Computational Universality in Symbolic Dynamical Systems*

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Abstract. Many different definitions of computational universality for various types of systems have flourished since Turing's work. In this paper, we propose a general definition of universality that applies to arbitrary discrete time symbolic dynamical systems. For Turing machines and tag systems, our definition coincides with the usual notion of universality. It however yields a new definition for cellular automata and subshifts. Our definition is robust with respect to noise on the initial condition, which is a desirable feature for physical realizability.
We derive necessary conditions for universality. For instance, a universal system must have a sensitive point and a proper subsystem. We conjecture that universal systems have an infinite number of subsystems. We also discuss the thesis that computation should occur at the 'edge of chaos' and we exhibit a universal chaotic system.

1 Introduction

Computability is often defined via universal Turing machines. A Turing machine is a dynamical system, i.e., a set of configurations together with a transformation of this set. Here a configuration is composed of the state of the head and the whole content of the tape. Computation is done by observing the trajectory of an initial point under iterated transformation.

However there is no reason why Turing machines should be the only dynamical systems capable of universal computation, and indeed we know that many systems may perform universal computations.

Artificial neural networks [1], cellular automata [2], billiard balls on a pool table of some complicated form, or a ray of light between a set of mirrors [3] are such systems.

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For all these systems, many particular definitions of universality have been proposed. Most of them mimic the definition of computation for Turing machines: an initial point is chosen, then we observe the trajectory of this point and see whether it reaches some 'halting' set. See for instance [4] and [5].

However it has been shown that the computational capabilities of many of these systems are strongly affected by the presence of noise [6,7]; fault-tolerant cellular automata are built in [8]. See also [9,10,11] for some definitions of analog computation and issues relative to noise and physical realizability.

Moreover, many variants of these definitions exist and lead to different classes of universal dynamical systems. In particular, there is no consensus for what it means for a cellular automaton to be universal.

Another field of investigation is to make a link between the computational properties of a system and its dynamical properties. For instance, attempts have been made to relate 'universal' cellular automata to Wolfram's classification. It has also been suggested that a 'complex' system must be on the 'edge of chaos': this means that the dynamical behavior of such a system is neither simple (i.e., an attracting fixed point) nor chaotic; see [2,12,13,14]. Other authors nevertheless argue that a universal system may be chaotic: see [1].

The basic questions we would like to address are the following:

- What is a computationally universal dynamical system?
- What are the dynamical properties of a universal system?

A long-term motivation is to answer these questions from the point of view of physics. What natural systems are universal? Is the gravitational N-body problem universal [3]? Are the Navier-Stokes equations universal [15]?

However in this paper we especially focus on *symbolic* dynamical systems, i.e., systems defined on the Cantor set $\{0,1\}^{\mathbb{N}}$ or a subset of it. Some motivating examples of dynamical systems are Turing machines, cellular automata and subshifts. Let us briefly describe our ideas.

Extending Davis' definition of universal Turing machine, we say that a system is universal if some property of its trajectories, such as reachability of the halting set, is r.e.-complete.

However, rather than considering point-to-point or point-to-set properties, we consider set-to-set properties. Typically, given an initial set and a halting set, we look whether there is at least one configuration in the initial set whose trajectory eventually reaches the halting set.

We require the initial and halting sets to be closed open sets of the Cantor space endowed with the usual product topology, which are sets that can be described with a finite number of bits in a natural standard way.

Finally, we do not restrict ourselves to the sole property 'Is there a trajectory going from A to B?' (where A and B are closed open sets), but to any property of closed open sets that can be described in temporal logic.

This definition addresses the two issues raised above. Firstly, it is a general definition directly transposable to any symbolic system. Secondly, dealing with open sets rather than points takes into account some constraints of physical realizability, such as robustness to noise.

With this definition in mind, we prove necessary conditions for a symbolic system to be universal. In particular, we show that a universal symbolic system is not minimal, not equicontinuous and does not satisfy the effective shadowing property. This last property is a variant of the usual shadowing property. We conjecture that a universal system must have infinitely many subsystems, and we show that there is a chaotic system that is universal, contradicting the idea that computation can only happen on the 'edge of chaos'.

The paper is organized as follows: in Section 2 we define effective symbolic systems; in Section 3 the syntax and semantics of temporal logic is exposed; in Section 4 the formal definition of universality is given, and simple examples are provided; this definition is discussed in Section 5; in Section 6 necessary conditions for a system to be universal are given, related to minimality, equicontinuity and effective shadowing property; in Section 7 we build a chaotic system that is universal, and briefly discuss the existence of the 'edge of chaos'; Section 8 discusses possible directions for future work.

2 Effective Symbolic Systems

Effective symbolic dynamical systems are computable continuous transformations of a symbolic space. In this section, we provide a formal definition and elementary examples.

A *symbolic* set is the Cantor set $\{0,1\}^{\mathbb{N}}$ or a subset of it. Some other sets deserve to be called symbolic, for instance $A^{\mathbb{N}}$, $Q \times A^{\mathbb{Z}}$, $A^{\mathbb{Z}^d}$, where A and Q are finite sets and d is a positive integer. But all these sets can be recoded into $\{0,1\}^{\mathbb{N}}$ with standard tricks. Thus, every time we deal with such a set we implicitly suppose that we deal with $\{0,1\}^{\mathbb{N}}$.

Another set of interest is the set of finite and infinite binary words $\{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$. This set can be recoded as a subset of $\{0,1,B\}^{\mathbb{N}}$, if we think of a finite word w as the infinite word $wBBBBBB\ldots$ This set can be again recoded as a subset of $\{0,1\}^{\mathbb{N}}$.

The Cantor set can be endowed with the product topology. This topology is given by the metric $d(x,y) = 0$ if $x = y$ and

$$d(x,y) = 2^{-n}$$

where n is the index of the first bit on which x and y differ.

If w is a word of $\{0,1\}^*$, then $[w]$ denotes the set of all sequences beginning by w. In fact, sets of this form, usually called *cylinders*, are exactly the balls of the metric space. Closed open sets (*clopen* sets for short) of $\{0,1\}^{\mathbb{N}}$ are exactly all finite unions of cylinders. Thus clopen sets are finitary objects that can be described by finite words in alphabet $\{0,1\} \cup \{,\}$.

A *symbolic space* is closed subset of $\{0,1\}^{\mathbb{N}}$. It is a topological space for the relative topology, whose clopen sets are the intersections of the closed subset with all clopen sets of Cantor space. A symbolic space is said to be *effective* if checking whether a given clopen set of the Cantor space intersects the symbolic space is decidable.

Definition 1 *An* effective symbolic dynamical system *is a continuous map from an effective symbolic space to itself, such that the inverse map restricted to clopen sets is computable.*

This definition of effective function in a Cantor space is equivalent to classical definitions in computable analysis, for instance [16].

An *effective subsystem* of an effective symbolic system is an effective closed subset that is invariant under the map.

For example, a cellular automaton is an effective symbolic system, acting on the space $A^{\mathbb{Z}^d}$, where A is the finite alphabet and d is the dimension. A Turing machine is an effective system acting on the space $Q \times A^{\mathbb{Z}}$, where Q is the finite set of states of the head and A is the finite tape alphabet.

Recall that a *shift* is a dynamical system on $A^{\mathbb{N}}$ or $A^{\mathbb{Z}}$ (where A is a finite alphabet) with the map $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}} : a_0a_1a_2a_3\ldots \mapsto a_1a_2a_3\ldots$ or $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}} : \ldots a_{-3}a_{-2}a_{-1}\underline{a_0}a_1a_2a_3\ldots \mapsto \ldots a_{-3}a_{-2}a_{-1}a_0\underline{a_1}a_2a_3\ldots$, where the symbol of index 0 is underlined. A *subshift* is a subsystem of a shift. A one-sided (two-sided) full shift is an effective system, and if a subshift is an effective closed subset of the Cantor space, then it is again an effective system.

A subshift can be seen as a set of infinite words over a finite alphabet. The set of all finite words appearing at least once in at least one of these words is called the *language* of the subshift. In fact it is easy to see that an effective subshift is exactly a subshift whose language is recursive.

3 Temporal Logic

Our goal is to describe properties of trajectories that may be useful in defining universal computation, such as 'starting from here, the system eventually goes there'. Temporal logic, developed by Prior in 1953, is appropriate to express such properties. It was later used by computer scientists to express and prove sentences such as, typically, 'the program will not reach a forbidden state'; see [17] for a reference book on modal and temporal logic.

Formally, we suppose that we have a set $\{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, ...\}$ of proposition symbols indexed by $\mathbb{N}$ including two propositions that we will denote $\perp$ and $\top$, and we form all temporal formulae by composing the propositions symbols with the boolean operators $\vee$ and $\neg$, the temporal unary operator $\circ$ (read 'next') and the temporal binary operator $\mathcal{U}$ ('until').

We can also add some usual abbreviations: $\phi \wedge \psi$ denotes $\neg(\neg\phi \vee \neg\psi)$, $\phi \Rightarrow \psi$ denotes $\neg\phi \vee \psi$, $\diamond\phi$ (read 'eventually ϕ') stands for $\top\mathcal{U}\phi$ and $\Box\phi$ (read 'always ϕ') for $\neg\diamond\neg\phi$.

We now give temporal formulae a semantics adapted to symbolic systems. Let (X, f) be an effective symbolic system. Recall that X is a symbolic space and $f : X \to X$ a continuous function. We suppose that clopen sets are numbered in an effective way $P_0, P_1, P_2, \ldots$ Then to each formula ϕ we assign a subset $|\phi|$ of X, called the *interpretation* of ϕ, in the following way.

- If ϕ is the proposition symbol $\mathcal{P}_n$, then $|\phi| = P_n$. Moreover we ask that $|\perp| = \emptyset$ and $|\top| = X$.

– If ϕ is $\phi_1 \vee \phi_2$ then $|\phi| = |\phi_1| \cup |\phi_2|$.
– If ϕ is $\neg\psi$ then $|\phi| = X \setminus |\psi|$.
– If ϕ is $\circ\psi$ then $|\phi| = f^{-1}(|\psi|)$.
– If ϕ is $\phi_1 \mathcal{U} \phi_2$ then $|\phi| = \bigcup_{n\in\mathbb{N}} A_n$, where $A_0 = |\phi_2|$ and $A_{n+1} = f^{-1}(A_n) \cap |\phi_1|$ for all n.

In particular, if ϕ is $\diamond\psi$ then $|\phi| = \bigcup_{n\in\mathbb{N}} f^{-n}(|\psi|)$.

We say that a formula is *satisfiable* if $|\phi| \neq \emptyset$.

Intuitively, we may think that a formula ϕ represents a statement about a point of X, which is seen as 'the current configuration of the system'. This statement may be true for some points of X and false everywhere else. For example, $\diamond\mathcal{P}_n$ means 'when applying f iteratively, the current configuration will eventually be in P_n'. The formula $\mathcal{P}_m\mathcal{U}\mathcal{P}_n$ means 'the configuration lies in P_m until it reaches P_n' or, in other words, 'the configuration will stay in P_m during a finite time and then get in P_n'.

Then $|\phi|$ is the set of points for which the assertion ϕ holds, and a satisfiable formula is verified by at least one configuration. Note that in the following, we will make no distinction between a proposition symbol $\mathcal{P}_n$ and the corresponding clopen set P_n.

4 Universal Systems

We are now ready to state the main definition. We define a universal system to be an effective system with some r.e.-complete temporal property. Then we show that most usual ways to define computability are particular examples of this definition.

Davis [18] proposed the following definition: a Turing machine is universal if the relation 'x_n is in the orbit of x_m' is r.e.-complete, where x_m and x_n are arbitrary finite configurations. Here we modify Davis' definition in order to be applied to any effective symbolic system. Our choices are justified in Section 5.

Definition 2 *An effective dynamical system is* universal *if there is a recursive family of temporal formulae such that knowing whether a given formula of the family is satisfiable is an r.e.-complete problem.*

An *r.e.-complete* problem, or Σ_1-complete problem, is a recursively enumerable problem, to which any recursively enumerable problem is Turing-reducible.

Note that this may be interpreted as a non-deterministic scheme of computation. The computation succeeds iff at least one trajectory exhibits a given behavior.

We may call *halting problem* for f, the satisfiability problem for formulae:

$$(P_n \wedge \diamond P_m)_{n,m\in\mathbb{N}},$$

which reads: 'There is a configuration in the clopen set P_n that eventually reaches the clopen set P_m'. We may think of P_n as an initial configuration of which we know only the first digits and P_m as the halting set. The unspecified digits of the initial configuration may be seen as encoding the non-deterministic choices occurring during the computation.

Turing machines are often described as working only on finite configurations. A finite configuration is an element of $Q \times \{0,1\}^* \times \{0,1\}^*$, where Q denotes the set of states of the head, the first binary word is content of the tape to the left of the head and the second binary word is the right part of the tape. The rest of the tape is supposed to be entirely filled with blank symbols. Such a Turing machine is universal if given two finite configurations u and v, checking whether u is in the trajectory of v is an r.e.-complete problem.

This is a particular case of our definition. Indeed, let $W = \{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$ the set of finite and infinite binary words. Then the Turing machine transition function is also defined on $Q \times W \times W$, which is a compact space, whose isolated points are $Q \times \{0,1\}^* \times \{0,1\}^*$. Isolated points are in fact clopen sets of $Q \times W \times W$. So the problem of checking whether the formula $P_n \wedge \diamond P_m$ is satisfiable, given two clopen sets P_n and P_m, is r.e.-complete. Indeed, it is already r.e.-complete if we restrict ourselves to clopen sets that are isolated points, and it is recursively enumerable (although perhaps not r.e.-complete) on non-isolated clopen sets.

Tag systems were introduced by Post in 1920. A *tag system* is a transformation rule acting on finite binary words. At each step, a fixed number of bits is removed from the beginning of the word and, depending on the values of these bits, a finite word is appended at the end of the word. Minsky proved in 1961 that there is a so-called universal tag system, for which checking that a given word will end up to the empty word when repeating the transformation is an r.e.-complete problem; see [2].

We can extend the rule of tag systems to infinite words, by just removing to them the fixed number of bits. Thus we have a dynamical system on the compact space $\{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$ of finite and infinite words, in which finite words are clopen sets. Again, if the tag system is universal for the word-to-word definition, then it is universal for our definition with the formulae $P_n \wedge \diamond P_m$.

We can also apply our definition to **functions on integers**. Let $\mathbb{N} \cup \{\infty\}$ be the topological space with the metric $d(n,m) = |\frac{1}{n+1} - \frac{1}{m+1}|$. This is effectively homeomorphic to the set $\{1^n 0^\infty | n \in \mathbb{N}\} \cup \{1^\infty\}$. Then a total computable map on $\mathbb{N}$ can be extended to an effective continuous map on $\mathbb{N} \cup \{\infty\}$ iff either it has a finite range and only one integer has an unbounded preimage set, or it has an infinite range and we can compute a (finite) bound on the largest preimage of every given integer.

For example, it is meaningful to ask whether the famous $3n+1$ function (which is effective) is universal. This is an unsettled question. But Conway [19] proved that similar functions, called Collatz functions, may be universal.

We now give an example of a universal **cellular automaton**.

Let us take a universal Turing machine with a blank symbol. We suppose that when the halting state is reached, then the head comes back to the cell of index 0. We can simulate it in an almost classic way with a one-dimensional cellular automaton. The alphabet of the automaton is $A \cup (A \times Q) \cup \{L, R, Error\}$, where A is the tape alphabet (including the blank symbol) and Q the set of states.

Let us take a point in the cylinder $[L, \text{initial data of the Turing machine}, R]$, and observe its trajectory. The symbol L moves to the left at the speed of light, leaving behind blank symbols. The symbol R moves to the right in a similar way. Meanwhile, the space between L and R is used to simulate the Turing machine and is composed of symbols from A and exactly one symbol from $(A \times Q)$, which denotes the position of the head. When L or R symbols meet each other, then a spreading $Error$ symbol is produced, that erases everything.

This cellular automaton is universal for formulae $P_n \wedge \diamond P_m$. Indeed, there is an orbit from the cylinder $[L, \text{initial data of the Turing machine}, R]$ to the cylinder [halting state] (both cylinders centered at cell of index zero) if and only if the universal Turing machine halts on the initial data.

5 Discussion on the Definition of Universality

Our definition of universality differs in several ways from what we could expect at first glance from a generalization of Turing machine universality. In this section we give various arguments to support the present definition against seemingly more obvious attempts. In particular, we justify the use of *set-to-set* properties, expressed in the formalism of *temporal logic*, on systems for which the transition function is *computable*.

Set-to-Set Properties. Many definitions of universality for particular systems propose to observe point-to-point properties. So it could seem that it is possible to build a general definition of universality with point-to-point properties.

The most natural idea would be to say that a metric space with a dense set of points $(x_n)_{n \in \mathbb{N}}$ is universal if the property 'x_n is in the trajectory of x_m' is r.e.-complete.

However, as remarked in [20], this leads to conclude that the shift is universal; a consequence that is counter-intuitive. It sounds unreasonable to admit the shift as universal, because it does not treat any information, but just reads the memory.

Indeed if instead of ultimately periodic points we choose configurations with primitive recursive digits, then we take as initial configurations the sequence of states of the head of a universal Turing machine during a computation. And we just have to shift to know whether the halting state will appear.

The definition presented in this text overcomes this problem in a simple manner: the user needs only to give a finite number of bits as an initial condition. Instead of initial *configurations* we shall rather talk about initial *sets*, which may be seen as 'fuzzy points', points defined with finite accuracy.

This solution is also more satisfactory from the point of view of physical realizability. Indeed, we expect the set of configurations of a physical system to be uncountable in general, and specifying an initial point for the computation means *a priori* that we must give an infinite amount of information. Preparing a physical system to be in a very particular configuration is likely to be impossible, because of the noise or finite precision inherent to every measure.

Temporal Properties. What kind of property are we going to test on clopen sets? Here again, we must avoid trivialities. Suppose that we look at identity on the Cantor space. We now choose to observe the following property: a clopen set satisfies the property iff its index (i.e., the integer describing the clopen set) satisfies some r.e.-complete property on $\mathbb{N}$. Then we find again that identity is computationally universal, which is desirable.

On the other hand, we see no reason to restrict ourselves to the sole halting property: 'there is a trajectory from this clopen set to that clopen set'. Any observable property could a priori be used as a basis for computation. For instance, the chaotic system built in Section 7 is universal but not for the halting property.

So we must precisely define a class of observable properties of clopen sets, not too large and not too restricted. Temporal logic, as defined above, has been widely used for decades to express expected properties of various transitions systems and seems to be a reasonable choice.

Effectiveness. Finally, we add an effectiveness structure on dynamical systems, because we want to be able to simulate the system step by step. Indeed, our informal goal is to study when universality emerges from the long-term dynamics. But if even a single step of the system is uncomputable, no surprise that the long-term dynamics is unpredictable!

We therefore restrict ourselves to systems such that the inverse image of a clopen set is computable. Note that for instance in [1] the author allows neural networks with non-recursive weights, leading to a non-computable transition function and to super-Turing capabilities.

6 Necessary Conditions for Universality

It has been highlighted in the Introduction that some attempts have been made to link computational capabilities of a system to its dynamical properties. This is also the purpose of this section.

For simplicity, we will write 'symbolic system' for 'effective symbolic dynamical system' — unless otherwise specified.

Minimality. A *minimal* dynamical system is a system with no subsystem (except the empty set and itself). It is characterized by the fact that all orbits are dense.

Proposition 1 *A minimal symbolic system is not universal.*

The proof shows by induction that the interpretation of a formula is always a clopen set, and that we can compute it.

Now suppose that the symbolic system is not minimal but consists of several minimal subsystems attracting the whole space of configurations. In other words, the limit set is made of finitely many minimal systems. Recall that the limit set of a dynamical system $f : X \to X$ is the set $\bigcap_{n\geq 0} f^n(X)$.

Proposition 2 *A symbolic system whose limit set is the finite union of minimal subsystems is not universal.*

For example, if all points uniformly converge to a periodic orbit, then the system is not universal. A stronger statement is suggested by the intuition that a universal system is able to simulate many other systems.

Conjecture 1 *A universal symbolic system has infinitely many minimal subsystems.*

Equicontinuity. A system $f : X \to X$ is *equicontinuous* if for all $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f^t(x), f^t(y)) < \epsilon$, for any points x, y and nonnegative t.

Proposition 3 *An equicontinuous symbolic system is not universal.*

Again, the proof shows that the interpretation of a formula is a computable clopen set.

We say that a point x of a dynamical system f is *sensitive* if there is an $\epsilon > 0$ such that for every $\delta > 0$ there is a point y with $d(x, y) < \delta$ and a nonnegative time t such that $d(f^t(x), f^t(y)) > \epsilon$.

It is easy to show from compactness that an equicontinuous dynamical system is exactly a system with no sensitive point. Hence, we can deduce from the above result that a universal symbolic system must have a sensitive point.

Equicontinuity in the case of cellular automata has been given a combinatorial characterization in [21]. It is also proved that equicontinuous cellular automata are eventually periodic, thus confirming in this particular case that equicontinuity prevents computational universality from arising.

Shadowing Property. We now define the *effective shadowing property* for a dynamical system.

Definition 3 *Let (X, f) be a dynamical system. A δ-*pseudo-orbit *is a (finite or infinite) sequence of points $(x_n)_{n\geq 0}$ such that $d(f(x_n), x_{n+1}) < \delta$ for all n.*

*A point x ϵ-*shadows *a (finite or infinite) sequence $(x_n)_{n\geq 0}$ if $d(f^n(x), x_n) < \epsilon$ for all n.*

The dynamical system is said to have the shadowing property *if for all $\epsilon > 0$ there is a $\delta > 0$ such that any δ-pseudo-orbit is ϵ-shadowed by some point.*

If moreover such a δ can be effectively computed from ϵ then we say that the system has the effective shadowing property.

We can give the following interpretation to this property: suppose that we want to compute numerically the trajectory of x such that at every step numerical errors amount to δ. The resulting sequence of points is a δ-pseudo-orbit, and the shadowing property ensures that this pseudo-orbit is indeed ϵ-close to an actual trajectory of the system.

Proposition 4 *A symbolic system that has the effective shadowing property is not universal.*

The proof shows that a formula is satisfiable iff it is satisfiable for δ-pseudo-orbits, with δ small enough; but the latter property is decidable.

In particular, the full shift is not universal.

The following proposition almost shows that we cannot lift effectiveness of the shadowing property in Proposition 4.

Proposition 5 *There is an undecidable symbolic system that has the shadowing property, but not the effective shadowing property.*

An *undecidable* system is a system for which satisfiability of a given temporal formula is undecidable. We don't know whether this undecidable system is in fact universal.

Note also that Turing machines that satisfy the shadowing property have been given a combinatorial characterization in [22]; in particular, the proof shows that the link between ϵ and δ (see Definition 3) is linear. Hence the effective shadowing property is not stronger than the shadowing property in the case of Turing machines.

7 A Universal Chaotic System and the Edge of Chaos

According to Devaney [23], a system is *chaotic* if it is infinite, topologically transitive and has a dense set of periodic points. We can prove that such a system is sensitive [24].

It is not difficult to prove the existence of a universal subshift. Indeed, consider all the forbidden words of the kind 01^n00^t1, where the universal Turing machine launched on data n does not halt in less than t steps. Then the subshift of all binary sequences avoiding this set of words is effective and universal.

Improving this construction, one gets the following result:

Proposition 6 *There exists an effective system on the Cantor space that is chaotic and universal.*

The central idea of the 'edge of chaos' is that a system that has a complex behavior should be neither too simple nor chaotic. There are several ways to understand that.

Here we interpret 'complex system' by 'universal symbolic system'. Then 'too simple' could refer to the situation treated in Proposition 2: one or several attracting minimal subsystems. This includes of course the case of a globally attracting fixed point.

If we take 'chaotic' as meaning 'Devaney-chaotic', then computational universality need not be on the 'edge of chaos', since we have just provided a chaotic system that is universal.

However, many examples of chaotic systems (whatever the exact meaning given to 'chaotic', and for symbolic systems as well as for analog ones), although not all of them, have the shadowing property and even the effective shadowing property. For instance the shift and Smale's horseshoe (present in some physical systems), as well as all hyperbolic systems, satisfy the effective shadowing property with a linear relation between ϵ and δ (see Definition 3).

Note nevertheless the 'edge of chaos' has been intensively studied for cellular automata. We don't know whether an example of chaotic universal cellular automaton exists.

8 Future Work

Many questions are yet to solve. For instance, we lack sufficient conditions of universality. We didn't investigate in depth what properties a universal cellular automaton satisfies. Let us mention three possible directions for future work.

All formulae involved in examples of universal systems in the preceding sections were quite simple; they were Σ_1 *formulae*, i.e., formulae where no 'until' is negated. We can see that the interpretation of such a formula is a Σ_1 Borel set (an open set) and has a satisfiability problem is recursively enumerable, thus in the level Σ_1 of the arithmetic hierarchy. How far does this correspondence go? Is it true that a universal system is always universal for a family of Σ_1 formulae?

If the symbolic space is endowed with a probability measure (not necessarily invariant for the map), then we would like to check whether a formula is satisfied with positive probability. This can yield a probabilistic definition of universality. How does it relate to the definition developed in this paper?

Can our definition be extended to analog systems? Do the results of Section 6 still apply in this context? For instance, hyperbolic systems are known to have the effective shadowing property. This would suggest that hyperbolic systems are not universal.

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