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Quasi-periodic configurations and undecidable dynamics for tilings, infinite words and Turing machines

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Abstract

We describe Turing machines, tilings and infinite words as dynamical systems and analyze some of their dynamical properties. It is known that some of these systems do not always have periodic configurations; we prove that they always have quasi-periodic configurations and we quantify quasi-periodicity. We then study the decidability of dynamical properties for these systems. In analogy to Rice's theorem for computable functions, we derive a theorem that characterizes dynamical system properties that are undecidable. As an illustration of this result, we prove that topological entropy is undecidable for Turing machines and for tilings.

Keywords: Tiling; Turing machine; Dynamical systems; Undecidability; Quasi-periodicity

1. Introduction

In this paper, we describe Turing machines, tilings and infinite words as dynamical systems and analyze some of their dynamical properties. Several questions and properties of these objects appear at first to be different and are given a unifying presentation in this paper. We do not provide explicit correspondences from one class to the other, but show instead that several properties for these dynamical systems can be proved in a common abstract context. In particular, we show that the domino problem for tilings and the mortality problem for Turing machines are the same conceptual problems, we prove that the dynamical systems we consider always have quasi-periodic

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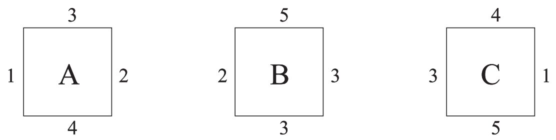

A	B	C	A	B	C	A	B	C
C	A	B	C	A	B	C	A	B
B	C	A	B	C	A	B	C	A
A	B	C	A	B	C	A	B	C
C	A	B	C	A	B	C	A	B
B	C	A	B	C	A	B	C	A
A	B	C	A	B	C	A	B	C
C	A	B	C	A	B	C	A	B
B	C	A	B	C	A	B	C	A

Fig. 1. A set of three Wang tiles and a piece of tiling generated by the tile set. Colors are represented by numbers. This is Wang's original example [17].

configurations, we quantify this quasi-periodicity with a function that generalizes those introduced for tilings and infinite words, and we prove a Rice-style undecidability result for dynamical systems. The dynamical systems that we consider are tilings of the plane, Turing machines and infinite words. We describe them briefly below.

A Wang tile is a unit square with colored borders. Given a finite set of Wang tiles, we consider the set of all tilings of the plane $\mathbb{Z}^2$. In a tiling of the plane every integer grid point of the plane is the center of a Wang tile and adjacent borders of tiles have the same color (see Fig. 1 for a simple example). Note that no rotation of the tiles is allowed. Once a tiling of the plane is given, other tilings can be obtained by horizontal and vertical shifts of the entire tiling. The dynamical system resulting from a set of Wang tiles is given by the set of all possible tilings of the plane, together with the horizontal and vertical shifts. The problem of determining if a given tile set can tile the plane (the domino problem) was proved undecidable by Berger [1]. Tiles sets that cannot tile the plane and those that can tile the plane in a periodic way are obviously recursively enumerable. From this it follows that some tile sets may tile the plane without being able to do so periodically; a fact that disproves the conjecture initially made by Wang that no such sets of tiles exist [17].

The Turing machine model that we consider in this paper is a Turing machine model with one or more tapes filled with symbols taken from a finite alphabet, and a head reading one symbol on each tape and able to write a new symbol and shift every tape independently to the left or to the right. This model differs from the more traditional one in that the machine is multi-tape, there is no "blank" symbol, and all tapes are *entirely* filled with symbols taken from the alphabet. The head is characterized by an internal state and acts deterministically. Among the possible states is the so-called halting state. An example is shown on Fig. 2. Let us fix now some Turing machine

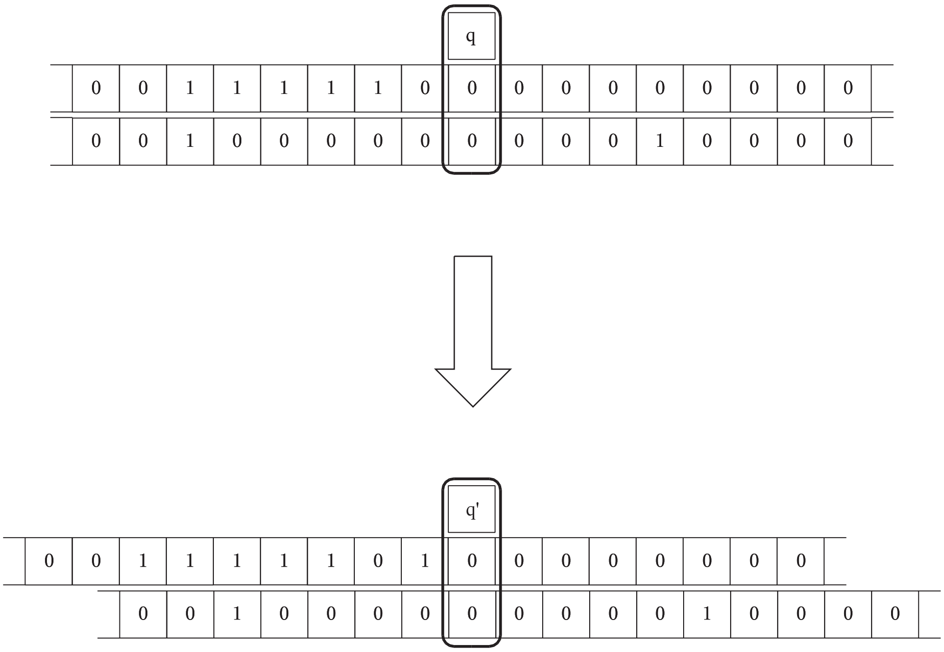

Fig. 2. A two-tape Turing machine. Here the head is jumping from state q to state q', writing a 1 in the cell of the upper tape, writing a 0 in the cell of the lower tape, then shifting the upper tape to the left and the lower tape to the right.

and consider the set of configurations of the machine that never halt (a Turing machine configuration is given by a tape content *and* a state). This is a dynamical system for the global transition function of the machine. The mortality problem for Turing machines is the problem of determining if the machine possesses a configuration that never halts, i.e., whether or not the corresponding dynamical system is empty. As explained below, this problem is the Turing machine equivalent to the domino problem and was proved undecidable by Hooper [8]. As for tiles, it follows from this result that there are Turing machines whose immortal configurations are all non-periodic. Kůrka has conjectured in [14] that no such Turing machine exists that has no halting state. This conjecture is disproved in [2] where a never-halting Turing machine with six states and four letters and no periodic configurations is constructed. For tilings, it is known that there is a set of 13 tiles with five colors that tile the plane and can only do so in a non-periodic way [5]. The decidability of the problem of determining if a Turing machine with no halting state possesses a periodic configuration is yet unsettled but is conjectured undecidable in [2].

Our final example is that of infinite words. A (right) infinite word on a finite alphabet Σ is a function $w:\mathbb{N}\to\Sigma$. The shift is the map σ defined by $\sigma(a_0a_1a_2a_3\ldots)=a_1a_2a_3\cdots$. Any set of infinite words that is closed in the product topology and that is stable under shift constitutes a dynamical system called *subshift*. In particular, the set of infinite words that do not contain as subwords words taken from a particular finite set is called a subshift of *finite type*.

Thus we have three natural examples of dynamical systems. All three systems are constructed in the following way: In the set of all possible configurations, we identify a subset of "bad" configurations Z. This subset Z is characterized by a finite number of constraints on symbols present in a finite number of places of the configurations. For Turing machines, Z is the set of configurations whose state is the halting state, for a tile set T, Z it is the set of sequences $\mathbb{Z}^2 \to T$ having a tiling error at the origin, and for a subshift, Z is the set of infinite words having a forbidden subword as a prefix. Then we consider the set of all configurations that never reach Z under the system transformations. The resulting sets are, respectively, the set of immortal configurations of a Turing machine, the set of tilings or the subshift of finite type. In this context Hooper's problem ('Is there an immortal configuration?') and Berger's problem ('Is there a tiling of the plane?') are the same questions. The corresponding question for infinite words is: 'Is a given subshift of finite type empty?' This problem is however easily decidable.

Motivated by those observations, we are lead to think that Turing machines, tilings and subshifts may be fruitfully studied in the common framework of topological dynamics. In this paper, we aim at unifying questions and theorems that are usually stated separately, and at deriving abstract theorems about dynamical systems that directly apply to tilings, Turing machines and infinite words.

In the first part of the paper, we prove from classical results of topological dynamics that the class of dynamical systems that we consider always have quasi-periodic configurations. A quasi-periodic configuration is a configuration whose associated trajectory regularly comes back near its starting point (in particular, periodic configurations are quasi-periodic). We also prove that dynamical systems that have a quasi-periodic configuration that is not periodic have uncountably many such configurations. We then define quasi-periodicity functions, which are a way to measure the regularity of quasi-periodic configurations. Abstracting from the case of tilings and subshifts we give a general definition, which can be applied to any dynamical system and prove that global convergence is a property that can be characterized in terms of quasi-periodicity function.

In the second part of the paper, we study the decidability of dynamical properties of dynamical systems. We are interested in this part by the following general question: *What properties of dynamical systems are decidable*? (A more precise meaning to this question is given in Section 4.) In the theory of computability the answer to the question: *What properties of computable functions are decidable*? is given by Rice's theorem: *All non-trivial*[1] *properties of computable functions are undecidable.* In this contribution we propose a Rice-style theorem for a class of dynamical systems that includes Turing machines and tilings. Here again we undertake a general approach that holds for any "rich enough" family of dynamical systems. Among others our result allows us to prove the undecidability of questions related to the number of invariant sets and the topological entropy of tilings and Turing machines.

[1] A property is *trivial* if it is verified by all computable functions or by none of them.

Let us note here that several results analogous to Rice's theorem are available in the literature for particular families of dynamical systems. Kari proved in [11] that every non-trivial property of the limit sets for cellular automata is undecidable. Cervelle and Durand show in [6] that it is undecidable to test whether two tile sets generate the same tilings, even when one of the tile sets is fixed and (almost) arbitrary. However, in this last reference "tilings" are understood in a way that is different from ours and the proofs given in [6] cannot be transposed to our set-up.

2. Dynamical systems and subsystems

A dynamical system is usually defined to be a continuous transformation of a compact metric (or Hausdorff) space. Here we need a slightly more general setting since we are interested in tilings for which there are two interesting maps: the horizontal and vertical shifts. We consider in this paper semigroups of transformations rather than just iterates of a single transformation. Some of the results and most definitions of this section are extracted from [3] and are reformulated here in the general context of semigroups.

A *dynamical system* is a couple (X, G), where X is a compact metric space and G is a semigroup acting on X such that the action of each element of G is continuous. By *semigroup* we mean a set endowed with an associative binary law and a unit element. By *G acts on X* we mean that each element of G corresponds to a transformation of X (called the *action* of the element), such that the unit element corresponds to the identity on X, and the composition on G corresponds to the composition on actions. Elements of X are called *points* or *configurations* of the dynamical system. The *orbit* of a configuration x is the set $\{g(x) | g \in G\}$. An *invariant* subset of X is a subset Y of X that is invariant for all transformations in G, i.e., $\forall g \in G\ g(Y) \subseteq Y$. A closed invariant subset Y defines a *subsystem* (Y, G). The three families of dynamical systems that we consider in this paper fit in this general definition.

2.1. Infinite words

Let A be a finite alphabet and let the set of (right) infinite words $X = A^{\mathbb{N}}$ be endowed with the product topology. This space is compact and is metrizable for the metric defined by $d(u, v) = 0$ if $u = v$ and

$$d(u, v) = 2^{-n},$$

where n is the index of the first letter on which u and v differ. Thus the set of configurations having a given word of length $s + 1$ as a prefix is an open ball of radius 2^{-s}. Conversely to an open ball of radius $r < 1$ is associated a finite word of length $\lfloor -\log_2 r \rfloor + 1$. Open balls are also closed, and conversely, every closed open set is a finite union of open balls. The *shift* is the map σ defined by $\sigma(a_0a_1a_2a_3\ldots) = a_1a_2a_3\ldots$. The pair $(A^{\mathbb{N}}, \sigma)$ is a dynamical system.[2] Its subsystems are called *subshifts*.

[2] Formally, we should write $(A^{\mathbb{N}}, G)$, where G is the free semigroup generated by σ.

We will be interested in particular subshifts. For that purpose, let us fix a closed open set Z of X. Then the set of infinite words whose orbit does not intersect Z defines a subsystem which we call a *subshift of finite type*.

2.2. Tilings

A Wang tile is a square of unit size with colored borders. Let T be a finite set of colored Wang tiles. The set $T^{\mathbb{Z}^2}$ may be endowed with the product topology. This space is compact for the metric $d(u,v)=0$ if $u=v$ and

$$d(u,v) = 2^{-n},$$

with n the smallest integer for which there are i and j such that $n = \min\{|i|,|j|\}$ and $u_{ij} \neq v_{ij}$. If a square pattern of odd size $2s+1$ is centered at the origin, then the set of configurations extending that pattern is an open ball of radius 2^{-s}. Conversely, to an open ball of radius $r<1$ is associated a square pattern of size $2\lfloor -\log_2 r \rfloor + 1$. Given a finite set of Wang tiles T, the set of all tilings of the plane is such that adjacent borders of tiles always have the same color. The set of tilings is a subset of $X = T^{\mathbb{Z}^2}$ that is invariant under the free semigroup generated by the north, south, west and east shifts (this semigroup is isomorphic to $(\mathbb{Z}^2,+)$). It is not difficult to see that this subset is closed (see [7]) and is therefore a dynamical system. Tilings may be thought of as subshifts of finite type in dimension two.

2.3. Turing machines

A (multi-tape) *Turing machine* is given by a finite set of bi-infinite tapes over finite alphabets. The tapes are handled by a head that can read the symbol on a cell, write a (possibly) new symbol, and make every tape move one cell to the right or one cell to the left. There is a finite set of internal states. The transition function tells the head which operations to perform on each tape and changes the internal state, given the set of symbols currently read on the tapes and the present internal state of the head. The space of configurations of a Turing machine is thus given by $X = Q \times A_1^{\mathbb{Z}} \times \cdots \times A_k^{\mathbb{Z}}$, where k is the number of tapes, $A_1,\dots,A_k$ the alphabets[3] and Q the set of internal states. This space is a compact metric space, with the metric defined by $d(u,v)=2$ if $u_0 \neq v_0$ (the states are different), $d(u,v)=0$ if $u=v$ (the states and tapes are identical), and

$$d(u,v) = 2^{-n}$$

if $u_0 = v_0$ and n is the smallest value of $|i|$ for which the ith entry of u and v differ on some tape. If f is the global transition function between configurations, then X with the free semigroup generated by f is a dynamical system. In the set of states we point

[3] In the sequel we shall assume for simplicity that all tapes operate with the same alphabet A of size two or more.

out a special state, called the *halting state*. Notice that the set of configurations with the halting state is a closed open set of X. The set of immortal configurations, i.e., configurations that never reach the halting state, is a closed subset of X that is invariant under f. This set defines a subsystem of X for the free semigroup generated by f.

Note, as already mentioned in the introduction, that these three dynamical systems are built on the same scheme. We have a dynamical system on a space X, we make a partition of X into two-closed open sets Z and $X \backslash Z$ and we consider the largest subsystem that is included in $X \backslash Z$.

3. Minimal dynamical systems and quasi-periodicity

Subsystems of dynamical systems can easily be constructed as follows: let (X, G) be some dynamical system and pick an arbitrary configuration x. The orbit of x is obviously an invariant set. Let Y be the topological closure of the orbit associated to x; then Y is a closed invariant subset of X, (Y, G) is a subsystem of (X, G), and any subsystem that contains x as a configuration also contains Y.

We prove in this section that three qualitatively different situations may occur. If (Y, G) is minimal, i.e., if (Y, G) contains no non-trivial proper subsystem, then either Y is finite, and x is periodic, or Y is infinite, and then Y is uncountably infinite and all configurations of (Y, G) are quasi-periodic; they repeatedly return arbitrary closely to configurations in their orbit. Finally, if Y is not minimal, then the dynamical system (Y, G) must contain a subsystem of one of the above two types. In this section, we make these notions precise, we describe minimal dynamical systems and prove that they coincide with closed orbits of quasi-periodic configurations.

3.1. Minimal dynamical systems

A *minimal* dynamical system is a dynamical system that has no proper subsystem, except the empty set. A point whose orbit is finite is said to be *eventually periodic*. If the orbit is minimal and finite, then the point is said to be *periodic*. Minimal dynamical systems can be characterized as follows:

Proposition 1. *For a dynamical system* (Y, G), *the following conditions are equivalent*:
(1) *the system is minimal*;
(2) *Y is the closed orbit of any of its configurations.*

Proof. If a dynamical system is minimal, every closed orbit (itself a subsystem) must be the system itself. Conversely, if a dynamical system is not minimal then there exists a proper non-empty subsystem; any configuration of this subsystem generates an orbit, the closure of which does not fill the whole space. □

Let us focus now on *finite-to-one* dynamical systems, i.e. systems for which every action of the semigroup is a finite-to-one transformation. Turing machines, words and tilings are finite-to-one systems (the latter is even a one-to-one system). We prove that minimal finite-to-one dynamical systems cannot contain countably many configurations. They must either be finite, or uncountably infinite. Recall that an *isolated* point is an open singleton, and a *perfect* space is a space without isolated point.

Proposition 2. *A minimal finite-to-one dynamical system is finite iff it has an isolated point. Infinite minimal finite-to-one dynamical systems are always uncountably infinite.*

Proof. Let (Y,G) be minimal. If $x \in Y$ is isolated then $\{x\}$ is an open set and, from minimality, $\bigcup_{g\in G} g^{-1}(x) = Y$ (otherwise $Y \setminus \bigcup_{g\in G} g^{-1}(x)$ would be a non-empty invariant closed subset of Y). From compactness, a finite covering may be extracted. But each set $g^{-1}(x)$ of the covering is finite, hence Y must be finite.

It is known that any perfect compact metric space is uncountable (as a corollary of Baire category theorem; see for instance [15]). We thus infer that a minimal finite-to-one dynamical system is either finite or uncountable. $\square$

Note that if we do not ask the system to be finite-to-one, the proposition is no longer valid, as shown by the following counterexample.

Example 1. Let Y be the set $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, 0\}$, f be the transformation $x \mapsto 1$, g be the transformation $x \mapsto x/2$ and G the semigroup generated by f and g. Notice that G is not finite-to-one because f is not, and (Y,G) is an infinite minimal system with isolated points.

3.2. Quasi-periodic configurations

A configuration x of a dynamical system (X,G) is said to be *quasi-periodic* (or *almost periodic*, or *uniformly recurrent*), if for every neighborhood U of x there is a finite set of transformations $H \subseteq G$ such that for every y in the orbit of x, the set Hy has some configuration in U. In other words, a configuration x is quasi-periodic if one can choose a finite number of transformations that send every point of the orbit near x. Notice in particular that periodic configurations are quasi-periodic. We can easily find combinatorial characterization of quasi-periodicity for our particular examples of dynamical systems. For this we need the notion of pattern. For an infinite word a *pattern* is a subword of finite length. For tilings on the set of tiles T, a *pattern* is a partial function $\mathbb{Z}^2 \to T$ of finite domain. For a Turing machine, a *pattern* is the conjunction of a finite set of constraints on the content of cells (i.e., a finite partial function $\mathbb{Z} \to A$ for each tape) and/or on the state.

Proposition 3 (Quasi-periodicity). *An infinite word w is quasi-periodic iff for every pattern u of w there is an integer k such that the pattern u appears in every pattern of length k.*

A configuration of the plane is quasi-periodic iff for every pattern u of the tiling there is an integer k such that u appears in every square $k \times k$ pattern.

A configuration of a Turing machine is quasi-periodic iff for every pattern u occurring in the configuration there is an integer k such that the pattern u occurs infinitely often, and the time between two successive occurrences is at most k.

The following proposition relates minimal subsystems and quasi-periodicity.

Proposition 4. *Let x be a configuration of a dynamical system. The following are equivalent*:
(1) *x is quasi-periodic*;
(2) *the closed orbit of x is minimal*;
(3) *for every neighborhood U of x, the preimages of U form a covering of the closed orbit of x.*

Proof. (1)$\Rightarrow$(2) Let x be a quasi-periodic configuration of (X,G) and y be in the closed orbit of x. Then for any closed neighborhood U of x there is a finite subset H of G such that $\bigcup_{h\in H} h^{-1}(U)$ covers the orbit of x. But this set, finite union of closed sets, is closed. It therefore also covers the closed orbit. Since this applies for any closed neighborhood U, we deduce that x is in the closed orbit of y. Thus the closed orbits of x and y are the same. As it is true for any y in the closed orbit of x, we find from Proposition 1 that the closed orbit of x is minimal.

(2)$\Rightarrow$(3) For any non-empty open set U of a dynamical system (X,G), the set $X\setminus\bigcup_{g\in G} g^{-1}(U)$ is invariant and closed. Thus it must be empty from minimality.

(3)$\Rightarrow$(1) If the preimages of a neighborhood of x cover the closed orbit of x, then from compactness, a finite number of preimages already cover the closed orbit. As it is true for any neighborhood of x, x is quasi-periodic. □

Consider now the set of all possible subsystems of a dynamical system and order them by inclusion. By Zorn's lemma and compactness, one of the subsystems is minimal. According to the above proposition, minimal subsystems exactly correspond to closed orbits of quasi-periodic configurations, and so we have the following well-known corollary.

Proposition 5 (Birkhoff). *Every non-empty dynamical system has a quasi-periodic configuration.*

From Proposition 2 we also get the following:

Proposition 6. *If a finite-to-one dynamical system has a quasi-periodic configuration that is non-periodic, then it has uncountably many such configurations.*

We can apply these result to tilings, infinite words and Turing machines.

Proposition 7. *Every tile set that can tile the plane can tile it in a quasi-periodic way. If a tile set can generate a non-periodic quasi-periodic tiling, then it generates an uncountable number of such tilings.*

Every Turing machine that has an immortal configuration has a quasi-periodic immortal configuration. If a Turing machine has an non-periodic quasi-periodic immortal configuration then it has an uncountable number of such configurations.

In every subshift there is a quasi-periodic word.

The result for infinite words is known for long; see [16] for instance. The result for tiles is proved by Durand in [7]; in particular, the uncountability part is established with combinatorial arguments. Our topological proof has the advantage that it is applicable to any dynamical system. In particular, the result for Turing machines appears here for the first time. It nicely complements the recent result proved in [2] that Turing machines do not always have periodic configurations.

It is interesting to note that, contrarily to tilings and Turing machines, subshifts of finite type always have a periodic configuration (as easily seen). This suggests that by some aspects at least, the dynamics of tilings is closer to that of Turing machines than to that of infinite words, even though tilings can be thought of as subshifts of finite type in two dimensions.

3.3. Quasi-periodicity functions

To study quasi-periodic configurations in greater length, a special function is introduced for tilings in [7] and for words in [4] (the function is called *recurrence function* in the latter reference). In both cases, these functions quantify how far a quasi-periodic configuration is from periodicity. In this subsection we introduce a quasi-periodicity function in the context of dynamical systems and prove that our definition essentially coincides with those for tilings and infinite words.

The *quasi-periodicity function* $Q_T : \mathbb{N} \to \mathbb{N}$ of a quasi-periodic tiling of the plane is the function that to n associates the smallest k such that every $(2k+1) \times (2k+1)$ pattern of the configuration contains all $(2n+1) \times (2n+1)$ patterns observable on the configuration. Note that this is not exactly the function introduced in [7]; we restrict ourselves to pattern of odd size, in order to simplify the presentation. Some properties of this function can be found in [7,6]. For instance, a quasi-periodic configuration is periodic iff its quasi-periodicity function is bounded by $n \mapsto n + c$ for some constant c. Moreover, for any increasing time-computable function $f : \mathbb{N} \to \mathbb{N}$, there is a tile set such that all tilings generated are quasi-periodic and have f for quasi-periodicity function.

An analogous definition is given in [4] for infinite words. The *recurrence function* $Q_W : \mathbb{N} \to \mathbb{N}$ of a quasi-periodic infinite word of $A^{\mathbb{N}}$ is defined there as the function that to n associates the smallest k such that every subword of length k contains all subwords of length n. Some properties of this function are derived in [4]. In particular, it is shown there that, as for tilings, a quasi-periodic configuration is periodic iff its recurrence function is bounded by $n \mapsto n + c$ for some constant c.

We now define quasi-periodicity functions for arbitrary dynamical systems.

Let (Y, G) be a minimal subsystem of the dynamical system (X, G). The *quasi-periodicity function* of Y relatively to (X, G) is the function $Q : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ defined by $\varepsilon \mapsto \inf_{y\in Y}\{\lambda_{y,\varepsilon}\}$, where

$$\lambda_{y,\varepsilon} = \sup\{\lambda > 0 | \forall x \in Y\ \exists g \in G : B(x,\lambda) \subseteq g^{-1}(B(y,\varepsilon))\}.$$

Notice that, thanks to Proposition 4, for every $y \in Y$ the collection of open sets $\{g^{-1}(B(y,\varepsilon)) : g \in G\}$ is an open covering of Y, and $\lambda_{y,\varepsilon}$ is a Lebesgue number of this covering.

Proposition 8. *When specialized to tilings and words the quasi-periodicity function defined above satisfies* $Q(2^{-n}) = 2^{-Q_T(n)}$ *and* $Q(2^{-n}) = 2^{-Q_W(n)}$.

Proof. We only prove $Q(2^{-n}) = 2^{-Q_T(n)}$; the proof for infinite words is similar. Let T be a finite set of tiles and suppose that $u \in T^{\mathbb{Z}^2}$ is a quasi-periodic configuration whose quasi-periodicity function relatively to $T^{\mathbb{Z}^2}$ is $\varepsilon \mapsto Q(\varepsilon)$ and fix a square pattern of size $2n+1$ present in u. If $k = Q_T(n)$ then this pattern is extended by every $(2k+1)\times(2k+1)$ square pattern of u. It is also extended by every $(2k+1)\times(2k+1)$ square pattern of every configuration in the closed orbit of u, because every such configuration is the limit of elements of the orbit of u.

This means exactly that for every point of the closed orbit of u, the open ball of radius 2^{-k} around this configuration is included in a preimage (by some shift) of the open ball of radius 2^{-n} around some (fixed) configuration of the orbit of u.

Thus $Q_T(n) = k$ means that for every configurations v, w of the closed orbit of u, $B(w, 2^{-k})$ is included in the preimage by some shift of $B(v, 2^{-n})$, and that k is the smallest integer to have this property. But this also means that $Q(2^{-n}) = 2^{-k}$. □

It would be interesting to see to what extent properties of the quasi-periodicity function for tilings and infinite words may be generalized to arbitrary dynamical systems. For instance, remark that for tilings and infinite words, the quasi-periodicity function is greater than the identity. Formulated in our context this would suggest that quasi-periodicity functions are bounded above by the identity function. However, this turns out to be false in general, as shown by the following example. Let

$$f : [-1,1] \to [-1,1] : x \mapsto \frac{x}{2}.$$

The only quasi-periodic point of this dynamical system is the origin, and the corresponding quasi-periodicity function is identically equal to infinity.

More generally, we can exactly characterize the dynamical systems that have a quasi-periodic configuration with infinite quasi-periodicity function. Let us say that a subsystem Y of a dynamical system (X, G) is *globally attractive* if for every open neighborhood U of Y, there is a $g \in G$ such that $Gg(X) \subseteq U$ (once g is applied the remaining part of the orbit remains in U). If G is generated by one transformation, this is equivalent to say that the trajectories are uniformly attracted to Y.

Proposition 9. *A minimal subsystem of a dynamical system with a commutative semigroup has its quasi-periodicity function identically equal to infinity iff it is a globally attractive point.*

Proof. Suppose x is a globally attractive point of (X, G), and choose an open neighborhood U of x. Then there is a $g \in G$ such that $g(X) \subseteq U$, i.e. $g^{-1}(U) = X$. Then $B(x, +\infty) = X$ is included in some preimage of U.

Conversely, if a minimal subsystem Y of (X, G) has a quasi-periodicity function $\varepsilon \mapsto \infty$, then for every $y \in Y$ and every $\varepsilon > 0$ there is a $g \in G$ such that $g^{-1}(B(y, \varepsilon)) = X$. Thus if there are distinct $y_1, y_2 \in Y$ then for any ε, there are g_1 and g_2 in G such that for all $x \in X$, $Gg_1(x) \subseteq B(y_1, \varepsilon)$ and $Gg_2(x) \subseteq B(y_2, \varepsilon)$. But then from commutativity $g_1 g_2(x)$ is in both $B(y_1, \varepsilon)$ and $B(y_2, \varepsilon)$, which is impossible if ε is small enough. Hence Y is a singleton which is, since $g(X) \subseteq U$ implies $Gg(X) = gG(X) \subseteq U$, globally attractive. □

In the proof of the result, we need to assume that the semigroup is commutative. We do not know if the statement remains valid when this assumption is removed.

The existence of a globally attractive point may thus be read in the quasi-periodicity functions of the system. We do not know what other properties of systems may be characterized in terms of their quasi-periodicity functions. Also, we do not know what conditions should be put on a system to ensure that the quasi-periodicity functions of minimal subsystems are bounded above by identity.

4. Undecidable properties

In this section we give sufficient conditions for dynamical properties of dynamical systems to be undecidable, and illustrate the interest of these conditions by deriving undecidability results for questions related to the topological entropy and the invariant subspaces of Turing machines and tilings.

4.1. Families of dynamical systems

A *family of dynamical systems* is a set $(Y_n, G_n)_{n \in \mathbb{N}}$ of dynamical systems, indexed by the positive integers. A system (Y, G) for which Y is empty is said to be an *empty system*. In the sequel we assume that the families we consider always contain an empty system. Examples of families of dynamical systems are given by tilings under vertical and horizontal shifts, subshifts of finite type and immortal configurations of Turing machines under transition function, as described in Section 2.

We say that a dynamical system property is *decidable* if for every given n, we can algorithmically decide whether or not (Y_n, G_n) satisfies the property. One of the most basic property we may want to test is whether the system is empty. As explained in Section 2, this problem was proved undecidable for tilings by Berger in [1], and for Turing machines (even with only one tape) by Hooper in [8]: there is no algorithm to decide whether the system of valid tilings built on a given tile set is empty, and there is no algorithm to decide whether the system of immortal configurations of a given Turing machine is empty. By contrast, the emptiness problem for subshifts of finite type is decidable. Motivated by this common feature of Turing machines and tilings, we investigate what properties are undecidable for dynamical systems.

4.2. A Rice-style theorem for dynamical systems

Not all dynamical systems have all their properties undecidable. In order to derive a general undecidability result we therefore need to impose conditions on the families and the properties that we consider.

Firstly, we will consider dynamical systems that have an undecidable emptiness problem and for which it is possible to effectively compute cartesian products (tilings and Turing machines satisfy these conditions, see below). Secondly, we will consider properties that are invariant under system isomorphisms (such as symbols renaming for example), and that are not affected by cartesian product. The first condition is quite natural; we do not know if our result remains valid if we remove the second condition.

Let us now formalize all this. We need several definitions. Consider the systems (X,G) and (Y,H) and assume that there is a continuous map $f : X \to Y$ and a semigroup isomorphism $s : G \to H$ such that for all $g \in G$, $s(g) \circ f = f \circ g$. If f is surjective then (Y,H) is a *factor* of (X,G). If f is bijective then (Y,H) and (X,G) are *isomorphic*. A system property that is invariant under isomorphism is said to be a *dynamical property*.

Given a semigroup isomorphism $s : G \to H$, the (cartesian) *product* of the two dynamical systems (X,G) and (Y,H) is the system $(X \times Y, J)$, where $J = \{(g, s(g)) : g \in G\}$ is isomorphic to G and H. Notice that products of systems are defined only for systems that have isomorphic semigroups and that the product of two systems is always empty when one of the systems is. Products of systems may or may not be effective. We say that *products are effective* for a family of dynamical systems if given two indices i, j, an index k_{ij} can be effectively computed such that the system of index k_{ij} is isomorphic to the product of the systems i and j. Products are effective for tilings as well as for Turing machines. Indeed, to make the product of two sets of tilings, just make the product of the corresponding sets of tiles: this is an effective operation. To make the product of two Turing machines, just make the disjoint union of the corresponding tapes and heads, and the product of the states.

Finally, we shall say that a dynamical property is *not affected by product* if the product of any system that has the property with a non-empty system is a system that has the property. Examples of such properties are given in Corollaries 1 and 2. We are now ready to state our result.

Theorem 1. *Consider a family of dynamical systems for which products are effective and emptiness is undecidable. Then, any dynamical property that is not affected by product and that is satisfied by at least one member of the family, but that is not satisfied by the empty system, is undecidable.*

Proof. As for Rice's theorem the proof is in fact very simple. The proof proceeds by reduction to the emptiness problem. Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a family satisfying the conditions of the theorem. Suppose we have a suitable property P, that is verified by the system Σ_i, and assume there is an algorithm that decides that property. Then the following algorithm decides emptiness of the system Σ_n:

```
Input n;
Compute an index for Σ_i × Σ_n;
If Σ_i × Σ_n has property P
then print 'Σ_n is not empty'
else print 'Σ_n is empty'.
```
□

We now apply this result to tilings and multi-tape Turing machines.

Corollary 1. *The following questions about tilings and immortal configurations of Turing machines are undecidable: Given a system of the family:*
(1) *Are there at least two disjoint invariant subspaces?.*
(2) *Is there an infinite number of disjoint invariant subspaces?.*
For any fixed non-empty system Σ of the family, the problem of determining, for a given system, if Σ is a factor of the system, is undecidable.

Proof. Properties (1) and (2) easily verify conditions of Theorem 1. In particular, a Turing machine with identity as transition function, and a set of tilings isomorphic to $2^{\mathbb{Z}^2}$ have both properties.

For a proof of the second statement, remark that Σ is not a factor of $\emptyset$, but it is a factor of Σ and if it is a factor of Σ' then it is a factor of any product of Σ' by a non-empty system. □

Note that the presence of two disjoint invariant subsets in a set of tilings exactly means that we can find two tilings of the plane and an integer N such that they have no finite pattern of size $N \times N$ in common.

4.3. Topological entropy

We apply Theorem 1 to derive results on the topological entropy of dynamical systems. Entropy is proved to be uncomputable for cellular automata in [9] and for piecewise affine maps in [13], we show that it is not computable for Turing machines and tilings. Let us first recall some definitions and basic properties (the material presented here is essentially taken from [10,3,14]).

Consider the set of tilings generated by some tile set. Call N_n the number of an $n \times n$ patterns that may be found in at least one of the tilings. Then the *topological entropy* of the tile set is given by

$$\lim_{n\to\infty} \frac{\log_2 N_n}{n^2}.$$

One can prove that this limit always exists. An analogous definition is possible for Turing machines. Consider the system (Y_T, f_T) with Y_T the set of immortal configurations of some Turing machine T with k tapes, set of states Q, alphabet A and global transition function f_T. For every n we can partition the set (Y_T, f_T) into open balls of radius 2^{-n}. In other words, we partition Y_T according to the internal state and the $2n+1$ symbols around the head on each tape. Hence the set Y_T is partitioned into at most $|Q||A|^{k(2n+1)}$ parts. This set of parts is seen as an alphabet α_n. Now suppose there is an immortal configuration $u \in Y_T$ such that $u \in a_0, f_T(u) \in a_1, \ldots, f_T^t(u) \in a_t$ (where $a_0, a_1, \ldots, a_t \in \alpha_n$ are balls of radius 2^{-n}). Then we say that $a_0 a_1 \cdots a_t$ is an *observed word*. Denote by $N_{n,t}$ the number of observed words of length $t+1$, the *topological entropy* of (Y_T, f_T) is given by

$$\lim_{n\to\infty} \lim_{t\to\infty} \frac{\log_2 N_{n,t}}{t+1}.$$

As for tilings, it can be shown that this limit always exists. One can also prove that in both cases the entropy of the product of two non-empty systems is the sum of the individual entropies, and the entropy of the sum of two non-empty systems is the maximum of the individual entropies. We define the entropy of the empty system to be equal to zero.

Corollary 2. *For any fixed $\alpha > 0$, the following questions about tilings and immortal configurations of Turing machines are undecidable: Given a system of the family,*
(1) *Is the topological entropy of the system greater or equal to α?*
(2) *Is the topological entropy of the system greater than α?*

Proof. The topological entropy of the product of two systems is the sum of the individual entropies, and is consequently greater than the individual entropies. Moreover, for any $\alpha \geqslant 0$, it is easy to build a tiling or a Turing machine, whose entropy is greater than α (just think of the Turing machine that does nothing but shifts $\lceil \alpha \rceil$ tapes to the left, with an alphabet of size two or more). This ensures undecidability of both questions. □

The reader might feel skeptical about the entropy argument: since entropy of the empty set is actually not defined, we could not blame an algorithm that correctly decides whether the entropy is greater than α *except for the empty set*, for which it would not halt. But we can easily modify the proof of Theorem 1 in a more satisfactory way: instead of testing the entropy of $\Sigma_i \times \Sigma_n$ we test $\Sigma_i \times \Sigma_n + \Sigma_k$, where '+' is the disjoint union and Σ_k is a system of the family with null entropy. For tilings, we can make the disjoint sum of tilings sets by making the disjoint sum of tile sets: this is an effective operation. For two Turing machines with the same number of tapes and the same alphabets, we can make the disjoint sum of immortal systems by making the disjoint sum of internal states: this is an effective operation. Thus in the latter case, we choose for Σ_k a Turing machine that has the same number of tapes and the same alphabets as $\Sigma_i \times \Sigma_n$, and whose entropy is zero.

5. Conclusion

This paper introduces a common method to formulate and solve questions about various objects such as tilings, Turing machines and infinite words. The method is to study them as dynamical systems rather than combinatorial objects.

We give a general result about the existence and cardinality of quasi-periodic points for dynamical systems. Then we introduce a quasi-periodicity function for dynamical systems that generalizes similar functions introduced in the literature for the specific case of tilings and infinite words. Finally, we study decidability of dynamical systems properties. A number of properties are shown to be undecidable for tilings and Turing machines. In particular topological entropy is shown to be uncomputable.

However many questions remain open. As explained in Section 3, the form of the quasi-periodicity functions of a dynamical system can be connected to the properties of the system. For example, what kinds of systems verify the properties of this function discussed in [4,6,7]? Also, it would desirable to characterize more completely what properties of dynamical systems are undecidable. A natural improvement would be to lift or weaken the hypothesis made in Theorem 1 that the property to be proved undecidable is not affected by product. Along this way, an ambitious goal would be to find general conditions for emptiness of a system to be undecidable. This could lead in particular to a unified proof of Berger's and Hooper's theorems that are shown in this paper to be two instances of a general question about dynamical systems.

Another direction of research would be to study which of our results apply to a broader set of dynamical systems, such as cellular automata, piecewise affine maps of $\mathbb{R}^n$ and ordinary differential equations.

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