ x_{\mathfrak{a}_1} - x_{\mathfrak{a}_2} \in \mathfrak{a}_1\}.$$

The family $\mathcal{F}$ is called a **(complete) defining set** for $\mathfrak{s}$. The Lie algebra $\mathfrak{s}$ is equipped with the **pro-topology** with respect to $\mathcal{F}$, namely the inverse limit topology provided by (b_3), where each $\mathfrak{s}/\mathfrak{a}$ is equipped with the discrete topology.

Given two pro-Lie algebras $(\mathfrak{s},\mathcal{F})$ and $(\mathfrak{s}',\mathcal{F}')$ over $\mathbb{K}$, a Lie algebra homomorphism $\phi\colon \mathfrak{s}\to\mathfrak{s}'$ is called a **pro-Lie algebra morphism** if $\phi^{-1}(\mathfrak{a}') \in \mathcal{F}$ for all $\mathfrak{a}'\in\mathcal{F}'$.

A pro-Lie algebra $\mathfrak{s}$ with defining set $\mathcal{F}$ is called a **pro-nilpotent Lie algebra** if each $\mathfrak{s}/\mathfrak{a}$, $\mathfrak{a}\in\mathcal{F}$, is a nilpotent Lie algebra. The category of pro-nilpotent Lie algebras over $\mathbb{K}$ with pro-Lie algebra morphisms is denoted $\mathcal{C}_{nil}$.

Example 6.10. If $\mathfrak{s}$ is a finite-dimensional Lie algebra, then it admits a unique structure of pro-Lie algebra with defining set $\mathcal{F}$ the set of ideals of $\mathfrak{s}$.

Example 6.11. Let $\{\mathfrak{s}_i\}_{i\in\mathbb{N}}$ be a family of finite-dimensional Lie algebras with surjective Lie algebra homomorphisms $\phi_i\colon \mathfrak{s}_{i+1} \twoheadrightarrow \mathfrak{s}_i$ for all $i\in\mathbb{N}$. Then $\mathfrak{s} = \varprojlim \mathfrak{s}_i$ is a pro-Lie algebra with defining set $\mathcal{F}$ the set of ideals of the form $\pi_i^{-1}(\mathfrak{a}_i)$, $i\in I$, for $\mathfrak{a}_i$ an ideal of $\mathfrak{s}_i$ and $\pi_i\colon \mathfrak{s}\twoheadrightarrow\mathfrak{s}_i$ the canonical projection. Alternatively, $\mathcal{F}$ is the set of ideals of $\mathfrak{s}$ containing $\ker\pi_i$ for some $i\in\mathbb{N}$.

6.3.2 Lie correspondence.

— Let $(S,\mathcal{F})$ be a pro-group over $\mathbb{K}$, and for all $N\in\mathcal{F}$ denote by $\mathfrak{s}_N$ the Lie algebra of the algebraic group S/N. Since $\{\mathfrak{s}_N\}_{N\in\mathcal{F}}$ is an inverse system of Lie algebras, one may define $\mathfrak{s} := \varprojlim \mathfrak{s}_N$. Then $\mathfrak{s}$ is a pro-Lie algebra with defining set $\tilde{\mathcal{F}}$ the set of ideals of $\mathfrak{s}$ containing $\ker\pi_N$ for some $N\in\mathcal{F}$, where $\pi_N\colon \mathfrak{s}\to\mathfrak{s}_N$ is the canonical projection. The Lie algebra $\mathfrak{s}$ is called the **pro-Lie algebra of the pro-group** S and is denoted by $\operatorname{Lie} S$.

Given a pro-group morphism $\phi\colon S\to S'$, there exists a unique pro-Lie algebra morphism $\dot\phi\colon \operatorname{Lie} S \to \operatorname{Lie} S'$ such that for all $N\in\mathcal{F}$, $N'\in\mathcal{F}'$ with $\phi(N)\subseteq N'$, the following diagram commutes (with obvious notations):

$$\begin{tikzcd} \operatorname{Lie} S \arrow[r, "\dot\phi"] \arrow[d, "\pi_N"'] & \operatorname{Lie} S' \arrow[d, "\pi'_{N'}"] \\ \mathfrak{s}_N \arrow[r, "\dot\phi_{N',N}"] & \mathfrak{s}_{N'} \end{tikzcd}$$

Note that the various exponential maps $\operatorname{Exp}_N\colon \mathfrak{s}_N\to S/N$ yield a continuous map $\operatorname{Exp}\colon \operatorname{Lie} S\to S$, which is the unique map such that $\operatorname{Exp}_N\circ\pi_N = \pi_N\circ\operatorname{Exp}$ for all $N\in\mathcal{F}$.

Note also that the covariant functor $S \mapsto \mathrm{Lie}\, S$ from the category of pro-groups with pro-group morphisms to the category of pro-Lie algebras with pro-Lie algebra morphisms does not yield an equivalence of categories, since there exist finite-dimensional Lie algebras $\mathfrak{s}$ for which there is no algebraic group S with $\mathrm{Lie}\, S = \mathfrak{s}$. However, we have the following result (see [Kum02, Theorem 4.4.19]).

Proposition 6.12. *The category $\mathcal{C}_{uni}$ is equivalent to $\mathcal{C}_{nil}$ under $S \mapsto \mathrm{Lie}\, S$, $\phi \mapsto \dot{\phi}$. Moreover,* $\mathrm{Exp}\colon \mathrm{Lie}\, S \to S$ *is bijective for all $S \in \mathcal{C}_{uni}$.*

6.3.3 Pro-unipotent groups. — Following S. Kumar, we consider a Kac–Moody root datum $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ of the form $\mathcal{D} = \mathcal{D}_{\mathrm{Kac}}$ (see Example 5.2), that is, we let $(\mathfrak{h} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{C}, \Pi = \{\alpha_i = c_i\}, \Pi^\vee = \{\alpha_i^\vee = h_i\})$ be a realisation of the generalised Cartan matrix $A = (a_{ij})_{i,j\in I}$ and $\mathfrak{g} = \mathfrak{g}_{\mathcal{D}} = \mathfrak{g}(A)$ be the associated Kac–Moody algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and with set of roots Δ. We identify the root lattice $Q = \sum_{i\in I} \mathbb{Z}\alpha_i$ with a subset of Λ and the root α with c_α (see Remark 5.4). Finally, we set for short $\mathfrak{n} = \mathfrak{n}_+$.

Consider, as in §4.2.3, the positive completion $\widehat{\mathfrak{g}}^p = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \widehat{\mathfrak{n}}$ of $\mathfrak{g}$, where $\widehat{\mathfrak{n}} = \prod_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$ is the completion of $\mathfrak{n}$. Recall also the definition of a closed set (respectively, an ideal) in Δ (see Definition 4.21). For each $k \in \mathbb{N}^*$, denote by $\Psi(k)$ the ideal of Δ_+ consisting of the roots α with $\mathrm{ht}(\alpha) \geq k$, and set

$$\widehat{\mathfrak{n}}(k) := \prod_{\alpha\in\Psi(k)} \mathfrak{g}_\alpha.$$

Then by Example 6.11, $\widehat{\mathfrak{n}}$ is a pro-nilpotent Lie algebra with defining set $\mathcal{F}$ the set of ideals of $\widehat{\mathfrak{n}}$ containing $\widehat{\mathfrak{n}}(k)$ for some $k \geq 1$. Denote by U^+ the pro-unipotent group with $\mathrm{Lie}\, U^+ = \widehat{\mathfrak{n}}$ provided by Proposition 6.12.

More generally, for a closed set of roots $\Theta \subseteq \Delta_+$, set $\widehat{\mathfrak{n}}_\Theta := \prod_{\alpha\in\Theta} \mathfrak{g}_\alpha$. This is a pro-Lie subalgebra of $\widehat{\mathfrak{n}}$, that is, it is closed in $\widehat{\mathfrak{n}}$ for the pro-topology and therefore also has the structure of a pro-Lie algebra. One can then define the subgroup $U_\Theta^+ := \mathrm{Exp}(\widehat{\mathfrak{n}}_\Theta)$ of U^+.

Lemma 6.13 ([Kum02, Lemmas 6.1.2 and 6.1.3])**.** *Let $\Theta \subseteq \Delta_+$ be closed. Then:*

(1) U_Θ^+ is a pro-subgroup of U^+, that is, it is closed in U^+ for the pro-topology and therefore also has the structure of a pro-group.

(2) If $\Delta_+ \setminus \Theta$ is closed, then the multiplication map $U_\Theta^+ \times U_{\Delta_+\setminus\Theta}^+ \to U^+$ is a bijection.

(3) If Θ is an ideal in Δ_+, then U_Θ^+ is normal in U^+.

Example 6.14. For a positive root $\alpha \in \Delta_+$, set $\Theta_\alpha = \{\alpha\}$ if α is real and $\Theta_\alpha = \mathbb{N}^*\alpha$ if α is imaginary. Then Θ_α is closed and hence one gets **root groups** $U_{\Theta_\alpha}^+$ associated to both real and imaginary roots, contrasting with minimal Kac–Moody groups where only real root spaces of the Kac–Moody algebra were exponentiated.

Example 6.15. For a subset $J \subseteq I$, set $\Delta(J) := \Delta \cap \oplus_{j\in J}\mathbb{Z}\alpha_j$ (thus $\Phi_J = \Delta(J) \cap \Delta^{\mathrm{re}}$ in the notations of §5.3.2). Then $\Delta_+(J) := \Delta_+ \cap \Delta(J)$ is closed, while $\Delta_+ \setminus \Delta_+(J)$ is an ideal in Δ_+. Hence $U^+ = U^+_{\Delta_+(J)} \ltimes U^+_{\Delta_+\setminus\Delta_+(J)}$ by Lemma 6.13.

6.3.4 Borel subgroups and minimal parabolics. — Let $J \subset I$ be a subset of I of finite type, that is, such that the corresponding parabolic subgroup W_J of $W = W(A)$ is finite. Set

$$\mathfrak{g}_J := \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta(J)} \mathfrak{g}_\alpha), \quad \mathfrak{u}_J := \bigoplus_{\alpha\in\Delta_+\setminus\Delta(J)} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{p}_J := \mathfrak{g}_J \oplus \mathfrak{u}_J.$$

Consider the (positive) completions

$$\widehat{\mathfrak{u}}_J = \widehat{\mathfrak{n}}_{\Delta_+\setminus\Delta_+(J)} \quad \text{and} \quad \widehat{\mathfrak{p}}_J = \mathfrak{g}_J \oplus \widehat{\mathfrak{u}}_J.$$

Then $\widehat{\mathfrak{p}}_J$ is a pro-Lie algebra, $\widehat{\mathfrak{n}}$ is a pro-Lie subalgebra of $\widehat{\mathfrak{p}}_J$ and $\widehat{\mathfrak{u}}_J$ is a pro-Lie ideal in $\widehat{\mathfrak{p}}_J$.

Let U_J denote the pro-unipotent group associated to $\widehat{\mathfrak{u}}_J$ and let G_J denote the unique (up to isomorphism) connected reductive linear algebraic group such that $\operatorname{Lie} G_J = \mathfrak{g}_J$. Then $T = \operatorname{Hom}_{\mathrm{gr}}(\Lambda, \mathbb{C}^*)$ is a maximal torus for G_J and $\operatorname{Lie} T = \mathfrak{h}$. As in Definition 5.5, we let r^h ($r \in \mathbb{C}^*$, $h \in \Lambda^\vee$) denote the element of T such that $r^h(\lambda) = r^{\langle\lambda,h\rangle}$ for all $\lambda \in \Lambda$. Recall that T acts on $\mathfrak{g}$ by $t(x) := t(\gamma)x$ for all $x \in \mathfrak{g}_\gamma$.

For each $k \geq 1$, set

$$M_k := \bigoplus_{\beta\in\Theta_+(J;k)} \mathfrak{g}_\beta, \quad \text{where} \quad \Theta_+(J;k) := \{\beta = \sum_{i\in I} n_i\alpha_i \in \Delta_+ \mid \sum_{i\notin J} n_i = k\}.$$

Thus M_k is a finite-dimensional $(\mathfrak{g}_J, T)$-submodule of $\mathfrak{g}$ for the adjoint representation. In particular, it can be integrated: M_k, and hence also $\widehat{\mathfrak{u}}_J = \prod_{k\geq1} M_k$, is a G_J-module. Since there is also an equivalence of categories similar to the one described in Lemma 6.12, but this time at the level of *pro-representations* (see [Kum02, Proposition 4.4.26]), this can be in turn integrated to get an action $\phi\colon G_J \to \operatorname{Aut} U_J$. The resulting semidirect decomposition $P_J = U_J \rtimes G_J$ is called the **standard parabolic subgroup of type** J. For $J = \emptyset$, we get the **Borel subgroup** $B := P_\emptyset$; for $J = \{i\}$, we get a **minimal parabolic** $P_i := P_{\{i\}}$.

Lemma 6.16 ([Kum02, Lemmas 6.1.14 and 6.1.15])**.** *Let J, J_1, J_2 be subsets of I of finite type such that $J_1 \subseteq J_2$. Then:*

(1) P_J is a pro-group with $\operatorname{Lie} P_J \approx \widehat{\mathfrak{p}}_J$.

(2) U_J is a normal pro-subgroup of P_J, and the pro-structure on U_J induced by P_J coincides with its original pro-structure.

(3) There is a unique injective pro-group morphism $\gamma_{J_2,J_1}\colon P_{J_1} \hookrightarrow P_{J_2}$ such that $\dot{\gamma}_{J_2,J_1}\colon \widehat{\mathfrak{p}}_{J_1} \hookrightarrow \widehat{\mathfrak{p}}_{J_2}$ is the inclusion.

6.3.5 The maximal group à la Kumar. — Write $S = \{s_i \mid i \in I\}$ for the set of fundamental reflections of $W = W(A)$. Let N be the group generated by the set $T \cup \{\tilde{s}_i \mid i \in I\}$ (for certain symbols $\tilde{s}_i$, $i \in I$) and subject to the following relations:

(c_1) system of relations defining T,

(c_2) $\tilde{s}_i t \tilde{s}_i^{-1} = s_i(t)$ for $i \in I$, $t \in T$,

(c_3) $\tilde{s}_i^2 = (-1)^{\alpha_i^\vee}$ for $i \in I$,

(c_4) $\underbrace{\tilde{s}_i \tilde{s}_j \tilde{s}_i \dots}_{m_{ij} \text{ factors}} = \underbrace{\tilde{s}_j \tilde{s}_i \tilde{s}_j \dots}_{m_{ij} \text{ factors}}$ for all $i, j \in I$ such that $s_i s_j$ has finite order m_{ij} in W.

Notice that (c_2) is the same as (5.2) from §5.1.2, whereas (c_3) and (c_4) are to be compared with Proposition 4.10 (3) and (4). Recalling the T-action on $\mathfrak{g}$, the following should then come as no surprise (see [Kum02, Lemma 6.1.7 and Corollary 6.1.8]).

Lemma 6.17. *Let $\theta\colon T \cup \{\tilde{s}_i\}_{i \in I} \to N$ denote the canonical map. Then:*

(1) The Kac–Moody algebra $\mathfrak{g}$ is an N-module, where for each $i \in I$, the element $\tilde{s}_i$ acts as $s_i^* = \exp(\operatorname{ad} f_i)\exp(\operatorname{ad} e_i)\exp(\operatorname{ad} f_i)$.

(2) The map θ is injective and there is an exact sequence of groups

$$1 \to T \xrightarrow{\theta|_T} N \xrightarrow{\pi} W \to 1$$

with $\pi(\tilde{s}_i) = s_i$ for all $i \in I$.

For each $i \in I$, let $\mathfrak{g}_i := \mathfrak{g}_{\{i\}} = \mathbb{C}e_i \oplus \mathfrak{h} \oplus \mathbb{C}f_i$ and let $\operatorname{Exp}\colon \mathfrak{g}_i \to G_i := G_{\{i\}}$ be the exponential map. Let also $N_i := T \cup T\tilde{s}_i$ be the subgroup of N generated by $T \cup \{\tilde{s}_i\}$, and let $\theta_i\colon N_i \hookrightarrow G_i \leq P_i$ be the embedding of N_i in G_i whose restriction to T is the identity and such that $\theta_i(\tilde{s}_i) = \operatorname{Exp}(f_i)\operatorname{Exp}(e_i)\operatorname{Exp}(f_i) \in G_i$. Finally, let $\gamma_i\colon B \to P_i$ be the embedding of the Borel subgroup B in P_i provided by Lemma 6.16 (3).

Define $Z := ((\coprod_{i \in I} P_i) \sqcup N)/\sim$, where $\sim$ identifies the following elements:

(d_1) $\gamma_i(b) \sim \gamma_j(b)$ for any $i, j \in I$, $b \in B$;

(d_2) $n \sim \theta_i(n)$ for $n \in N_i \subset N$.

Then each P_i and N inject in Z, and $B \cap N = T$.

Definition 6.18. The **maximal Kac–Moody group à la Kumar**, denoted G^{ku}, is the amalgamated product of the system of groups $\{N, P_i \mid i \in I\}$ (see [Kum02, Definition 5.1.6]), where each of the groups N and P_i is thought of as a subset of Z.

Proposition 6.19 ([Kum02, Theorem 6.1.17]). *The canonical map $Z \to G^{ku}$ is injective. In particular, the canonical group homomorphisms $P_i \to G^{ku}$, $i \in I$, and $N \to G^{ku}$ are all injective. Hence, each of P_i, B and N can be considered as a subgroup of G^{ku}. Furthermore, (G^{ku}, B, N, S) is a Tits system, where $S := \{\tilde{s}_i T \in N/T \mid i \in I\}$.*

Note that G^{ku} in fact even possesses a refined Tits system (see [Kum02, Theorem 6.2.8]).

The group à la Mathieu–Rousseau

We now describe the group à la Mathieu–Rousseau, which was introduced by O. Mathieu (([Mat88b, XVIII §2], [Mat88a], [Mat89, I and II]) and further developed by G. Rousseau ([Rou12]). It generalises the group à la Kumar over arbitrary fields. The reference we will follow in the following paragraphs is [Rou12, Sections 2,3 and 6].

For the rest of this section, we will consider a Kac–Moody root datum $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ of the form $\mathcal{D} = \mathcal{D}_{\mathrm{Kac}}$ (see Example 5.2): this is the setting considered by O. Mathieu, and it allows to simplify the notations by viewing the root lattice Q of $\mathfrak{g}_{\mathcal{D}}$ as a subset of Λ (see Remark 5.4). Note, however, that the group à la Mathieu–Rousseau can also be constructed for a general Kac–Moody root datum (see [Rou12, Section 3.19]).

Let thus $(\mathfrak{h} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{C}, \Pi, \Pi^\vee)$ be a realisation of A, where $\Pi = \{\alpha_i = c_i \mid i \in I\}$, $\Pi^\vee = \{\alpha_i^\vee = h_i \mid i \in I\}$ and $\mathfrak{h}^* = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. As before, we identify the root lattice $Q = \sum_{i\in I} \mathbb{Z}\alpha_i$ of the Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}_{\mathcal{D}} = \mathfrak{g}(A)$ with a subset of Λ. We let $W = W(A)$ be the Weyl group of $\mathfrak{g}$, generated by the fundamental reflections s_i or r_i or $r_i^\vee$ ($i \in I$), depending on whether W is viewed as an abstract Coxeter group, or as a subgroup of $\mathrm{GL}(\mathfrak{h})$, or as a subgroup of $\mathrm{GL}(\mathfrak{h}^*)$, respectively. As usual, Δ is the set of roots, and we set $\Phi := \Delta^{\mathrm{re}}$ (consistently with the notations of §5.3.2). Finally, $\mathcal{U} = \mathcal{U}_{\mathcal{D}}$ is the $\mathbb{Z}$-form of the enveloping algebra $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ of $\mathfrak{g}$ defined in Section 4.2, and we keep all the notations from this section, which we will freely use without further reference in what follows.

6.3.6 Twisted exponentials in $\widehat{\mathcal{U}}^+$. — We first wish to be able to "exponentiate" imaginary root spaces, in the same way as real root spaces could be exponentiated to get root groups in the minimal Kac–Moody group $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$. Over $\mathbb{K} = \mathbb{C}$, this could be achieved using the exponential map Exp (see §6.3.3). Over an arbitrary field $\mathbb{K}$, however, the exponential map Exp is not well-defined anymore, and we have to consider "twisted" versions of the exponentials.

The following is [Rou12, Proposition 2.4].

Lemma 6.20. *Let $x \in \mathfrak{g}_{\mathbb{Z}}$. Then there exist elements $x^{[n]} \in \mathcal{U}$ for $n \in \mathbb{N}$ such that $x^{[0]} = 1$, $x^{[1]} = x$ and $x^{[n]} - x^{(n)}$ has filtration less than n in $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$, where $x^{(n)} := x^n/n!$. If moreover $x \in \mathfrak{g}_{\alpha\mathbb{Z}}$, $\alpha \in Q$, then one may choose $x^{[n]} \in \mathcal{U}_{n\alpha}$.*

Recall from §4.2.4 the bialgebra structure on $\mathcal{U}$.

Lemma 6.21 ([Rou12, Proposition 2.7])**.** *For $x \in \mathfrak{g}_{\alpha\mathbb{Z}}$, $\alpha \in \Delta \cup \{0\}$, one can modify the elements $x^{[n]}$ from Lemma 6.20 such that they moreover satisfy the following relations:*

$$\nabla(x^{[n]}) = \sum_{k+l=n} x^{[k]} \otimes x^{[l]} \quad \textit{and} \quad \epsilon(x^{[n]}) = 0 \textit{ for } n > 0.$$

Definition 6.22. For $x \in \mathfrak{g}_{\alpha\mathbb{Z}}$, $\alpha \in \Delta \cup \{0\}$, the sequence $(x^{[n]})_{n \in \mathbb{N}}$ is called an **exponential sequence** associated to x if it satisfies the conditions of Lemmas 6.20 and 6.21 (including $x^{[n]} \in \mathcal{U}_{n\alpha}$).

Definition 6.23. For a ring R and an element $x \in \mathfrak{g}_{\alpha\mathbb{Z}}$, $\alpha \in \Delta_+$, we define for all $\lambda \in R$ the **twisted exponential**

$$[\exp]\lambda x := \sum_{n \geq 0} \lambda^n x^{[n]} \in \widehat{\mathcal{U}}_R^+$$

associated to the exponential sequence $(x^{[n]})_{n \in \mathbb{N}}$. As mentioned in [Rou12, Section 2.9], such a twisted exponential for x is unique up to modifying each $x^{[n]}$ of the exponential sequence ($n \geq 2$) by an element of $\mathfrak{g}_{n\alpha\mathbb{Z}}$.

6.3.7 Poincaré–Birkhoff–Witt for $\mathcal{U}$.

— For $\alpha \in \Delta \cup \{0\}$, let $\mathcal{B}_\alpha$ be a basis of $\mathfrak{g}_{\alpha\mathbb{Z}}$ over $\mathbb{Z}$. For $\alpha = \alpha_i$ a simple root, we choose $\mathcal{B}_\alpha = \{e_i\}$ and $\mathcal{B}_{-\alpha} = \{f_i\}$. Using the W-action (or more precisely, the W^*-action), one may also choose $\mathcal{B}_\alpha = \{e_\alpha\}$ and $\mathcal{B}_{-\alpha} = \{f_\alpha\}$ for all $\alpha \in \Phi_+ = \Delta_+^{\mathrm{re}}$, so that $[e_\alpha, f_\alpha] = -\alpha^\vee$ (see §4.2.5). We put an arbitrary order on the $\mathbb{Z}$-basis $\mathcal{B} = \bigcup_\alpha \mathcal{B}_\alpha$ of $\mathfrak{g}_\mathbb{Z}$.

Let $\mathbb{N}^{(\mathcal{B})}$ denote the set of sequences of natural numbers indexed by $\mathcal{B}$ with only finitely many nonzero components. For $x \in \mathcal{B}$, we choose exponential sequences $(x^{[n]})_{n \in \mathbb{N}}$. For $N = (N_x)_{x \in \mathcal{B}} \in \mathbb{N}^{(\mathcal{B})}$, define the element

$$[N] := \prod_{x \in \mathcal{B}} x^{[N_x]}$$

of $\mathcal{U}$, where the product is taken in the chosen order on $\mathcal{B}$.

The following lemma is an immediate consequence of Lemma 6.20 together with [Bou75, VIII §12 n°3 Théorème 1].

Lemma 6.24. *The elements $[N]$ form a basis of $\mathcal{U}$ over $\mathbb{Z}$.*

6.3.8 Pro-unipotent groups.

— Let $\Psi \subseteq \Delta_+$ be a closed set of roots. Consider the algebra $\mathcal{U}(\Psi)$ from §4.2.3: it is a $\mathbb{Z}$-bialgebra which is co-invertible, cocommutative and graded by Q_+:

$$\mathcal{U}(\Psi) = \bigoplus_{\alpha \in \mathbb{N}\Psi} \mathcal{U}(\Psi)_\alpha.$$

Moreover, all its root spaces are free $\mathbb{Z}$-modules of finite rank. Consider its restricted dual

$$\mathbb{Z}[\mathfrak{U}_\Psi^{ma}] = \bigoplus_{\alpha\in\mathbb{N}\Psi} \mathcal{U}(\Psi)^*_\alpha.$$

It is a co-invertible and commutative $\mathbb{Z}$-bialgebra, hence the algebra of an affine group scheme $\mathfrak{U}_\Psi^{ma}$, which we view as a group functor:

$$\mathfrak{U}_\Psi^{ma}(R) = \mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\mathfrak{U}_\Psi^{ma}], R) \quad \text{for any ring } R.$$

For $\alpha \in \Delta_+^{\mathrm{im}}$, we write $\mathfrak{U}_{(\alpha)} = \mathfrak{U}^{ma}_{\{n\alpha \mid n\geq 1\}}$, and for $\alpha \in \Phi_+$, we write $\mathfrak{U}_\alpha = \mathfrak{U}_{(\alpha)} = \mathfrak{U}^{ma}_{\{\alpha\}}$; then $\mathfrak{U}_\Psi^{ma}$ and $\mathfrak{U}_{(\alpha)}$ are sub-group schemes of

$$\mathfrak{U}^{ma+} := \mathfrak{U}^{ma}_{\Delta_+}.$$

Set $\mathcal{B}_\Psi := \bigcup_{\alpha\in\psi}\mathcal{B}_\alpha$. Then the basis of $\mathcal{U}(\Psi)$ indexed by $\mathbb{N}^{(\mathcal{B}_\Psi)}$ provided by Lemma 6.24 gives, by duality, a basis $(Z^N)_{N\in\mathbb{N}^{(\mathcal{B}_\Psi)}}$ of $\mathbb{Z}[\mathfrak{U}_\Psi^{ma}]$. Since

$$\nabla[N] = \sum_{K+L=N} [K]\otimes[L]$$

by Lemma 6.21, one has $Z^K \cdot Z^L = Z^{K+L}$ for all $K, L \in \mathbb{N}^{(\mathcal{B}_\Psi)}$. In particular, $\mathbb{Z}[\mathfrak{U}_\Psi^{ma}]$ is a polynomial algebra over $\mathbb{Z}$ whose indeterminates Z_x are indexed by the elements $x \in \mathcal{B}_\Psi$.

Proposition 6.25 ([Rou12, Proposition 3.2])**.** *Let $\Psi \subseteq \Delta_+$ be closed and let R be a ring. Then $\mathfrak{U}_\Psi^{ma}(R)$ can be identified to the multiplicative subgroup of $\widehat{\mathcal{U}}_R(\Psi)$ consisting of the products*

$$\prod_{x\in\mathcal{B}_\Psi} [\exp]\lambda_x x$$

for $\lambda_x \in R$, where the product is taken in the (arbitrary) chosen order on $\mathcal{B}$. The expression of an element of $\mathfrak{U}_\Psi^{ma}(R)$ in the form of such a product is unique.

Note that for each $n \in \mathbb{N}$, all the factors (in the infinite products above) but finitely many are equal to 1 modulo terms of total degree at least n. Hence these infinite products indeed belong to $\widehat{\mathcal{U}}_R(\Psi)$.

Example 6.26. If Ψ is finite, and hence contains only real roots, we get back the group scheme $\mathfrak{U}_\Psi$ from Lemma 5.7. In particular, for $\alpha \in \Phi_+$, the choice of e_α as a $\mathbb{Z}$-basis element of $\mathfrak{g}_{\alpha\mathbb{Z}}$ determines an isomorphism x_α from the additive group $\mathbb{G}_a$ onto $\mathfrak{U}_\alpha$ given by $x_\alpha(r) = \exp(re_\alpha)$ (see §5.1.3). What we get in addition now, is a **root group** $\mathfrak{U}_{(\alpha)}$ for each imaginary root $\alpha \in \Delta_+^{\mathrm{im}}$, which was our original motivation.

Using the above proposition and Lemma 6.13, one can then prove an analogue to this lemma.

Lemma 6.27 ([Rou12, Lemme 3.3])**.** *Let $\Psi' \subseteq \Psi \subseteq \Delta_+$ be closed subsets of roots. Then:*

(1) $\mathfrak{U}^{ma}_{\Psi'}$ is a closed subgroup of $\mathfrak{U}^{ma}_{\Psi}$ and $\mathbb{Z}[\mathfrak{U}^{ma}_{\Psi}/\mathfrak{U}^{ma}_{\Psi'}]$ is a polynomial algebra with indeterminates indexed by $\mathcal{B}_\Psi \setminus \mathcal{B}_{\Psi'}$.

(2) If $\Psi\setminus\Psi'$ is also closed, then there is a unique decomposition $\mathfrak{U}^{ma}_{\Psi} = \mathfrak{U}^{ma}_{\Psi'}.\mathfrak{U}^{ma}_{\Psi\setminus\Psi'}$.

(3) If Ψ' is an ideal in Ψ, then $\mathfrak{U}^{ma}_{\Psi'}$ is normal in $\mathfrak{U}^{ma}_{\Psi}$ and one has a semidirect decomposition if, moreover, $\Psi \setminus \Psi'$ is closed.

(4) If $\Psi \setminus \Psi'$ consists of a single root α, then the quotient $\mathfrak{U}^{ma}_{\Psi}/\mathfrak{U}^{ma}_{\Psi'}$ is isomorphic to the additive group $\mathbb{G}_a$ for α a real root, and to the commutative unipotent group $\mathbb{G}_a^{\dim \mathfrak{g}_\alpha}$ for α imaginary.

(5) If $\alpha, \beta \in \Psi$, $\alpha + \beta \in \Psi$ implies $\alpha + \beta \in \Psi'$, then $\mathfrak{U}^{ma}_{\Psi}/\mathfrak{U}^{ma}_{\Psi'}$ is commutative. The group $\mathfrak{U}^{ma}_{\Psi}/\mathfrak{U}^{ma}_{\Psi'}(R)$ is then isomorphic to the additive group $\prod_{\alpha\in\Psi\setminus\Psi'} \mathfrak{g}_{\alpha R}$.

(6) If $\Psi'' \subseteq \Psi'$ is closed and if $\alpha \in \Psi'$, $\beta \in \Psi$, $\alpha + \beta \in \Delta$ implies $\alpha + \beta \in \Psi''$, then $\mathfrak{U}^{ma}_{\Psi''}$ contains the commutator group $[\mathfrak{U}^{ma}_{\Psi}, \mathfrak{U}^{ma}_{\Psi'}]$.

6.3.9 Borel subgroups and minimal parabolics. — Let as before $\mathfrak{T} = \mathfrak{T}_\Lambda$ denote the split torus scheme associated to $\mathcal{D}$, that is, $\mathfrak{T}(R) = \mathrm{Hom}_{\mathrm{gr}}(\Lambda, R^\times)$ for any ring R. The **Borel subgroup** $\mathfrak{B}^{ma+}$ is by definition the semidirect product $\mathfrak{T} \ltimes \mathfrak{U}^{ma+}$ for the following action of $\mathfrak{T}$ on $\mathfrak{U}^{ma+}$. For a ring R, $\mathfrak{T}(R)$ acts on $\mathcal{U}_R$ by bialgebra automorphisms: the action $\mathrm{Ad}_R(t)$ of $t \in \mathfrak{T}(R)$ on $\mathcal{U}_{\alpha R} = \mathcal{U}_\alpha \otimes_{\mathbb{Z}} R$ is the multiplication by $t(\alpha) \in R^\times$ (see Proposition 5.10). One thus gets the following action of $\mathfrak{T}(R)$ on $\mathfrak{U}^{ma+}(R)$ by conjugation:

$$\mathrm{Int}(t) \cdot [\exp]\lambda x = [\exp]t(\alpha)\lambda x \quad \text{if } x \in \mathfrak{g}_{\alpha R}.$$

Note that the groups $\mathfrak{U}^{ma}_{\Psi}$ from Proposition 6.25 are stabilised by this $\mathfrak{T}$-action.

For $\alpha = \alpha_i$ a simple root, we have $\mathfrak{U}^{ma+} = \mathfrak{U}_\alpha \ltimes \mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}}$ by Lemma 6.27 (3). Denote by $\mathfrak{U}_{-\alpha}$ the affine group scheme with algebra $\mathbb{Z}[\mathfrak{U}_{-\alpha}]$, contained in the dual of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g}_{-\alpha}) \cap \mathcal{U}$, exactly as in §6.3.8. In other words, $\mathfrak{U}_{-\alpha}$ is the group scheme isomorphic to $\mathbb{G}_a$ by $x_{-\alpha}\colon\ \mathbb{G}_a(R) \xrightarrow{\sim} \mathfrak{U}_{-\alpha}(R) : r \mapsto \exp(rf_i)$ for any ring R, as in §5.1.3. Let also $\mathfrak{A}_i^\Lambda$ denote the unique connected affine algebraic group associated to the Kac–Moody root datum $(\{1\}, (2), \Lambda, \{\alpha_i\}, \{\alpha_i^\vee\})$ (see [Spr98, Theorem 10.1.1]). It contains $\mathfrak{T}$, $\mathfrak{U}_\alpha$ and $\mathfrak{U}_{-\alpha}$ as closed subgroups and is generated by them.

We define an action of $\mathfrak{A}_i^\Lambda$ on $\mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}}$ as follows. Since we already have an action by conjugation of $\mathfrak{U}_\alpha$ and $\mathfrak{T}$ on $\mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}}$, it suffices to define for any ring R a conjugation action of $\mathfrak{U}_{-\alpha}(R) = \{\exp(rf_i) \mid r \in R\}$ on $\mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}}(R)$. Notice first that, by Lemma 4.9 (1), the conjugation action of $\mathfrak{U}_\alpha(R)$ on $\mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}}(R) \subseteq \widehat{\mathcal{U}}^+_R$ is given by

$$(\exp e_i).z.(\exp -e_i) = \sum_{k,l} (-1)^l e_i^{(k)} z e_i^{(l)} = \sum_{m=k+l\in\mathbb{N}} (\mathrm{ad}(e_i)^m/m!)(z)$$

for all $z \in \mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}}(R) \subset \widehat{\mathcal{U}}^+_R = \widehat{\mathcal{U}}_R(\Delta_+)$. As noted at the end of §4.2.3, one can replace Δ_+ by $s_i(\Delta_+)$ and work in $\widehat{\mathcal{U}}_R(s_i(\Delta_+))$ instead of $\widehat{\mathcal{U}}_R(\Delta_+)$. Of course, all the results from the previous paragraphs remain valid in this setting. In particular, $\mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}} = \mathfrak{U}^{ma}_{s_i(\Delta_+\setminus\{\alpha\})}$ is a closed normal subgroup of $\mathfrak{U}^{ma}_{s_i(\Delta_+)}$ by Lemma 6.27 and is therefore normalised by $\mathcal{U}_{-\alpha}$. Moreover, the conjugation action of $\mathcal{U}_{-\alpha}(R)$ on $\mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}}(R)$ is given as above by

$$\mathrm{Int}(\exp(f_i))(z) = \sum_{m\geq 0} (\mathrm{ad}(f_i)^m/m!)(z)$$

for all $z \in \mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}}(R) \subset \widehat{\mathcal{U}}_R(s_i(\Delta_+))$.

The resulting semidirect product $\mathfrak{B}^{ma+}_i = \mathfrak{A}^\Lambda_i \ltimes \mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}}$ is an affine group scheme, called a **minimal parabolic**. It contains $\mathfrak{B}^{ma+}$ as a closed subgroup.

Denote by $\tilde{s}_i$ the element

$$\tilde{s}_i = x_\alpha(1)x_{-\alpha}(1)x_\alpha(1) = \exp(e_i)\exp(f_i)\exp(e_i)$$

of $\mathfrak{A}^\Lambda_i$. Then $\tilde{s}_i$ normalises $\mathfrak{T}$, satisfies $\tilde{s}_i^2 = (-1)^{\alpha_i^\vee} \in \mathfrak{T}(\mathbb{Z})$ and acts on $\mathfrak{g}_\mathbb{Z}$ as s_i^* (see Proposition 4.10 and Lemma 6.17).

6.3.10 The maximal group à la Mathieu–Rousseau. — Write for short $\mathfrak{B} = \mathfrak{B}^{ma+}$ and $\mathfrak{B}_i = \mathfrak{B}^{ma+}_i$, $i \in I$. For an element $w \in W$ with reduced decomposition $w = s_{i_1}\dots s_{i_n}$, let $\mathfrak{C}(w) = \mathfrak{B}_{i_1} \times^\mathfrak{B} \mathfrak{B}_{i_2} \times^\mathfrak{B} \cdots \times^\mathfrak{B} \mathfrak{B}_{i_n}$ denote the scheme over $\mathbb{Z}$ which is the n-fold product over $\mathfrak{B}$ of the corresponding minimal parabolics, where $\mathfrak{B}_i \times^\mathfrak{B} \mathfrak{B}_j$ denotes the quotient of $\mathfrak{B}_i \times \mathfrak{B}_j$ by the $\mathfrak{B}$-action $(p_1, p_2) \mapsto (p_1 b^{-1}, bp_2)$. Let $\mathfrak{B}(w)$ denote the affine scheme $\mathrm{Spec}(\mathbb{Z}[\mathfrak{C}(w)])$.

If $w' \leq w$ for the Bruhat–Chevalley order (see e.g. [Kum02, Definition 1.3.15]), then one can obtain a reduced decomposition of w' by deleting some generators s_{i_j} in a reduced decomposition of w. Replacing the factors $\mathfrak{B}_{i_j}$ by $\mathfrak{B}$ in the corresponding expression of $\mathfrak{C}(w)$, one gets a closed sub-scheme of $\mathfrak{C}(w)$ isomorphic to $\mathfrak{C}(w')$. The closed immersion $\mathfrak{C}(w') \to \mathfrak{C}(w)$ induces a closed immersion $\mathfrak{B}(w') \to \mathfrak{B}(w)$, yielding an inductive system of $\mathbb{Z}$-schemes $\{\mathfrak{B}(w) \mid w \in W\}$.

Definition 6.28. The **(positive) maximal Kac–Moody group à la Mathieu–Rousseau** is the ind-scheme $\mathfrak{G}^{pma} = \mathfrak{G}^{pma}_\mathcal{D}$ which is the inductive limit of the above inductive system. Each morphism $\mathfrak{B}(w) \to \mathfrak{G}^{pma}$ is a closed immersion. We view $\mathfrak{G}^{pma}$ as a functor on the category of rings:

$$\mathfrak{G}^{pma}(R) = \varinjlim \mathfrak{B}(w)(R)$$

for any ring R. For each $w, w' \in W$, there is a multiplication morphism $\mathfrak{B}(w) \times \mathfrak{B}(w') \to \mathfrak{B}(\psi(w,w'))$ for some $\psi(w,w') \leq ww'$, turning $\mathfrak{G}^{pma}$ into a group scheme (the definition of the inverse being clear).

For a subset $J \subseteq I$, let W_J be the standard parabolic subgroup of W of type J and consider the Kac–Moody root datum $\mathcal{D}(J) = (J, A|_J, \Lambda, (\alpha_j)_{j\in J}, (\alpha_j^\vee)_{j\in J})$. Define also $\Delta_+(J) := \Delta_+ \cap \oplus_{j\in J}\mathbb{Z}\alpha_j$ as in Example 6.15. We then have Levi decompositions:

Proposition 6.29 ([Rou12, 3.10])**.** *The ind-group scheme which is the inductive limit of the $\mathfrak{B}(w)$, $w \in W_J$, is the parabolic subgroup*

$$\mathfrak{P}(J) = \mathfrak{G}^{pma}_{\mathcal{D}(J)} \ltimes \mathfrak{U}^{ma}_{\Delta_+\setminus\Delta_+(J)}.$$

Note also that in case W is finite, we get back the Tits functor $\mathfrak{G}_{\mathcal{D}}$:

Proposition 6.30 ([Rou12, Proposition 3.9])**.** *If W is finite and w_0 is its longest element (see [AB08, 1.5.2]), then the Tits functor $\mathfrak{G}_{\mathcal{D}}$ identifies with $\mathfrak{B}(w_0)$, which is equal to $\mathfrak{G}^{pma}$.*

Finally, we have a nice behaviour of the W-action on root groups: for an element $w \in W$ with reduced decomposition $w = s_{i_1}\dots s_{i_n}$, write $\overline{w} := \tilde{s_{i_1}}\dots\tilde{s_{i_n}}$ for the corresponding element of $\mathfrak{G}^{pma}$, which only depends on w and not on the choice of a reduced decomposition for w. Recall the definitions of the root groups $\mathfrak{U}_{(\alpha)}$, $\alpha \in \Delta_+$ from §6.3.8. One then has the following.

Proposition 6.31. *For any $\alpha \in \Delta_+$ and any $w \in W$ such that $w\alpha \in \Delta_+$:*

$$\overline{w}\mathfrak{U}_{(\alpha)}\overline{w}^{\,-1} = \mathfrak{U}_{(w\alpha)}.$$

Proof. For α a real root, this is [Rou12, 3.11]. In any case, this amounts to show that, whenever $s_i \in W$ is such that $s_i(\alpha) \in \Delta_+$, one has

$$\tilde{s_i} \cdot ([\exp]x) \cdot \tilde{s_i}^{-1} = [\exp](s_i^* x)$$

for any homogenous $x \in \oplus_{n\geq 1}\mathfrak{g}_{n\alpha R}$, with R an arbitrary ring. This last statement follows from the definition of the semidirect product $\mathfrak{B}_i^{ma+} = \mathfrak{A}_i^{\Lambda} \ltimes \mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha_i\}}$ (see §6.3.9). □

6.3.11 Comparison with $\mathfrak{G}_{\mathcal{D}}$ and refined Tits system for $\mathfrak{G}^{pma}$. —

Let $\mathfrak{G} = \mathfrak{G}_{\mathcal{D}}$ be the Tits functor associated to $\mathcal{D}$, and let e_α, f_α for $\alpha \in \Phi_+$ be the elements introduced in §6.3.7. For a ring k, the generators of $\mathfrak{G}(k)$ are the $x_\alpha(r)$, $\alpha \in \Phi$, $r \in k$, and the elements $t \in \mathfrak{T}_\Lambda(k)$ (see §5.1.2 and §5.1.3). We map each of these elements to the element of $\mathfrak{G}^{pma}(k)$ of the same name (see §6.3.8 and §6.3.9).

Proposition 6.32 ([Rou12, 3.12]). *The map $\iota\colon \mathfrak{G} \to \mathfrak{G}^{pma}$ defined on the generators as above is a homomorphism of group functors.*

Note that the homomorphism ι also maps each $\tilde{s}_i$ from §5.1.2 to the element of the same name of $\mathfrak{G}^{pma}(\mathbb{Z})$ (see §6.3.9). Let $\mathfrak{N}$ denote the sub-functor of $\mathfrak{G}$ such that $\mathfrak{N}(k)$ is the subgroup generated by $\mathfrak{T}_\Lambda(k)$ and the $\tilde{s}_i$, $i \in I$ (see Proposition 5.13). Then ι is an isomorphism of $\mathfrak{N}$ on $\iota(\mathfrak{N})$, which is thus generated by $\mathfrak{T}_\Lambda$ and the $\tilde{s}_i$, $i \in I$. Moreover, for each $\alpha \in \Phi$, ι induces an isomorphism of the group $\mathfrak{U}_\alpha$ from §5.1.2 onto the subgroup of $\mathfrak{G}^{pma}$ of the same name (see §6.3.8).

Proposition 6.33 ([Rou12, Proposition 3.13]). *If k is a field, then ι_k is injective.*

One can thus see the minimal Kac–Moody group $\mathfrak{G}(k)$ as a subgroup of $\mathfrak{G}^{pma}(k)$.

Assume now that k is a field, and let $\mathfrak{U}^+$ (respectively, $\mathfrak{U}^-$) denote the subgroup functor of $\mathfrak{G}^{pma}$ generated by the subgroups $\mathfrak{U}_\alpha$ for $\alpha \in \Phi_+$ (respectively, $\alpha \in \Phi_-$); again, it identifies with the subgroup of $\mathfrak{G}$ of the same name (see §5.1.2). Note also that $\mathfrak{U}^+ \subset \mathfrak{U}^{ma+}$. Denote again by $\mathfrak{N}$ its image $\iota(\mathfrak{N})$ in $\mathfrak{G}^{pma}$, and let $\mathfrak{B}^+ = \mathfrak{T}_\Lambda \ltimes \mathfrak{U}^+$. Finally, write $G^{pma}, N, B^{ma+}, U^{ma+}, U^\pm, T$ for the evaluation on k of the corresponding functors $\mathfrak{G}^{pma}, \mathfrak{N}, \mathfrak{B}^{ma+}, \mathfrak{U}^{ma+}, \mathfrak{U}^\pm, \mathfrak{T}_\Lambda$. Recalling that $S = \{s_i \mid i \in I\}$ is the fundamental system of generators of W, we then have the following awaited result.

Proposition 6.34 ([Rou12, 3.16]). *Assume that k is a field. Then the sextuple $(G^{pma}, N, U^{ma+}, U^-, T, S)$ is a refined Tits system.*

Corollary 6.35 ([Rou12, Corollary 3.17]). *If k is a field, $U^+ = \mathfrak{U}^+(k) = \mathfrak{G}(k) \cap \mathfrak{U}^{ma+}(k)$ and $B^+ = \mathfrak{B}^+(k) = \mathfrak{G}(k) \cap \mathfrak{B}^{ma+}(k)$.*

Corollary 6.36 ([Rou12, Corollary 3.18]). *If k is a field, then the injection ι_k induces a simplicial $\mathfrak{G}(k)$-equivariant isomorphism from the building X_+ associated to $\mathfrak{G}(k)$ to the building $\widehat{X_+}$ associated to the Tits system (G^{pma}, B^{ma+}, N, S). More precisely, these two buildings have the same simplices and the apartments of X_+ are apartments of $\widehat{X_+}$.*

Finally, one would like to check that G^{pma} is indeed a "complete" Kac–Moody group, and that it contains $G = \mathfrak{G}(k)$ as a dense subgroup. For this, we need to define a topology on G^{pma}. As we did for the previously defined complete Kac–Moody groups G^{crr} and G^{cgr}, we consider a filtration on U^{ma+}. For each $n \in \mathbb{N}^*$, consider the ideal $\Psi(n) := \{\alpha \in \Delta_+ \mid \mathrm{ht}(\alpha) \geq n\}$ of Δ_+, and write

$$U_n^{ma+} := \mathfrak{U}^{ma}_{\Psi(n)}(k).$$

Proposition 6.37 ([Rou12, Remarque 6.3.6 a)]). *With the notations above:*

(1) The filtration $(U_n^{ma+})_{n \in \mathbb{N}^*}$ of U^{ma+} defines a left-invariant metric on G^{pma} whose open balls are the translates gU_n^{ma+}, $g \in G^{pma}$, turning G^{pma} into a complete topological group. In particular, U^{ma+} is open in G^{pma}.

(2) *The closure* $\overline{U}^+$ *of* U^+ *in* U^{ma+} *is the completion of* U^+ *with respect to the induced filtration by the* $U_n^+ = U^+ \cap U_n^{ma+}$.

(3) *The closure* $\overline{G}$ *of* G *in* G^{pma} *is the completion of* G *for the same filtration.*

Remark 6.38. Over a finite field $\mathbb{F}_q$, the topological group $G^{pma}(\mathbb{F}_q)$ is locally compact (because $U^{ma+}(\mathbb{F}_q)$ is compact open) and totally disconnected (because the filtration $(U_n^{ma+}(\mathbb{F}_q))_{n\in\mathbb{N}^*}$ of $U^{ma+}(\mathbb{F}_q)$ is separated). This is to be compared with Proposition 6.3 (1).

Unfortunately, it turns out that $\overline{G}$ does not always coincide with G^{pma}: indeed, G. Rousseau showed that the minimal Kac–Moody group over $\mathbb{F}_2$ associated to the generalised Cartan matrix $A = \left(\begin{smallmatrix} 2 & -m \\ -m & 2 \end{smallmatrix}\right)$ with $m \geq 3$ is not dense in the corresponding Mathieu–Rousseau group. We will give other examples of this phenomenon, over $\mathbb{F}_3$ this time, in Section 8.2 (see Corollary 8.54). Note, however, that G is dense in G^{pma} in most cases. Let M denote the maximum (in absolute value) of the non-diagonal entries of the generalised Cartan matrix A of G.

Proposition 6.39 ([Rou12, Proposition 6.11])**.** *Assume that the characteristic* p *of the field* k *is either zero or strictly bigger than* M. *Then* U^+ *is dense in* U^{ma+} *and* G *is dense in* G^{pma}.

6.3.12 Comparison with the group à la Kumar. — Recall from the first paragraphs of this section the definition of the group G^{ku}. In particular, we assume that the ground field is $k = \mathbb{C}$ and that $\mathcal{D} = \mathcal{D}_{\mathrm{Kac}}$.

Then the twisted exponentials in Proposition 6.25 can be replaced by the usual exponentials from §6.3.3. Hence, for a closed set of roots $\Theta \subseteq \Delta_+$, the group U_Θ^+ from §6.3.3 identifies with $\mathfrak{U}_\Theta^{ma}(\mathbb{C})$. Clearly, we also have an identification of the groups T and N.

Similarly, for $i \in I$, the minimal parabolic $P_i = G_{\{i\}} \ltimes U_{\{i\}}$ from §6.3.4 identifies with the minimal parabolic $B_i^{ma+} = \mathfrak{B}_i^{ma+}(\mathbb{C}) = \mathfrak{A}_i^\Lambda(\mathbb{C}) \ltimes \mathfrak{U}_{\Delta_+\setminus\{\alpha\}}^{ma}(\mathbb{C})$ from §6.3.9.

Since, as we saw, $(G^{pma}, B^{ma+} = TU^{ma+}, N, S)$ is a Tits system, it follows from [Kum02, Proposition 5.1.7] that G^{pma} is the amalgamated product of the subgroups N and B_i^{ma+}, $i \in I$. But this was precisely the definition of G^{ku}.

Corollary 6.40. *Over* $k = \mathbb{C}$, *the group à la Mathieu–Rousseau and the group à la Kumar coincide:* $\mathfrak{G}^{pma}(\mathbb{C}) = G^{ku}$.

6.3.13 A simplicity result. — Finally, we mention a simplicity result for the group $\mathfrak{G}^{pma}(k)$, which was established over fields k of characteristic 0 by R. Moody in an unpublished note [Moo82], and which was extended by G. Rousseau to the fields k of positive characteristic p that are not algebraic over $\mathbb{F}_p$.

Let k be a field and let $G^{pma} = \mathfrak{G}^{pma}(k)$ denote the Mathieu–Rousseau group over k (of simply connected type) associated to the generalised Cartan matrix

A. Denote by $Z' := \bigcap_{g\in G^{pma}} gB^{ma+}g^{-1}$ the kernel of the action of G^{pma} on its associated positive building. Recall that a group is said to be **(abstractly) simple** if it does not possess any nontrivial (abstract) normal subgroup. Note that Z' is a normal subgroup of G^{pma}.

Proposition 6.41 ([Rou12, Théorème 6.19]). *Assume that A is indecomposable and not of affine type, and that k is either a field of characteristic 0, or a field of characteristic $p > 0$ that is not algebraic over $\mathbb{F}_p$. Then G^{pma}/Z' is abstractly simple.*

We will take care of the case $k = \mathbb{F}_p$ later, in Section 8.2.

6.4 Comparisons and GK-simplicity

We conclude this chapter by comparing the different completions of the previous sections, following [Rou12, Section 6].

6.4.1 Comparison of the filtrations. — Let k be a fixed ground field and consider the Kac–Moody root datum $\mathcal{D}$ of simply connected type. Let $G = \mathfrak{G}_{\mathcal{D}}(k)$ be the corresponding minimal Kac–Moody group, which injects in the different complete groups G^{crr}, G^{cgr} and G^{pma}.

Recall that $U^{rr+} \subset G^{crr}$ is the completion of $U^+ \subset G$ for the filtration by the $U^+_{D(n)}$, that U^{cgr+} is the completion of U^+ for the filtration by the $U^+_{V(n)}$, and that $\overline{U}^+ \subset G^{pma}$ is the completion of U^+ for the filtration by the U^+_n. Let M be as in Proposition 6.39 and set $M' := \max\{\lambda(\alpha_i^\vee) \mid i \in I\}$, where λ is any of the regular dominant weights $\lambda_1, \dots, \lambda_r$ considered in Definition 6.5.

Lemma 6.42 ([Rou12, 6.3]). *One has $U^+_{n+1} \subset U^+_{V(n)} \subset U^+_{D(d)}$ for each $n, d \in \mathbb{N}$ such that $n \geq \frac{M'}{M}(1+M)^d$.*

Note that the inclusion $U^+_{n+1} \subset U^+_{D(d)}$, which holds for $n \geq (1+M)^d$, implies in particular the following fact: *given a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of positive real roots such that $\mathrm{ht}(\alpha_n) \geq n$ for all $n \in \mathbb{N}$, the corresponding root groups $U_{\alpha_n} \subset U^+_n$ fix bigger and bigger balls around the fundamental chamber of the positive building X_+ associated to G, as n goes to infinity.* This is a reformulation of Property (FPRS) from [CR09, 2.1], which will play a crucial role in the original results presented in Section 8.1. We now reproduce Rousseau's proof of the above inclusion, since it is quite easy and hence provides a nice alternative to the proof given in [CR09, Proposition 4] that minimal Kac–Moody groups satisfy property (FPRS).

For $i \in I$ and $\alpha \in \Delta_+ \setminus \{\alpha_i\}$, one has $s_i(\alpha) \in \Delta_+$ and $\mathrm{ht}(s_i(\alpha)) \leq (1+M)\,\mathrm{ht}(\alpha)$, or else $\mathrm{ht}(\alpha) \leq (1+M)\,\mathrm{ht}(s_i(\alpha))$ by replacing α with $s_i(\alpha)$. Hence, as soon as $\mathrm{ht}(\alpha) \geq (1+M)^d$, one has $\mathrm{ht}(s_{i_1}\dots s_{i_d}(\alpha)) \geq \frac{\mathrm{ht}(\alpha)}{(1+M)^d}$ for all $i_1, \dots, i_d \in I$. It follows from Proposition 6.31 that for $n \geq (1+M)^d$ and for each $\overline{w} = \tilde{s_{i_1}} \dots \tilde{s_{i_d}}$ of length at most d, one has $U_n^{ma+} \subset \overline{w}\, U^{ma+} \overline{w}^{\,-1}$. Hence U_n^{ma+} is contained in the fixator $U_{C(d)}^{ma+}$ of all the chambers of the fundamental apartment of X_+ that are at distance at most d from the fundamental chamber C_0. Since moreover U_n^{ma+} is normal in U^{ma+} by Lemma 6.27, it is contained in the intersection of all the U^{ma+}-conjugates of $U_{C(d)}^{ma+}$, that is, in the fixator $U_{D(d)}^{ma+}$ of the ball in X_+ of radius d and centered at C_0. Intersecting with U^+, we get the desired result.

Back to our original concern of comparing the different complete Kac–Moody groups, we remark that the comparison of the filtrations in Lemma 6.42 yields continuous homomorphisms

$$\phi\colon\ \overline{U}^+ \xrightarrow{\gamma} U^{cgr+} \xrightarrow{\rho} U^{rr+}.$$

Moreover, since $\overline{G}$, G^{cgr}, G^{crr} are the completions of G for the corresponding filtrations, these extend to continuous homomorphisms (again denoted by the same letter)

$$\phi\colon\ \overline{G} \xrightarrow{\gamma} G^{cgr} \xrightarrow{\rho} G^{crr}.$$

Note that one also has a continuous homomorphism $G^{cg\lambda} \to G^{rr}$.

Since the problem of comparing $\overline{G}$ with G^{pma} was already addressed at the end of §6.3.11, the problem of comparing the different "complete" Kac–Moody groups now amounts to see when the homomorphisms ϕ, γ and ρ are isomorphisms of topological groups. This is the object of the following paragraph.

6.4.2 GK-simplicity. — Consider first the question of the surjectivity of ϕ, γ and ρ. At first glance, not much can be said, except when the ground field k is finite. Indeed, in this case, U^{ma+} and hence also $\overline{U}^+$ are compact (this follows from Proposition 6.25). Since U^{cgr+} and U^{rr+} are Hausdorff, this implies that the homomorphisms ϕ, γ and ρ are surjective, open and closed. In particular, over a finite field, the only question that remains open is the injectivity of ϕ.

Back to an arbitrary field k, notice that the kernel $\ker\phi$ of $\phi\colon\ \overline{G} \to G^{crr}$ is by definition

$$\ker\phi = \overline{U}^+ \cap Z'(G^{pma}) \subseteq U^{ma+} \cap Z'(G^{pma}) = \bigcap_{r\in\mathbb{N}} U_{D(r)}^{ma+},$$

where $Z'(G^{pma}) := \bigcap_{g\in G^{pma}} gB^{ma+}g^{-1}$ is the kernel of the action of G^{pma} on the building X_+.

Lemma 6.43 ([Rou12, Proposition 6.4])**.** *Let $Z(G)$ and $Z(G^{pma})$ denote the respective centers of G and G^{pma}. Then:*

(1) $Z(G) \subseteq Z(G^{pma}) \subseteq Z'(G^{pma})$.

(2) $Z'(G^{pma}) = Z(G) \cdot (Z'(G^{pma}) \cap U^{ma+})$ *and* $Z(G^{pma}) = Z(G) \cdot (Z(G^{pma}) \cap U^{ma+})$.

(3) $Z'(G^{pma}) \cap U^{ma+}$ *is normal in* G^{pma}.

Thus ϕ is injective when $Z'(G^{pma}) \cap U^{ma+} = \{1\}$, that is, when $Z'(G^{pma}) = Z(G)$. One also says in this case that G^{pma} is *GK-simple*. We now explain this terminology.

Recall from Proposition 4.14 the *Gabber–Kac theorem*, which states that when the generalised Cartan matrix A is symmetrisable, the Kac–Moody algebra $\mathfrak{g}(A)$ can be obtained as the quotient of $\tilde{\mathfrak{g}}(A)$ by its unique maximal ideal intersecting $\mathfrak{h}$ trivially. Thus, every graded ideal of $\mathfrak{g}_A$ intersecting $\mathfrak{h}$ trivially is reduced to $\{0\}$. Equivalently, every graded sub-$\mathfrak{g}_A$-module of $\mathfrak{g}_A$ that is contained in $\mathfrak{n}_+$ is reduced to $\{0\}$.

Definition 6.44. Let k be an arbitrary field and set $\mathfrak{g} = \mathfrak{g}_A$. The Lie algebra $\mathfrak{g}_k$ is *simple in the sense of the Gabber–Kac theorem*, or simply **GK-simple**, if every graded sub-$\mathcal{U}_k$-module of $\mathfrak{g}_k$ that is contained in $\mathfrak{n}_k^+$ is reduced to $\{0\}$. Equivalently, $\mathfrak{g}_k$ is GK-simple if and only if for all $\alpha \in \Delta_+^{\mathrm{im}}$, any element $x \in \mathfrak{g}_{\alpha k}$ such that $\mathrm{ad}(f_j^{(q)})(x) = 0$ for all $j \in I$ and $q \in \mathbb{N}^*$ must be zero.

Similarly, the group G^{pma} is said to be **GK-simple** if every normal subgroup of G^{pma} that is contained in U^{ma+} is reduced to $\{1\}$, that is, if $Z'(G^{pma}) \cap U^{ma+} = \{1\}$.

In characteristic zero, $\mathfrak{g}_k$ is thus GK-simple in the symmetrisable case, and conjecturally in all cases. In positive characteristic p however, there are counter-examples: for example, the affine Kac–Moody algebra $\mathfrak{g}_k = \mathfrak{sl}_m(k) \otimes_k k[t, t^{-1}]$ is not GK-simple when p divides m. Note however that the corresponding Kac–Moody group $G^{pma} = \mathrm{SL}_m(k((t)))$ is GK-simple (see [Rou12, Exemple 6.8]).

Conjecture 6.45. *G^{pma} is always GK-simple.*

We now state the main result obtained by G. Rousseau regarding the comparison of the several complete Kac–Moody groups, and hence also regarding the above conjecture. Let M denote as before the maximum (in absolute value) of the non-diagonal entries of the generalised Cartan matrix A of G.

Proposition 6.46 ([Rou12, Proposition 6.7])**.** *Assume that $\mathfrak{g}_k$ is GK-simple and that the field k is infinite. Then ϕ, γ and ρ are homeomorphisms, and hence the topological groups $\overline{G}$, G^{cgr} and G^{crr} are isomorphic. In particular, if* $\mathrm{char}(k) = 0$ *or if* $\mathrm{char}(k) = p > M$*, the topological groups G^{pma}, G^{cgr} and G^{crr} are isomorphic.*

Chapter 7

Kac–Moody groups in characteristic zero

The results presented in this chapter are part of the original work of this thesis.

7.1 Infinitesimal structure of minimal Kac–Moody groups

In this section, we give a proof of Theorem C from the introduction, as well as of other related results of independent interest (see in particular Theorem 7.1 and Corollary 7.6 below). The essential tool will be the fact that almost connected Lie groups possess property (FB), as was established in Section 3.3. The content of the present section is the second part of the paper [Mar12b].

Recall from Proposition 5.13 and §2.4.3 that a minimal Kac–Moody group $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ acts strongly transitively on a twin building (Δ_+, Δ_-). A subgroup of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ is called **bounded** if it is contained in the intersection of two finite type parabolic subgroups of opposite signs, that is, in the intersection of the stabilisers in $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ of a spherical residue of Δ_+ and of a spherical residue of Δ_-.

The key result of this section, which gives a partial answer to a problem stated in [Cap09a, page xi], is the following. In this statement, we equip a Kac–Moody group with the σ-algebra generated by the finite type parabolic subgroups of each sign; measurability in Lie groups is understood as Haar measurability.

Theorem 7.1. *Any measurable homomorphism of an almost connected Lie group into a Kac–Moody group has bounded image.*

Proof. Since a measurable homomorphism from an almost connected Lie group H into a Kac–Moody group G yields measurable actions of H on the associated positive and negative buildings of G, the result readily follows from Corollary 3.13. $\square$

Let now $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and consider a minimal Kac–Moody group $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ over $\mathbb{K}$, which we will assume to be of simply connected type. Moreover, since we will want to consider one-parameter subgroups of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ of the form $\alpha(t) = \exp \operatorname{ad}(tx)$ for some ad-locally finite x in the corresponding Kac–Moody algebra, it is convenient to rather consider the adjoint form $G = \operatorname{Ad}(\mathfrak{G}_{\mathcal{D}}(\mathbb{K}))$ of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ (see §5.1.3 and Remark 5.15). In this section, we denote by $\operatorname{Lie} G$ the corresponding Kac–Moody algebra (that is, the Kac–Moody algebra of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$), and we endow G with the Kac–Peterson topology (see Section 5.2). Thus G is a connected Hausdorff topological group by Proposition 5.12. For the rest of this section, we will refer to a Kac–Moody group G as above as a *real or complex Kac–Moody group*.

Let G be a real or complex Kac–Moody group. A **one-parameter subgroup** of G is a continuous homomorphism from $\mathbb{R}$ to G. The set of all one-parameter subgroups of G is denoted by $\operatorname{Hom}_c(\mathbb{R}, G)$. Notice that, since G lies inside $\operatorname{Aut}(\operatorname{Lie} G)$, one can surely describe examples of such one-parameter subgroups. Indeed, recall from §4.1.2 that for a given ad-locally finite $x \in \operatorname{Lie} G$, the exponential $\exp \operatorname{ad}(x) = \sum_{n \geq 0} (\operatorname{ad}(x))^n/n!$ makes sense in $\operatorname{Aut}(\operatorname{Lie} G)$. Thus, for any ad-locally finite $x \in \operatorname{Lie} G$, we have a one-parameter subgroup α given by

$$\alpha(t) = \exp \operatorname{ad}(tx) \quad \text{for all } t \in \mathbb{R}.$$

One can verify it is indeed continuous (see Lemma 7.5 below).

As is well known, in case G is a Lie group, the set $\operatorname{Hom}_c(\mathbb{R}, G)$ can be given a Lie algebra structure so that it identifies with the Lie algebra of G (see e.g. [HM07, Proposition 2.10]). In fact, by the solution to Hilbert's fifth problem (see Lemma 3.3), this construction extends to connected locally compact groups as well. The following result, which completely describes the set of one-parameter subgroups of a real or complex Kac–Moody group G, opens the way for analogues to these classical results within Kac–Moody theory, as it should then be possible to reconstruct the Lie algebra $\operatorname{Lie} G$ from $\operatorname{Hom}_c(\mathbb{R}, G)$ by generators and relations.

Proposition 7.2. *Every continuous one-parameter subgroup α of a real or complex Kac–Moody group G is of the form $\alpha(t) = \exp \operatorname{ad}(tx)$ for some* ad*-locally finite $x \in \operatorname{Lie}(G)$.*

To prove this, we need three additional lemmas. First, we have to check that the σ-algebra on G considered in Theorem 7.1 is compatible with the σ-algebra of Borel sets of G for the Kac–Peterson topology.

Lemma 7.3. *The finite type parabolic subgroups of G of each sign are borelian in G. In particular, the σ-algebra on G generated by these parabolics is contained in the σ-algebra of Borel sets of G.*

Proof. Recall from §5.3.2 that a finite type parabolic subgroup P of G has a Levi decomposition $P = L \ltimes U$, with L the Levi factor and U the unipotent radical of P.

Note that L is locally compact and σ-compact by [HKM12, Remark 5.13]. Moreover, by a straightforward adaptation of the proof of [HKM12, Proposition 5.11], the subgroup U is closed. Writing L as a countable union $L = \bigcup_{k\geq 1} L_k$ of compact sets, it follows that P is the countable union $P = \bigcup_{k\geq 1} L_k.U$ of closed sets, and hence a Borelian, as desired.

We remark that in the 2-spherical case this is in fact an immediate consequence of [HKM12, Theorem 1 and Proposition 3.20]. □

Second, we formally recall that Proposition 7.2 is well-known in the case of finite-dimensional Lie groups (see e.g. [HM07, Proposition 2.10]).

Lemma 7.4. *Let H be a Lie group. Then every $\alpha \in \mathrm{Hom}_c(\mathbb{R}, H)$ is of the form $\alpha(t) = \exp \mathrm{ad}(tx)$ for some $x \in \mathrm{Lie}(H)$.*

Third, we show that a maximal bounded subgroup of the Kac–Moody group G inherits the structure of a (finite-dimensional) almost connected Lie group. Let (X_+, X_-) denote the twin building associated to G.

Lemma 7.5. *Let x_+ and x_- be spherical residues in X_+ and X_-, respectively. Then the stabiliser G_{x_+,x_-} in G of these two residues is closed in G. Moreover, with the induced topology, it has the structure of a finite-dimensional almost connected Lie group $H \cong G_{x_+,x_-}$, and every vector $x \in \mathrm{Lie}(H)$ can be identified with an* ad*-locally finite vector of* $\mathrm{Lie}(G)$*.*

Proof. It follows from [CM06, Proposition 3.6] that G_{x_+,x_-} possesses a Levi decomposition $G_{x_+,x_-} = L \ltimes U$ with Levi factor L and with unipotent radical U associated to parallel residues in X_+ and X_- (see [CM06, 3.2.1] for precise definitions). We show that L and U are both almost connected Lie groups, whence the structure of almost connected Lie group on G_{x_+,x_-}.

By [CM06, Lemma 3.3] together with [CM06, Corollary 3.5], the subgroup U is also the unipotent radical associated to a pair of chambers in X_+ and X_-. It follows by [CM06, Lemma 3.2] that U is a bounded product of finitely many root groups, which carry the Lie group topology by [HKM12, Corollary 5.12]. In particular, U is connected. Moreover, the Lie algebra of U is the direct sum of the finitely many Lie algebras of the root groups which boundedly generate U, and is therefore finite-dimensional. Hence U is a connected Lie group, as desired.

The Levi factor L is a Lie group because of [HKM12, Corollary 5.12 and Remark 5.13]. It is almost connected, since it decomposes as a product of a torus T (homeomorphic to $(\mathbb{K}^*)^{|I|}$, where I is the indexing set for the generalised Cartan matrix of G) with a subgroup generated by root groups.

The claim about ad-local finiteness can be found in [KP87, Theorems 1 and 2]. □

Proof of Proposition 7.2. Let $\alpha \in \mathrm{Hom}_c(\mathbb{R}, G)$. By Theorem 7.1 together with Lemma 7.3, we know that $\alpha(\mathbb{R})$ is bounded in G. The conclusion then follows from Lemmas 7.4 and 7.5. $\square$

We now extend the automatic continuity of measurable homomorphisms between Lie groups (see e.g. [Kle89, Theorem 1]) to the case of real or complex Kac–Moody groups. In the two following statements, by a **measurable** homomorphism between two real or complex Kac–Moody groups, we mean a homomorphism $\phi\colon G_1 \to G_2$ between these groups such that the preimage of an open set of G_2 by the restriction of ϕ to any Lie subgroup of G_1 (that is, any closed subgroup of G_1 with a Lie group structure) is Haar measurable. Note that Borel homomorphisms are examples of measurable homomorphisms in this sense.

Corollary 7.6. *Every measurable homomorphism between real or complex Kac–Moody groups is continuous.*

Proof. Let $\alpha\colon G_1 \to G_2$ be a measurable homomorphism between real or complex Kac–Moody groups G_1, G_2. Note first that by Theorem 7.1 together with Lemma 7.3, the image of any measurable homomorphism $\beta\colon \mathrm{SL}(2,\mathbb{K}) \to G_2$ is contained in a bounded subgroup of G_2, and hence in a Lie group by Lemma 7.5. In particular such a β must be continuous by [Kle89, Theorem 1]. It follows that for any copies $X_{i_1}, \dots, X_{i_n}$ of $\mathrm{SL}(2,\mathbb{K})$ in G_1, the map $\overline{\alpha}$ in the following commutative diagram is continuous:

$$\begin{array}{ccc} X_{i_1} \times \cdots \times X_{i_n} & \longrightarrow & G_1 \\ {\scriptstyle\overline{\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ \alpha(X_{i_1}) \times \cdots \times \alpha(X_{i_n}) & \longrightarrow & G_2. \end{array}$$

Continuity of α then follows by definition of the Kac–Peterson topology. $\square$

Finally, as a last consequence of Theorem 7.1, we get the following classification of measurable isomorphisms between real or complex Kac–Moody groups. Without using the assumption of measurability, this has been proved by P-E. Caprace ([Cap09a]). For almost split (hence not necessarily split) Kac–Moody groups of 2-spherical type, the corresponding result has been obtained by G. Hainke ([Hai12]). Note however that our proof relying on Theorem 3.12 is substantially shorter.

Corollary 7.7. *Let α be a measurable isomorphism between real or complex Kac–Moody groups. Then α is continuous and standard, that is, it induces an isomorphism of the corresponding twin root data.*

Proof. Let $\alpha\colon G_1 \to G_2$ be a measurable isomorphism between real or complex Kac–Moody groups G_1, G_2. Recall from [CM06, Theorem 6.3] that α is standard whenever it maps bounded subgroups to bounded subgroups. Let thus H be a bounded subgroup of G_1. Then H is contained in an almost connected Lie group by Lemma 7.5. Thus $\alpha(H)$ is bounded by Theorem 7.1 together with Lemma 7.3, as desired. $\square$

7.2 Free groups in complete Kac–Moody groups

In this section, we explain how the free Lie subalgebras of a Kac–Moody algebra $\mathfrak{g}(A)$ that were considered in Section 4.3 can be integrated to free groups inside Kac–Moody groups.

Recall that these free Lie algebras were contained in the positive imaginary algebra $\mathfrak{n}_+^{\mathrm{im}}$ and were of the form $\bigoplus_{n\geq 1}\mathfrak{g}_{n\beta}$ for some positive imaginary root $\beta\in\Delta_+^{\mathrm{im}}$. In particular, in order to integrate these Lie algebras, one surely has to consider *complete* Kac–Moody groups.

In what follows, $G=\mathfrak{G}_{\mathcal{D}}^{pma}(\mathbb{K})$ is the maximal Kac–Moody group à la Mathieu–Rousseau of simply connected type over a field $\mathbb{K}$, and A is its generalised Cartan matrix. Integrating $\mathfrak{n}_+^{\mathrm{im}}$ then yields the unipotent subgroup $U^{im+}:=\mathfrak{U}^{ma}_{\Delta_+^{\mathrm{im}}}(\mathbb{K})$ of G (see §6.3.8), which we call the **positive imaginary subgroup** of G.

We begin by giving a geometric interpretation of U^{im+} and by proving some non-degeneracy property for its action on the building X_+ associated to G. Let $W=W(A)$ be the Weyl group of A, and choose a representative $\overline{w}$ in G of each $w\in W$, as in Proposition 6.31. Let T be the split torus of G, set $U^{ma+}:=\mathfrak{U}^{ma+}(\mathbb{K})$ and let $B^{ma+}=T\ltimes U^{ma+}$ be the Borel subgroup of G (see §6.3.9). Note then that the kernel of the G-action on X_+ is $Z'(G):=\bigcap_{g\in G} gB^{ma+}g^{-1}$.

Lemma 7.8. *One has $U^{im+}=\bigcap_{w\in W}\overline{w}\,U^{ma+}\overline{w}^{\,-1}$. In particular, $T\ltimes U^{im+}$ coincides with the pointwise fixator of the fundamental apartment of X_+.*

Proof. This follows from Proposition 6.31 together with the fact that $\mathfrak{n}_+^{\mathrm{im}}=\bigcap_{w^*\in W^*}w^*(\mathfrak{n}_+)$ (see Section 4.3). $\square$

Proposition 7.9. *Assume that A is (indecomposable) of indefinite type and that $\operatorname{char}\mathbb{K}\neq 2$. Then U^{im+} does not act trivially on X_+, that is, it is not contained in $Z'(G)$.*

Proof. Let $p=\operatorname{char}\mathbb{K}$ denote the characteristic of $\mathbb{K}$. Thus $p=0$ or $p\geq 3$. Assume for a contradiction that U^{im+} is contained in $Z'(G)$. It then follows from Lemma 7.8 that $U^{im+}=Z'(G)\cap U^{ma+}$, and hence U^{im+} is normal in G by Lemma 6.43 (3).

Fix a realisation $(\mathfrak{h},\Pi,\Pi^\vee)$ of A with simple roots α_i and Chevalley generators e_i, f_i, $i\in I$. Remember from §4.2.1 the definitions of $\mathcal{U}$, $\mathfrak{g}_{\mathbb{Z}}$ and $\mathfrak{g}_{\mathbb{K}}$, where $\mathfrak{g}=\mathfrak{g}_A$. We will show that there exist an imaginary root $\delta\in\Delta_+^{\mathrm{im}}$, a simple root α_i, and a nonzero element $x\in\mathfrak{g}_{\delta\mathbb{K}}$ such that $\delta-\alpha_i\in\Delta_+^{\mathrm{re}}$ and $[f_i,x]$ is nonzero in $\mathfrak{g}_{\mathbb{K}}$. Recalling from §6.3.9 the definition of the semidirect product $\mathfrak{B}_i^{ma+}=\mathfrak{A}_i^{\Lambda}\ltimes\mathfrak{U}^{ma}_{\Delta_+\backslash\{\alpha_i\}}$, this will imply that the root group $U_{-\alpha_i}$ conjugates the imaginary root group $U_{(\delta)}$ outside U^{im+}, so that U^{im+} cannot be normal in G, yielding the desired contradiction.

Assume first that the size of A is at least 3. Since A is of indefinite type, it then follows from the classification of finite and affine Dynkin diagrams (see §4.1.4) that the Coxeter group $W = W(A)$ is (irreducible) non-spherical and non-affine. By Corollary 1.32, there thus exist nested roots $-\beta \subsetneq \gamma$ of the Coxeter complex of W (corresponding to positive real roots $\beta, \gamma \in \Delta_+^{\mathrm{re}}$) such that $\mathrm{Pc}(r_\beta, r_\gamma) = W$. Set $m := \langle \beta, \gamma^\vee \rangle$ and $n := \langle \gamma, \beta^\vee \rangle$. Then Proposition 4.31 implies that $m, n < 0$ and $mn > 4$. Up to interchanging m and n, we may then assume that $m \leq -3$. Set $\beta_1 := r_\beta(\gamma) = \gamma - n\beta \in \Delta_+^{\mathrm{re}}$ and $\gamma_1 := r_\gamma(\beta) = \beta - m\gamma \in \Delta_+^{\mathrm{re}}$, so that

$$\langle \gamma_1, \beta^\vee \rangle = \langle \beta, \gamma_1^\vee \rangle = \langle \beta_1, \gamma^\vee \rangle = \langle \gamma, \beta_1^\vee \rangle = 2 - mn$$

and

$$\langle \gamma_1, \beta_1^\vee \rangle = m(mn-3), \quad \langle \beta_1, \gamma_1^\vee \rangle = n(mn-3).$$

If p does not divide $2 - mn$, we set $\delta := w(\beta + \gamma_1) \in \Delta_+^{\mathrm{im}}$ where $w \in W$ is such that $w\beta = \alpha_i$ for some i, and $x := [e_i, e_{\gamma'}] \in \mathfrak{g}_{\mathbb{K}}$ where $\gamma' = w\gamma_1$ and $e_{\gamma'}$ is a basis element of $\mathfrak{g}_{\gamma'\mathbb{K}}$ as in §4.2.5. Since $\beta - \gamma_1 = m\gamma \notin \Delta$ and hence also $\alpha_i - \gamma' \notin \Delta$, we deduce that

$$[f_i, x] = \langle \gamma', \alpha_i^\vee \rangle \cdot e_{\gamma'} = \langle \gamma_1, \beta^\vee \rangle \cdot e_{\gamma'} = (2 - mn) \cdot e_{\gamma'} \neq 0 \quad \text{in } \mathfrak{g}_{\mathbb{K}},$$

as desired.

If p divides $2 - mn$, then since $p \neq 2$, it does not divide $m(mn-3)$. We then set $\delta := w(\beta_1 + \gamma_1) \in \Delta_+^{\mathrm{im}}$ where $w \in W$ is such that $w\beta_1 = \alpha_i$ for some i, and $x := [e_i, e_{\gamma'}] \in \mathfrak{g}_{\mathbb{K}}$ where $\gamma' = w\gamma_1$ and $e_{\gamma'}$ is a basis element of $\mathfrak{g}_{\gamma'\mathbb{K}}$ as above. Since $\beta_1 - \gamma_1 = (m+1)\gamma - (n+1)\beta \notin \Delta$ and hence also $\alpha_i - \gamma' \notin \Delta$, we deduce that

$$[f_i, x] = \langle \gamma', \alpha_i^\vee \rangle \cdot e_{\gamma'} = \langle \gamma_1, \beta_1^\vee \rangle \cdot e_{\gamma'} = m(mn-3) \cdot e_{\gamma'} \neq 0 \quad \text{in } \mathfrak{g}_{\mathbb{K}},$$

as desired.

Finally, if A is of size 2 and of the form $A = \left(\begin{smallmatrix} 2 & m \\ n & 2 \end{smallmatrix}\right)$, then $m, n < 0$ and $mn > 4$ since A is of indefinite type, and we may proceed as above with $(\beta, \gamma) = (\alpha_1, \alpha_2)$. $\square$

As promised, we next integrate the free Lie subalgebras of $\mathfrak{g}(A)$ to free groups inside U^{im+}, when the characteristic of $\mathbb{K}$ is zero.

Proposition 7.10. *Assume that the field $\mathbb{K}$ has characteristic zero. Let $\beta \in \Delta_+^{\mathrm{im}}$ be such that the subalgebra $\mathfrak{g}_{(\beta)} := \oplus_{n \geq 1} \mathfrak{g}_{n\beta}$ of $\mathfrak{g} = \mathfrak{g}_A$ contains a free Lie algebra L on a finite set X. Then the root subgroup $U_{(\beta)} := \mathfrak{U}_{(\beta)}(\mathbb{K})$ of G (see §6.3.8) contains a free group on the set X.*

Proof. We may assume without loss of generality that $X \subset \mathfrak{g}_{\mathbb{Z}}$ is contained in a $\mathbb{Z}$-basis (of homogeneous elements) of the free $\mathbb{Z}$-Lie algebra $L_{\mathbb{Z}} := L \cap \mathcal{U} \subseteq \mathfrak{g}_{(\beta)\mathbb{Z}} = \mathfrak{g}_{(\beta)} \cap \mathcal{U}$, where $\mathcal{U}$ is the $\mathbb{Z}$-form of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ introduced in §4.2.1. For a ring R, let $A_R(X)$ denote the free associative algebra on X over R, and let $\widehat{A}_R(X)$ denote its completion with respect to its canonical $\mathbb{N}$-filtration: this is the so-called **Magnus algebra** of X over R (see [Bou72, II §5 n°1]). Since $A_{\mathbb{Z}}(X)$ is isomorphic to the enveloping algebra of $L_{\mathbb{Z}}$ by [Bou72, II §3 n°1 Théorème 1], there is an injection $A_{\mathbb{Z}}(X) \hookrightarrow \mathcal{U}^+$, and we identify $A_{\mathbb{Z}}(X)$ with its image in $\mathcal{U}^+$. Since the field $\mathbb{K}$ has characteristic zero, the extension of scalars to $\mathbb{K}$ also yields an identification of $A_{\mathbb{K}}(X) = A_{\mathbb{Z}}(X) \otimes_{\mathbb{Z}} \mathbb{K}$ with a subset of $\mathcal{U}^+_{\mathbb{K}}$. Finally, since X is finite, the respective canonical $\mathbb{N}$-filtrations on $A_{\mathbb{K}}(X)$ and $\mathcal{U}^+_{\mathbb{K}}$ are equivalent, and one may thus identify $\widehat{A}_{\mathbb{K}}(X)$ with a subalgebra of $\widehat{\mathcal{U}}^+_{\mathbb{K}}$.

Let $\Gamma_{\mathbb{K}}(X)$ denote the so-called **Magnus group**, that is, the multiplicative group of formal sums in $\widehat{A}_{\mathbb{K}}(X)$ with constant term 1 (see [Bou72, II §5 n°2]). Thus $\Gamma_{\mathbb{K}}(X)$ can be identified with a subgroup of $(\widehat{\mathcal{U}}^+_{\mathbb{K}})^{\times}$. It then follows from [Bou72, II §5 n°3 Théorème 1] that the unique group homomorphism $g\colon F(X) \to \Gamma_{\mathbb{K}}(X) \cap \mathfrak{U}_{(\beta)}(\mathbb{K}) \subset \widehat{\mathcal{U}}^+_{\mathbb{K}}$ from the free group over X to the Magnus group such that $g(x) = \exp(x) = \sum_{n \geq 0} x^{(n)}$ for all $x \in X$ is injective, whence the claim. $\square$

The following is then an immediate consequence of Theorem 4.30.

Theorem 7.11. *Let $G = \mathfrak{G}^{pma}_{\mathcal{D}}(\mathbb{K})$ be a complete Kac–Moody group of (indecomposable) simply connected type over a field $\mathbb{K}$ of characteristic zero, and let U^{im+} denote its positive imaginary subgroup. Then exactly one of the following holds.*

(1) G is of spherical type and $U^{im+} = \{1\}$.

(2) G is of affine type and U^{im+} is infinite abelian.

(3) G is of indefinite type and U^{im+} contains a nonabelian free subgroup.

Chapter 8

Kac–Moody groups in positive characteristic

In this chapter, we study the structure of complete Kac–Moody groups over finite fields. Note that these are locally compact and totally disconnected topological groups by Proposition 6.3 (1) and Remark 6.38. In this thesis, they will be called **locally compact Kac–Moody groups**.

As mentioned at the end of §3.2.1, the interest in the structure of those groups is motivated by the fact that they constitute a prominent family of locally compact groups which are simultaneously *topologically simple* and *non-linear over any field* (see [Rém04] and [CR09]). They also show some resemblance with the simple linear locally compact groups arising from semi-simple algebraic groups over local fields of positive characteristic.

The results presented in this chapter are part of the original work of this thesis. The results from Section 8.1 were jointly obtained with Pierre-Emmanuel Caprace.

8.1 On open subgroups of locally compact Kac–Moody groups

In this section, we present a proof of Theorem D from the introduction and give several corollaries to this theorem. As mentioned at the beginning of Section 1.4, Theorem A will be an essential tool in the proof of Theorem D. The content of the present section is the second part of the joint paper [CM12].

8.1.1 Statement of the results. — Let $G = \mathfrak{G}^{crr}(\mathbb{F}_q)$ be a complete Kac–Moody group à la Caprace–Rémy–Ronan over a finite field $\mathbb{F}_q$ (see Section 6.1). One of the most natural questions one might ask about the structure of the topological group G is: What are its open subgroups? A related, more specific, question is: How many open subgroups of G are there? Let us introduce some terminology providing possible answers to this more specific question. We say that G **has few open subgroups** if every proper open subgroup of G is compact. We say that G is **Noetherian** if G satisfies an ascending chain condition on open subgroups. Equivalently G is Noetherian if and only if every open subgroup of G is compactly generated (see Lemma 8.26 below). Clearly, if G has few open subgroups, then it is Noetherian. Basic examples of locally compact groups that are Noetherian — and in fact, even have few open subgroups — are connected groups and compact groups. Noetherianity can thus be viewed as a finiteness condition which generalises simultaneously the notion of connectedness and of compactness. It is highlighted in [CM11b], where it is notably shown that a Noetherian group admits a subnormal series with every subquotient compact, or abelian, or simple. An example of a non-Noetherian group is given by the additive group $\mathbb{Q}_p$ of the p-adics. Other examples, including simple ones, can be constructed as groups acting on trees.

According to a theorem of G. Prasad [Pra82] (which he attributes to Tits), simple locally compact groups arising from algebraic groups over local fields have few open subgroups. Note however that this is not the case in general for locally compact Kac–Moody groups G: indeed, as we saw in Section 6.1, parabolic subgroups of G are open, but they are non-compact as soon as they are not of spherical type.

Back to our first question of determining the open subgroups of G, our main result is that parabolic subgroups in locally compact Kac–Moody groups are essentially the only source of open subgroups.

Theorem 8.1. *Every open subgroup of a complete Kac–Moody group $G = \mathfrak{G}^{crr}(\mathbb{F}_q)$ over a finite field has finite index in some parabolic subgroup.*

Moreover, given an open subgroup O, there are only finitely many distinct parabolic subgroups of G containing O as a finite index subgroup.

A more precise statement of this theorem will be given later, see Theorem 8.7. This theorem then allows one to give answers to our second, more specific question.

Corollary 8.2. *Complete Kac–Moody groups over finite fields are Noetherian.*

Corollary 8.3. *Let G be a complete Kac–Moody group of irreducible type over a finite field. Then G has few open subgroups if and only if the Weyl group of G is of affine type, or of compact hyperbolic type.*

We recall that the Weyl groups of affine or compact hyperbolic type are precisely those all of whose proper parabolic subgroups are finite. Notice that the list of all compact hyperbolic types of Weyl groups is finite and contains diagrams of rank at most 5 (see *e.g.* Exercise V.4.15 on p. 133 in [Bou68]). The groups in Corollary 8.3 include in particular all complete Kac–Moody groups of rank two.

Another application of Theorem 8.1 is that it shows how the BN-pair structure is encoded in the topological group structure of a Kac–Moody group. Here is a precise formulation of this.

Corollary 8.4. *Let G be a complete Kac–Moody group over a finite field and $P < G$ be an open subgroup. If P is maximal in its commensurability class, then P is a parabolic subgroup of G.*

We recall that two subgroups H_1, H_2 of a group G are **commensurable** if their intersection $H_1 \cap H_2$ has finite index in both of them.

8.1.2 Notations. — For the rest of this section, we fix a minimal Kac–Moody group $\mathcal{G} = \mathfrak{G}_{\mathcal{D}}(\mathbb{F}_q)$, say of simply connected type, over a finite field $\mathbb{F}_q$ of order q. As usual, we also fix a realisation of the associated generalised Cartan matrix A, yielding in particular a set $\Phi := \Delta^{\mathrm{re}}$ of real roots, and we let (W, S) be the corresponding Coxeter system. Recall from §5.3.1 that $\mathcal{G}$ is endowed with a twin group datum $\{U_\alpha \mid \alpha \in \Phi\}$, which yields a twin BN-pair $(\mathcal{B}_+, \mathcal{B}_-, \mathcal{N})$ with associated twin building (Δ_+, Δ_-). Let C_0 be the fundamental chamber of $\Delta := \Delta_+$, namely the chamber such that $\mathcal{B}_+ = \mathrm{Stab}_{\mathcal{G}}(C_0)$, and let A_0 be the fundamental apartment of Δ, so that $\mathcal{N} = \mathrm{Stab}_{\mathcal{G}}(A_0)$ and $\mathcal{H} := \mathcal{B}_+ \cap \mathcal{N} = \mathrm{Fix}_{\mathcal{G}}(A_0)$. We identify Φ with the set of half-spaces of A_0.

We next let $G = \mathfrak{G}^{crr}(\mathbb{F}_q)$ be the complete Kac–Moody group à la Caprace–Rémy–Ronan over $\mathbb{F}_q$ (see Section 6.1). Thus G contains $\mathcal{G}$ as a dense subgroup, is locally compact and totally disconnected, and acts properly and continuously on Δ by automorphims. Let $B = \mathcal{H} \ltimes U^{rr+}$ be the closure of $\mathcal{B}_+$ in G, and set $N := \mathrm{Stab}_G(A_0)$ and $H := B \cap N = \mathrm{Fix}_G(A_0)$. [We warn the reader that $\mathcal{N}$ and $\mathcal{H}$ are discrete, whence closed in G while N and H are non-discrete closed subgroups.] Then (B, N) is a BN-pair of type (W, S) for G; in particular $N/H \cong W$. Moreover, the group B is a compact open subgroup, and every standard parabolic subgroup $P_J = BW_JB$ for some $J \subseteq S$ is thus open in G. Important to our later purposes is the fact that the group G acts transitively on the *complete* apartment system of Δ. In particular B acts transitively on the apartments containing C_0.

For a root $\alpha \in \Phi$, we denote as in Section 1.4 the unique reflection of W fixing the wall $\partial\alpha$ pointwise by r_α. In addition, we choose some element $n_\alpha \in N \cap \langle U_\alpha \cup U_{-\alpha} \rangle$ which maps onto r_α under the quotient map $N \to N/H \cong W$ (see axiom (TRD2) from Definition 2.19).

Finally, recall from Section 1.4 the notation $J^\perp := \{s \in S \setminus J \mid sj = js \ \forall j \in J\}$ for a subset $J \subseteq S$. Recall also that we call a parabolic subgroup of W of **essential type** if its irreducible components are all non-spherical, and that $J \subseteq S$ is **essential** if W_J is of essential type.

For the rest of this section, we fix for each parabolic subgroup P of W a subset J of S, which we call the **type** of P, such that P can be written in the form $P = wW_Jw^{-1}$ for some $w \in W$. When P is standard of the form W_I, we let I be its type. Note that this choice is in general not unique as there might be conjugate subsets of S. However, P is of essential type if and only if its type is essential, independently of the above choice of a type for P.

8.1.3 On Levi decompositions in complete Kac–Moody groups. — We first describe "complete" versions of the Levi decompositions of the parabolic subgroups of $\mathcal{G}$ described in §5.3.2, which we use to give a sufficient condition for an open subgroup O of G to be contained with finite index in a parabolic subgroup.

Given $J \subseteq S$, we denote by $\mathcal{P}_J = \mathcal{B}_+ W_J \mathcal{B}_+$ (respectively $P_J = BW_JB$) the standard parabolic subgroup of $\mathcal{G}$ (respectively G) of type J and by $R_J(C_0)$ the J-residue of Δ containing the chamber C_0. Thus $\mathcal{P}_J = \mathrm{Stab}_{\mathcal{G}}(R_J(C_0))$, $P_J = \mathrm{Stab}_G(R_J(C_0))$ and $\mathcal{P}_J$ is dense in P_J. Recall from §5.3.2 the definition of $\Phi_J \subseteq \Phi$ and define as in this paragraph the subgroups

$$\mathcal{L}_J^+ = \langle U_\alpha \mid \alpha \in \Phi_J \rangle, \quad \mathcal{L}_J = \mathcal{H} \cdot \mathcal{L}_J^+$$

and

$$\mathcal{U}_J = \ll \langle U_\alpha \mid \alpha \in \Phi, \ \alpha \supset R_J(C_0) \cap A_0 \rangle \gg_{\mathcal{B}_+}$$

of $\mathcal{G}$. Then by Proposition 5.16, we have a semidirect decomposition

$$\mathcal{P}_J = \mathcal{L}_J \ltimes \mathcal{U}_J$$

with unipotent radical $\mathcal{U}_J$ and Levi factor $\mathcal{L}_J$.

We next define

$$L_J^+ = \overline{\mathcal{L}_J^+}, \qquad L_J = \overline{\mathcal{L}_J} \qquad \text{and} \qquad U_J = \overline{\mathcal{U}_J}.$$

Thus U_J and L_J are closed subgroups of P_J, respectively called the **unipotent radical** and the **Levi factor** of P_J.

Lemma 8.5. *With the notations above:*

(1) U_J is a compact normal subgroup of P_J, and we have $P_J = L_J \cdot U_J$.

(2) L_J^+ is normal in L_J and we have $L_J = \mathcal{H} \cdot L_J^+$.

Proof. Since $\mathcal{U}_J$ is normal in $\mathcal{P}_J$, which is dense in P_J, it is clear that U_J is normal in P_J. Moreover U_J is compact (since it is contained in B) and the product $L_J \cdot U_J$ is thus closed in P_J. Assertion (1) follows since $L_J \cdot U_J$ contains $\mathcal{P}_J$.

For assertion (2), we remark that $\mathcal{H}$ normalises $\mathcal{L}_J^+$ and hence also L_J^+. Moreover, since $\mathcal{H}$ is finite, hence compact, the product $\mathcal{H} \cdot L_J^+$ is closed. Since $\mathcal{H} \cdot \mathcal{L}_J^+$ is dense in L_J, the conclusion follows. □

Notice that the decomposition $P_J = L_J \cdot U_J$ is even semidirect when J is spherical, see [RR06, Section 1.C.]. It is probably also the case in general, but this will not be needed here. Note that the corresponding statement for the Mathieu–Rousseau group is always true by Proposition 6.29.

Lemma 8.6. *Let $J \subseteq S$. Then every open subgroup O of P_J that contains the product $L_J^+ \cdot U_{J \cup J^\perp}$ has finite index in P_J.*

Proof. Set $K := J^\perp$ and $U := U_{J\cup J^\perp}$. Note that $U \lhd P_{J\cup K} = L_{J\cup K} \cdot U$. Moreover, L_J^+ is normal in $L_{J\cup K}$. Indeed, as $[U_\alpha, U_\beta] = 1$ for all $\alpha \in \Phi_J$ and $\beta \in \Phi_K$, the subgroups $\mathcal{L}_J^+$ and $\mathcal{L}_K^+$ centralise each other. Since in addition $\mathcal{H}$ normalises each root group, we get a decomposition $\mathcal{L}_{J\cup K} = \mathcal{H} \cdot \mathcal{L}_J^+ \cdot \mathcal{L}_K^+$. In particular, $\mathcal{L}_{J\cup K}$ normalises $\mathcal{L}_J^+$, whence also L_J^+. As the normaliser of a closed subgroup is closed, this implies that $L_{J\cup K}$ normalises L_J^+, as desired.

Let $\pi_1\colon P_{J\cup K} \to P_{J\cup K}/U$ denote the natural projection. Then $\pi_1(L_J^+)$ is normal in $P_{J\cup K}/U$, since it is the image of L_J^+ under the composition map

$$L_{J\cup K} \to \frac{L_{J\cup K}}{L_{J\cup K}\cap U} \overset{\cong}{\to} \frac{P_{J\cup K}}{U} : l \mapsto l(L_{J\cup K}\cap U) \mapsto lU.$$

Let $\pi\colon P_{J\cup K} \to \pi_1(P_{J\cup K})/\pi_1(L_J^+)$ denote the composition of π_1 with the canonical projection onto $\pi_1(P_{J\cup K})/\pi_1(L_J^+)$. Note that π is an open continuous group homomorphism. Then $\pi(P_J) = \pi_1(L_J^+ \cdot U_J \cdot \mathcal{H})/\pi_1(L_J^+)$ is compact. Indeed, it is homeomorphic to the quotient of the compact group $\pi_1(U_J \cdot \mathcal{H})$ by the normal subgroup $\pi_1(L_J^+ \cap U_J \cdot \mathcal{H})$ under the map

$$\frac{\pi_1(L_J^+ \cdot U_J \cdot \mathcal{H})}{\pi_1(L_J^+)} \overset{\cong}{\to} \frac{\pi_1(U_J \cdot \mathcal{H})}{\pi_1(L_J^+ \cap U_J \cdot \mathcal{H})} : \pi_1(l\cdot u)\pi_1(L_J^+) \mapsto \pi_1(u)\pi_1(L_J^+ \cap U_J \cdot \mathcal{H}).$$

In particular, since $\pi(O)$ is open in $\pi(P_J)$, it has finite index in $\pi(P_J)$. But then since $O = \pi^{-1}(\pi(O))$ by hypothesis, O has finite index in $\pi^{-1}(\pi(P_J)) = P_J$, as desired. □

8.1.4 A refined version of Theorem 8.1. — We will prove the following statement, having Theorem 8.1 as an immediate corollary.

Theorem 8.7. *Let O be an open subgroup of G. Let $J \subseteq S$ be the type of a residue which is stabilised by some finite index subgroup of O and minimal with respect to this property.*

Then there exist a spherical subset $J' \subseteq J^\perp$ and an element $g \in G$ such that

$$L_J^+ \cdot U_{J\cup J^\perp} < gOg^{-1} < P_{J\cup J'}.$$

In particular, gOg^{-1} has finite index in $P_{J\cup J'}$.

Moreover, any subgroup of G containing gOg^{-1} as a finite index subgroup is contained in $P_{J\cup J''}$ for some spherical subset $J'' \subseteq J^\perp$. In particular, only finitely many distinct parabolic subgroups contain O as a finite index subgroup.

8.1.5 Proof of Theorem 8.7: outline and first observations. — This paragraph and the next ones are devoted to the proof of Theorem 8.7 itself.

Let thus O be an open subgroup of G. We define the subset J of S as in the statement of the theorem, namely, J is minimal amongst the subsets L of S for which there exists a $g \in G$ such that $O \cap g^{-1}P_L g$ has finite index in O. For such a $g \in G$, we set $O_1 = gOg^{-1} \cap P_J$. Thus O_1 stabilises $R_J(C_0)$ and is an open subgroup of G contained in gOg^{-1} with finite index.

We first observe that the desired statement is essentially empty when O is compact. Indeed, in that case the Bruhat–Tits fixed point theorem (see §1.2.2) ensures that O stabilises a spherical residue of G, and hence Theorem 8.7 stands proven with $J = \varnothing$. It thus remains to prove the theorem when O, and hence also O_1, is non-compact, which we assume henceforth.

We begin with the following simple observation.

Lemma 8.8. *J is essential.*

Proof. Let $J_1 \subseteq J$ denote the union of the non-spherical irreducible components of J. As P_{J_1} has finite index in P_J, the subgroup $O_1 \cap P_{J_1}$ is open of finite index in O_1 and stabilises $R_{J_1}(C_0)$. The definition of J then yields $J_1 = J$. □

Let us now describe the outline of the proof. Our first task will be to show that O_1 contains L_J^+. We will see that this is equivalent to prove that O_1 acts transitively on the standard J-residue $R_J(C_0)$, or else that the stabiliser in O_1 of any apartment A containing C_0 is transitive on $R_J(C_0) \cap A$. Since each group $\mathrm{Stab}_{O_1}(A)/\mathrm{Fix}_{O_1}(A)$ can be identified with a subgroup of the Coxeter group W acting on A, we will be in a position to apply the results on Coxeter groups from Section 1.4. This will allow us to show that each $\mathrm{Stab}_{O_1}(A)/\mathrm{Fix}_{O_1}(A)$ contains a finite index parabolic subgroup of type $I_A \subseteq J$, and hence acts transitively on the corresponding residue.

We thus begin by defining some "maximal" subset I of J such that $\mathrm{Stab}_{O_1}(A_1)$ acts transitively on $R_I(C_0) \cap A_1$ for a suitably chosen apartment A_1 containing C_0. We then establish that I contains all the types I_A when A varies over all apartments containing C_0. This eventually allows us to prove that in fact $I = J$, so that $\mathrm{Stab}_{O_1}(A_1)$ is transitive on $R_J(C_0) \cap A_1$, or else that O_1 contains L_J^+, as desired.

We next show that O_1 contains the unipotent radical $U_{J \cup J^\perp}$. Finally, we make use of the transitivity of O_1 on $R_J(C_0)$ to prove that O is contained in the desired parabolic subgroup.

8.1.6 Proof of Theorem 8.7: O_1 contains L_J^+. — We first need to introduce some additional notation which we will retain until the end of the proof.

Let $\mathcal{A}_{\geq C_0}$ denote the set of apartments of Δ containing C_0. For $A \in \mathcal{A}_{\geq C_0}$, set $N_A := \mathrm{Stab}_{O_1}(A)$ and $\overline{N}_A = N_A/\mathrm{Fix}_{O_1}(A)$, which one identifies with a subgroup of W. Finally, for $h \in N_A$, denote by $\overline{h}$ its image in $\overline{N}_A \leq W$. Here is the main tool developed in Section 1.4.

Lemma 8.9. *For all $A \in \mathcal{A}_{\geq C_0}$, there exists $h \in N_A$ such that*

$$\mathrm{Pc}(\overline{h}) = \langle r_\alpha \mid \alpha \textit{ is an } \overline{h}\textit{-essential root of } \Phi \rangle$$

and such that $\mathrm{Pc}(\overline{h})$ is of essential type and has finite index in $\mathrm{Pc}(\overline{N}_A)$.

Proof. This is an immediate consequence of Corollary 1.35 and Lemma 1.25. □

Lemma 8.10. *Let $(g_n)_{n\in\mathbb{N}}$ be an infinite sequence of elements of O_1. Then there exist an apartment $A \in \mathcal{A}_{\geq C_0}$, a subsequence $(g_{\psi(n)})_{n\in\mathbb{N}}$ and elements $z_n \in O_1$, $n \in \mathbb{N}$, such that for all $n \in \mathbb{N}$ we have*

(1) $h_n := z_0^{-1} z_n \in N_A$,

(2) $\mathrm{d}(C_0, z_n R) = \mathrm{d}(C_0, g_{\psi(n)} R)$ *for every residue R containing C_0 and*

(3) $|\,\mathrm{d}(C_0, h_n C_0) - \mathrm{d}(C_0, g_{\psi(n)} C_0)| < \mathrm{d}(C_0, z_0 C_0)$.

Proof. As O_1 is open, it contains a finite index subgroup $K := \mathrm{Fix}_G(B(C_0, r))$ of B for some $r \in \mathbb{N}$. Since B is transitive on the set $\mathcal{A}_{\geq C_0}$, we deduce that K has only finitely many orbits in $\mathcal{A}_{\geq C_0}$, say $\mathcal{A}_1, \dots, \mathcal{A}_k$. So, up to choosing a subsequence, we may assume that all chambers $g_n C_0$ belong to the same K-orbit $\mathcal{A}_{i_0}$ of apartments. Hence there exist elements $x_n \in K \subset O_1$ and an apartment $A' \in \mathcal{A}_{i_0}$ containing C_0 such that $g'_n := x_n g_n \in O_1$, $g'_n C_0 \in A'$ and $\mathrm{d}(C_0, g'_n C_0) = \mathrm{d}(C_0, g_n C_0)$. For each n, we now choose an element of G stabilising A' and mapping C_0 to $g'_n C_0$. Thus such an element is in the same right coset modulo B as g'_n. In particular, up to choosing a subsequence, we may assume it has the form $g'_n y_n b \in \mathrm{Stab}_G(A')$ for some $y_n \in K$ and some $b \in B$ independant of n. Denote by $\{\psi(n) \mid n \in \mathbb{N}\}$ the resulting indexing set for the subsequence. Then setting $A := bA' \in \mathcal{A}_{\geq C_0}$, the sequence $z_n := g'_{\psi(n)} y_{\psi(n)} \in O_1$ is such that $h_n := z_0^{-1} z_n \in b\,\mathrm{Stab}_G(A') b^{-1} \cap O_1 = \mathrm{Stab}_{O_1}(A) = N_A$ and

$$|\,\mathrm{d}(C_0, h_n C_0) - \mathrm{d}(C_0, g_{\psi(n)} C_0)| = |\,\mathrm{d}(z_0 C_0, z_n C_0) - \mathrm{d}(C_0, z_n C_0)| < \mathrm{d}(C_0, z_0 C_0).$$

□

Lemma 8.11. *There exists an apartment $A \in \mathcal{A}_{\geq C_0}$ such that the orbit $N_A \cdot C_0$ is unbounded. In particular, the parabolic closure in W of $\overline{N}_A$ is non-spherical.*

Proof. Since O_1 is non-compact, the orbit $O_1 \cdot C_0$ is unbounded in Δ. For $n \in \mathbb{N}$, choose $g_n \in O_1$ such that $\mathrm{d}(C_0, g_n C_0) \geq n$. Then by Lemma 8.10, there exist an apartment $A \in \mathcal{A}_{\geq C_0}$ and elements $h_n \in N_A$ for n in some unbounded subset of $\mathbb{N}$ such that $\mathrm{d}(C_0, h_n C_0)$ is arbitrarily large when n varies. This proves the lemma. □

Let $A_1 \in \mathcal{A}_{\geq C_0}$ be an apartment such that the type of the product of the non-spherical irreducible components of $\mathrm{Pc}(\overline{N}_{A_1})$ is nonempty and maximal for this property. Such an apartment exists by Lemma 8.11. Now choose $h_{A_1} \in N_{A_1}$ as in Lemma 8.9, so that in particular $[\mathrm{Pc}(\overline{N}_{A_1}) : \mathrm{Pc}(\overline{h}_{A_1})] < \infty$. Up to conjugating O_1 by an element of P_J, we may then assume without loss of generality that $\mathrm{Pc}(\overline{N}_{A_1})$ is standard, non-spherical, and that $\mathrm{Pc}(\overline{h}_{A_1})$ is the product of its non-spherical irreducible components. Thus $\mathrm{Pc}(\overline{h}_{A_1})$ is of the form W_I, where I is essential. Moreover, I is maximal in the following sense: if $A \in \mathcal{A}_{\geq C_0}$ is such that $\mathrm{Pc}(\overline{N}_A)$ contains a parabolic subgroup of essential type I_A with $I_A \supseteq I$, then $I = I_A$.

Now that I is defined, we need some tool to show that O_1 contains sufficiently many root groups U_α. This will ensure that O_1 is "transitive enough" in two ways: first on residues in the building by showing it contains subgroups of the form $\mathcal{L}_T^+$, and second on residues in apartments by establishing the presence in O_1 of enough $n_\alpha \in \langle U_\alpha \cup U_{-\alpha} \rangle$, since these lift reflections r_α in stabilisers of apartments. This tool is provided by the so-called (FPRS) property from [CR09, 2.1] which we alluded to in §6.4.1, and which we now state formally. Note for this that as O_1 is open, it contains the fixator in G of a ball of Δ: we fix $r \in \mathbb{N}$ such that $O_1 \supset K_r := \mathrm{Fix}_G(B(C_0, r))$.

Lemma 8.12. *There exists a constant $N = N(W, S, r) \in \mathbb{N}$ such that for every root $\alpha \in \Phi$ with $\mathrm{d}(C_0, \alpha) > N$, the root group $U_{-\alpha}$ is contained in $\mathrm{Fix}_G(B(C_0, r)) = K_r$.*

Proof. See [CR09, Proposition 4]. □

We also record a version of this result in a slightly more general setting.

Lemma 8.13. *Let $g \in G$ and let $A \in \mathcal{A}_{\geq C_0}$ containing the chamber $D := gC_0$. Also, let $b \in B$ such that $A = bA_0$, and let $\alpha = b\alpha_0$ be a root of A, with $\alpha_0 \in \Phi$. Then there exists $N = N(W, S, r) \in \mathbb{N}$ such that if $\mathrm{d}(D, -\alpha) > N$ then $bU_{\alpha_0}b^{-1} \subseteq gK_rg^{-1}$.*

Proof. Take for $N = N(W, S, r)$ the constant of Lemma 8.12 and suppose that $\mathrm{d}(D, -\alpha) > N$. Let $h \in \mathrm{Stab}_G(A_0)$ be such that $hC_0 = b^{-1}D$. Then

$$N < \mathrm{d}(D, -\alpha) = \mathrm{d}(bhC_0, -b\alpha_0) = \mathrm{d}(hC_0, -\alpha_0) = \mathrm{d}(C_0, -h^{-1}\alpha),$$

and so Lemma 8.12 implies $h^{-1}U_{\alpha_0}h = U_{h^{-1}\alpha_0} \subseteq K_r$. Let $b_1 \in B$ such that $bh = gb_1$. Then

$$bU_{\alpha_0}b^{-1} \subseteq bhK_rh^{-1}b^{-1} = gb_1K_rb_1^{-1}g^{-1} = gK_rg^{-1}.$$

□

This will prove especially useful in the following form, when we will use the description of the parabolic closure of some $w \in W$ in terms of w-essential roots as in Lemma 8.9.

Lemma 8.14. *Let $A \in \mathcal{A}_{\geq C_0}$ and $b \in B$ such that $A = bA_0$. Also, let $\alpha = b\alpha_0$ ($\alpha_0 \in \Phi$) be a w-essential root of A for some $w \in \mathrm{Stab}_G(A)/\mathrm{Fix}_G(A)$, and let $g \in \mathrm{Stab}_G(A)$ be a representative of w. Then there exists $n \in \mathbb{Z}$ such that for $\epsilon \in \{+,-\}$ we have $U_{\epsilon\alpha_0} \subseteq b^{-1}g^{\epsilon n}K_r g^{-\epsilon n}b$.*

Proof. Choose $n \in \mathbb{Z}$ such that $\mathrm{d}(g^{\epsilon n}C_0, -\epsilon\alpha) > N$ for $\epsilon \in \{+,-\}$, where $N = N(W,S,r)$ is the constant appearing in the statement of Lemma 8.12. Thus, for $\epsilon \in \{+,-\}$ we have $\mathrm{d}(b^{-1}g^{\epsilon n}C_0, -\epsilon\alpha_0) > N$, and so $\mathrm{d}(C_0, -\epsilon(b^{-1}g^{-\epsilon n}b)\alpha_0) > N$. Lemma 8.12 then yields

$$(b^{-1}g^{-\epsilon n}b)U_{\epsilon\alpha_0}(b^{-1}g^{-\epsilon n}b)^{-1} = U_{\epsilon(b^{-1}g^{-\epsilon n}b)\alpha_0} \subseteq K_r,$$

and so

$$U_{\epsilon\alpha_0} \subseteq (b^{-1}g^{\epsilon n}b)K_r(b^{-1}g^{-\epsilon n}b) = (b^{-1}g^{\epsilon n})K_r(g^{-\epsilon n}b).$$

□

We are now ready to prove how the different transitivity properties of O_1 are related.

Lemma 8.15. *Let $T \subseteq S$ be essential, and let $A \in \mathcal{A}_{\geq C_0}$. Then the following are equivalent:*

(1) O_1 contains $\mathcal{L}_T^+$;

(2) O_1 is transitive on $R_T(C_0)$;

(3) N_A is transitive on $R_T(C_0) \cap A$;

(4) $\overline{N}_A$ contains the standard parabolic subgroup W_T of W.

Proof. The equivalence (3) ⇔ (4), as well as the implications (1) ⇒ (2), (3) are trivial.

To see that (4) ⇒ (2), note that if $b \in B$ maps A_0 onto A, then for each $\alpha_0 \in \Phi_T$, we have $bU_{\pm\alpha_0}b^{-1} \subseteq O_1$, and so $O_1 \supseteq b\mathcal{L}_T^+b^{-1}$ is transitive on $R_T(C_0)$. Indeed, let $\alpha_0 \in \Phi_T$ and consider the corresponding root $\alpha := b\alpha_0 \in \Phi_T(A)$ of A. By Lemma 1.37, there exists $w \in W_T \subseteq \overline{N}_A$ such that α is w-essential. Then if $g \in O_1$ is a representative for w, Lemma 8.14 yields an $n \in \mathbb{Z}$ such that for $\epsilon \in \{+,-\}$ we have $U_{\epsilon\alpha_0} \subseteq b^{-1}g^{\epsilon n}K_r g^{-\epsilon n}b \subseteq b^{-1}O_1 b$.

Finally, we show (2) ⇒ (1). Again, it is sufficient to check that if $\alpha \in \Phi_T$, then O_1 contains $U_{\epsilon\alpha}$ for $\epsilon \in \{+,-\}$. By Lemma 1.37, there exists $g \in \mathrm{Stab}_G(A_0)$ stabilising $R_T(C_0) \cap A_0$ such that α is $\overline{g}$-essential, where $\overline{g}$ denotes the image of g in the quotient group $\mathrm{Stab}_G(A_0)/\mathrm{Fix}_G(A_0)$. Then, by Lemma 8.14, one can find an $n \in \mathbb{Z}$ such that $U_{\epsilon\alpha} \subseteq g^{\epsilon n}K_r g^{-\epsilon n}$ for $\epsilon \in \{+,-\}$. Now, since O_1 is transitive on

$R_T(C_0)$, there exist $h_\epsilon \in O_1$ such that $h_\epsilon C_0 = g^{\epsilon n} C_0$, and so we find $b_\epsilon \in B$ such that $g^{\epsilon n} = h_\epsilon b_\epsilon$. Therefore

$$U_{\epsilon\alpha} \subseteq h_\epsilon b_\epsilon K_r b_\epsilon^{-1} h_\epsilon^{-1} = h_\epsilon K_r h_\epsilon^{-1} \subseteq O_1.$$

□

Now, to ensure that O_1 indeed satisfies one of those properties for some "maximal T", we use Lemma 8.9 to show that stabilisers in O_1 of apartments contain finite index parabolic subgroups.

Lemma 8.16. *Let $A \in \mathcal{A}_{\geq C_0}$. Then there exists an essential subset $I_A \subseteq S$ such that $\overline{N}_A$ contains a parabolic subgroup P_{I_A} of W of type I_A as a finite index subgroup.*

Proof. Choose $h \in N_A$ as in Lemma 8.9, so that in particular $\mathrm{Pc}(\overline{h})$ is generated by the reflections r_α with α an $\overline{h}$-essential root of A. Let $\alpha = b\alpha_0$ be such a root ($\alpha_0 \in \Phi$), where $b \in B$ maps A_0 onto A. By Lemma 8.14, we then find $K \in \mathbb{Z}$ such that for $\epsilon \in \{+,-\}$,

$$U_{\epsilon\alpha_0} \subseteq (b^{-1}h^{\epsilon K})K_r(h^{-\epsilon K}b) \subseteq b^{-1}O_1 b.$$

In particular, $n_{\alpha_0} \in \langle U_{\alpha_0} \cup U_{-\alpha_0}\rangle \subseteq b^{-1}O_1 b$. As r_{α_0} is the image in W of n_{α_0} and since $r_\alpha = b r_{\alpha_0} b^{-1}$, we finally obtain $\mathrm{Pc}(\overline{h}) \subseteq \overline{N}_A$. Then $P_{I_A} := \mathrm{Pc}(\overline{h})$ is the desired parabolic subgroup, of type I_A. □

For each $A \in \mathcal{A}_{\geq C_0}$, we fix such an $I_A \subseteq S$ and we consider the corresponding parabolic P_{I_A} contained in $\overline{N}_A$. Note then that $P_{I_{A_1}}$ has finite index in $\mathrm{Pc}(\overline{N}_{A_1})$ by Lemma 1.22, and so $I = I_{A_1}$.

Lemma 8.17. *O_1 contains L_I^+.*

Proof. As noted above, we have $I = I_{A_1}$ and $P_I = W_I$. Since O_1 is closed in G, Lemma 8.15 allows us to conclude. □

We now have to show that I is "big enough", that is, $I = J$. For this, we first need to know that I is "uniformly" maximal amongst all apartments containing C_0.

Lemma 8.18. *Let $A \in \mathcal{A}_{\geq C_0}$. Then $I_A \subseteq I$.*

Proof. Set $R_1 := R_I(C_0) \cap A$ and let R_2 be an I_A-residue in A on which N_A acts transitively and that is at minimal distance from R_1 amongst such residues. Note that N_A is transitive on R_1 as well by Lemma 8.15.

If $R_1 \cap R_2$ is nonempty, then N_A is also transitive on the standard $I \cup I_A$-residue of A and so $\overline{N}_A$ contains $W_{I \cup I_A}$. By maximality of I and since $I \cup I_A$ is again essential, this implies $I_A \subseteq I$, as desired.

We henceforth assume that $R_1 \cap R_2 = \varnothing$. Let $b \in B$ such that $bA_0 = A$. Consider a root $\alpha = b\alpha_0$ of A, $\alpha_0 \in \Phi$, whose wall $\partial\alpha$ separates R_1 from R_2.

If both R_1 and R_2 are at unbounded distance from $\partial\alpha$, then the transitivity of N_A on R_1 and R_2 together with Lemma 8.13 yield $bU_{\pm\alpha_0}b^{-1} \subseteq K_r \subseteq O_1$. Since $r_{\alpha_0} \in \langle U_{\alpha_0} \cup U_{-\alpha_0}\rangle$, we thus have $r_\alpha := br_{\alpha_0}b^{-1} \in O_1$ and so $r_\alpha \in \overline{N}_A$. But then $\overline{N}_A = r_\alpha \overline{N}_A r_\alpha^{-1}$ is also transitive on the I_A-residue $r_\alpha R_2$ which is closer to R_1, a contradiction.

If R_2 is at bounded distance from $\partial\alpha$ then by Lemma 1.38, r_α centralises the stabiliser P in W of R_2, that is, $P = r_\alpha P r_\alpha^{-1}$. Note that $\overline{N}_A$ contains P since it is transitive on R_2. Thus N_A is transitive on the I_A-residue $r_\alpha R_2$, which is closer to R_1, again a contradiction.

Thus we are left with the case where R_1 is contained in a tubular neighbourhood of every wall $\partial\alpha$ separating R_1 from R_2. But in that case, Lemma 1.38 again yields that W_I is centralised by every reflection r_α associated to such walls. Choose chambers C_i in R_i, $i = 1, 2$, such that $\mathrm{d}(C_1, C_2) = \mathrm{d}(R_1, R_2)$, and let $\partial\alpha_1, \dots, \partial\alpha_k$ be the walls separating C_1 from C_2, crossed in that order by a minimal gallery from C_1 to C_2. Then each α_i, $1 \le i \le k$, separates R_1 from R_2 and so $w := r_{\alpha_k} \dots r_{\alpha_1}$ centralises W_I and maps C_1 to C_2. So $W_I = wW_Iw^{-1} \subseteq \overline{N}_A$ is transitive on wR_1 and R_2, and hence also on $R_{I \cup I_A}(C_2) \cap A$. Therefore $\overline{N}_A$ contains a parabolic subgroup of essential type $I \cup I_A$, so that $I \supseteq I_A$ by maximality of I, as desired. □

Lemma 8.19. *Let $A \in \mathcal{A}_{\ge C_0}$. Then $\overline{N}_A$ contains W_I as a subgroup of finite index.*

Proof. We know by Lemmas 8.15 and 8.17 that $\overline{N}_A$ contains W_I. Also, by Lemma 8.16, $\overline{N}_A$ contains a finite index parabolic subgroup $P_{I_A} = wW_{I_A}w^{-1}$ of type I_A, for some $w \in W$. Since $I_A \subseteq I$ by Lemma 8.18, we get $W_{I_A} \subseteq \overline{N}_A$ and so the parabolic subgroup $P := W_{I_A} \cap wW_{I_A}w^{-1}$ has finite index in W_{I_A}. As I_A is essential, [AB08, Proposition 2.43] then yields $P = W_{I_A}$ and so $W_{I_A} \subseteq wW_{I_A}w^{-1}$. Finally, since the chain $W_{I_A} \subseteq wW_{I_A}w^{-1} \subseteq w^2W_{I_A}w^{-2} \subseteq \dots$ stabilises, we find that $W_{I_A} = P_{I_A}$ has finite index in $\overline{N}_A$. The result follows. □

We are now ready to make the announced connection between I and J.

Lemma 8.20. $I = J$.

Proof. Let $\mathcal{R}$ denote the set of I-residues of Δ containing a chamber of $O_1 \cdot C_0$, and set $R := R_I(C_0)$. We first show that the distance from C_0 to the residues of $\mathcal{R}$ is bounded, and hence that $\mathcal{R}$ is finite.

Indeed, suppose for a contradiction that there exists a sequence of elements $g_n \in O_1$ such that $\mathrm{d}(C_0, g_nR) \ge n$ for all $n \in \mathbb{N}$. Then, up to choosing a subsequence and relabeling, Lemma 8.10 yields an apartment $A \in \mathcal{A}_{\ge C_0}$ and a sequence $(z_n)_{n \ge n_0}$ of elements of O_1 such that $h_n := z_{n_0}^{-1}z_n \in N_A$ and $\mathrm{d}(C_0, z_nR) = \mathrm{d}(C_0, g_nR)$. Moreover by Lemma 8.19, we have a finite coset decomposition of the form $\overline{N}_A = \coprod_{j=1}^t v_jW_I$. Denote by $\pi\colon N_A \to \overline{N}_A$ the natural projection. Again up to choosing a subsequence and relabeling, we may assume that $\pi(h_n) = v_{j_0}u_n$ for all $n \geq n_1$ (for some fixed $n_1 \in \mathbb{N}$), where each $u_n \in W_I$ and where j_0 is independant of n. Then the elements $w_n := \pi(h_{n_1}^{-1}h_n) = \pi(z_{n_1}^{-1}z_n)$ belong to W_I. Thus the chambers $z_{n_1}C_0$ and z_nC_0 belong to the same I-residue since z_{n_1} maps an I-gallery between C_0 and w_nC_0 to an I-gallery between $z_{n_1}C_0$ and z_nC_0. Therefore

$$\mathrm{d}(C_0, g_nR) = \mathrm{d}(C_0, z_nR) \leq \mathrm{d}(C_0, z_{n_1}C_0)$$

and so $\mathrm{d}(C_0, g_nR)$ is bounded, a contradiction.

So $\mathcal{R}$ is finite and is stabilised by O_1. Hence the kernel O' of the induced action of O_1 on $\mathcal{R}$ is a finite index subgroup of O_1 stabilising an I-residue. Up to conjugating by an element of O_1, we thus have $O' < P_I$ and $[O_1 : O'] < \infty$. Then $O'' := O_1 \cap P_I$ is open and contains O', and has therefore finite index in O_1. The definition of J finally implies that $I = J$. □

In particular, Lemmas 8.17 and 8.20 yield the following.

Corollary 8.21. *O_1 contains L_J^+.*

8.1.7 Proof of Theorem 8.7: O_1 contains the unipotent radical $U_{J\cup J^\perp}$.

— To show that O_1 contains the desired unipotent radical, we again make use of the (FPRS) property.

Lemma 8.22. *O_1 contains the unipotent radical $U_{J\cup J^\perp}$.*

Proof. By definition of $U_{J\cup J^\perp}$, we just have to check that for every $b \in B$ and every $\alpha \in \Phi$ containing $R_{J\cup J^\perp}(C_0) \cap A_0$, we have $bU_\alpha b^{-1} \in O_1$. Fix such b and α. In particular, α contains $R := R_J(C_0) \cap A_0$. We claim that R is at unbounded distance from the wall $\partial\alpha$ associated to α. Indeed, if it were not, then as J is essential by Lemma 8.8, the reflection r_α would centralise W_J by Lemma 1.38, and hence would belong to $W_{J^\perp}$ by Lemma 1.19, contradicting $\alpha \supset R_{J^\perp}(C) \cap A_0$.

Set now $A = bA_0$. Then $\alpha' = b\alpha$ is a root of A containing $R' := R_J(C_0) \cap A$. Moreover, R' is at unbounded distance from $-\alpha'$. Since O_1 is transitive on $R_J(C_0)$ by Corollary 8.21, there exists $g \in O_1$ such that $D := gC_0 \in R_J(C_0) \cap A$ and $\mathrm{d}(D, -\alpha') > N$, where N is provided by Lemma 8.13. This lemma then implies that $bU_\alpha b^{-1} \subseteq gK_rg^{-1} \subseteq O_1$, as desired. □

8.1.8 Proof of Theorem 8.7: endgame.

— We can now prove that gOg^{-1} is contained in a parabolic subgroup that has P_J as a finite index subgroup.

Lemma 8.23. *Every subgroup H of G containing O_1 as a subgroup of finite index is contained in some standard parabolic $P_{J\cup J'}$ of type $J \cup J'$, with J' spherical and $J' \subseteq J^\perp$.*

Proof. Recall that O_1 stabilises the J-residue $R := R_J(C_0)$ and acts transitively on its chambers by Corollary 8.21. Let $\mathcal{R}$ be the (finite) set of J-residues of Δ containing a chamber in the orbit $H \cdot C_0$.

We first claim that for any $R' \in \mathcal{R}$ there is a constant M such that R is contained in an M-neighbourhood of R' (and since $\mathcal{R}$ is finite we may then as well assume that this constant M is independant of R'). Indeed, because $\mathcal{R}$ is finite, there is a finite index subgroup H' of H which stabilises R'. In particular $\mathrm{d}(D, R') = \mathrm{d}(H' \cdot D, R')$ for any chamber D of R. Moreover, the chambers of R are contained in finitely H'-orbits since H acts transitively on R. The claim follows.

Let now $J' \subseteq S \setminus J$ be minimal such that $\mathbf{R} := R_{J \cup J'}(C_0)$ contains the reunion of the residues of $\mathcal{R}$. In other words, $H < P_{J \cup J'}$ with J' minimal for this property.

We next show that $J' \subseteq J^{\perp}$. For this, it is sufficient to see that H stabilises $R_{J \cup J^{\perp}}(C_0)$.

Note that, given $R' \in \mathcal{R}$, if A is an apartment containing some chamber C_0' of R', then every chamber D in $R \cap A$ is at distance at most M from $R' \cap A$. Indeed, if $\rho = \rho_{A, C_0'}$ is the retraction of Δ onto A centered at C_0' (see Lemma 2.3), then for every $D' \in R'$ such that $\mathrm{d}(D, D') \leq M$, the chamber $\rho(D')$ belongs to $R' \cap A$ and is at distance at most M from $D = \rho(D)$ since ρ is distance decreasing (see [Dav98, Lemma 11.2]).

Let now $g \in H$ and set $R' := gR \in \mathcal{R}$. Let Γ be a minimal gallery from C_0 to its combinatorial projection onto R', which we denote by C_0'. Let A be an apartment containing Γ. Finally, let $w \in W = \mathrm{Stab}_G(A)/\operatorname{Fix}_G(A)$ such that $wC_0 = C_0'$. We want to show that Γ is a $J^{\perp}$-gallery, that is, $w \in W_{J^{\perp}}$.

To this end, we first observe that, since Γ joins C_0 to its projection onto R', it does not cross any wall of $R' \cap A$. We claim that Γ does not cross any wall of $R \cap A$ either. Indeed, assume on the contrary that Γ crosses some wall m of $R \cap A$. Then by Lemma 1.37 we would find a wall $m' \neq m$ intersecting $R \cap A$ and parallel to m, and therefore also chambers of $R \cap A$ at unbounded distance from $R' \cap A$, a contradiction.

Thus every wall crossed by Γ separates $R \cap A$ from $R' \cap A$. In particular, $R \cap A$ is contained in an M-neighbourhood of any such wall m since it is contained in an M-neighbourhood of $R' \cap A$ and since every minimal gallery between a chamber in $R \cap A$ and a chamber in $R' \cap A$ crosses m. Then, by Lemmas 1.19 and 1.38, the reflection associated to m belongs to $W_{J^{\perp}}$. Therefore w is a product of reflections that belong to $W_{J^{\perp}}$, as desired.

Finally, we show that J' is spherical. As $\mathbf{R}$ splits into a product of buildings $\mathbf{R} = R_J \times R_{J'}$, where $R_J := R_J(C_0)$ and $R_{J'} := R_{J'}(C_0)$, we get a homomorphism $H \to \mathrm{Aut}(R_J) \times \mathrm{Aut}(R_{J'})$. As O_1 stabilises R_J and has finite index in H, the image of H in $\mathrm{Aut}(R_{J'})$ has finite orbits in $R_{J'}$. In particular, by the Bruhat–Tits fixed point theorem, H fixes a point in the Davis realisation of $R_{J'}$, and thus stabilises a spherical residue of $R_{J'}$. But this residue must be the whole of $R_{J'}$ by minimality of J'. This concludes the proof of the lemma. $\square$

Proof of Theorem 8.7. The first statement summarises Corollary 8.21 and Lemmas 8.22 and 8.23, since some conjugate gOg^{-1} of O contains O_1 as a finite index subgroup. The second statement then follows from Lemma 8.6 applied to the open subgroup O_1 of P_J. Finally, the two last statements are a consequence of Lemma 8.23. Indeed, any subgroup H containing gOg^{-1} with finite index also contains O_1 with finite index. Then H is a subgroup of some standard parabolic $P_{J\cup J'}$ for some spherical subset $J' \subset J^{\perp}$. Moreover, since the index of O_1 in $P_{J\cup J'}$ is finite, and since there are only finitely many spherical subsets of $J^{\perp}$, it follows that there are only finitely many possibilities for H. □

Remark 8.24. Let O be a subgroup of G, and let $J \subseteq S$ be as in the statement of Theorem 8.7. Assume that $J^{\perp}$ is spherical. Then $L_J^+ \cdot U_{J\cup J^{\perp}}$ has finite index in $P_{J\cup J^{\perp}}$ and is thus open since it is closed. Thus, in that case, O is open if and only if $L_J^+ \cdot U_{J\cup J^{\perp}} < gOg^{-1} < P_{J\cup J^{\perp}}$ for some $g \in G$.

Corollary 8.25. *Let O be an open subgroup of G and let $J \subseteq S$ be minimal such that O virtually stabilises a J-residue. If $J^{\perp} = \varnothing$, then there exists some $g \in G$ such that $L_J^+ \cdot U_J < gOg^{-1} < P_J = \mathcal{H} \cdot L_J^+ \cdot U_J$.*

Proof. This readily follows from Theorem 8.7. □

To prove Corollary 8.2, we use the following general fact, which is well known in the discrete case.

Lemma 8.26. *Let G be a locally compact group. Then G is Noetherian if and only if every open subgroup is compactly generated.*

Proof. Assume that G is Noetherian and let $O < G$ be open. Let $U_1 < O$ be the subgroup generated by some compact identity neighbourhood V in O. If $U_1 \neq O$, there is some $g_1 \in O \setminus U_1$ and we let $U_2 = \langle U_1 \cup \{g_1\}\rangle$. Proceeding inductively we obtain an ascending chain of open subgroups $U_1 < U_2 < \cdots < O$, and the ascending chain condition ensures that $O = U_n$ for some n. In other words O is generated by the compact set $V \cup \{g_1, \dots, g_n\}$.

Assume conversely that every open subgroup is compactly generated, and let $U_1 < U_2 < \dots$ be an ascending chain of open subgroups. Then $U = \bigcup_n U_n$ is an open subgroup. Let C be a compact generating set for U. By compactness, the inclusion $C \subset \bigcup_n U_n$ implies that C is contained in U_n for some n since every U_j is open. Thus $U = \langle C\rangle < U_n$, whence $U = U_n$ and G is Noetherian. □

Proof of Corollary 8.2. By Theorem 8.1, every open subgroup of a complete Kac–Moody group G over a finite field is contained as a finite index subgroup in some parabolic subgroup. Notice that parabolic subgroups are compactly generated by the Svarc–Milnor Lemma since they act properly and cocompactly on the residue of which they are the stabiliser. Since a cocompact subgroup of a group acting cocompactly on a space also acts cocompactly on that space, it follows for the same reason that all open subgroups of G are compactly generated; hence G is Noetherian by Lemma 8.26. $\square$

Proof of Corollary 8.3. Immediate from Theorem 8.1 since Coxeter groups of affine and compact hyperbolic type are precisely those Coxeter groups all of whose proper parabolic subgroups are finite. $\square$

8.2 Abstract simplicity of locally compact Kac–Moody groups

In this section, we give a proof of Theorem E from the introduction, as well as of other related results of independent interest (see in particular Theorem 8.46 and Corollaries 8.51, 8.53, 8.54 and 8.55 below).

8.2.1 Description of the problem. — In this section, we investigate whether complete Kac–Moody groups G are abstractly simple, that is, whether they only admit trivial (abstract) normal subgroups.

Let k be a field, $\mathcal{D}$ be a Kac–Moody root datum with generalised Cartan matrix $A = (a_{ij})$, and fix as usual a realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A with associated set of real roots Δ^{re}. Let $\{U_\alpha \mid \alpha \in \Delta^{\mathrm{re}}\}$ denote the twin root datum associated to the minimal Kac–Moody group $\mathfrak{G}_{\mathcal{D}}(k)$, as in §5.3.1. Let also $\widehat{G} \supset \mathfrak{G}_{\mathcal{D}}(k)$ be one of the corresponding complete Kac–Moody groups $\mathfrak{G}^{pma}_{\mathcal{D}}(k)$ or $\mathfrak{G}^{crr}_{\mathcal{D}}(k)$ and let $\widehat{U}^+ \subset \widehat{G}$ denote either the subgroup $\mathfrak{U}^{ma+}(k)$ or U^{rr+}. Finally, let N and T be the subgroups of $\mathfrak{G}_{\mathcal{D}}(k)$ as in the statement of Proposition 5.13, which we view as subgroups of $\widehat{G}$.

Remark 8.27. Notice first that the subgroup $\widehat{G}_{(1)}$ of $\widehat{G}$ generated by $\widehat{U}^+$ and all U_α, $\alpha \in \Delta^{\mathrm{re}}$, is normal in $\widehat{G}$. Indeed, since $(T\widehat{U}^+, N)$ is a BN-pair for $\widehat{G}$, the subgroups $T\widehat{U}^+$ and N generate $\widehat{G}$ and hence $\widehat{G} = T\widehat{G}_{(1)}$. We should thus restrict our attention to $\widehat{G}_{(1)}$ for simplicity questions. Another way to do it, which simplifies the notations, is to consider a root datum $\mathcal{D}$ such that $\mathfrak{G}_{\mathcal{D}}(k)$ is already generated by its root subgroups, so that $\widehat{G} = \widehat{G}_{(1)}$ (see Remark 5.14). Consistently with Remark 5.4, we will even consider in this section the Kac–Moody root datum $\mathcal{D} = \mathcal{D}^A_{\mathrm{sc}}$ of simply connected type.

Observe also that the kernel $Z' = Z'(\widehat{G})$ of the action of $\widehat{G}$ on its associated positive building is a normal subgroup of $\widehat{G}$ as well, hence only the quotient $\widehat{G}/Z'$ may be abstractly simple.

Finally, notice that we should also assume the generalised Cartan matrix A to be indecomposable, since otherwise the BN-pair $(T\widehat{U}^+, N)$ for $\widehat{G}$ splits as a product of BN-pairs, and hence $\widehat{G}/Z'$ cannot be simple.

8.2. ABSTRACT SIMPLICITY OF LOCALLY COMPACT KAC–MOODY GROUPS

With the obvious restrictions of Remark 8.27 in mind, we now consider the question of the abstract simplicity of the complete Kac–Moody groups $\mathfrak{G}_A^{rr}(k) := \mathfrak{G}_{\mathcal{D}_{\mathrm{sc}}^A}^{rr}(k) = \mathfrak{G}_{\mathcal{D}_{\mathrm{sc}}^A}^{crr}(k)/Z'$ and $\mathfrak{G}_A^{pma}(k)/Z'$ where $\mathfrak{G}_A^{pma}(k) := \mathfrak{G}_{\mathcal{D}_{\mathrm{sc}}^A}^{pma}(k)$, of simply connected type over the field k, with indecomposable generalised Cartan matrix A.

The question whether $\mathfrak{G}_A^{rr}(k)$ is (abstractly) simple for A indecomposable and k arbitrary was explicitly addressed by J. Tits [Tit89]. Recall from §6.3.13 that thanks to the work of R. Moody and G. Rousseau, we already know that $\mathfrak{G}_A^{pma}(k)/Z'$ is abstractly simple when the field k is of characteristic 0 or of positive characteristic p but is not algebraic over $\mathbb{F}_p$. The abstract simplicity of $\mathfrak{G}_A^{rr}(k)$ when k is a finite field was shown in [CER08] in some important special cases, including groups of 2-spherical type over fields of order at least 4, as well as some other hyperbolic types under additional restrictions on the order of the ground field.

Our contribution is to establish the abstract simplicity of both locally compact Kac–Moody groups $\mathfrak{G}_A^{rr}(k)$ and $\mathfrak{G}_A^{pma}(k)/Z'$ of indecomposable type over arbitrary finite fields, without any restriction. Our proof relies on an approach which is completely different from the one used in [CER08].

Theorem 8.28. *Let G be either $\mathfrak{G}_A^{rr}(\mathbb{F}_q)$ or $\mathfrak{G}_A^{pma}(\mathbb{F}_q)$, where $\mathbb{F}_q$ is an arbitrary finite field. Assume that A is indecomposable of indefinite type. Then $G/Z'(G)$ is abstractly simple.*

After completion of this work, I was informed by Bertrand Rémy that, in a recent joint work [CR13] with I. Capdeboscq, they obtained independently a special case of this theorem, namely the abstract simplicity over finite fields of order at least 4 and of characteristic p in case p is greater than $M = \max_{i\neq j}|a_{ij}|$. Their approach is similar to the one used in [CER08].

Note that the topological simplicity of $\mathfrak{G}_A^{rr}(\mathbb{F}_q)$ (that is, all closed normal subgroups are trivial), which we will use in our proof of Theorem 8.28, was previously established by B. Rémy when $q > 3$ (see [Rém04, Theorem 2.A.1]); the tiniest finite fields were later covered by P-E. Caprace and B. Rémy (see [CR09, Proposition 11]). As for the topological simplicity of $\mathfrak{G}_A^{pma}(\mathbb{F}_q)/Z'$, it was not previously known in full generality (see [Rou12, Lemme 6.14 and Proposition 6.16] for known results), and we will need to establish it first (see Proposition 8.48 below).

Note also that for minimal groups, abstract simplicity fails in general since groups of affine type admit numerous congruence quotients. However, it has been shown by P-E. Caprace and B. Rémy ([CR09]) that $\mathfrak{G}_{\mathcal{D}_{\mathrm{sc}}^A}(\mathbb{F}_q)$ is abstractly simple provided A is indecomposable, $q > n > 2$ and A is not of affine type. They also recently covered the rank 2 case for matrices A of the form $A = \left(\begin{smallmatrix} 2 & -m \\ -1 & 2 \end{smallmatrix}\right)$ with $m > 4$ (see [CR12, Theorem 2]).

Remark 8.29. Let X_+ denote the positive building associated to $\mathfrak{G}(k) = \mathfrak{G}_A(k)$. When the field k is finite, the several group homomorphisms $\overline{\mathfrak{G}(k)} \to \mathfrak{G}^{cgr}(k) \to \mathfrak{G}^{crr}(k) \to \mathfrak{G}^{rr}(k) \leq \mathrm{Aut}(X_+)$ are all surjective (see §6.4.2), and if G is either

$\overline{\mathfrak{G}(k)}$ or $\mathfrak{G}^{cg\lambda}(k)$ or $\mathfrak{G}^{crr}(k)$, the effective quotient of G by the kernel $Z'(G)$ of its action on X_+ thus coincides with $\mathfrak{G}^{rr}(k)$. If moreover the characteristic p of k is greater than the maximum M (in absolute value) of the non-diagonal entries of A, one has $\overline{\mathfrak{G}(k)} = \mathfrak{G}^{pma}(k)$ by Proposition 6.39, and hence in that case there is only one simple group $G/Z'(G)$. If $p \leq M$, it is possible that the effective quotient of $\mathfrak{G}^{pma}(k)$ inside $\mathrm{Aut}(X_+)$ properly contains $\mathfrak{G}^{rr}(k)$ (see Corollary 8.54 below). When this happens, Theorem 8.28 thus asserts the abstract simplicity of two different groups $\mathfrak{G}^{rr}_A(\mathbb{F}_q)$ and $\mathfrak{G}^{pma}_A(\mathbb{F}_q)/Z'$.

8.2.2 Notations. — For the rest of this section, we let k be an arbitrary field and $A = (a_{ij})_{1\leq i,j\leq n}$ be a generalised Cartan matrix. We fix a realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A, where $\Pi = \{\alpha_1, \dots, \alpha_n\}$. As usual, $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$ is the associated root lattice and Δ the set of roots. We will view the real roots of Δ^{re} either as elements of Q or as half-spaces in the corresponding Coxeter complex $\Sigma = \Sigma(W, S)$, where $W = W(A)$ is the Weyl group of A and $S = \{s_1, \dots, s_n\}$ is such that $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ for all $i, j \in \{1, \dots, n\}$.

We let $\mathfrak{G} = \mathfrak{G}_A$ denote the Tits functor of simply connected type associated to A, and we define the sub-functors $\mathfrak{B}^+$, $\mathfrak{U}^+$, $\mathfrak{N}$ and $\mathfrak{T}$ of $\mathfrak{G}$ as in §6.3.11. In particular, $\mathfrak{N}(k)/\mathfrak{T}(k) \cong W$, and we fix a section $W \cong \mathfrak{N}(k)/\mathfrak{T}(k) \to \mathfrak{N}(k) : w \mapsto \overline{w}$.

We write $\mathfrak{U}^{rr+}(k)$ and $\mathfrak{B}^{crr+}(k) = \mathfrak{T}(k) \ltimes \mathfrak{U}^{rr+}(k)$ for the respective closures in $\mathfrak{G}^{crr}(k) = \mathfrak{G}^{crr}_A(k)$ of $\mathfrak{U}^+(k)$ and $\mathfrak{B}^+(k)$, and we let $\mathfrak{B}^{rr+}(k)$ denote the image of $\mathfrak{B}^{crr+}(k)$ in $\mathfrak{G}^{rr}(k) = \mathfrak{G}^{rr}_A(k) = \mathfrak{G}^{crr}(k)/Z'(\mathfrak{G}^{crr}(k))$.

We view $\mathfrak{G}(k)$ as a subgroup of both $\mathfrak{G}^{pma}(k) = \mathfrak{G}^{pma}_A(k)$ and $\mathfrak{G}^{crr}(k)$, and we denote by $\overline{\mathfrak{G}(k)}$ (respectively, $\overline{\mathfrak{U}^+(k)}$) the closure of $\mathfrak{G}(k)$ (respectively, $\mathfrak{U}^+(k)$) in $\mathfrak{G}^{pma}(k)$ (see §6.3.11). We recall from §6.3.8 that to any closed set of positive roots $\Psi \subseteq \Delta_+$ is attached a pro-unipotent sub-group scheme $\mathfrak{U}^{ma}_\Psi$ of $\mathfrak{U}^{ma+} \subset \mathfrak{G}^{pma}$, and we write again $\mathfrak{U}^{ma}_{(\alpha)}$ for the root group attached to $\alpha \in \Delta_+$. We let also $\mathfrak{U}^{ma}_n$, $n \in \mathbb{N}^*$, denote the filtration of $\mathfrak{U}^{ma+}$ defining the topology on $\mathfrak{G}^{pma}$ (see §6.3.11), and we write $\mathfrak{B}^{ma+} = \mathfrak{T} \ltimes \mathfrak{U}^{ma+}$ for the Borel subgroup of $\mathfrak{G}^{pma}$ (see §6.3.9).

Finally, recall from §6.4.1 the definition of the continuous group homomorphism $\phi\colon \overline{\mathfrak{G}(k)} \to \mathfrak{G}^{crr}(k)$. We will write

$$\varphi\colon \overline{\mathfrak{G}(k)} \to \mathfrak{G}^{rr}(k)$$

for the composition of ϕ with the canonical projection $\mathfrak{G}^{crr}(k) \to \mathfrak{G}^{rr}(k)$. As we saw in §6.4.2, the map φ then has the following properties:

Lemma 8.30. *The kernel of φ is contained in $\mathfrak{T}(k) \ltimes \overline{\mathfrak{U}^+(k)}$ and the restriction of φ to $\overline{\mathfrak{U}^+(k)}$ is surjective onto $\mathfrak{U}^{rr+}(k)$ when the field k is finite.*

Given a topological group H and an element $a \in H$, we define the **contraction group** $\mathrm{con}^H(a)$, or simply $\mathrm{con}(a)$, as the set of elements $g \in H$ such that $a^n g a^{-n} \overset{n\to\infty}{\to} 1$. Note then that for any $a \in \overline{\mathfrak{G}(k)} \subseteq \mathfrak{G}^{pma}(k)$, one has

$\varphi(\mathrm{con}^{\mathfrak{G}^{pma}(k)}(a) \cap \overline{\mathfrak{G}(k)}) \subseteq \mathrm{con}^{\mathfrak{G}^{rr}(k)}(\varphi(a))$. To avoid cumbersome notation, we will write $\mathrm{con}(a)$ for both contraction groups $\mathrm{con}^{\mathfrak{G}^{pma}(k)}(a)$ and $\mathrm{con}^{\mathfrak{G}^{rr}(k)}(a)$, as k is fixed and as it will be always clear in which group we are working.

8.2.3 Outline of the proof of Theorem 8.28. — Our proof relies on the following result of Caprace–Reid–Willis, whose proof can be found in the appendix of [Mar12a].

Theorem 8.31 (Caprace–Reid–Willis)**.** *Let G be a totally disconnected locally compact group and let $f \in G$. Any abstract normal subgroup of G containing f also contains the closure $\overline{\mathrm{con}(f)}$.*

Let G be either $\mathfrak{G}^{rr}_A(\mathbb{F}_q)$ or $\mathfrak{G}^{pma}_A(\mathbb{F}_q)$ and set $U^+ := \mathfrak{U}^{rr+}(\mathbb{F}_q)$ or $U^+ := \mathfrak{U}^{ma+}(\mathbb{F}_q)$ accordingly. Recall from Proposition 6.3 and Remark 6.38 that G is locally compact and totally disconnected. We first prove that $G/Z'(G)$ is always topologically simple, so that it suffices to consider a dense normal subgroup K of G. Our strategy will then be to show that each (positive) root group in G is contracted by some suitable $a \in K$. Since U^+ is topologically generated by these root groups, Theorem 8.31 will imply that it is contained in K, which imposes that $K = G$, as desired.

8.2.4 Coxeter groups and root systems. — In this paragraph, we prepare the ground for the proof of Theorem 8.28 by establishing several results which concern the Coxeter group W and the set of roots Δ.

Throughout this paragraph, we let X denote the Davis realisation of $\Sigma = \Sigma(W, S)$, or else the Davis complex of W (see Section 1.3). Also, we let C_0 be the fundamental chamber of Σ. With the exception of Lemma 8.32 below where no particular assumption on W is made, we will always assume that W is infinite irreducible. Note that this is equivalent to saying that A is indecomposable of non-finite type. We consider both the actions of W on the root lattice Q and on the Coxeter complex Σ.

Lemma 8.32. *Let $w = s_1 \dots s_n$ be the Coxeter element of W, viewed as an element of $\mathrm{GL}(\oplus_{i=1}^n \mathbb{C}\alpha_i)$. Let $A = A_1 + A_2$ be the unique decomposition of A as a sum of matrices A_1, A_2 such that A_1 (respectively, A_2) is an upper (respectively, lower) triangular matrix with 1's on the diagonal. Then the matrix of w in the basis $\{\alpha_1, \dots, \alpha_n\}$ of simple roots is $-A_1^{-1}A_2 = I_n - A_1^{-1}A$.*

Proof. For a certain property P of two integer variables i, j (e.g. $\mathrm{P}(i,j) \equiv j \leq i$), we introduce for short the Kronecker symbol $\delta_{\mathrm{P}(i,j)}$ taking value 1 if $\mathrm{P}(i,j)$ is satisfied and 0 otherwise.

Let $B = (b_{ij})$ denote the matrix of w in the basis $\{\alpha_1, \dots, \alpha_n\}$. Thus, b_{ij} is the coefficient of α_i in the expression of $s_1 \dots s_n \alpha_j$ as a linear combination of the simple roots, which we will write for short as $[s_1 \dots s_n\alpha_j]_i$. Thus $b_{ij} = [s_1 \dots s_n\alpha_j]_i = [s_i \dots s_n\alpha_j]_i$. Note that

$$s_{i+1} \dots s_n\alpha_j = \sum_{k=i+1}^{n} [s_{i+1} \dots s_n\alpha_j]_k\alpha_k + \delta_{j\leq i}\alpha_j = \sum_{k=i+1}^{n} b_{kj}\alpha_k + \delta_{j\leq i}\alpha_j.$$

Whence

$$\begin{aligned} b_{ij} &= [s_i(\sum_{k=i+1}^{n} b_{kj}\alpha_k + \delta_{j\leq i}\alpha_j)]_i = -\sum_{k=i+1}^{n} a_{ik}b_{kj} - \delta_{j\leq i}a_{ij} + \delta_{i=j} \\ &= (-\sum_{k=1}^{n}(A_1)_{ik}b_{kj} + b_{ij}) + (\delta_{j>i}a_{ij} - a_{ij}) + \delta_{i=j} \\ &= -\sum_{k=1}^{n}(A_1)_{ik}b_{kj} + b_{ij} - a_{ij} + \sum_{k=1}^{n}(A_1)_{ik}(I_n)_{kj}. \end{aligned}$$

Thus $A = -A_1B + A_1$, so that $B = -A_1^{-1}A_2$, as desired. □

Lemma 8.33. *Let $w = s_1 \dots s_n$ be the Coxeter element of W. Then w acts on X as a hyperbolic isometry. Moreover, there exists some $v \in W$ such that $w_1 := vwv^{-1}$ possesses an axis D going through some point $x_0 \in X$ whose support is C_0. In particular, x_0 does not lie on any wall of X (see §1.3.2).*

Proof. Note first that w is indeed hyperbolic, for otherwise it would be elliptic by Lemma 1.17 and hence would be contained in a spherical parabolic subgroup of W, contradicting the fact that its parabolic closure is the whole of W (see [Par07, Theorem 3.4]).

Note also that w does not stabilise any wall of X. Indeed, suppose to the contrary that there exists some positive real root $\alpha \in \Delta_+$ such that $w\alpha = \pm\alpha$. Recall the decomposition $A = A_1 + A_2$ from Lemma 8.32. Viewing w as an automorphism of the root lattice, it follows from this lemma that $A_2\alpha = \mp A_1\alpha$. If $w\alpha = \alpha$, this implies that $A\alpha = A_1\alpha + A_2\alpha = 0$, hence that α is an imaginary root by Proposition 4.18, a contradiction. Assume now that $w\alpha = -\alpha \in \Delta_-$. Then by Lemma 4.8 (2), there is some $t \in \{1, \dots, n\}$ such that $\alpha = s_n \dots s_{t+1}\alpha_t$. Hence $w\alpha = s_1 \dots s_{t-1}(-\alpha_t)$ and thus $s_n \dots s_{t+1}\alpha_t = s_1 \dots s_{t-1}\alpha_t$. Writing these expressions in the basis $\{\alpha_1, \dots, \alpha_n\}$ yields $n = t = 1$ or $a_{it} = 0$ for all $i \neq t$, contradicting the fact that W is infinite irreducible.

It follows from Lemma 1.18 that for any wall m of X and any w-axis D, either $m \cap D$ is empty or consists of a single point. Thus any w-axis contains a point which does not belong to any wall. Since the W-action is transitive on the chambers, the conclusion follows. □

Lemma 8.34. *Let w_1 be as in Lemma 8.33. Let $t_1t_2 \dots t_k$ be a reduced expression for w_1, where $t_j \in S$ for all $j \in \{1, \dots, k\}$. Then for all $l \in \mathbb{N}$ and $j \in \{1, \dots, k\}$, one has $\ell(t_jt_{j+1} \dots t_kw_1^l) = \ell(t_{j+1} \dots t_kw_1^l) + 1$ and $\ell(t_jt_{j-1} \dots t_1w_1^{-l}) = \ell(t_{j-1} \dots t_1w_1^{-l}) + 1$.*

Proof. Note that since $\ell(sv) \leq \ell(v) + 1$ for $s \in S$ and $v \in W$, it is sufficient to show that $\ell(w_1^l) = l\ell(w_1) = lk$ for all $l \in \mathbb{N}^*$. Let x_0 be as in Lemma 8.33. Then $\ell(w_1^l)$ coincides with the number of walls separating x_0 from w_1x_0 in X (see Proposition 1.4). In particular, k walls separate x_0 from w_1x_0, and the claim then follows from Lemma 8.33. □

For $\omega \in W$ and $\alpha \in \Delta_+$, define the function $f_\alpha^\omega \colon \mathbb{Z} \to \{\pm 1\} : k \mapsto \mathrm{sign}(\omega^k\alpha)$, where $\mathrm{sign}(\Delta_\pm) = \pm 1$.

Lemma 8.35. *Let $w = s_1 \dots s_n$ be the Coxeter element of W, and let w_1 be as in Lemma 8.33. Then the following hold.*

(1) Let $\omega \in W$ be such that $\ell(\omega^l) = |l|\ell(\omega)$ for all $l \in \mathbb{Z}$. Then f_α^ω is monotonic for all $\alpha \in \Delta_+$.

(2) f_α^w and $f_\alpha^{w_1}$ are monotonic for all $\alpha \in \Delta_+$.

Proof. Let $\omega \in W$ be such that $\ell(\omega^l) = |l|\ell(\omega)$ for all $l \in \mathbb{Z}$ and let $\omega = t_1t_2 \dots t_k$ be a reduced expression for ω, where $t_j \in S$ for all $j \in \{1, \dots, k\}$. Let $\alpha \in \Delta_+$ and assume that f_α^ω is not constant. Then α is a real root because $W.\Delta_+^{\mathrm{im}} = \Delta_+^{\mathrm{im}}$ by Proposition 4.18. Let $k_\alpha \in \mathbb{Z}^*$ be minimal (in absolute value) so that $f_\alpha^\omega(k_\alpha) = -1$. We deal with the case when $k_\alpha > 0$; the same proof applies for $k_\alpha < 0$ by replacing ω with its inverse. We have to show that $\omega^l\alpha \in \Delta_-$ if and only if $l \geq k_\alpha$.

Let $\beta := \omega^{k_\alpha - 1}\alpha$. Thus $\beta \in \Delta_+^{\mathrm{re}}$ and $\omega\beta \in \Delta_-^{\mathrm{re}}$. It follows that there is some $i \in \{1, \dots, k\}$ such that $\beta = t_kt_{k-1} \dots t_{i+1}\alpha_{t_i}$. In other words, β is one of the n positive roots whose wall $\partial\beta$ in the Coxeter complex Σ of W separates the fundamental chamber C_0 from $\omega^{-1}C_0$. We want to show that $\omega^l\beta \in \Delta_-$ if and only if $l \geq 1$.

Assume first for a contradiction that there is some $l \geq 1$ such that $\omega^{l+1}\beta \in \Delta_+$, that is, $\omega^{l+1}\beta$ contains C_0. Since $\omega^{l+1}\beta$ contains $\omega^{l+1}C_0$ but not ω^lC_0, its wall $\omega^{l+1}\partial\beta$ separates ω^lC_0 from $\omega^{l+1}C_0$ and C_0. In particular, any gallery from C_0 to $\omega^{l+1}C_0$ going through ω^lC_0 cannot be minimal. This contradicts the assumption that $\ell(\omega^l) = |l|\ell(\omega)$ for all $l \in \mathbb{Z}$ since this implies that the product of $l+1$ copies of $t_1 \dots t_k$ is a reduced expression for ω^{l+1}.

Assume next for a contradiction that there is some $l \geq 1$ such that $\omega^{-l}\beta \in \Delta_-$. Then as before, $\omega^{-l}\partial\beta$ separates $\omega^{-l}C_0$ from $\omega^{-l-1}C_0$ and C_0. Again, this implies that any gallery from C_0 to $\omega^{-l-1}C_0$ going through $w^{-l}C_0$ cannot be minimal, a contradiction. This proves the first statement.

The second statement is then a consequence of the first and of [Spe09] in case $\omega = w$ (respectively, and of Lemma 8.34 in case $\omega = w_1$). □

Lemma 8.36. *Let $w = s_1 \dots s_n$ be the Coxeter element of W. Let $\alpha \in \Delta_+$. Assume that A is of indefinite type. Then $w^l\alpha \neq \alpha$ for all nonzero integers l.*

Proof. Assume for a contradiction that $w^k\alpha = \alpha$ for some $k \in \mathbb{N}^*$. It then follows from Lemma 8.35 that $w^i\alpha \in \Delta_+$ for all $i \in \{0, \dots, k-1\}$. Viewing w as an automorphism of the root lattice, we get that

$$(w - \mathrm{Id})(w^{k-1} + \cdots + w + \mathrm{Id})\alpha = 0.$$

Moreover, $\beta := (w^{k-1} + \cdots + w + \mathrm{Id})\alpha$ is a sum of positive roots, and hence can be viewed as a nonzero vector of $\mathbb{R}^n$ with nonnegative entries. Recall from Lemma 8.32 that w is represented by the matrix $-A_1^{-1}A_2$. Thus, multiplying the above equality by $-A_1$, we get that $A\beta = 0$. Since A is indecomposable of indefinite type, this gives the desired contradiction by Proposition 4.15. □

Lemma 8.37. *Let $\omega \in W$ and $\alpha \in \Delta_+$ be such that $\omega^l\alpha \neq \alpha$ for all positive integers l. Then $|\operatorname{ht}(\omega^l\alpha)|$ goes to infinity as l goes to infinity.*

Proof. If $|\operatorname{ht}(\omega^l\alpha)|$ were bounded as l goes to infinity, the set of roots $\{\omega^l\alpha \mid l \in \mathbb{N}\}$ would be finite, and so there would exist an $l \in \mathbb{N}^*$ such that $\omega^l\alpha = \alpha$, a contradiction. □

Lemma 8.38. *Let w_1 be as in Lemma 8.33. Let $\alpha \in \Delta_+$ and let $\epsilon \in \pm$ be such that $w_1^{\epsilon k}\alpha \in \Delta_+$ for all $k \in \mathbb{N}$. Assume that A is of indefinite type. Then $\operatorname{ht}(w_1^{\epsilon k}\alpha)$ goes to infinity as k goes to infinity.*

Proof. Writing $w_1 = v^{-1}wv$ for some $v \in W$, where $w = s_1 \dots s_n$ is the Coxeter element of W, we notice that $w_1^l\alpha = \alpha$ for some integer l if and only if $w^l\beta = \beta$, where $\beta = v\alpha$. Thus the claim follows from Lemmas 8.36 and 8.37. □

8.2.5 Contraction groups.

— In this paragraph, we make use of the results proven so far to establish, under suitable hypotheses, that the subgroups $\mathfrak{U}^{ma+}(k)$ of $\mathfrak{G}^{pma}(k)$ and $\mathfrak{U}^{rr+}(k)$ of $\mathfrak{G}^{rr}(k)$ are contracted. Moreover, we give some control on the contraction groups involved using building theory.

Throughout this paragraph, we let X_+ denote the positive building associated to $\mathfrak{G}(k)$ and we write Σ_0 and C_0 for the fundamental apartment and chamber of X_+, respectively. We will write $|X_+| = |X_+|_{\mathrm{CAT}(0)}$ and $|\Sigma_0| = |\Sigma_0|_{\mathrm{CAT}(0)}$ for the corresponding Davis realisations. Finally, we again assume that W is infinite irreducible and we fix an element w_1 of W as in Lemma 8.33.

Lemma 8.39. *Let H be a topological group acting on a set E with open stabilisers. Then any dense subgroup of H is orbit-equivalent to H.*

Proof. Let N be a dense subgroup of H. Let x, y be two points of E in the same H-orbit, say $y = hx$ for some $h \in H$. As the stabiliser H_x of x in H is open, the open neighbourhood hH_x of h in H must intersect N, whence the result. □

Lemma 8.40. *Let G be either $\mathfrak{G}^{rr}(k)$ or $\mathfrak{G}^{pma}(k)$, and set $B := \mathfrak{B}^{rr+}(k)$ or $B := \mathfrak{B}^{ma+}(k)$ accordingly. Let K be a dense normal subgroup of G. Then there exist an element $a \in K$ and elements $b_l \in B$ for $l \in \mathbb{Z}$ such that $a^l = b_l \overline{w_1}^l$ for all $l \in \mathbb{Z}$.*

Proof. Let $x_0 \in |\Sigma_0|$ be as in Lemma 8.33. By Lemma 8.39 applied to the action of G on the set of couples of points in $|X_+|$, one can find some $a_1 \in K$ such that $a_1 \overline{w_1}^{-1} x_0 = x_0$ and $a_1 x_0 = \overline{w_1} x_0$. By Lemma 1.15 together with Lemma 8.33, we know that a_1 is hyperbolic and that $D := \bigcup_{l\in\mathbb{Z}} [a_1^l x_0, a_1^{l+1} x_0]$ is an axis of a_1. In particular, D is contained in the Davis realisation of an apartment $b\Sigma_0$ for some $b \in B$ (see Lemma 2.5). Thus $a := b^{-1} a_1 b$ is a hyperbolic element of K possessing $b^{-1}D \subseteq |\Sigma_0|$ as a translation axis.

Note that since $a_1 x_0 = \overline{w_1} x_0$ we have $aC_0 = b^{-1}\overline{w_1} C_0$ and so a belongs to the double coset $B\overline{w_1}B$. It follows from Lemma 2.15 together with Lemma 8.34 that $a^l \in B\overline{w_1}^l B$ for all $l \in \mathbb{Z}$. Since $a^l C_0 \in \Sigma_0$ and hence $a^l C_0 = \overline{w_1}^l C_0$ for all $l \in \mathbb{Z}$, one can then find elements $b_l \in B$, $l \in \mathbb{Z}$, such that $a^l = \overline{w_1}^l b_{-l}^{-1}$ for all $l \in \mathbb{Z}$. Taking inverses, this yields $a^l = b_l \overline{w_1}^l$ for all $l \in \mathbb{Z}$, as desired. □

Lemma 8.41. *Let $\Psi_1 \subseteq \Psi_2 \subseteq \cdots \subseteq \Delta_+$ be an increasing sequence of closed subsets of Δ_+ and set $\Psi = \bigcup_{i=1}^{\infty} \Psi_i$. Then the corresponding increasing union of subgroups $\bigcup_{i=1}^{\infty} \mathfrak{U}^{ma}_{\Psi_i}(k)$ is dense in $\mathfrak{U}^{ma}_{\Psi}(k)$.*

Proof. This follows from Proposition 6.25. □

Proposition 8.42. *Let $\Psi \subseteq \Delta_+$ be closed. Let $\omega \in W$ be such that $\omega\Psi \subseteq \Delta_+$. Then $\overline{\omega}\mathfrak{U}^{ma}_{\Psi}\overline{\omega}^{-1} = \mathfrak{U}^{ma}_{\omega\Psi}$.*

Proof. It follows from Proposition 6.31 that

$$\overline{\omega}\langle \mathfrak{U}^{ma}_{(\alpha)} \mid \alpha \in \Psi\rangle \overline{\omega}^{-1} \subseteq \langle \mathfrak{U}^{ma}_{(\omega\alpha)} \mid \alpha \in \Psi\rangle.$$

Passing to the closures, Lemma 8.41 then yields $\overline{\omega}\mathfrak{U}^{ma}_{\Psi}\overline{\omega}^{-1} \subseteq \mathfrak{U}^{ma}_{\omega\Psi}$, as desired. □

Lemma 8.43. *Let $\Psi \subseteq \Delta_+$ be the set of positive roots α such that $w_1^l\alpha \in \Delta_+$ for all $l \in \mathbb{N}$. Then both Ψ and $\Delta_+ \setminus \Psi$ are closed. In particular, one has a unique decomposition $\mathfrak{U}^{ma+} = \mathfrak{U}^{ma}_{\Psi}.\mathfrak{U}^{ma}_{\Delta_+\setminus\Psi}$.*

Proof. Clearly, Ψ is closed. Let now $\alpha, \beta \in \Delta_+ \setminus \Psi$ be such that $\alpha + \beta \in \Delta$. Thus there exist some positive integers l_1, l_2 such that $w_1^{l_1}\alpha \in \Delta_-$ and $w_1^{l_2}\beta \in \Delta_-$. Then $w_1^l(\alpha+\beta) \in \Delta_-$ for all $l \geq \max\{l_1, l_2\}$ by Lemma 8.35 and hence $\alpha + \beta \in \Delta_+ \setminus \Psi$. Thus $\Delta_+ \setminus \Psi$ is closed, as desired. The second statement follows from Lemma 6.27. □

Remark 8.44. Let $\Psi \subseteq \Delta_+$ be as in Lemma 8.43. Put an arbitrary order on Δ_+. This yields enumerations $\Psi = \{\beta_1, \beta_2, \dots\}$ and $\Delta_+ \setminus \Psi = \{\alpha_1, \alpha_2, \dots\}$. For each $i \in \mathbb{N}^*$, we let Ψ_i (respectively, Φ_i) denote the closure in Δ_+ of $\{\beta_1, \dots, \beta_i\}$ (respectively, of $\{\alpha_1, \dots, \alpha_i\}$). It follows from Lemma 8.43 that $\Psi = \bigcup_{i=1}^{\infty} \Psi_i$ and that $\Delta_+ \setminus \Psi = \bigcup_{i=1}^{\infty} \Phi_i$.

Lemma 8.45. *Fix $i \in \mathbb{N}^*$, and let $\Psi_i, \Phi_i \subseteq \Delta_+$ be as in Remark 8.44. Assume that A is of indefinite type. Then there exists a sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ going to infinity as k goes to infinity, such that $\overline{w_1}^k \mathfrak{U}^{ma}_{\Psi_i} \overline{w_1}^{-k} \subseteq \mathfrak{U}^{ma}_{n_k}$ and $\overline{w_1}^{-k} \mathfrak{U}^{ma}_{\Phi_i} \overline{w_1}^{k} \subseteq \mathfrak{U}^{ma}_{n_k}$ for all $k \in \mathbb{N}$.*

Proof. Let $\alpha_j, \beta_j \in \Delta_+$ be as in Remark 8.44. By Lemma 8.38 together with Lemma 8.35, one can find for each $j \in \{1, \dots, i\}$ sequences of positive integers $(m^j_k)_{k\in\mathbb{N}}$ and $(n^j_k)_{k\in\mathbb{N}}$ going to infinity as k goes to infinity, such that $\mathrm{ht}(w_1^{-k}\alpha_j) \geq m^j_k$ and $\mathrm{ht}(w_1^k\beta_j) \geq n^j_k$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, set $n_k = \min\{m^k_j, n^k_j \mid 1 \leq j \leq i\}$. Then the sequence $(n_k)_{k\in\mathbb{N}}$ goes to infinity as k goes to infinity. Moreover, $\mathrm{ht}(\alpha) \geq n_k$ for all $\alpha \in w_1^{-k}\Phi_i$ and $\mathrm{ht}(\beta) \geq n_k$ for all $\beta \in w_1^k\Psi_i$. The conclusion then follows from Proposition 8.42. □

Theorem 8.46. *Let $a \in \mathfrak{G}^{pma}_A(k)$ be such that $a^l = b_l\overline{w_1}^l$ for all $l \in \mathbb{Z}$, for some $b_l \in \mathfrak{B}^{ma+}(k)$. Let Ψ, Ψ_i, Φ_i be as in Remark 8.44 and assume that A is of indefinite type. Then the following hold.*

(1) $\mathfrak{U}^{ma}_{\Psi_i}(k) \subseteq \mathrm{con}(a)$ and $\mathfrak{U}^{ma}_{\Phi_i}(k) \subseteq \mathrm{con}(a^{-1})$ for all $i \in \mathbb{N}^*$.

(2) $\mathfrak{U}^{ma}_{\Psi}(k) \subseteq \overline{\mathrm{con}(a)}$ and $\mathfrak{U}^{ma}_{\Delta_+\setminus\Psi}(k) \subseteq \overline{\mathrm{con}(a^{-1})}$.

(3) $\mathfrak{U}^{ma+}(k) \subseteq \langle\overline{\mathrm{con}(a)} \cup \overline{\mathrm{con}(a^{-1})}\rangle$.

Proof. Note that $\mathfrak{U}^{ma}_n(k)$ is normal in $\mathfrak{U}^{ma+}(k)$ by Lemma 6.27, and thus also in $\mathfrak{B}^{ma+}(k)$, for all $n \in \mathbb{N}$. The first statement then follows from Lemma 8.45. The second statement is a consequence of the first together with Lemma 8.41. The third statement follows from the second together with Lemma 8.43. □

Recall the definition and properties of the map φ from §8.2.2.

Lemma 8.47. *Let K be a dense normal subgroup of $\mathfrak{G}^{rr}_A(k)$. Assume that A is of indefinite type. Assume moreover that the continuous homomorphism $\varphi\colon \overline{\mathfrak{U}^+(k)} \to \mathfrak{U}^{rr+}(k)$ is surjective (e.g. k finite). Then there exists some $a \in K$ such that the following hold.*

(1) The subgroups $U_1 := \varphi(\mathfrak{U}^{ma}_{\Psi}(k) \cap \overline{\mathfrak{U}^+(k)})$ and $U_2 := \varphi(\mathfrak{U}^{ma}_{\Delta_+\setminus\Psi}(k) \cap \overline{\mathfrak{U}^+(k)})$ of $\mathfrak{U}^{rr+}(k)$ are respectively contained in $\overline{\mathrm{con}(a)}$ and $\overline{\mathrm{con}(a^{-1})}$.

(2) $\mathfrak{U}^{rr+}(k) \subseteq \langle\overline{\mathrm{con}(a)} \cup \overline{\mathrm{con}(a^{-1})}\rangle$.

Proof. Let $a \in K$ and $b_l \in \mathfrak{B}^{rr+}(k)$ for $l \in \mathbb{Z}$ be as in Lemma 8.40, so that $a^l = b_l\overline{w_1}^l$ for all $l \in \mathbb{Z}$. For each $l \in \mathbb{Z}$, let $\tilde{b_l} \in \mathfrak{T}(k) \ltimes \overline{\mathfrak{U}^+(k)} \subseteq \mathfrak{B}^{ma+}(k)$ be such that $\varphi(\tilde{b_l}) = b_l$. Set $\tilde{a} = \tilde{b_1}\overline{w_1} \in \overline{\mathfrak{G}(k)} \subseteq \mathfrak{G}^{pma}(k)$. Then $\varphi(\tilde{b_l}\overline{w_1}^l) = a^l = \varphi(\tilde{a}^l)$ for all $l \in \mathbb{Z}$. As the kernel of $\varphi\colon \overline{\mathfrak{G}(k)} \to \mathfrak{G}^{rr}(k)$ lies in $\mathfrak{T}(k) \ltimes \overline{\mathfrak{U}^+(k)}$ by Lemma 8.30, we may assume up to modifying the elements $\tilde{b_l}$ that $\tilde{a}^l = \tilde{b_l}\overline{w_1}^l$ for all $l \in \mathbb{Z}$.

Since φ is continuous, both statements are then a consequence of Theorem 8.46 and of the surjectivity of $\varphi\colon \overline{\mathfrak{U}^+(k)} \to \mathfrak{U}^{rr+}(k)$. □

8.2.6 Proof of Theorem 8.28. — We now let $k = \mathbb{F}_q$ be a finite field, A be an indecomposable generalised Cartan matrix of indefinite type, and we let G be one of the complete Kac–Moody groups $\mathfrak{G}_A^{rr}(\mathbb{F}_q)$ or $\mathfrak{G}_A^{pma}(\mathbb{F}_q)$. We also set $U^+ := \mathfrak{U}^{rr+}(\mathbb{F}_q)$ or $U^+ := \mathfrak{U}^{ma+}(\mathbb{F}_q)$ accordingly. Then, as mentioned in §8.2.3, G is a locally compact totally disconnected topological group, and U^+ is a compact open subgroup of G.

We first need to establish the topological simplicity of $\mathfrak{G}_A^{pma}(\mathbb{F}_q)$ in full generality.

Proposition 8.48. *Assume that the generalised Cartan matrix A is indecomposable of indefinite type. Then $\mathfrak{G}_A^{pma}(\mathbb{F}_q)/Z'(\mathfrak{G}_A^{pma}(\mathbb{F}_q))$ is topologically simple over any finite field $\mathbb{F}_q$.*

Proof. Set $G := \mathfrak{G}_A^{pma}(\mathbb{F}_q)$ and $Z' := Z'(\mathfrak{G}_A^{pma}(\mathbb{F}_q))$. It follows from [CM11b, Corollary 3.1] that G possesses a closed cocompact normal subgroup H containing Z' and such that H/Z' is topologically simple. It thus remains to see that in fact $H = G$. Let $\pi\colon G \to G/H$ denote the canonical projection. Let also w_1 be as in Theorem 8.46, and set $a := \overline{w_1} \in \mathfrak{N}(\mathbb{F}_q) \subset G$. Since G/H is compact and totally disconnected, its contraction groups are trivial (this is because G/H admits a basis of neighbourhoods of the identity consisting of compact open normal subgroups; see also [BW04]). In particular,

$$\pi(\mathrm{con}(a^{\pm 1})) \subseteq \mathrm{con}(\pi(a^{\pm 1})) = \{1\},$$

and hence the closures of the contraction groups $\mathrm{con}(a)$ and $\mathrm{con}(a^{-1})$ are contained in $\ker \pi = H$. It follows from Theorem 8.46 that H contains $\mathfrak{U}^{ma+}(\mathbb{F}_q)$. But G normalises H and contains $\mathfrak{N}(\mathbb{F}_q)$, and hence H also contains all real root groups. Therefore $H = G$, as desired. □

Proof of Theorem 8.28. Let K be a nontrivial normal subgroup of $G/Z'(G)$. Since $G/Z'(G)$ is topologically simple (see [CR09, Proposition 11] for $\mathfrak{G}_A^{rr}(\mathbb{F}_q)$ and Proposition 8.48 for $\mathfrak{G}_A^{pma}(\mathbb{F}_q)$), K must be dense in G. Since G is locally compact and totally disconnected, it then follows from Theorem 8.46 and Lemma 8.47, together with Theorem 8.31, that K contains U^+. Since U^+ is open, K is open as well, and hence closed in G. Therefore $K = G$, as desired. □

Remark 8.49. Notice that the simplicity results we have proven remain of course valid if one replaces the Kac–Moody root datum of simply connected type by an arbitrary Kac–Moody root datum $\mathcal{D}$: one just has to consider a subquotient of the complete Kac–Moody group involved, to account for the fact that $\mathfrak{G}_{\mathcal{D}}(k)$ might not be generated by its root subgroups (see Remark 8.27).

8.2.7 Applications of Theorem 8.46. — We conclude by mentioning some applications of Theorem 8.46. More precisely, we will make use of the following lemma. Recall from [Wil12, Section 3] the definition of the **nub** of an automorphism α of a totally disconnected locally compact group G. It possesses many equivalent definitions (see [Wil12, Theorem 4.12]), and given an element $a \in G$ (viewed as a conjugation automorphism), it can be characterised as $\mathrm{nub}(a) = \overline{\mathrm{con}(a)} \cap \overline{\mathrm{con}(a^{-1})}$ (see [Wil12, Remark 3.3 (b) and (d)]).

Lemma 8.50. *Let $G = \mathfrak{G}_A^{pma}(\mathbb{F}_q)$ be a complete Kac–Moody group over a finite field $\mathbb{F}_q$, with indecomposable generalised Cartan matrix A of indefinite type. Let $U^{im+} = \mathfrak{U}^{ma}_{\Delta_+^{\mathrm{im}}}(\mathbb{F}_q)$ denote its positive imaginary subgroup, let $w \in W = W(A)$ denote the Coxeter element of W, and set $a := \overline{w} \in \mathfrak{N}(\mathbb{F}_q)$. Then*

$$U^{im+} \subseteq \mathrm{nub}(a) = \overline{\mathrm{con}(a)} \cap \overline{\mathrm{con}(a^{-1})}.$$

Proof. Notice that Lemma 8.45 remains valid if one replaces Ψ by its subset Δ_+^{im} and w_1 by $w^{\pm 1}$. The lemma thus follows from Theorem 8.46. □

The first application of Theorem 8.46 concerns the existence of non-closed contraction groups in complete Kac–Moody groups of non-affine type. Recall that in simple algebraic groups over local fields, contraction groups are always closed (they are in fact either trivial, or coincide with the unipotent radical of some parabolic subgroup). In particular they are closed in a complete Kac–Moody group $\mathfrak{G}_A^{rr}(k)$ over a finite field k as soon as the defining generalised Cartan matrix A is of affine type (see e.g. [BRR08]). It has been shown in [BRR08] that, on the other hand, if A is indecomposable non spherical, non affine and of size at least 3, then the contraction group $\mathrm{con}(a)$ of some element $a \in G$ must be non-closed. The following result shows that this also holds when A is indecomposable non spherical, non affine and of size 2.

Corollary 8.51. *Let A denote an $n \times n$ generalised Cartan matrix of indecomposable indefinite type, let $W = W(A)$ be the associated Weyl group, and let $w = s_1 \dots s_n$ denote the Coxeter element of W. Let also G be one the complete Kac–Moody groups $\mathfrak{G}_A^{rr}(\mathbb{F}_q)$ or $\mathfrak{G}_A^{pma}(\mathbb{F}_q)$ of simply connected type. Then the contraction group $\mathrm{con}^G(w)$ is not closed in G, unless maybe if $G = \mathfrak{G}_A^{rr}(\mathbb{F}_q)$ and $\mathfrak{U}^{ma}_{\Delta_+^{\mathrm{im}}}(\mathbb{F}_q) \cap \overline{\mathfrak{G}(\mathbb{F}_q)}$ is contained in the kernel of φ.*

Note that $\mathfrak{U}^{ma}_{\Delta_+^{\mathrm{im}}}(\mathbb{F}_q) \cap \overline{\mathfrak{G}(\mathbb{F}_q)}$ is not often contained in the kernel of φ (and probably never is), see Proposition 7.9.

Proof. Set $a := \overline{w} \in \mathfrak{N}(\mathbb{F}_q)$. It then follows from Lemma 8.50 that

$$U_{im}^{ma+} := \mathfrak{U}^{ma}_{\Delta_+^{\mathrm{im}}}(\mathbb{F}_q) \subseteq \overline{\mathrm{con}(a)} \quad \text{in} \quad \mathfrak{G}_A^{pma}(\mathbb{F}_q)$$

and that

$$U_{im}^{rr+} := \varphi(\mathfrak{U}^{ma}_{\Delta^{\mathrm{im}}_+}(\mathbb{F}_q) \cap \overline{\mathfrak{U}^+(\mathbb{F}_q)}) \subseteq \overline{\mathrm{con}(a)} \quad \text{in} \quad \mathfrak{G}^{rr}_A(\mathbb{F}_q).$$

Since U_{im}^{ma+} is closed in $\mathfrak{U}^{ma+}(\mathbb{F}_q)$ which is compact (see §6.4.2), both groups U_{im}^{ma+} and U_{im}^{rr+} are compact. Moreover, they are normalised by a by Proposition 8.42. Hence they cannot be contracted by a because of Lemma 8.52 below, since by assumption U_{im}^{rr+} is nontrivial. In particular, $\mathrm{con}(a) \neq \overline{\mathrm{con}(a)}$ and hence $\mathrm{con}(a)$ cannot be closed.

Note that one could also directly use the fact that $\mathrm{con}(a)$ is closed if and only if $\mathrm{nub}(a) = \{1\}$ (see [Wil12, Remark 3.3 (b)]) together with Lemma 8.50. We prefered however to present a more elementary proof as well. □

The proof of the following lemma is an adaptation of the proof of Proposition 2.1 in [Wan84].

Lemma 8.52. *Let G be a locally compact group, let a be an element of G, and let Q be a compact subset of G such that $Q \subseteq \mathrm{con}(a)$. Then Q is uniformly contracted by a, that is, for every open neighbourhood U of the identity one has $a^nQa^{-n} \subset U$ for all large enough n.*

Proof. Fix an open neighbourhood U of the identity, and let V be a compact neighbourhood of the identity such that $V^2 \subset U$. By hypothesis, for all $x \in Q$ there exists an N_x such that $a^nxa^{-n} \in V$ for all $n \geq N_x$. In other words,

$$Q \subset \bigcup_{N\geq 0} \bigcap_{n \geq N} a^{-n}Va^n.$$

Note that the sets $C_N = \bigcap_{n\geq N} a^{-n}Va^n$ form an ascending chain of compact sets. It follows from Baire theorem that $Q \cap C_N$ has nonempty interior in Q for a large enough N.

By compacity of Q, one then finds a finite subset F of Q such that

$$Q \subset F.C_N.$$

Since F is finite and contained in $\mathrm{con}(a)$, we know that $a^nFa^{-n} \subset V$ for all large enough n. Moreover, by construction, $a^nC_Na^{-n} \subset V$ for $n \geq N$, and hence

$$a^nQa^{-n} = (a^nFa^{-n}).(a^nC_Na^{-n}) \subset V^2 \subset U$$

for all large enough n, as desired. □

The second application of Theorem 8.46 concerns isomorphism classes of Kac–Moody groups and their completions. While over infinite fields, it is known that two minimal Kac–Moody groups can be isomorphic only if their ground field are isomorphic and their underlying generalised Cartan matrix coincide up to a row-column permutation (see [Cap09a, Theorem A]), this fails to be true over finite fields. Indeed, over a given finite field, two minimal Kac–Moody groups associated with two different generalised Cartan matrices of size 2 can be isomorphic, as noticed in [Cap09a, Lemma 4.3]. The following result shows that, however, the corresponding Mathieu–Rousseau completions should not be expected to be isomorphic as topological groups.

Corollary 8.53. *There exist minimal Kac–Moody groups $G_1 = \mathfrak{G}_{A_1}(\mathbb{F}_3)$, $G_2 = \mathfrak{G}_{A_2}(\mathbb{F}_3)$ over $\mathbb{F}_3$ associated to 2×2 generalised Cartan matrices A_1, A_2, such that G_1 and G_2 are isomorphic as abstract groups, but their Mathieu–Rousseau completions $\mathfrak{G}^{pma}_{A_1}(\mathbb{F}_3)$ and $\mathfrak{G}^{pma}_{A_2}(\mathbb{F}_3)$ are not isomorphic as topological groups.*

Corollary 8.53 should be viewed as an instance of a phenomenon that holds most probably in a greater level of generality, i.e. for other types of Kac–Moody groups over fields of possibly larger order.

Proof. It follows from [Cap09a, Lemma 4.3] that the minimal Kac–Moody group $G_1 = \mathfrak{G}_{A_1}(\mathbb{F}_3)$ over $\mathbb{F}_3$ of simply connected type with generalised Cartan matrix $A_1 = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ (hence of affine type) is isomorphic to any minimal Kac–Moody group $G_2 = \mathfrak{G}_{A_2}(\mathbb{F}_3)$ over $\mathbb{F}_3$ of simply connected type with generalised Cartan matrix $A_2 = \left(\begin{smallmatrix} 2 & -2m \\ -2n & 2 \end{smallmatrix}\right)$ for $m, n > 1$ (hence of indefinite type). Fix such a group G_2.

For $i = 1, 2$ set $\widehat{G}_i := \mathfrak{G}^{pma}_{A_i}(\mathbb{F}_3)$ and let Z'_i denote the kernel of the action of $\widehat{G}_i$ on its associated positive building. Assume for a contradiction that there is an isomorphism $\psi\colon \widehat{G}_1 \to \widehat{G}_2$ of topological groups. As noticed in [Rou12, Remarque 6.20 (4)], the quotient $\widehat{G}_1/Z'_1$ is a simple algebraic group over the local field $\mathbb{F}_3(\!(t)\!)$. In particular, all the contraction groups of $\widehat{G}_1/Z'_1$ are closed. Moreover, $\psi(Z'_1)$ is the unique maximal proper normal subgroup of $\widehat{G}_2$, and it is compact. It follows that $\psi(Z'_1) = Z'_2$, for otherwise by Tits' lemma (see [AB08, Lemma 6.61]), the group $\widehat{G}_2$ would be compact, a contradiction. Hence ψ induces an isomorphism of topological groups between $\widehat{G}_1/Z'_1$ and $\widehat{G}_2/Z'_2$, so that in particular all contraction groups of $\widehat{G}_2/Z'_2$ are closed. Let $\pi\colon \widehat{G}_2 \to \widehat{G}_2/Z'_2$ denote the canonical projection, and let a be any element of $\widehat{G}_2$. Then

$$\pi(\overline{\mathrm{con}(a)}) \subseteq \overline{\pi(\mathrm{con}(a))} \subseteq \overline{\mathrm{con}(\pi(a))} = \mathrm{con}(\pi(a)).$$

It follows from Lemma 8.50 that the subgroup $U^+_{im} := \mathfrak{U}^{ma}_{\Delta^{\mathrm{im}}_+}(\mathbb{F}_3)$ of $U^{ma+} := \mathfrak{U}^{ma}_{\Delta_+}(\mathbb{F}_3)$ in $\widehat{G}_2$ is such that

$$\pi(U^+_{im}) \subseteq \pi(\overline{\mathrm{con}(a)}) \subseteq \mathrm{con}(\pi(a))$$

for a suitably chosen $a \in \widehat{G}_2$ normalising U^+_{im}. Thus Lemma 8.52 implies that $\pi(U^+_{im}) = \{1\}$, that is, $U^+_{im} \subseteq Z'_2$. But this contradicts Proposition 7.9, as desired. □

The following consequence of Corollary 8.53 was announced at the end of §6.3.11.

Corollary 8.54. *Let A be a generalised Cartan matrix of the form $A = \left(\begin{smallmatrix} 2 & -2m \\ -2n & 2 \end{smallmatrix}\right)$ for some $m, n > 1$. Then the minimal Kac–Moody group $\mathfrak{G}_A(\mathbb{F}_3)$ is not dense in $\mathfrak{G}_A^{pma}(\mathbb{F}_3)$.*

Proof. Let $G_1 = \mathfrak{G}_{A_1}(\mathbb{F}_3)$ be as in Corollary 8.53, that is, G_1 is the minimal Kac–Moody group over $\mathbb{F}_3$ with generalised Cartan matrix $A_1 = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. Set also $G_2 = \mathfrak{G}_{A_2}(\mathbb{F}_3)$ where $A_2 := A$. For each $i = 1, 2$, consider as in the proof of Corollary 8.53 the Mathieu–Rousseau completion $\widehat{G}_i$ of G_i, as well as the corresponding kernel Z_i'. What we proved in this corollary is that $\widehat{G}_1/Z_1'$ and $\widehat{G}_2/Z_2'$ are not isomorphic as topological groups.

Notice that the isomorphism between G_1 and G_2 provided by [Cap09a, Lemma 4.3] maps the twin BN-pair of G_1 to that of G_2. In particular, the Rémy–Ronan completions $\mathfrak{G}_{A_1}^{rr}(\mathbb{F}_3)$ of G_1 and $\mathfrak{G}_{A_2}^{rr}(\mathbb{F}_3)$ of G_2 are isomorphic as topological groups.

Notice also that G_1 is dense in $\widehat{G}_1$ by Proposition 6.39. Assume for a contradiction that G_2 is also dense in $\widehat{G}_2$. Then the surjective continuous homomorphisms $\varphi_i \colon \widehat{G}_i \to \mathfrak{G}_{A_i}^{rr}(\mathbb{F}_3)$ yield isomorphisms of topological groups

$$\widehat{G}_1/Z_1' \cong \mathfrak{G}_{A_1}^{rr}(\mathbb{F}_3) \cong \mathfrak{G}_{A_2}^{rr}(\mathbb{F}_3) \cong \widehat{G}_2/Z_2',$$

a contradiction. □

Finally, as a last application of Theorem 8.46, we mention one more "affine versus non-affine" alternative, in the spirit of Theorem 7.11. Recall that the action of a Borel isomorphism T on a measure space (X, μ) is called **ergodic** if every T-stable Borel subset of X has either zero or full measure.

Corollary 8.55. *Let $G = \mathfrak{G}_A^{pma}(\mathbb{F}_q)$ be a complete Kac–Moody group of simply connected type over a finite field, with indecomposable generalised Cartan matrix A. Let $U^{im+} = \mathfrak{U}^{ma}_{\Delta_+^{\mathrm{im}}}(\mathbb{F}_q)$ denote its positive imaginary subgroup, and let $w \in W = W(A)$ denote the Coxeter element of W. Then the type of G is characterised as follows.*

(1) G is of spherical type if and only if G is finite if and only if U^{im+} is trivial.

(2) G is of affine type if and only if w normalises a basis of identity neighbourhoods in U^{im+} (and U^{im+} is nontrivial).

(3) G is of indefinite type if and only if the conjugation action of w on U^{im+} possesses a dense orbit in U^{im+} (and U^{im+} is nontrivial) if and only if the conjugation action of w on U^{im+} is ergodic (and U^{im+} is nontrivial).

Proof. The fact that G is of finite type if and only if it is finite follows for example from Proposition 6.34. The fact that this is equivalent to the triviality of U^{im+} follows from Proposition 4.19.

If G is of affine type, then $\Delta_+^{\mathrm{im}} = \mathbb{N}\delta$ for some $\delta \in \Delta_+$ such that $W.\delta = \delta$ (see Propositions 4.15 and 4.19) and thus w normalises the normal subgroups $\mathfrak{U}^{ma}_{\{n\delta|n\in\mathbb{N}\}}(\mathbb{F}_q) = U_n^{ma+} \cap U^{im+}$ of $U^{im+} = \mathfrak{U}^{ma}_{(\delta)}(\mathbb{F}_q)$ by Proposition 8.42.

If G is of indefinite type, then U^{im+} is contained in $\mathrm{nub}(\overline{w})$ by Lemma 8.50. The statement about ergodicity is then a consequence of [Wil12, Proposition 4.8], while its equivalent formulation in terms of dense orbits is for example mentioned in [Wil12, Section 2.3].

Notice to conclude that the statements "w normalises a basis of identity neighbourhoods in U^{im+}" and "the conjugation action of w on U^{im+} possesses a dense orbit in U^{im+}" are mutually exclusive, provided U^{im+} is nontrivial. $\square$

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Bibliography

[AB08] Peter Abramenko and Kenneth S. Brown, *Buildings*, Graduate Texts in Mathematics, vol. 248, Springer, New York, 2008, Theory and applications.

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[Bor95] Armand Borel, *Sous-groupes compacts maximaux des groupes de Lie*, Séminaire Bourbaki, Vol. 1, Exp. No. 33, Soc. Math. France, Paris, 1995, pp. 271–279.

[Bou68] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[Bou72] ———, *Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Chapitre III: Groupes de Lie*, Actualités Scientifiques et Industrielles, No. 1349, Hermann, Paris, 1972.

[Bou75] ———, *Éléments de mathématique. Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées*, Actualités Scientifiques et Industrielles, No. 1364, Hermann, Paris, 1975.

[Bri99] Martin R. Bridson, *On the semisimplicity of polyhedral isometries*, Proc. Amer. Math. Soc. **127** (1999), no. 7, 2143–2146.

[BRR08] Udo Baumgartner, Jacqui Ramagge, and Bertrand Rémy, *Contraction groups in complete Kac–Moody groups*, Groups Geom. Dyn. **2** (2008), no. 3, 337–352.

[BW88] Richard E. Block and Robert Lee Wilson, *Classification of the restricted simple Lie algebras*, J. Algebra **114** (1988), no. 1, 115–259.

[BW04] Udo Baumgartner and George A. Willis, *Contraction groups and scales of automorphisms of totally disconnected locally compact groups*, Israel J. Math. **142** (2004), 221–248.

[Cap06] Pierre-Emmanuel Caprace, *Conjugacy of 2-spherical subgroups of Coxeter groups and parallel walls*, Algebr. Geom. Topol. **6** (2006), 1987–2029.

[Cap09a] ______, *"Abstract" homomorphisms of split Kac–Moody groups*, Mem. Amer. Math. Soc. **198** (2009), no. 924, xvi+84.

[Cap09b] ______, *Buildings with isolated subspaces and relatively hyperbolic Coxeter groups*, Innov. Incidence Geom. **10** (2009), 15–31.

[CER08] Lisa Carbone, Mikhail Ershov, and Gordon Ritter, *Abstract simplicity of complete Kac–Moody groups over finite fields*, J. Pure Appl. Algebra **212** (2008), no. 10, 2147–2162.

[CF10] Pierre-Emmanuel Caprace and Koji Fujiwara, *Rank-one isometries of buildings and quasi-morphisms of Kac–Moody groups*, Geom. Funct. Anal. **19** (2010), no. 5, 1296–1319.

[CG03] Lisa Carbone and Howard Garland, *Existence of lattices in Kac–Moody groups over finite fields*, Commun. Contemp. Math. **5** (2003), no. 5, 813–867.

[CH09] Pierre-Emmanuel Caprace and Frédéric Haglund, *On geometric flats in the CAT(0) realization of Coxeter groups and Tits buildings*, Canad. J. Math. **61** (2009), no. 4, 740–761.

[CL10] Pierre-Emmanuel Caprace and Alexander Lytchak, *At infinity of finite-dimensional CAT(0) spaces*, Math. Ann. **346** (2010), no. 1, 1–21.

[CL11] Pierre-Emmanuel Caprace and Jean Lécureux, *Combinatorial and group-theoretic compactifications of buildings*, Ann. Inst. Fourier, Grenoble **61** (2011), no. 2, 619–672.

[CM06] Pierre-Emmanuel Caprace and Bernhard Mühlherr, *Isomorphisms of Kac–Moody groups which preserve bounded subgroups*, Adv. Math. **206** (2006), no. 1, 250–278.

[CM11a] Pierre-Emmanuel Caprace and Timothée Marquis, *Can an anisotropic reductive group admit a Tits system?*, Pure Appl. Math. Q. **7** (2011), no. 3, Special Issue: In honor of Jacques Tits, 539–557.

[CM11b] Pierre-Emmanuel Caprace and Nicolas Monod, *Decomposing locally compact groups into simple pieces*, Math. Proc. Cambridge Philos. Soc. **150** (2011), no. 1, 97–128.

[CM12] Pierre-Emmanuel Caprace and Timothée Marquis, *Open subgroups of locally compact Kac–Moody groups*, Math. Z. (2012), ISSN (Online) 1432-1823, ISSN (Print) 0025-5874, DOI: 10.1007/s00209-012-1070-4, available at http://link.springer.com/article/10.1007/s00209-012-1070-4.

[CR09] Pierre-Emmanuel Caprace and Bertrand Rémy, *Simplicity and superrigidity of twin building lattices*, Invent. Math. **176** (2009), no. 1, 169–221.

[CR12] ———, *Simplicity of twin tree lattices with non-trivial commutation relations*, preprint (2012), http://arxiv.org/abs/1209.5372.

[CR13] Inna Capdeboscq and Bertrand Rémy, *On some pro-p groups from infinite-dimensional Lie theory*, preprint (2013), http://arxiv.org/abs/1302.4174.

[Dav98] Michael W. Davis, *Buildings are* CAT(0), Geometry and cohomology in group theory (Durham, 1994), London Math. Soc. Lecture Note Ser., vol. 252, Cambridge Univ. Press, Cambridge, 1998, pp. 108–123.

[DdSMS99] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, second ed., Cambridge Studies in Advanced Mathematics, vol. 61, Cambridge University Press, Cambridge, 1999.

[Deo82] Vinay V. Deodhar, *On the root system of a Coxeter group*, Comm. Algebra **10** (1982), no. 6, 611–630.

[DJ99] A. Dranishnikov and T. Januszkiewicz, *Every Coxeter group acts amenably on a compact space*, Proceedings of the 1999 Topology and Dynamics Conference (Salt Lake City, UT), vol. 24, 1999, pp. 135–141.

[Hai12] Guntram Hainke, *The isomorphism problem for almost split Kac–Moody groups*, PhD thesis (2012), http://nbn–resolving.de/urn:nbn:de:hbz:6–86499481671.

[Hée93] J.-Y. Hée, *Sur la torsion de Steinberg–Ree des groupes de Chevalley et des groupes de Kac–Moody*, Thèse d'État de l'Université Paris 11 Orsay, 1993.

[HKM12] Tobias Hartnick, Ralf Köhl, and Andreas Mars, *On topological twin buildings and topological split Kac–Moody groups*, preprint (2012), http://arxiv.org/abs/1201.3562.

[HM98] Karl H. Hofmann and Sidney A. Morris, *The structure of compact groups*, de Gruyter Studies in Mathematics, vol. 25, Walter de Gruyter & Co., Berlin, 1998, A primer for the student—a handbook for the expert.

[HM07] ______, *The Lie theory of connected pro-Lie groups*, EMS Tracts in Mathematics, vol. 2, European Mathematical Society (EMS), Zürich, 2007, A structure theory for pro-Lie algebras, pro-Lie groups, and connected locally compact groups.

[HR79] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. I*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, Berlin, 1979, Structure of topological groups, integration theory, group representations.

[Jac37] Nathan Jacobson, *Abstract derivation and Lie algebras*, Trans. Amer. Math. Soc. **42** (1937), no. 2, 206–224.

[Jac79] ______, *Lie algebras*, Dover Publications Inc., New York, 1979, Republication of the 1962 original.

[Kac90] Victor G. Kac, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990.

[Kle89] Adam Kleppner, *Measurable homomorphisms of locally compact groups*, Proc. Amer. Math. Soc. **106** (1989), no. 2, 391–395.

[Kle99] Bruce Kleiner, *The local structure of length spaces with curvature bounded above*, Math. Z. **231** (1999), no. 3, 409–456.

[KP85] V. G. Kac and D. H. Peterson, *Defining relations of certain infinite-dimensional groups*, Astérisque (1985), no. Numero Hors Serie, 165–208, The mathematical heritage of Élie Cartan (Lyon, 1984).

[KP87] Victor G. Kac and Dale H. Peterson, *On geometric invariant theory for infinite-dimensional groups*, Algebraic groups Utrecht 1986, Lecture Notes in Math., vol. 1271, Springer, Berlin, 1987, pp. 109–142.

[Kra09] Daan Krammer, *The conjugacy problem for Coxeter groups*, Groups Geom. Dyn. **3** (2009), no. 1, 71–171.

[Kra12] Linus Kramer, *Tits systems in Lie groups*, talk at the "Buildings 2012" conference in Münster (2012).

[Kum02] Shrawan Kumar, *Kac–Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

[Mar12a] Timothée Marquis, *Abstract simplicity of locally compact Kac–Moody groups*, preprint (2012), http://arxiv.org/abs/1301.3681.

[Mar12b] ———, *A fixed point theorem for Lie groups acting on buildings and applications to Kac–Moody theory*, Forum Mathematicum (2012), ISSN (Online) 1435-5337, ISSN (Print) 0933-7741, DOI: 10.1515/forum-2012-0111, Published ahead of print, October 2012.

[Mat88a] Olivier Mathieu, *Construction du groupe de Kac–Moody et applications*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), no. 5, 227–230.

[Mat88b] ———, *Formules de caractères pour les algèbres de Kac–Moody générales*, Astérisque (1988), no. 159-160, 267.

[Mat89] ———, *Construction d'un groupe de Kac–Moody et applications*, Compositio Math. **69** (1989), no. 1, 37–60.

[Mat96] ———, *On some modular representations of affine Kac–Moody algebras at the critical level*, Compositio Math. **102** (1996), no. 3, 305–312.

[Moo82] Robert Moody, *A simplicity theorem for Chevalley groups defined by generalized Cartan matrices*, preprint (April 1982).

[MZ55] Deane Montgomery and Leo Zippin, *Topological transformation groups*, Interscience Publishers, New York-London, 1955.

[NV02] Guennadi A. Noskov and Èrnest B. Vinberg, *Strong Tits alternative for subgroups of Coxeter groups*, J. Lie Theory **12** (2002), no. 1, 259–264.

[Par07] Luis Paris, *Irreducible Coxeter groups*, Internat. J. Algebra Comput. **17** (2007), no. 3, 427–447.

[PK83] Dale H. Peterson and Victor G. Kac, *Infinite flag varieties and conjugacy theorems*, Proc. Nat. Acad. Sci. U.S.A. **80** (1983), no. 6 i., 1778–1782.

[Pra82] Gopal Prasad, *Elementary proof of a theorem of Bruhat-Tits-Rousseau and of a theorem of Tits*, Bull. Soc. Math. France **110** (1982), no. 2, 197–202.

[Pyb02] L. Pyber, *Bounded generation and subgroup growth*, Bull. London Math. Soc. **34** (2002), no. 1, 55–60.

[Rém02] Bertrand Rémy, *Groupes de Kac–Moody déployés et presque déployés*, Astérisque (2002), no. 277, viii+348.

[Rém04] B. Rémy, *Topological simplicity, commensurator super-rigidity and non-linearities of Kac–Moody groups*, Geom. Funct. Anal. **14** (2004), no. 4, 810–852, With an appendix by P. Bonvin.

[Rou12] Guy Rousseau, *Groupes de Kac–Moody déployés sur un corps local, II masures ordonnées*, preprint (2012), http://arxiv.org/abs/1009.0138.

[RR06] Bertrand Rémy and Mark Ronan, *Topological groups of Kac–Moody type, right-angled twinnings and their lattices*, Comment. Math. Helv. **81** (2006), no. 1, 191–219.

[Ser77] Jean-Pierre Serre, *Arbres, amalgames,* SL_2, Société Mathématique de France, Paris, 1977, Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.

[Spe09] David E. Speyer, *Powers of Coxeter elements in infinite groups are reduced*, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1295–1302.

[Spr98] T. A. Springer, *Linear algebraic groups*, second ed., Progress in Mathematics, vol. 9, Birkhäuser Boston Inc., Boston, MA, 1998.

[Str04] Helmut Strade, *Simple Lie algebras over fields of positive characteristic. I*, de Gruyter Expositions in Mathematics, vol. 38, Walter de Gruyter & Co., Berlin, 2004, Structure theory.

[Tit74] Jacques Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin, 1974.

[Tit87] ______, *Uniqueness and presentation of Kac–Moody groups over fields*, J. Algebra **105** (1987), no. 2, 542–573.

[Tit89] ______, *Groupes associés aux algèbres de Kac–Moody*, Astérisque (1989), no. 177-178, Exp. No. 700, 7–31, Séminaire Bourbaki, Vol. 1988/89.

[Wan84] John S. P. Wang, *The Mautner phenomenon for p-adic Lie groups*, Math. Z. **185** (1984), no. 3, 403–412.

[Wil12] George Willis, *The nub of an automorphism of a totally disconnected, locally compact group*, Ergodic Theory & Dynamical Systems (to appear) (2012).