Distortion theorems for rational functions without poles or zeros in simply connected domains

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Abstract

We prove distortion theorems for rational functions without poles or zeros in simply connected domains. The distance between the values $u(z_1)$ and $u(z_2)$ assumed by such a rational function is limited by its degree and by the hyperbolic distance between z_1 and z_2. For discs and halfplanes, the case of equality in these estimates is fully explored. These results have applications in control theory which are illustrated by examples.

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1 Preliminaries

We denote the open unit disc in the complex plane $\mathcal{C}$ by $D := \{z \in \mathcal{C} : |z| < 1\}$ and the open right half plane by $\mathcal{C}_+ := \{s \in \mathcal{C} : \Re(s) > 0\}$. The hyperbolic distance ρ_D on D between z_1 and z_2 in D is given by

$$\rho_D(z_1, z_2) = \log \frac{1 + \alpha(z_1, z_2)}{1 - \alpha(z_1, z_2)}$$

where

$$\alpha(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|.$$

The quantity $\alpha(z_1, z_2)$ is always less than 1 and is invariant under conformal mapping. If $\mu : D \to D$ is a conformal map from the unit disc onto itself, then $\alpha(z_1, z_2) = \alpha(\mu(z_1), \mu(z_2))$. As a consequence, the hyperbolic distance between two points is invariant under conformal mapping. For all theoretic aspects we refer the reader to [1].

For a simply connected region Ω of the complex plane, the hyperbolic distance is defined by $\rho_\Omega(z_1, z_2) = \rho_D(\mu(z_1), \mu(z_2))$, where $\mu : \Omega \to D$ is any conformal map from Ω onto D. Since ρ_D is invariant under conformal mapping, the quantity ρ_Ω is well-defined independent of the mapping μ chosen to compute it. When the conformal map μ is unknown, we may also consider the quasi-hyperbolic distance ρ^*_Ω defined by

$$\rho^*_\Omega(z_1, z_2) = \min_\Gamma \int_\Gamma \frac{|dz|}{\operatorname{dist}(z, \partial\Omega)} \ ,$$

where the minimum is taken over all curves $\Gamma \subset \Omega$ connecting z_1 to z_2, and $\operatorname{dist}(z, \partial\Omega)$ denotes the distance of z to the boundary of Ω. From [9] we have the following estimates

$$\frac{1}{2}\rho_\Omega(z_1, z_2) \leq \rho^*_\Omega(z_1, z_2) \leq 2\rho_\Omega(z_1, z_2) \ .$$

Note the different meaning of the hyperbolic distance in the book of Pommerenke [9]. His hyperbolic distance is just one half of our distance. Thus taking any path Γ, we are able to estimate the pseudo-hyperbolic distance, hence the hyperbolic distance.

An important region for control purposes is the open right half plane $\mathcal{C}_+$ (for control theoretic aspects we refer to [11]). The map

$$\mu : \mathcal{C}_+ \to D : s \mapsto \frac{s-1}{s+1}$$

is a conformal map from the open right half plane onto the unit disc. From this we compute the hyperbolic distance on the open right half plane by

$$\rho_{\mathcal{C}_+}(s_1, s_2) = \log \frac{1 + \alpha(s_1, s_2)}{1 - \alpha(s_1, s_2)} \ ,$$

where

$$\alpha(s_1, s_2) = \left| \frac{s_2 - s_1}{s_2 + \overline{s_1}} \right| .$$

When s_1 and s_2 are real and positive, the hyperbolic distance on $\mathcal{C}_+$ specializes to

$$\rho_{\mathcal{C}_+}(s_1, s_2) = \left| \log \frac{s_2}{s_1} \right| .$$

A Blaschke factor $S(z)$ is an analytic automorphism of the unit disc D. Blaschke factors have the form

$$S(z) = e^{i\phi}\frac{z-a}{1-\overline{a}z} \ ,$$

for some $\phi \in [0, 2\pi)$ and $a \in D$. Let z_1 and z_2 be distinct points of D. The unique Blaschke factor $S(z)$ mapping 0 to z_1 and $\alpha(z_1, z_2)$ to z_2 is given by

$$S(z) = \frac{ze^{i\phi} + z_1}{1 + \overline{z_1}e^{i\phi}z} \ ,$$

where the angle ϕ is given by

$$\phi = \arg \frac{z_2 - z_1}{1 - \overline{z_1}z_2} \ .$$

We need one more definition. Let $f(z)$ and $F(z)$ be analytic functions on D. The function $f(z)$ is called subordinate to $F(z)$, for short, $f(z) \prec F(z)$, if $f(z) = F(\mu(z))$, where $\mu(z)$ is some analytic map of the disc D into itself and $\mu(0) = 0$. The following proposition is important in the sequel. It follows easily from Schwarz's lemma and from the maximum modulus principle.

Proposition 1 (Subordination principle). *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $F(z) = \sum_{n=0}^{\infty} A_n z^n$ *be analytic functions on* D *and suppose* $f(z)$ *is subordinate to* $F(z)$*. Then* $a_0 = A_0$*,* $|a_1| \leq |A_1|$ *and*

$$M(r, f) \leq M(r, F), \quad 0 \leq r < 1,$$

where $M(r, f)$ *denotes the maximum modulus of* $f(z)$ *on* $\{z : |z| = r\}$*.*

2 Distortion theorems for discs and halfplanes

The next theorem was proved by the authors in [2]. The proof given there hinges on a result of Dieudonné [7] for zero-free polynomials in the disc D. Here we present an elementary proof and solve the question of equality in the estimate. In particular, we do not use the subordination principle.

Theorem 1: *Let* $u(z)$ *be a rational function with no poles or zeros in* D *and let* d *denote the degree of* $u(z)$*. For all* $z \in D$ *we have*

$$\left|\frac{u'(z)}{u(z)}\right| \leq \frac{2d}{1-|z|^2} \ .$$

Equality holds for a point z_0 *if and only if there exists a Blaschke factor* $S(z)$ *and a nonzero constant* C *satisfying* $S(0) = z_0$ *and*

$$u(S(z)) = C\left(\frac{1+z}{1-z}\right)^d \ .$$

Proof: Let $a_j, j = 1, \ldots, n$ and $b_k, k = 1, \ldots, m$ denote the zeros and poles of $u(z)$ outside D, repeated according to their multiplicity. Differentiating the product representation of $u(z)$ gives

$$\frac{u'(z)}{u(z)} = \sum_{j=1}^{n} \frac{1}{z - a_j} - \sum_{k=1}^{m} \frac{1}{z - b_k} \ .$$

(If there are no poles or zeros the corresponding sum is taken to be zero.)

Part 1: Assume that $z = 0$. Since all zeros and poles have modulus not less than 1 we obtain

$$\begin{aligned} \left|\frac{u'(0)}{u(0)}\right| &= \left|\sum_{j=1}^{n} \frac{1}{(-a_j)} + \sum_{k=1}^{m} \frac{1}{b_k}\right| \\ &\leq \sum_{j=1}^{n} \frac{1}{|a_j|} + \sum_{k=1}^{m} \frac{1}{|b_k|} \\ &\leq 2d \ . \end{aligned}$$

Equality in the first estimate occurs if and only if the arguments of all summands are equal, i. e.

$$\arg \frac{1}{(-a_1)} = \ldots = \arg \frac{1}{(-a_n)} = \arg \frac{1}{b_1} = \ldots = \arg \frac{1}{b_m} \ .$$

Thus there is a $\theta \in [0, 2\pi)$ such that for all j, k

$$a_j = -|a_j| e^{i\theta} \ , \ b_k = |b_k| e^{i\theta} \ .$$

But the second estimate forces the moduli to be 1 in case of equality. Moreover, we must have $m = n = d$. Hence

$$a_j = -e^{i\theta} \ , \ b_k = e^{i\theta}$$

for all j and k. Integrating the identity for $u'(z)/u(z)$, we obtain

$$u(z) = C\left(\frac{z + e^{i\theta}}{z - e^{i\theta}}\right)^d = C\left(\frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z}\right)^d$$

where C is denotes a nonzero constant. The desired Blaschke factor is just the rotation $S(z) := e^{i\theta}z$.

Part 2: Fix an arbitrary point $z \in D$ and take $T(\zeta) := (\zeta + z)/(1 + \overline{z}\zeta)$. It is a rational map of degree 1 of the disc D onto itself with $T(0) = z$. Then $r(\zeta) := u(T(\zeta))$ is rational of degree d and all zeros and poles of $r(\zeta)$ are outside D. Since $T'(\zeta) = (1 - |z|^2)/(1 + \overline{z}\zeta)^2$,

$$r'(0) = u'(T(0))T'(0) = u'(z)(1 - |z|^2) \ .$$

Part 1 now gives

$$\left| \frac{u'(z)}{u(z)} \right| = \left| \frac{r'(0)}{r(0)} \right| \frac{1}{1 - |z|^2} \leq \frac{2d}{1 - |z|^2} \ .$$

Equality at z_0 implies equality at the origin for the composition $u(T(\zeta))$. By Part 1 we have a nonzero constant C and a $\theta \in [0, 2\pi)$ such that

$$u(T(z)) = C \left(\frac{1 + e^{-i\theta} z}{1 - e^{-i\theta} z} \right)^d \ .$$

Hence the Blaschke factor $S(z) := T(e^{i\theta} z)$ gives

$$u(S(z)) = C \left(\frac{1 + z}{1 - z} \right)^d \ .$$

If on the other hand the identity in the assertion of the theorem is satisfied, and $S(0) = z_0$, we have $u(z_0) = u(S(0)) = C$ and

$$|u'(z_0)|(1 - |z_0|^2) = |(u \circ S)'(0)| = 2dC \ , \text{ hence}$$

$$\left| \frac{u'(z_0)}{u(z_0)} \right| = \frac{2d}{1 - |z_0|^2} \ .$$

■

An integration leads to the following:

Theorem 2: *Let $u(z)$ be a rational function with no poles or zeros in D and let d denote the degree of $u(z)$. For all $z_1, z_2 \in D$ we have*

$$\left| \log \left| \frac{u(z_1)}{u(z_2)} \right| \right| \leq d\rho_D(z_1, z_2) \ .$$

Equality holds for two different points z_1, z_2 if and only if

$$u(S(z)) = C \left(\frac{1 + z}{1 - z} \right)^d \ ,$$

where C is a nonzero constant, and $S(z)$ is the unique Blaschke factor mapping 0 to z_1 and $\alpha(z_1, z_2)$ to z_2.

Proof: Write $u(z) = \exp(h(z))$, where $h(z)$ is a single-valued branch of the logarithm of $u(z)$. Then $u'(z)/u(z) = h'(z)$ and the inequality in Theorem 1 now reads

$$|h'(z)| \leq \frac{2d}{1-|z|^2} \ . \tag{1}$$

Part 1: Suppose $z_1 = 0, z_2 = \alpha > 0$. Inequality (1) and integration along the segment from α to 0 yields

$$|h(0) - h(\alpha)| \leq \int_0^\alpha |h'(x)| \, dx \leq 2d \int_0^\alpha \frac{dx}{1-x^2} \ .$$

A short calculation shows that

$$|h(0) - h(\alpha)| \leq d \log \frac{1+\alpha}{1-\alpha} = d\rho_D(0, \alpha) \ , \text{ hence}$$

$$\left| \log \left| \frac{u(0)}{u(\alpha)} \right| \right| = |\Re(h(0) - h(\alpha))| \leq |h(0) - h(\alpha)| \leq d\rho_D(0, \alpha) \ . \tag{2}$$

Equality in (2) for $\alpha_0 > 0$ implies equality in (1) for all $z \in [0, \alpha_0]$, because

$$\int_0^{\alpha_0} \left(|h'(x)| - \frac{2d}{1-x^2} \right) dx = 0 \ .$$

Since the integrand is nonpositive, we obtain equality in (1) for all $z \in [0, \alpha_0]$. Theorem 1 now shows that there is a nonzero constant C and a rotation $S(z) := e^{i\theta} z$ such that

$$u(S(z)) = C \left(\frac{1+z}{1-z} \right)^d \text{ ,i. e. } u(z) = C \left(\frac{1+e^{-i\theta} z}{1-e^{-i\theta} z} \right)^d \ .$$

However, equality in (2) for 0 and $\alpha_0 > 0$ now implies that $\theta = 0$, i. e. $S(z) = z$. Hence, in this case, the assertions follow.

Part 2: Suppose z_1, z_2 are abitrary points in D. Take the unique Blaschke factor $S(z)$ mapping 0 to z_1 and $\alpha(z_1, z_2)$ to z_2. Then apply Part 1 to the composition $q(z) := u(S(z))$. Note that the hyperbolic distance is invariant under $S(z)$, i. e.

$$\rho_D(z_1, z_2) = \rho_D(S(0), S(\alpha)) = \rho_D(0, \alpha) \ .$$

Hence the conclusion follows. ■

The usual mapping to the right half plane now gives a simple proof of the following distortion theorem. Other proofs of the same result can be found in [2] and [8]. The discussion of equality given here is new:

Theorem 3: *Let $u(s)$ be a rational function with no poles or zeros in $\mathcal{C}_+$ and let d denote the degree of $u(s)$. For all $s, s_1, s_2 \in \mathcal{C}_+$ we have*

$$\left|\frac{u'(s)}{u(s)}\right| \le d\frac{1}{\Re(s)}\ , \qquad \left|\log\left|\frac{u(s_1)}{u(s_2)}\right|\right| \le d\,\rho_{\mathcal{C}_+}(s_1,s_2)\ .$$

$\rho_{\mathcal{C}_+}$ denotes the hyperbolic distance in $\mathcal{C}_+$, and equality holds for two different points s_1, s_2 if and only if there is a nonzero constant C such that

$$u(T(s)) = Cs^d\ ,$$

where $T : \mathcal{C}_+ \to \mathcal{C}_+$ denotes the unique Blaschke factor for $\mathcal{C}_+$ mapping 1 to s_1 and $\alpha(s_1, s_2)$ to s_2.

As a corollary we obtain an estimate which can be seen as the counterpart of estimates for zero-free polynomials in D given by Rivlin in [10]. However, since we do not know where the maximum modulus for $u(z)$ is assumed, we cannot discuss the case of equality:

Corollary 1: *Let $u(z)$ denote a rational function of degree d without poles or zeros in the disc $\{|z| < R\}$ and define*

$$M(u,r) = \sup_{|z|=r} |u(z)|.$$

Then for $0 \le R_1 \le R_2 \le R$

$$\left|\log\frac{M(u,R_1)}{M(u,R_2)}\right| \le d\log\frac{1+R\frac{R_1+R_2}{R^2+R_1R_2}}{1-R\frac{R_1+R_2}{R^2+R_1R_2}}\ .$$

Proof: Consider the function $v(z) := u(Rz)$ defined on the unit disc D and use the estimate

$$\rho_D(z_1,z_2) \le \frac{|z_1|+|z_2|}{1+|z_1||z_2|}\ .$$

■

The Bloch norm of an analytic function $f(z)$ in D is given by

$$\|f(z)\|_{\mathcal{B}} := \sup_{z\in D}\left(1-|z|^2\right)|f'(z)|\ ;$$

see, for example, [9]:

Corollary 2: *Let $u(z)$ denote a rational function of degree d without poles or zeros in D. Then for every single-valued branch $\log u(z)$*

$$\|\log u(z)\|_{\mathcal{B}} \le 2d\ .$$

3 Distortion theorems in certain simply connected domains

In this section we generalize the previous results to more general simply connected domains.

Theorem 4: *Let Ω be a simply connected region of the complex plane having the following property: To each point $a \in \mathcal{C} \setminus \Omega$ there exists a straight line $L_a \subset \mathcal{C} \setminus \Omega$ connecting ∞ and a. Let $u(z)$ be a rational function with no poles or zeros in Ω, let d be the degree of $u(z)$, and let $\mu : \Omega \to D$ denote any conformal map from Ω onto D. Then the following estimates hold for all $z, z_1, z_2 \in \Omega$:*

$$\left|\frac{u'(z)}{u(z)}\right| \leq 16d\frac{|\mu'(z)|}{1-|\mu(z)|^2} \ , \quad \left|\log\left|\frac{u(z_1)}{u(z_2)}\right|\right| \leq 8d\rho_\Omega(z_1, z_2) \ .$$

Proof: Write $q(z) := u(z)/u(z_2)$ and let $a_1, \dots, a_n, b_1, \dots, b_m$, $n, m \leq d$ denote the zeros and poles of $u(z)$ outside Ω repeated according to their multiplicity. Since Ω is simply connected, there exist single-valued branches of $\log u(z)$ in Ω. Obviously we have

$$\log q(z) = \sum_{j=1}^{n}(\log(z-a_j) - \log(z_2-a_j)) - \sum_{k=1}^{m}(\log(z-b_k) - \log(z_2-b_k)) \ ,$$

where we can take any single-valued branch of the logarithmns. Hence we conclude that

$$\Im(\log q(z)) = \sum_{j=1}^{n}(\arg(z-a_j) - \arg(z_2-a_j)) - \sum_{k=1}^{m}(\arg(z-b_k) - \arg(z_2-b_k)),$$

where arg denotes any branch of the argument, arbitrarily chosen for every index. By assumption there exists a straight line $L_a \subset \mathcal{C} \setminus \Omega$ joining a to infinity. Taking the branch whose slit is just L_a, we conclude that

$$|\arg(z-a_j) - \arg(z_2-a_j)| \leq 2\pi \ , \ |\arg(z-b_k) - \arg(z_2-b_k)| \leq 2\pi \ .$$

Thus we determined a single-valued branch of $\log q(z)$ on Ω with $\Im(\log q(z_2)) = 0$ and

$$|\Im(\log q(z))| \leq 4d\pi.$$

The function

$$v(z) := \exp\left(\frac{1}{8d}\log q(z)\right) = (q(z))^{\frac{1}{8d}}$$

maps Ω into $\mathcal{C}_+$ and is such that $v(z_2) = 1$. Hence

$$w(z) := \frac{v(z)-1}{v(z)+1}$$

maps Ω into D with $w(z_2) = 0$. We obtain

$$v(z) = \frac{1 + w(z)}{1 - w(z)}.$$

Fixing a conformal map ϕ from D onto Ω with $\phi(0) = z_2$, we have

$$v(\phi(z)) = \frac{1 + w(\phi(z))}{1 - w(\phi(z))},$$

hence

$$q(\phi(z)) = (v(\phi(z)))^{8d} = \left(\frac{1 + w(\phi(z))}{1 - w(\phi(z))}\right)^{8d}.$$

Since $w \circ \phi$ maps D into D with $w(\phi(0)) = 0$, we arrive at

$$q(\phi(z)) \prec \left(\frac{1+z}{1-z}\right)^{8d}.$$

The desired estimates now follow from the subordination principle (Proposition 1) as follows:

$$|q(z_1)| \le M(|\phi^{-1}(z_1)|, q \circ \phi) \le \left(\frac{1 + |\phi^{-1}(z_1)|}{1 - |\phi^{-1}(z_1)|}\right)^{8d}.$$

The right hand side is just the term needed to compute the hyperbolic distance $\rho_\Omega(z_1, z_2)$. Using the subordination principle again, we conclude that for all $z_2 \in \Omega$

$$|q'(z_2)\phi'(0)| \le 16d \, ;$$

recall that $\phi(0) = z_2$. Now the composition $\sigma := \mu \circ \phi$ must be a conformal map from the disc onto itself. Hence

$$|\sigma'(0)| = 1 - |\sigma(0)|^2 = 1 - |\mu(\phi(0))|^2 = 1 - |\mu(z_2)|^2.$$

By the chain rule

$$|\phi'(0)| = \frac{1 - |\mu(z_2)|^2}{|\mu'(z_2)|}\ , \text{ hence } |q'(z_2)| \le 16d\, \frac{|\mu'(z_2)|}{1 - |\mu(z_2)|^2}\ .$$

Remembering that $q'(z_2) = u'(z_2)/u(z_2)$ the result follows. ■

Important examples for the domains above are simply connected convex domains Ω: For every point a outside Ω, there exist a straight line g_a such that a and Ω lie entirely on different halfplanes determined by g_a. Given $z_2 \in \Omega$, the convexity of Ω implies that a segment L_a of the straight line through a and z_2 belongs to $\mathcal{C} \setminus \Omega$, connecting a and ∞.

However, in the convex case, the bounds can be sharpened. The proof given next follows the reasoning leading to Theorem 4, so we just emphasize the differences.

Theorem 5: *Let Ω be a simply connected convex domain of the complex plane, let $u(z)$ be a rational function with no poles or zeros in Ω, let d be the degree of $u(z)$ and let $\mu : \Omega \to D$ denote any conformal map from Ω onto D. Then the following estimates hold for all $z, z_1, z_2 \in D$:*

$$\left|\frac{u'(z)}{u(z)}\right| \leq 8d \, \frac{|\mu'(z)|}{1-|\mu(z)|^2} \ , \qquad \left|\log\left|\frac{u(z_1)}{u(z_2)}\right|\right| \leq 4d \; \rho_\Omega(z_1, z_2) \ .$$

Proof: Using the notations introduced in the proof of Theorem 4, we again conclude that

$$\Im(\log q(z)) = \sum_{j=1}^{n}(\arg(z-a_j) - \arg(z_2 - a_j)) - \sum_{k=1}^{m}(\arg(z-b_k) - \arg(z_2 - b_k)),$$

where $q(z) := u(z)/u(z_2)$ and arg denotes any branch of the argument, arbitrarily chosen for every index. Since Ω is convex there exists a straight line g_a, such that Ω and $a = a_j$ (or $a = b_k$) lie on different halfplanes determined by g_a. Because z and z_2 are on the same side of the halfplane, the modulus of the differences of the above arguments are less than π for a branch of $\log(z-a)$ with slit along a segment of the straight line L_a through a and z_2; see the discussion preceeding Theorem 5. Hence we end up with a single-valued branch of $\log q(z)$ on Ω such that $\Im(\log q(z_2)) = 0$ and

$$|\Im(\log q(z))| \leq 2d\pi.$$

In the course of the proof of Theorem 4, we just replace the factors $4d$ by the smaller factor $2d$. Thus the result follows. ■

4 Examples

Example 1: We consider the region of stability given in [11]: $\Omega := \{z \in \mathcal{C} : -\pi + \Theta < \arg z < \pi - \Theta\} \cup \{z \in \mathcal{C} : \Re z > -\sigma\}$. All we have to do is to compute a conformal mapping $\mu : \Omega \to \mathcal{C}_+$. Because Ω is a triangle, the Schwarz-Christoffel-formula determines the conformal mapping $f : H \to \Omega$, H upper halfplane, satisfying $f(0) = w_1 = -\sigma + i\sigma \tan\Theta$, $f(1) = w_2 = -\sigma - i\sigma\tan\Theta$ and $f(\infty) = \infty$. The angles are $\alpha_1 = \alpha_2 = 3\pi/2 - \Theta$. There exists a constant A such that:

$$f'(z) = Az^{1/2-\Theta/\pi}(z-1)^{1/2-\Theta/\pi} \ .$$

Using well-known techniques to determine A, we end up with

$$A = -e^{i\Theta} \frac{2\sigma \tan \Theta}{B(3/2 - \Theta/\pi, 3/2 - \Theta/\pi)} \ ,$$

where we used

$$B(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \ .$$

Hence

$$f(w) = A \int_0^w (z(z-1))^{1/2 - \Theta/\pi} dz - \sigma + i\sigma \tan \Theta \ ,$$

which can be calculated numerically, if necessary.
Changing to $\mathcal{C}_+$ gives a function $g : \mathcal{C}_+ \to \Omega, g(s) := f(is)$. The mapping μ is just the inverse of g. We obtain

$$|\mu'(f(is))| = \frac{1}{|f'(is)|} \ .$$

According to Theorem 4, rephrased for the case $\mathcal{C}_+$, the final estimate for a rational function u of degree d having no poles or zeros in Ω is

$$\left| \frac{u'(g(s))}{u(g(s))} \right| \leq 4d \frac{B(3/2 - \Theta/\pi, 3/2 - \Theta/\pi)}{\sigma \tan \Theta} \frac{(|s||s+i|)^{\Theta/\pi - 1/2}}{\Re s} \ .$$

Example 2: We illustrate the use of Theorem 1 for obtaining bounds on the degree of a simultaneous stabilizing controller for two systems (for background information on simultaneous stabilization and its relation to interpolation by rational functions, see [3,4,5,6]). Let β_1 and β_2 be two real numbers and consider the two discrete time systems

$$p_1(z) = \frac{1}{z} + \beta_1 \ , p_2(z) = \frac{1}{z} + \beta_2.$$

A coprime factorization of $p_1(z)$ in the ring of rational functions with no poles in the unit disc D is given by

$$p_1(z) = n_1(z)d_1(z)^{-1} = (1 + \beta_1 z)(z)^{-1}.$$

A solution of the Bezout identity $n_1 x + d_1 y = 1$ is given by

$$n_1.1 + d_1.(-\beta_1) = 1 \ ;$$

the Youla-Kucera parameterization of the set of all stabilizing controllers of p_1 is then given by

$$\frac{x + qd_1}{y - qn_1} = \frac{1 + zq(z)}{-\beta_1 - (1 + \beta_1 z)q(z)},$$

where $q(z)$ is an arbitrary rational function with no poles in D. For a controller in this set to stabilize the second system p_2, we need to satisfy

$$(1 + \beta_2 z)(1 + zq(z)) + z(-\beta_1 - (1 + \beta_1 z)q(z)) = u(z)$$

for some unit $u(z)$. This equality can also be written

$$q(z) = \frac{u(z) - (1 + (\beta_2 - \beta_1)z)}{z^2(\beta_2 - \beta_1)}.$$

The corresponding interpolation conditions on $u(z)$ are

$$u(0) = 1$$

and

$$u'(0) = \beta_2 - \beta_1 ,$$

which, according to Theorem 1, can be satisfied only if

$$|\beta_2 - \beta_1| = \left|\frac{u'(0)}{u(0)}\right| \leq 2d .$$

Thus, the two systems can always be simultaneously stabilized but the degree of the unit must be larger than the difference between β_1 and β_2 divided by 2. To prove the sharpness of our estimate we look at the case

$$\beta_1 := 0 \ , \ \beta_2 := 2d .$$

According to the reasonning above any controler $c(z)$ has the form

$$c(z) = -\frac{u(z) - 1}{(u(z) - 1 - 2dz)/z} ,$$

where $u(z)$ is a unit satisfying

$$u(0) = 1 \, , u'(0) = 2d .$$

Our choice for $u(z)$ is

$$u(z) := \left(\frac{1 + z}{1 - z}\right)^d .$$

Working with a coprime decomposition of the rational function $u(z) - 1$ (of the same degree as $u(z)$) the formula for the controller $c(z)$ yields that the degree of the controller $c(z)$ is always equal to the degree of $u(z) - 1$, hence of $u(z)$.

Therefore, in this special case, we obtain equality in our degree estimate.

Example 3: We illustrate the use of Theorem 3 for obtaining bounds on the degree of a strong stabilizing controller. Consider the system

$$p(s) = \frac{(s-1)^2}{(s+1)(s-(1+\alpha))}, \qquad \alpha \neq 0,$$

which can be factorized into

$$p(s) = n_p(s)d_p(s)^{-1} = \left(\frac{s-1}{s+1}\right)^2 \left(\frac{s-(1+\alpha)}{s+1}\right)^{-1}.$$

Following the factorization approach described, e.g., in Vidyasagar [11], the stable stabilizing controllers for $p(s)$ are given by

$$c(s) = \frac{u(s) - d_p(s)}{n_p(s)} \ ,$$

where $u(s)$ is any unit satisfying the interpolations constraints

$$u(1) = d_p(1) = -\frac{\alpha}{2} \ , u'(1) = d'_p(1) = \frac{2+\alpha}{4} \ .$$

These interpolation conditions can always be satisfied and the system is therefore strongly stabilizable. By Theorem 3, the degree d of the interpolating unit is such that

$$|\frac{2+\alpha}{4}\frac{2}{\alpha}| \leq d.$$

When $\alpha > 0$, this inequality gives

$$1/2 + 1/\alpha \leq d.$$

The degree of $c(s)$ is again equal to the degree of $u(s)$, we therefore conclude that any stable stabilizing controller for the system has degree no less than $1/2 + 1/\alpha$. For example, all the stable stabilizing controllers for the system

$$p(s) = \frac{(s-1)^2}{(s+1)(s-1.01)}$$

have degree larger than 100.

5 An open problem

Are the constants 16 and 8 (respectively 8 and 4) in Theorem 4 (respectively Theorem 5) the best possible constants?

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