Méthodes numériques d'ordre élevé pour l'océan : est-ce vraiment utile ?

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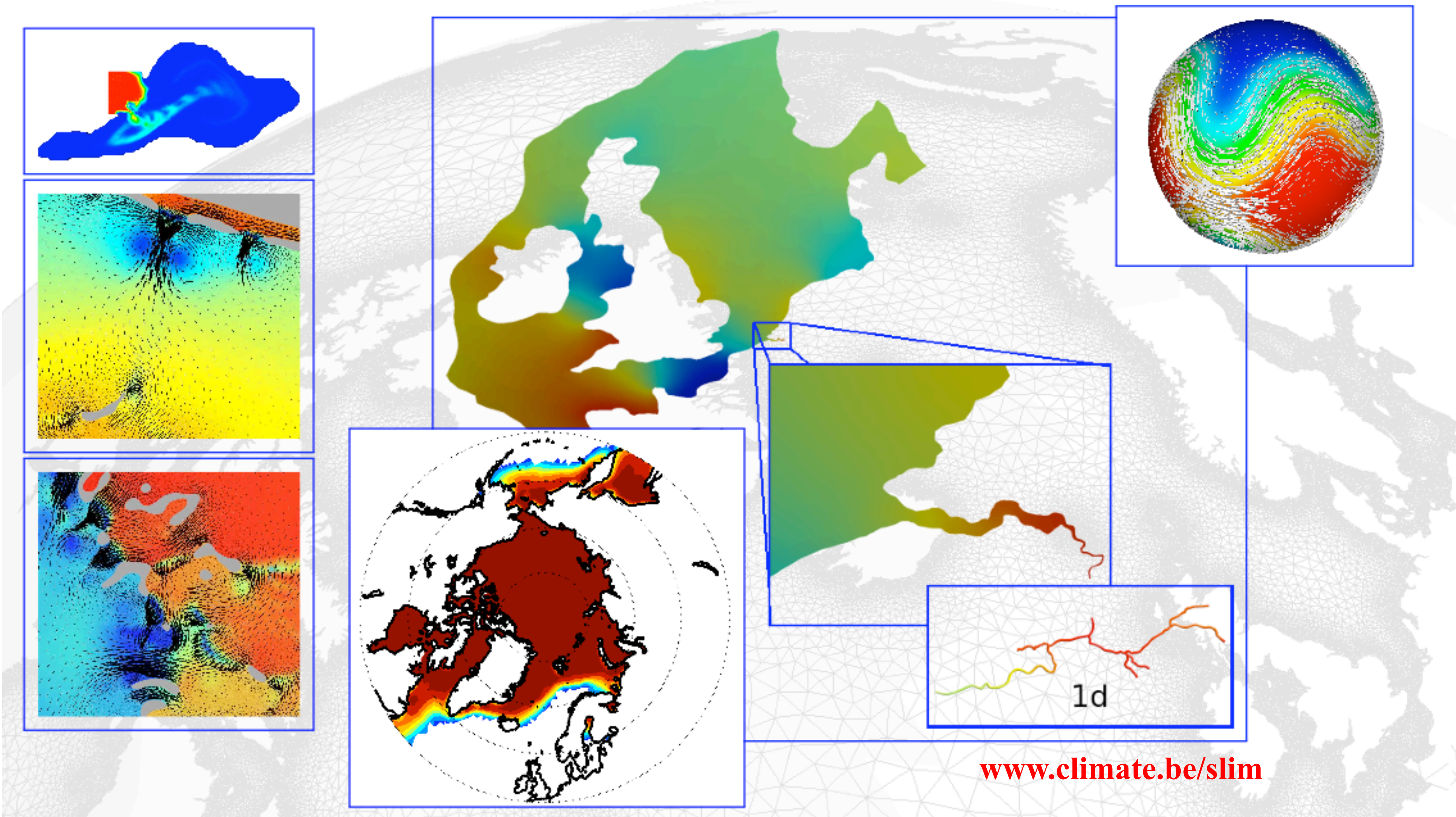
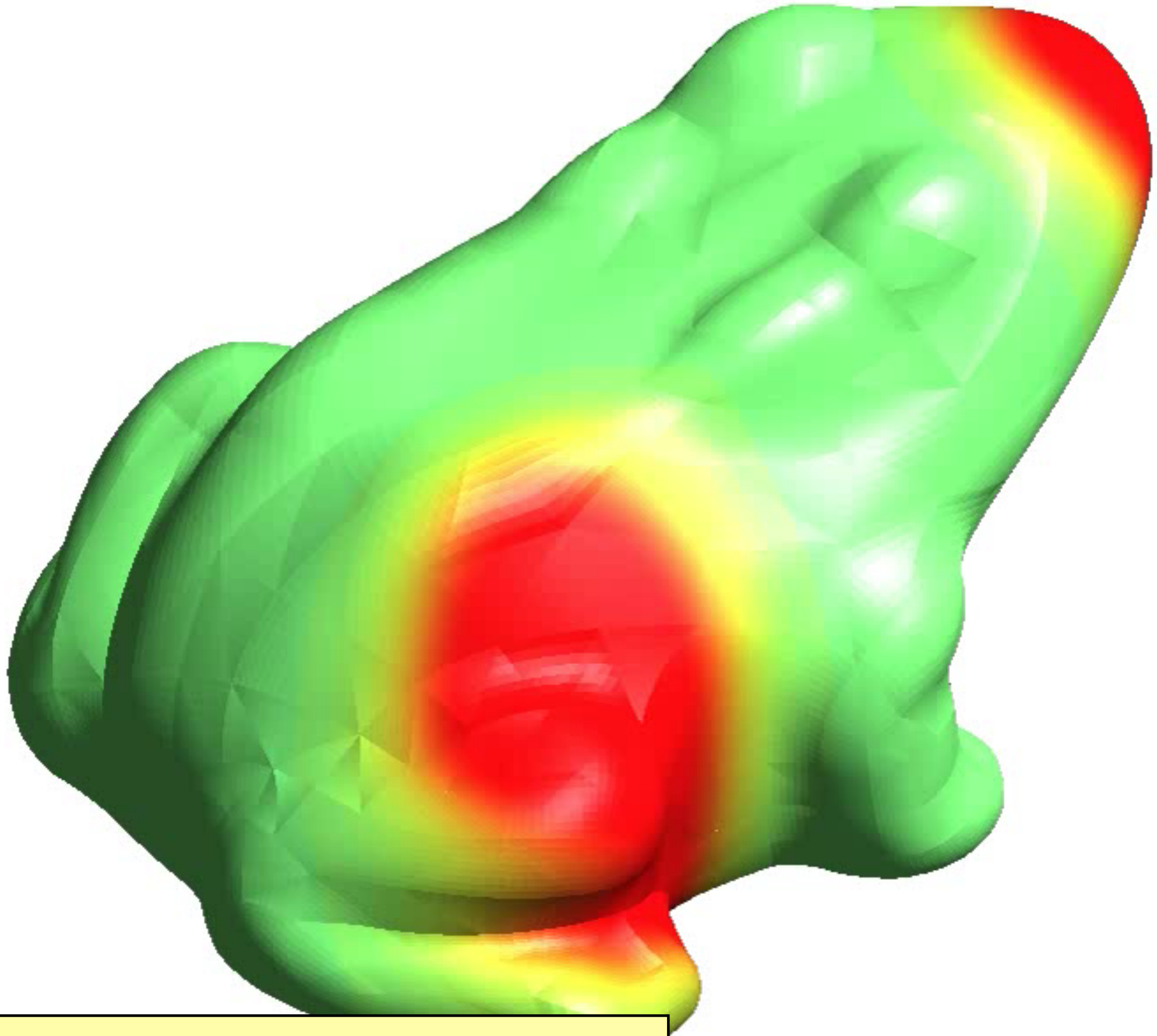
Colloque MathOcéan LAMA, Université de Savoie

Slim : a multi-scale model for the ocean, coaslines and rivers



www.climate.be/slim

Gravity waves on a froggy planet



Numbers are fun !

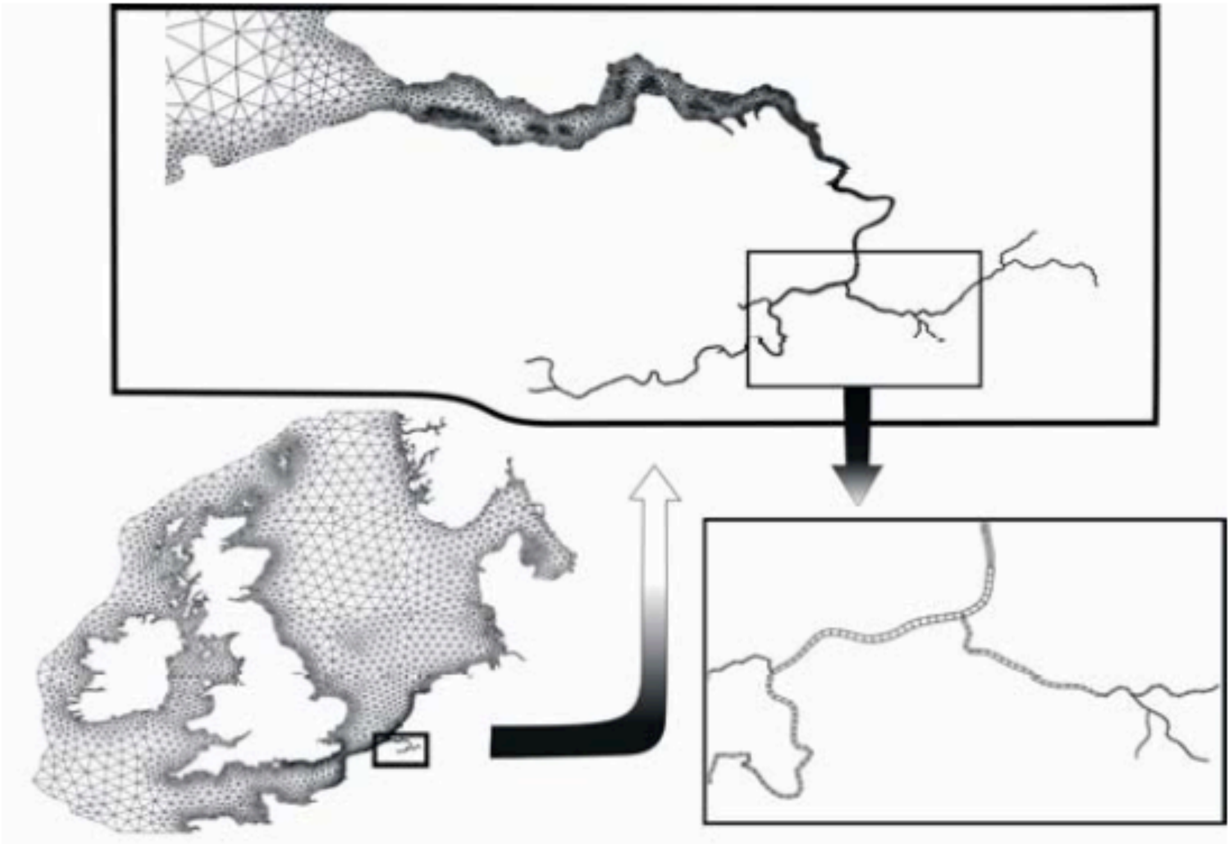
- **The method is independent of the manifold**
- **It must be easy to implement**
- **It must be robust to handle such a funny benchmark**

Multi-scale modelling of coastal, shelf and global ocean dynamics

Developing mathematical models and numerical technique to solve them is still a huge challenge today !

Models are not done !

Scheldt river, estuary and North Sea

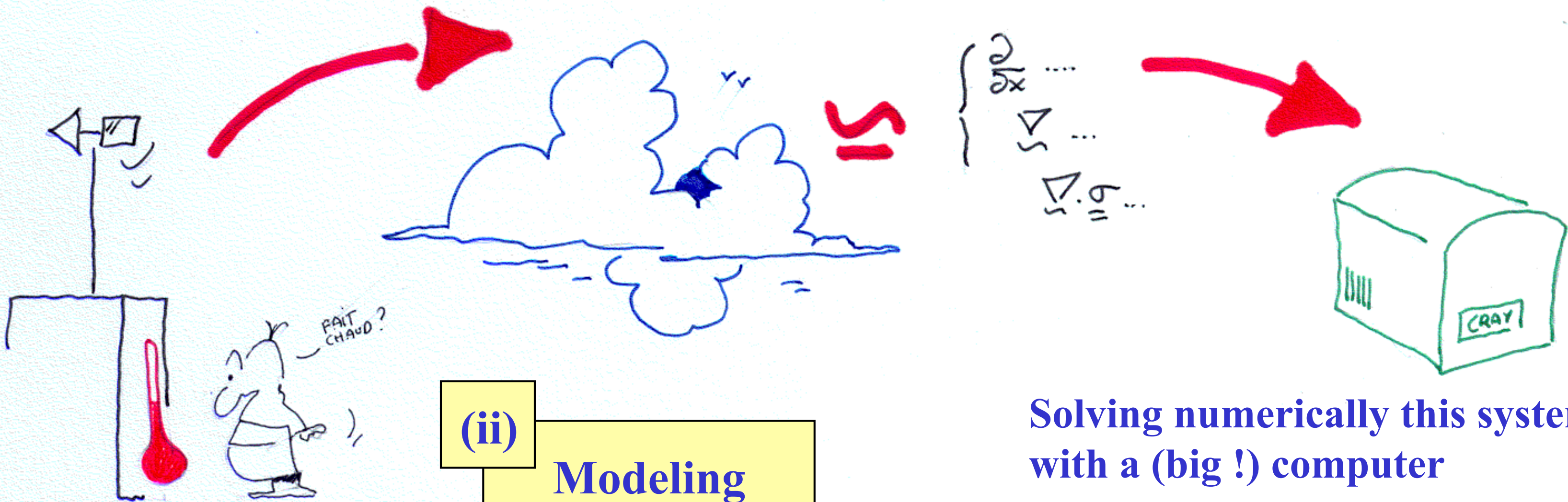


- Validated hydrodynamics with wetting/drying processes.
- Validated salinity and tracer transport.
- Model for E. coli in the tidal rivers upstream of Antwerp.
- Computation of water renewal diagnostics.
- Simulation of virtual radioactive tracer release.

Weather forecasts are not always reliable…

(iii) Computation error

Modeling the atmosphere evolution by a system of PDE’s



(ii) Modeling error

Solving numerically this system with a (big !) computer

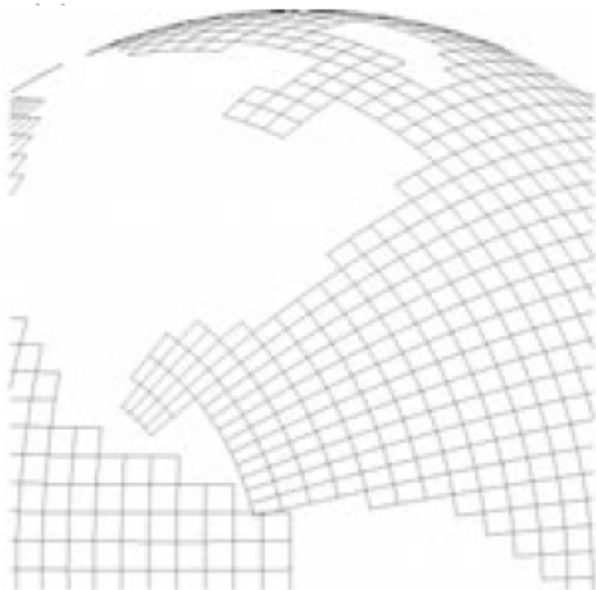
(i) Data error

Typical example of mathematical modeling and computer simulation

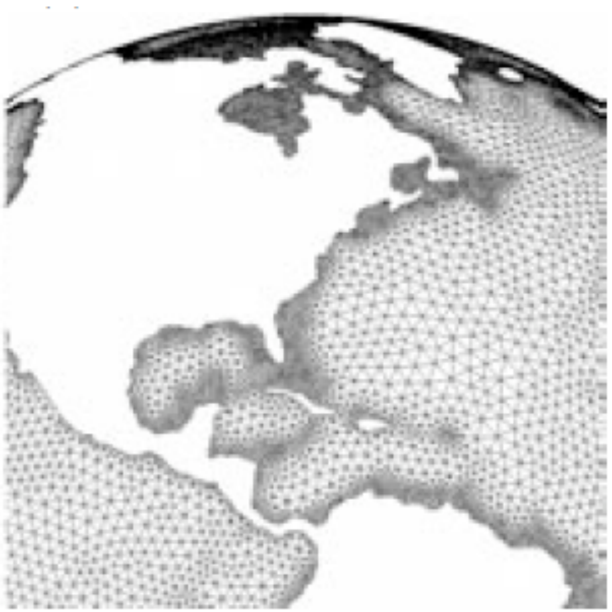
Measuring the current weather

Structured grid …

- Finite differences are easy to implement
- Programming is easy
- Well known in the world of oceanography
- Bad representation of the coastlines
- Difficult to enhance locally the resolution
- Poles singularity

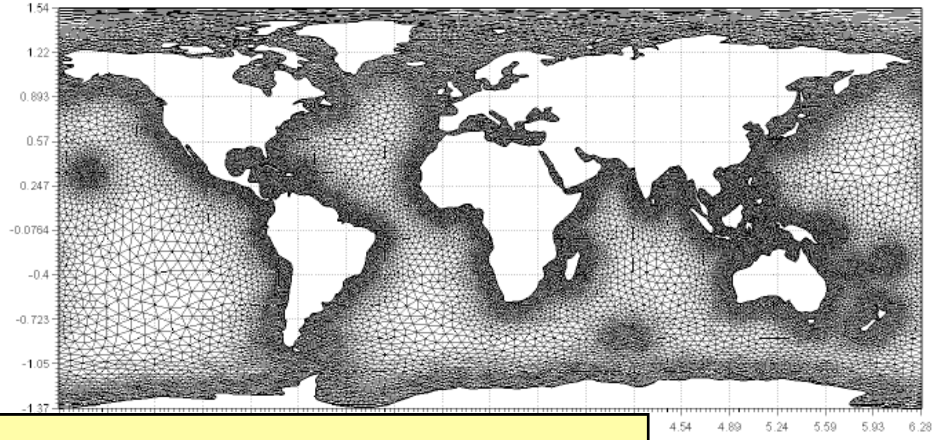


…versus unstructured grid



- Numerical methods are more complicated
- Programming is more complicated
- Not well known in the world of oceanography
- Accurate representation of the coastlines
- Enhancing the resolution is flexible
- No singular points

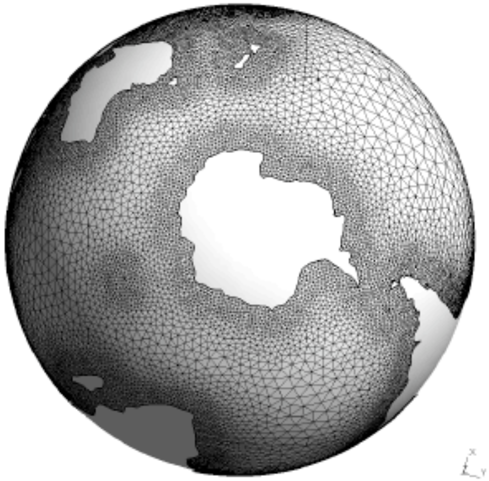
Coordinate systems for the sphere



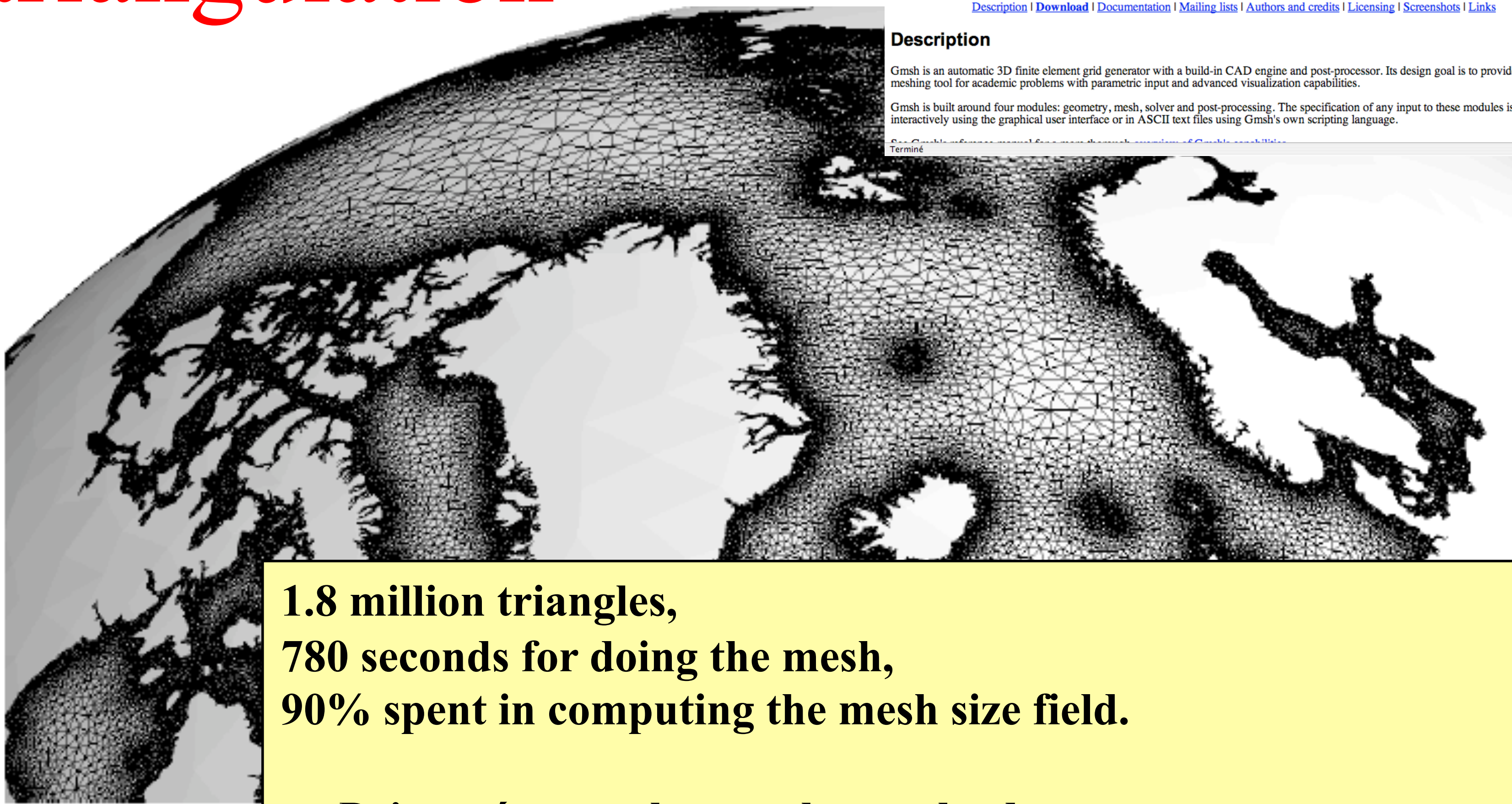
Geographical coordinates in CAD modelers and in geoscience

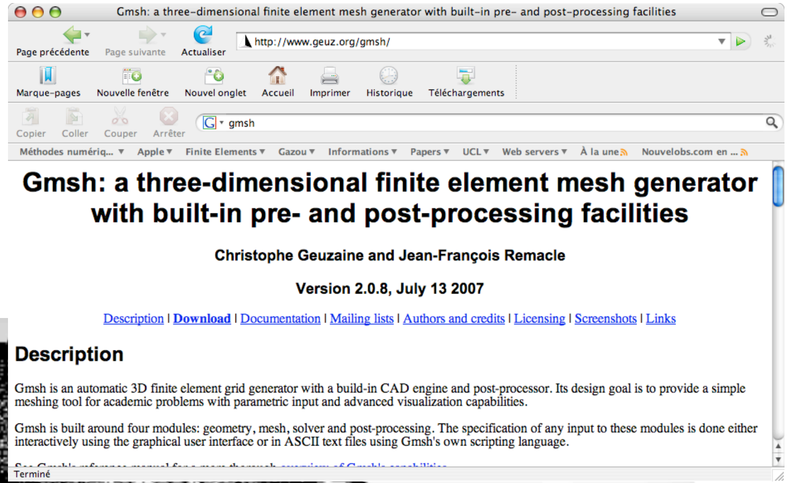
But :-(

- **It is a non-conformal mapping**
- **One seam edge is required**
- **Two degenerated positions on the poles**



Delaunay based triangulation

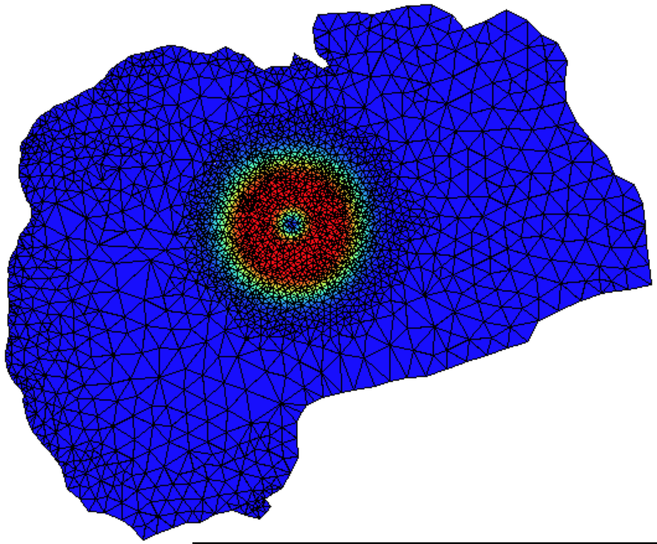




1.8 million triangles,
780 seconds for doing the mesh,
90% spent in computing the mesh size field.

- **Poincaré waves have to be resolved**
- **Mesh size smaller along coastlines**
- **Geometry of the coastlines has to be represented**

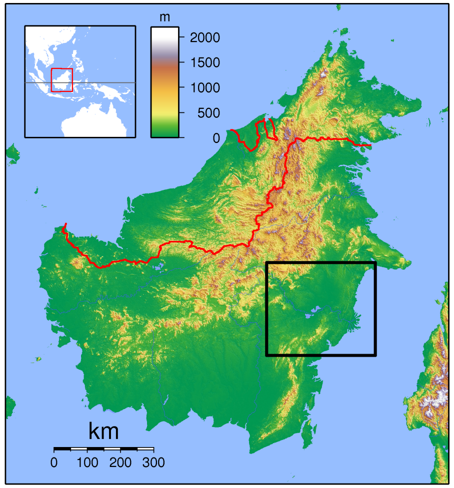
Are adaptive unstructured-grid models coming of age ?

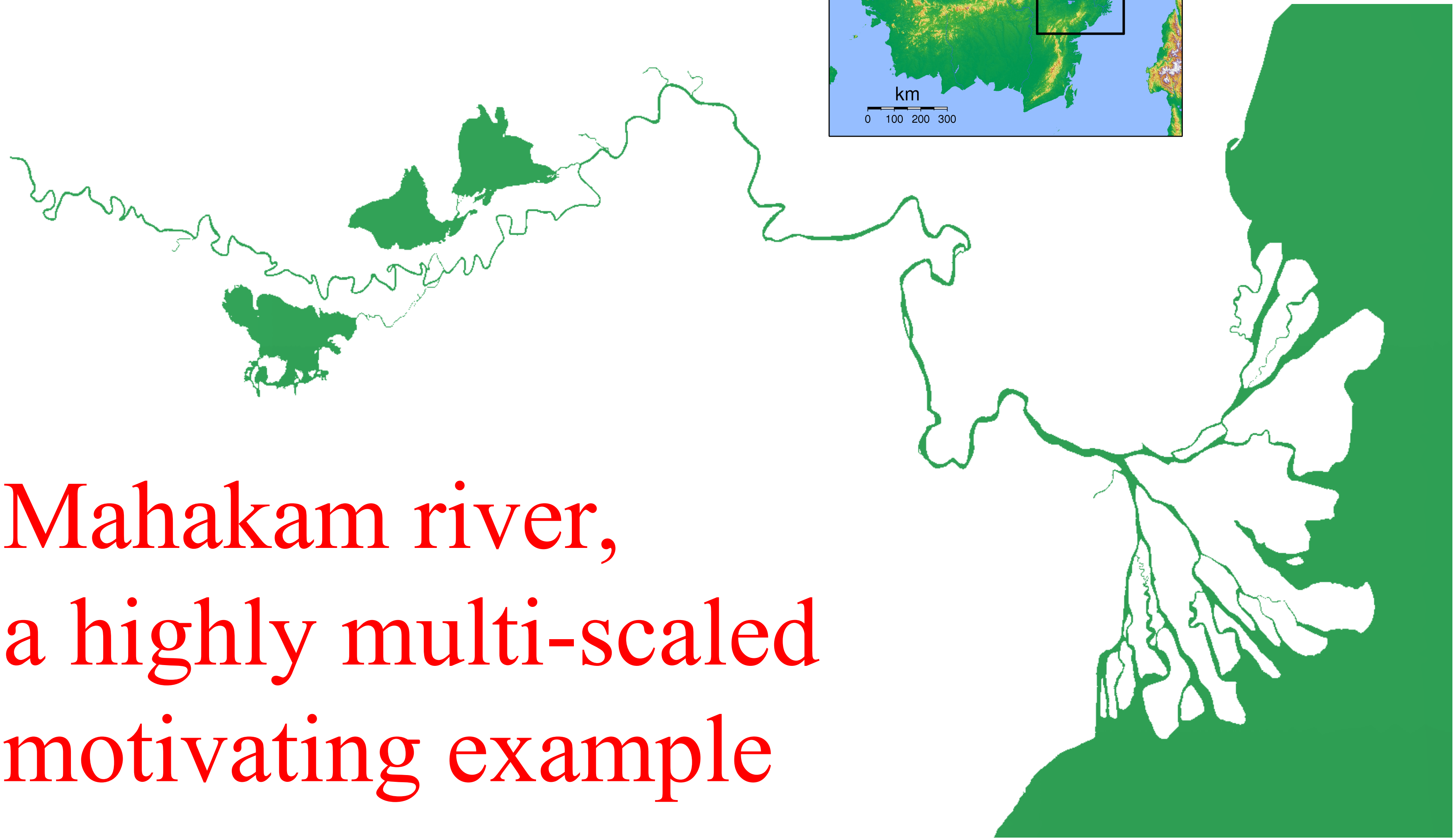


Reduced-gravity simulation of a baroclinic eddy in the Gulf of Mexico.

This simulation is several orders of magnitude cheaper than a constant resolution one of the same accuracy ! (Bernard, 2007)

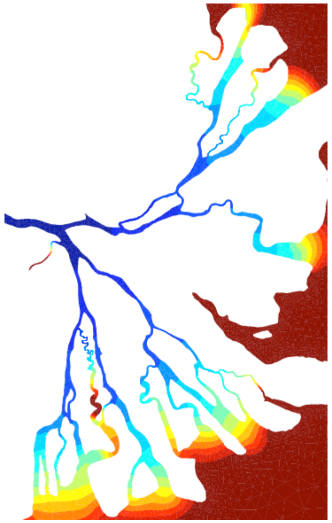
- **Numerical models of marine systems should be able to explicitly represent the broadest possible range of scales.**
- **Increasing the resolution everywhere is not the best option as this often results in a very inefficient use of the computational resources.**
- **The idea is to increase the resolution where and when it is needed !**

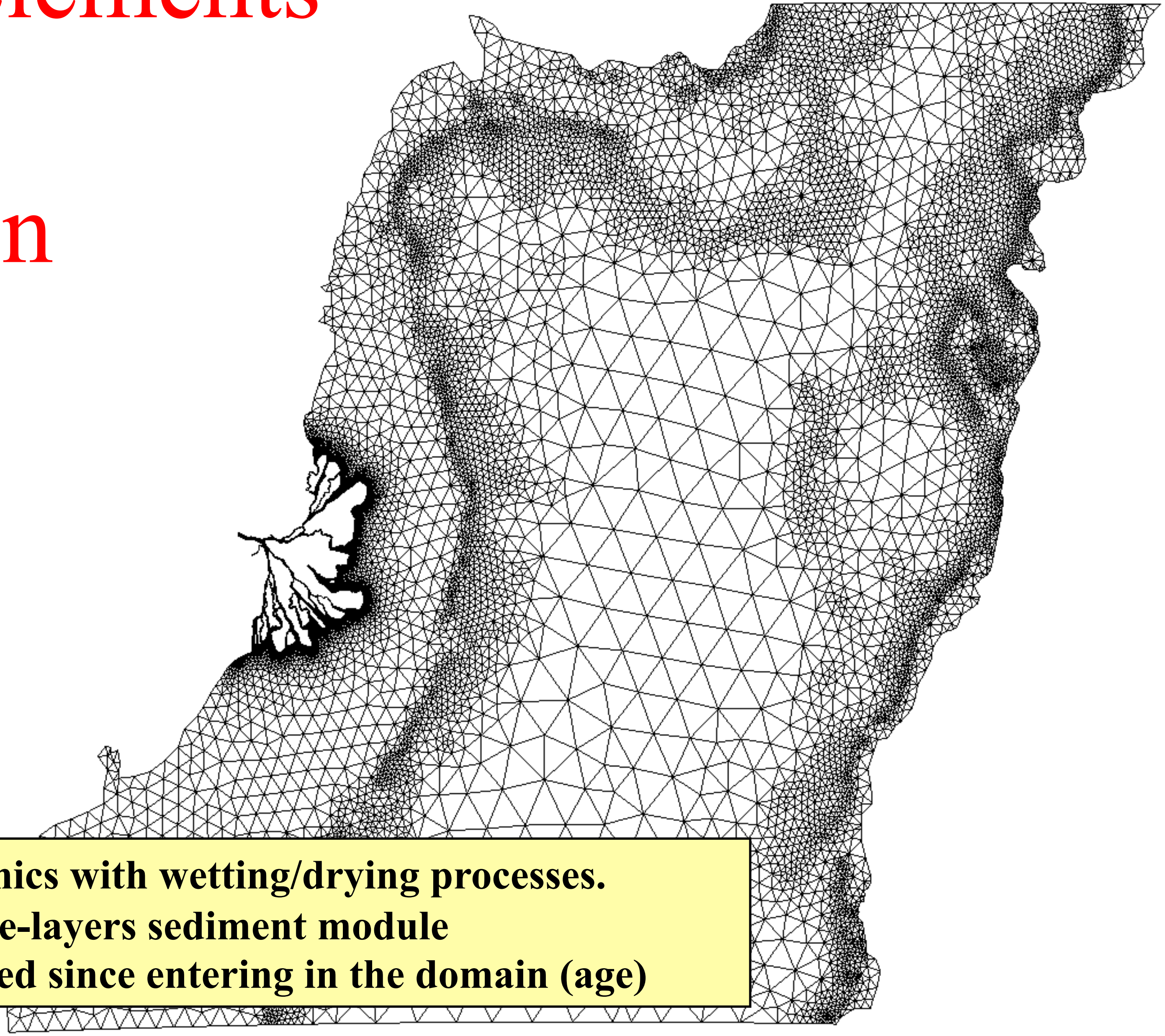




Mahakam river,
a highly multi-scaled
motivating example

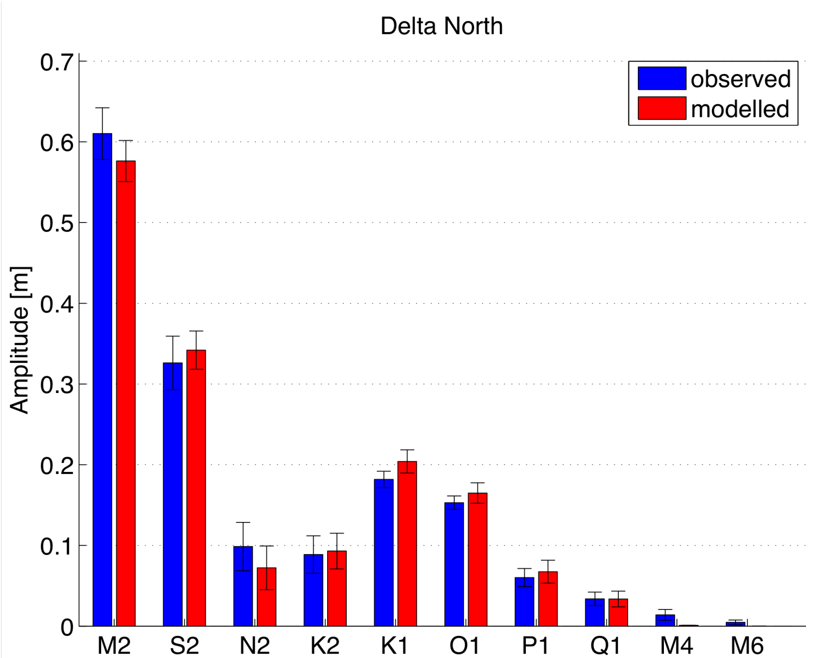
72% of the elements are in 1.4% of the domain

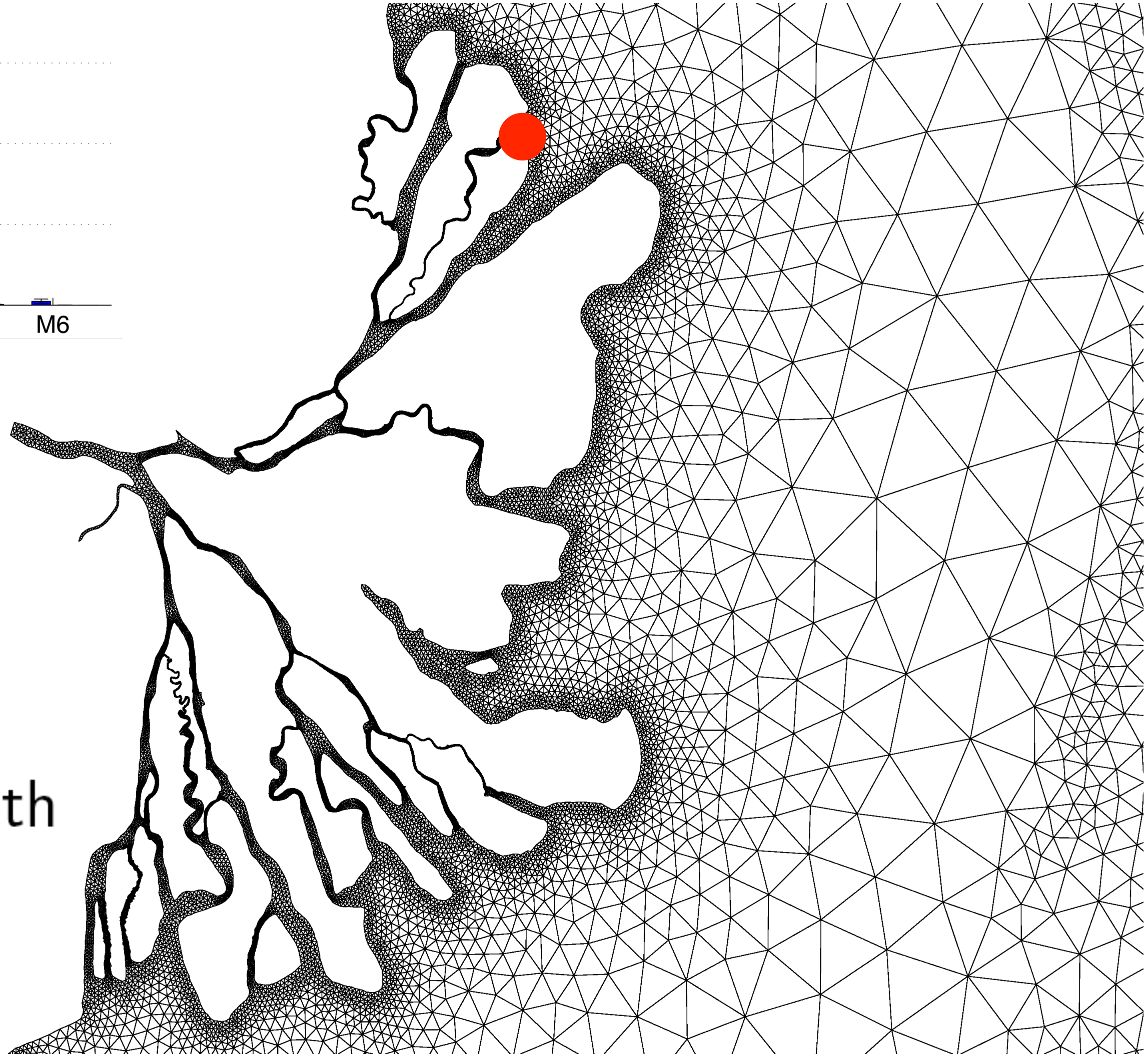




- **Validated hydrodynamics with wetting/drying processes.**
- **Development of a three-layers sediment module**
- **Computing time elapsed since entering in the domain (age)**

Hydrodynamics





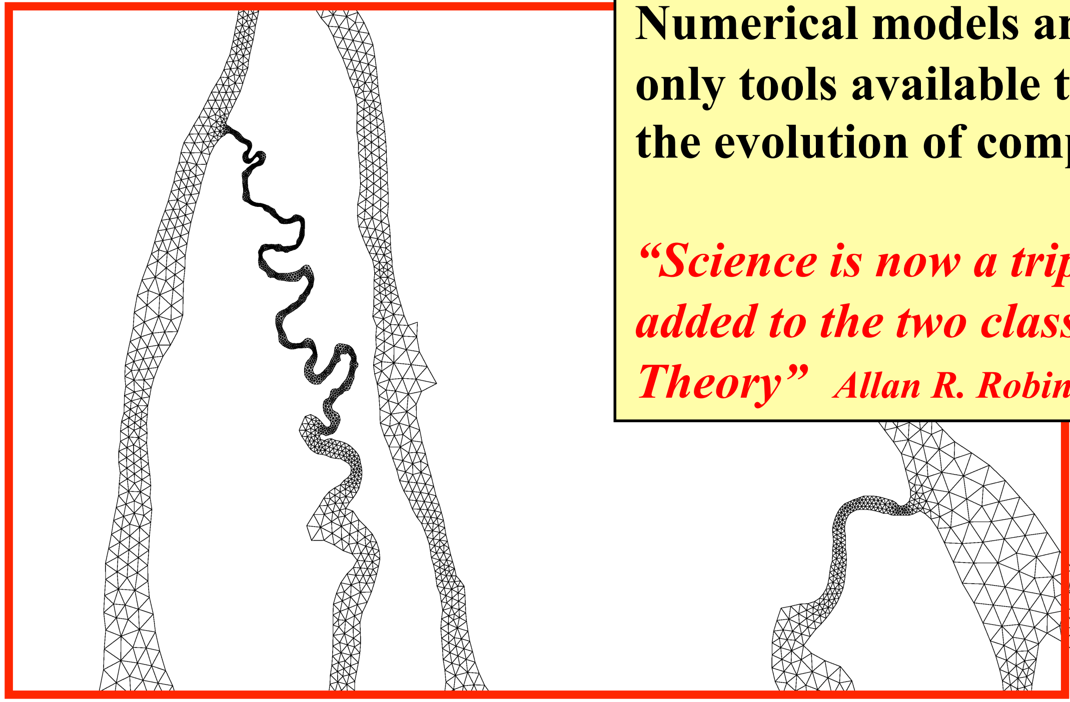
$\Delta \propto \sqrt{gH}$

$\Delta \propto$ distance to coast

$\Delta \propto ||\nabla H||^{-1}$

$\Delta \propto$ delta channels width

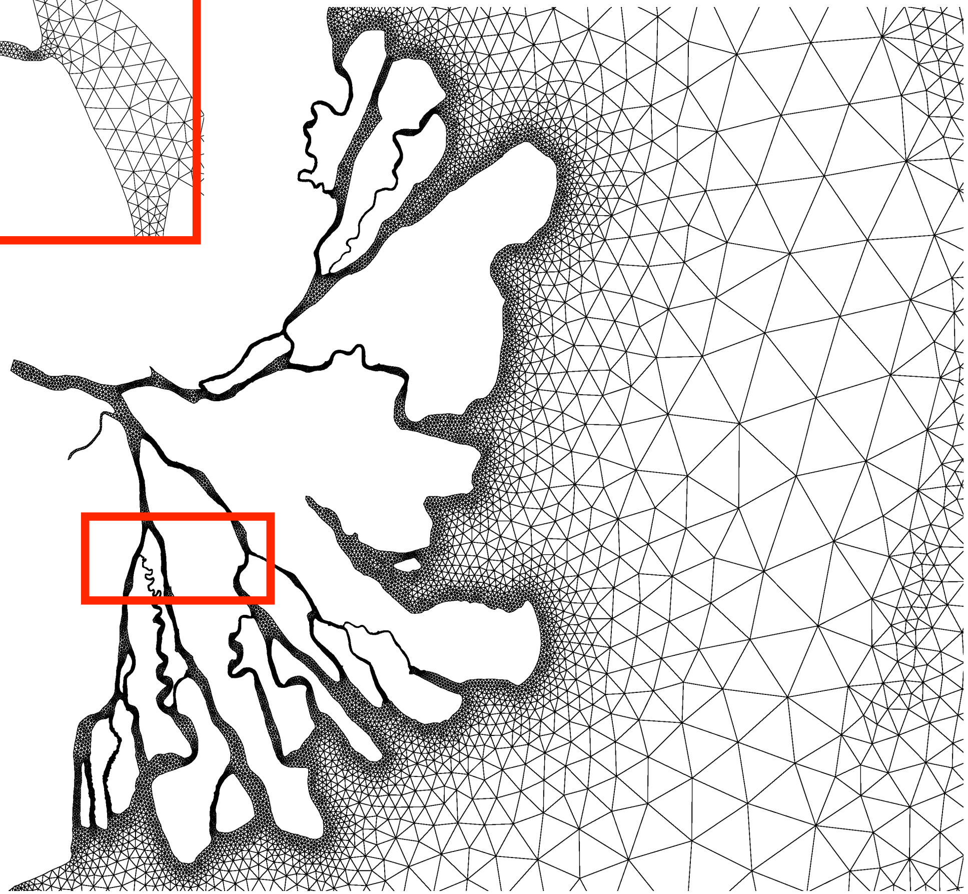
$N \approx 50\ 000$ triangles



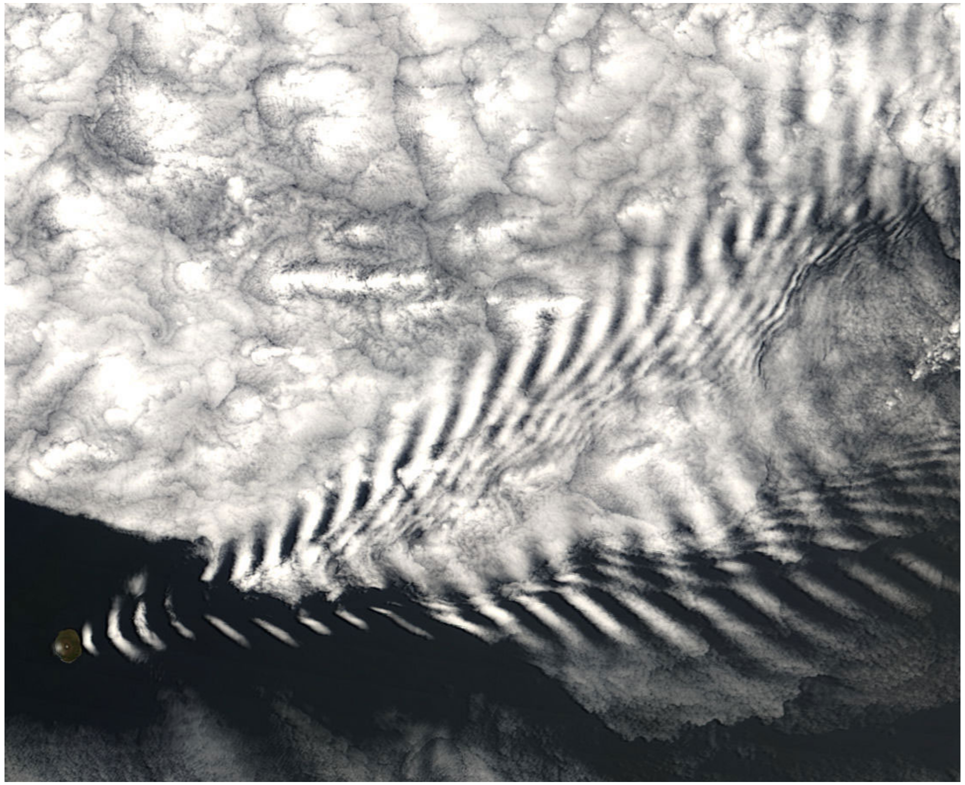
Numerical models and computer simulations are the only tools available to understand in detail and predict the evolution of complex environmental systems.

"Science is now a tripartite endeavour, with Simulation added to the two classical components, Experiment and Theory" ***Allan R. Robinson***

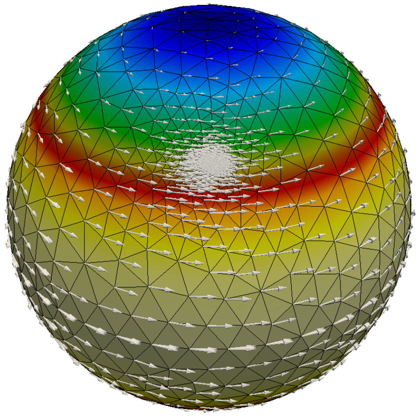
Size of the smallest element is 7 m



Internal waves in the lee of a moderately tall seamount

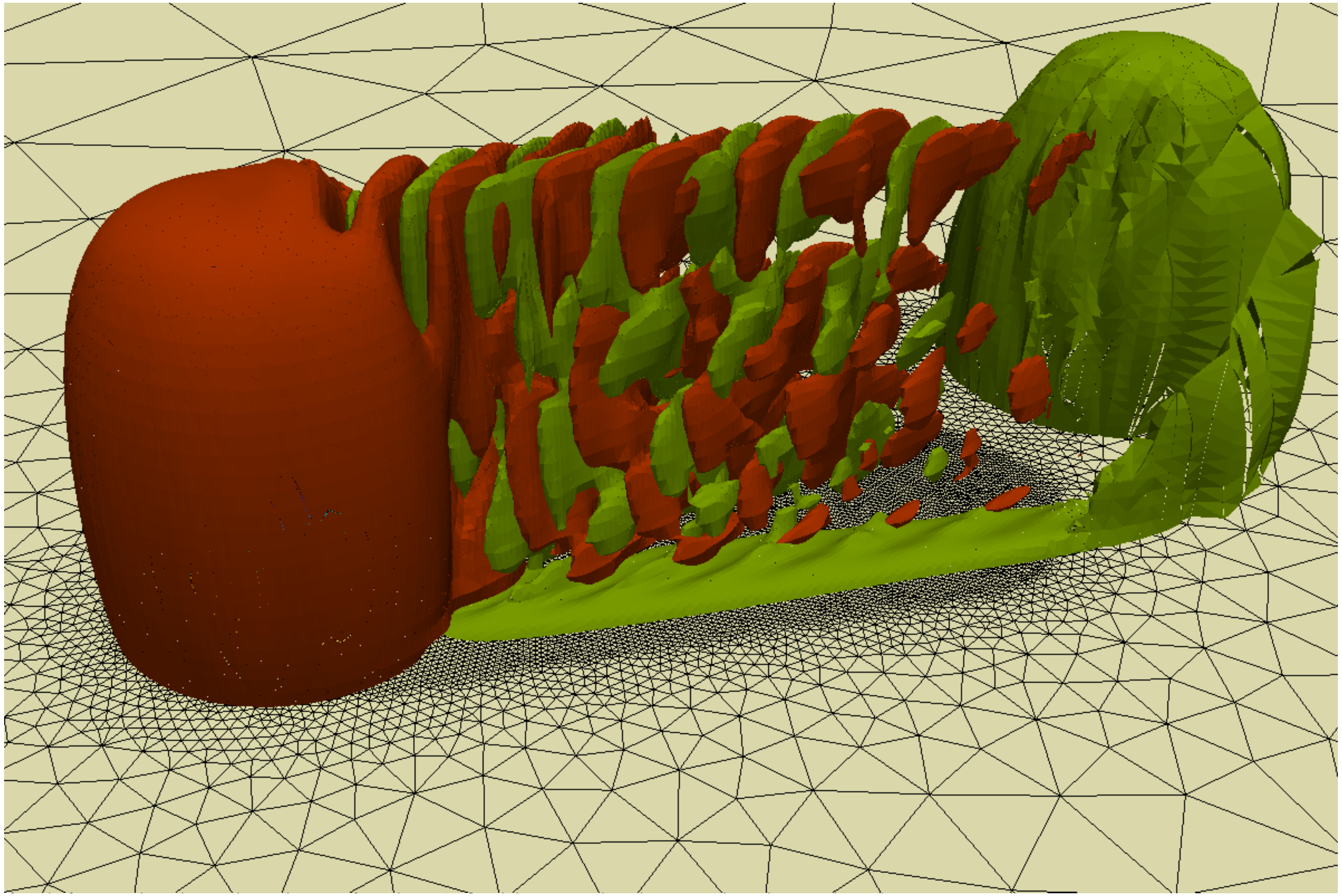


Cloud waves in the lee of Amsterdam island (NASA image from J. Schmalz)

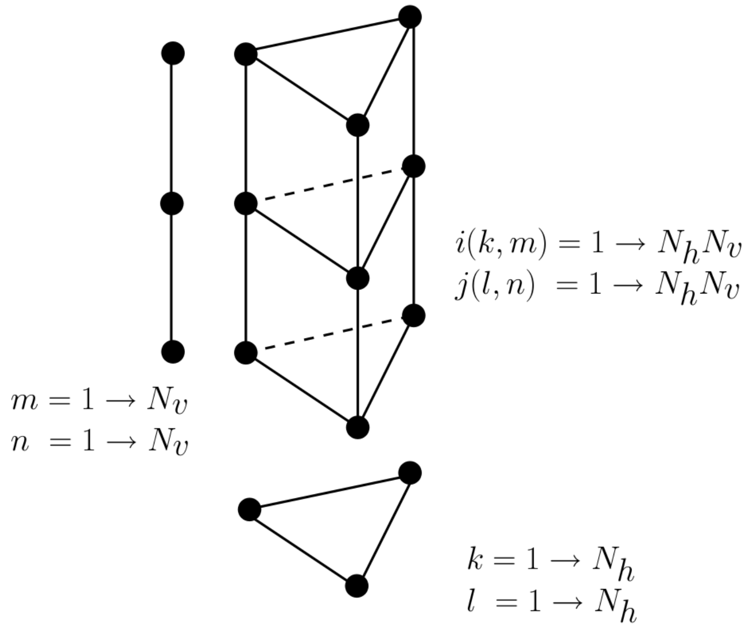


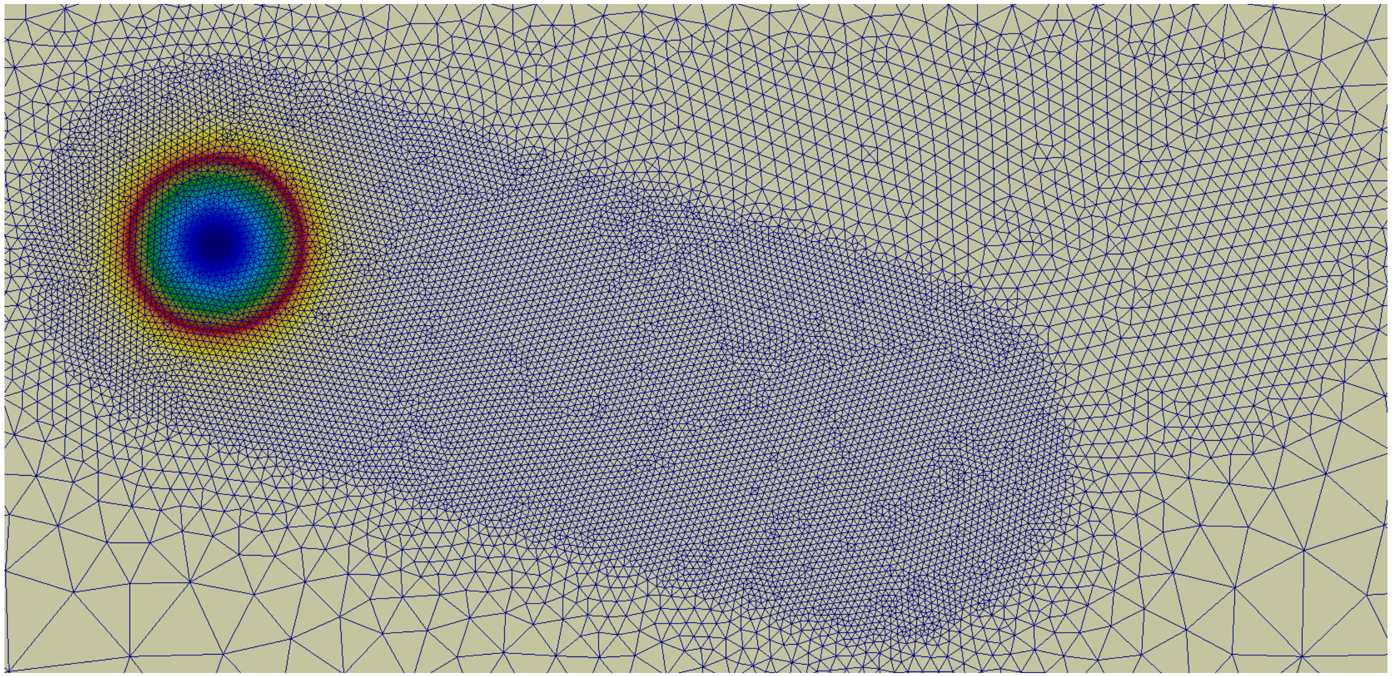
The computation starts with a global zonal geostrophic equilibrium ignoring the seamount as in Williamson testcase 5

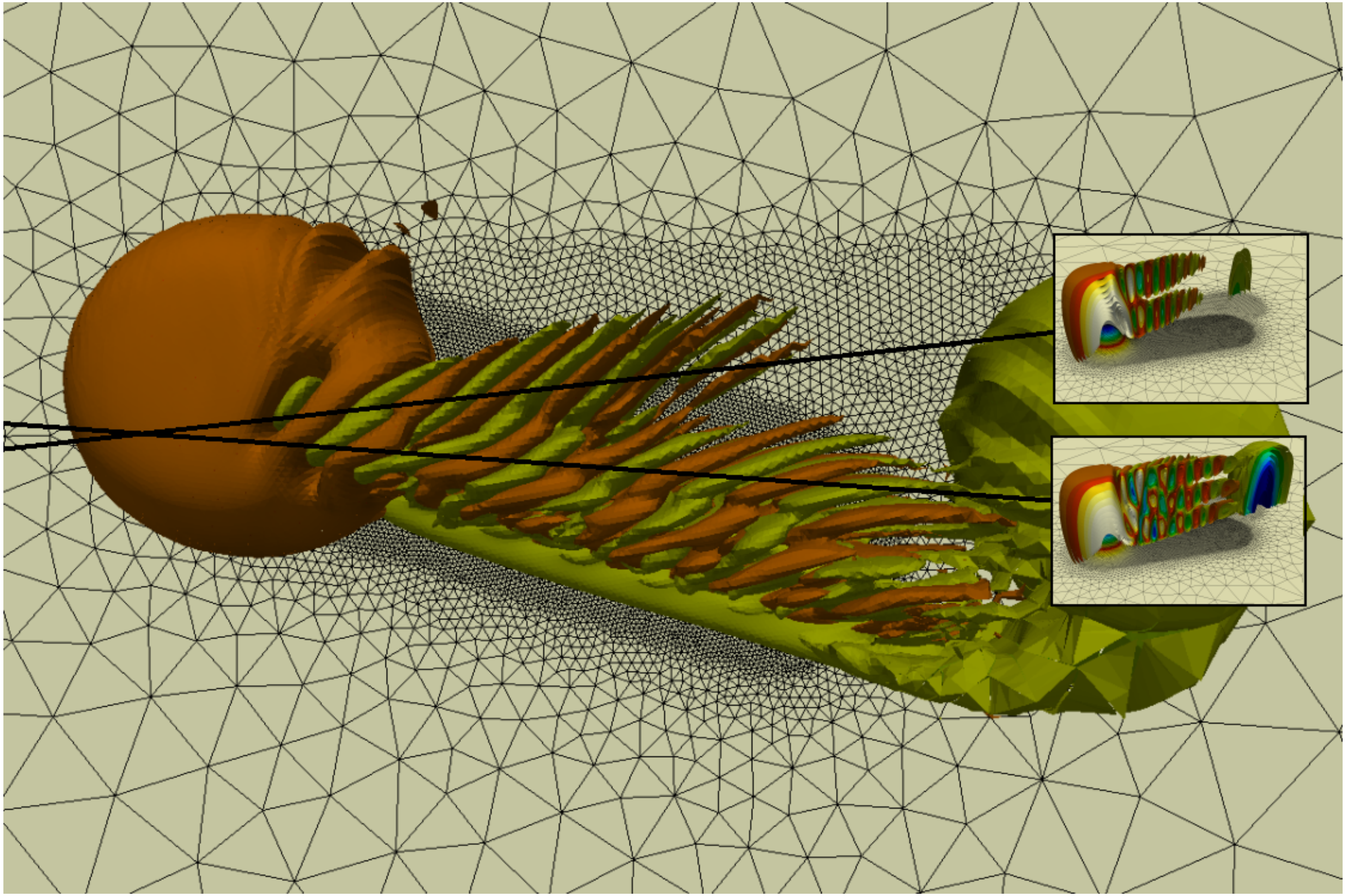
7 days evolution of density deviation field



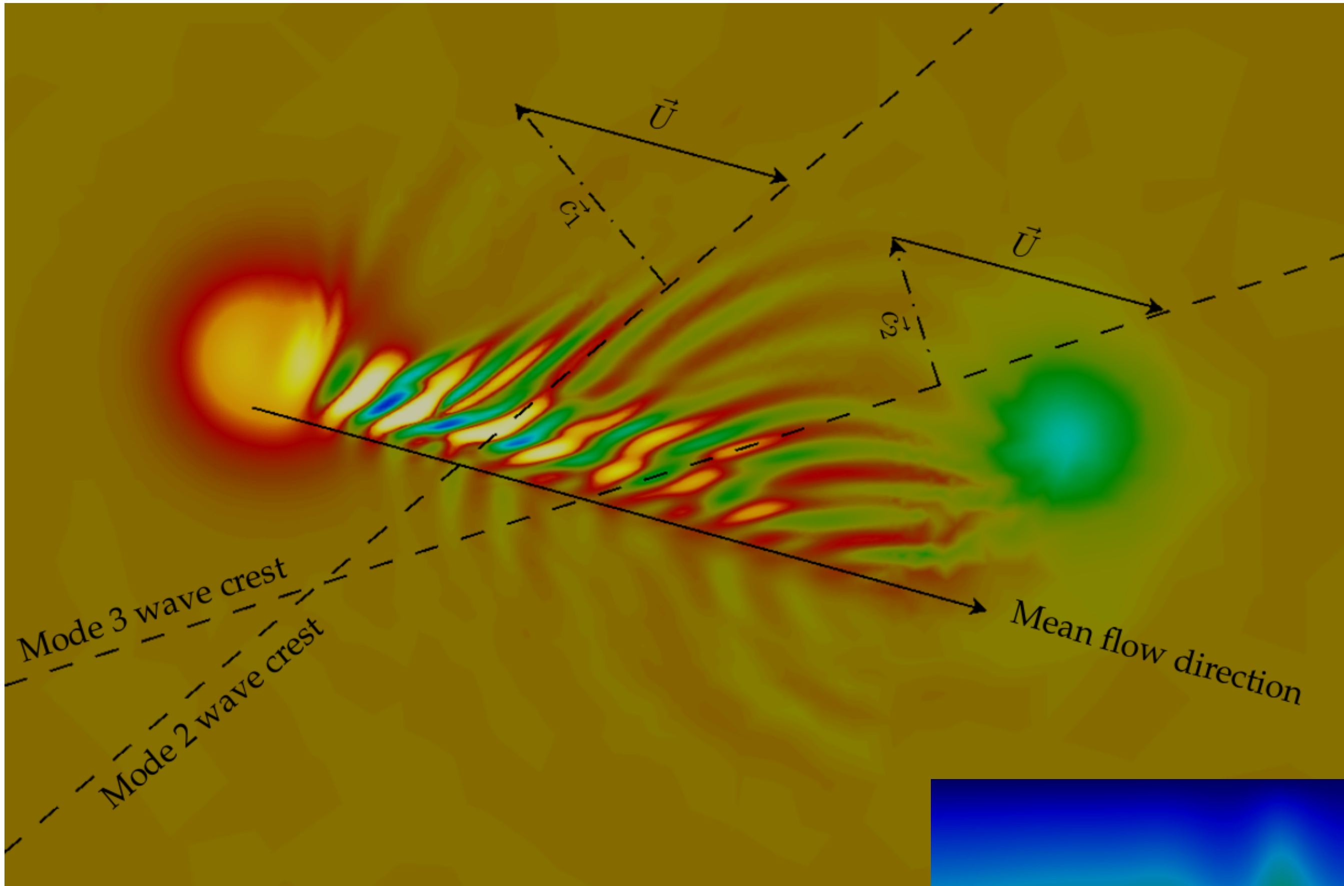
Mesh of 23562 triangles extruded into 25 σ layers



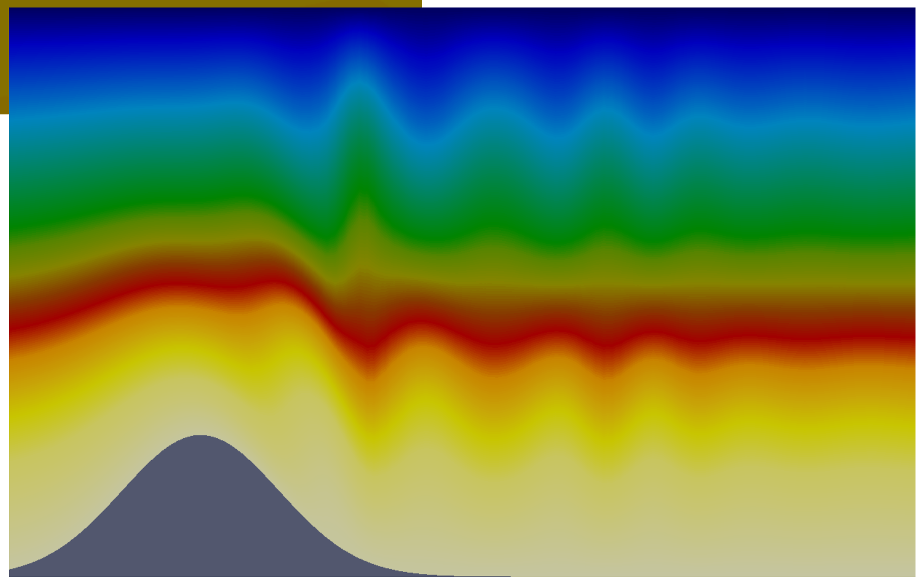




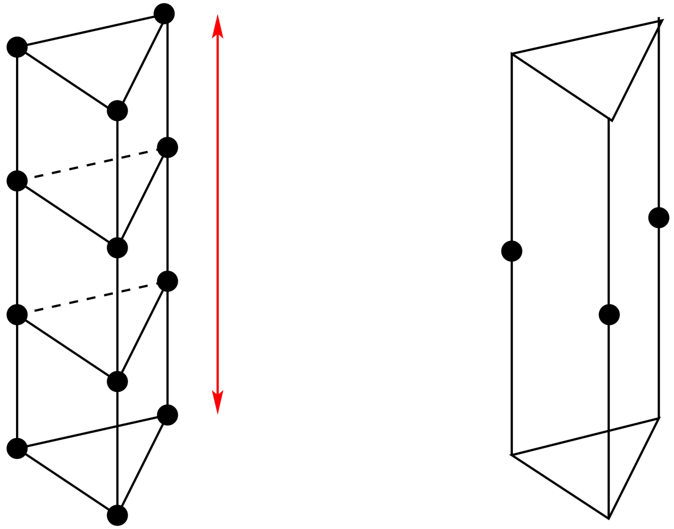
Two well separated modes at day 7

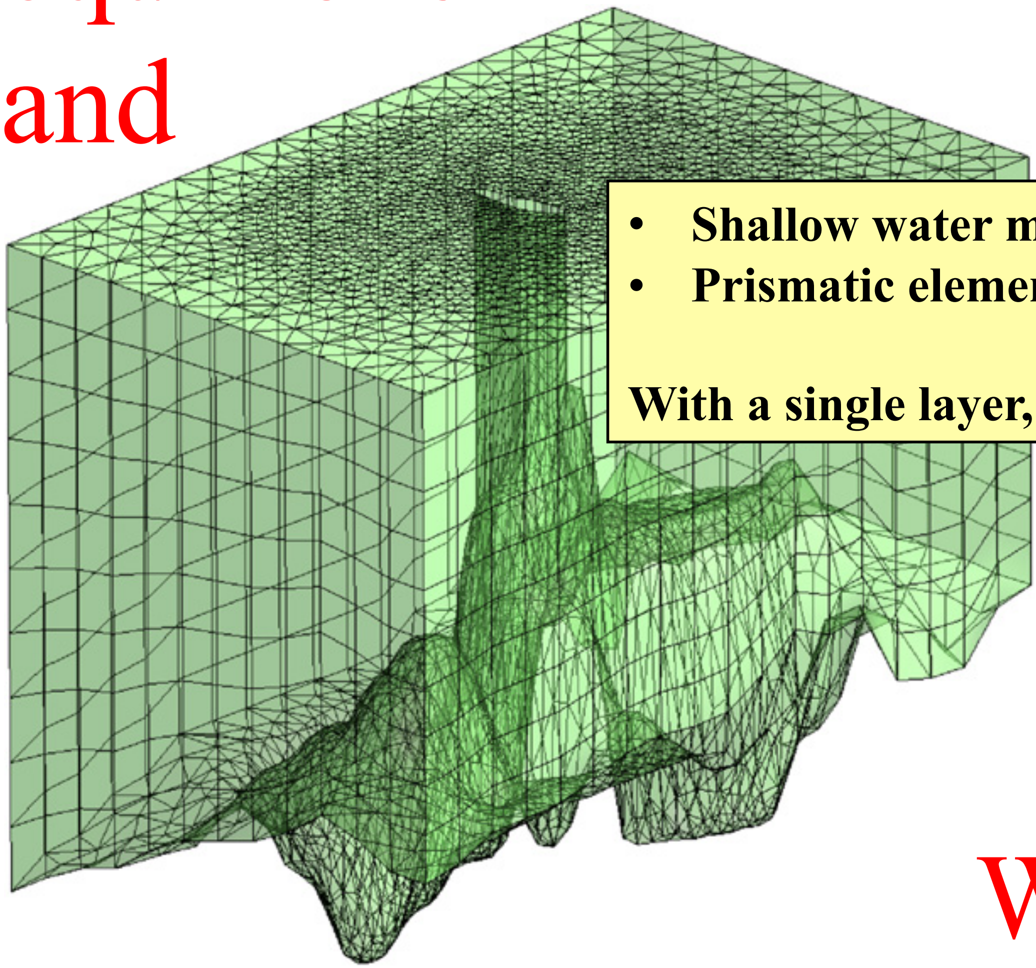


Cut in the density field at day 7



The hydrostatic Boussinesq equations and





- **Shallow water model is the depth-integrated 3D model**
- **Prismatic elements appear as a natural choice**

With a single layer, we solve the shallow water model !

… the Shallow Water Equations

A lot of physical processes inside the Shallow Water Equations

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left((h + \eta)\, \boldsymbol{u} \right) = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot (\nabla \boldsymbol{u}) + f\boldsymbol{k} \times \boldsymbol{u} + g \nabla \eta = \frac{1}{H} \nabla \cdot \left(H\nu (\nabla \boldsymbol{u}) \right) + \frac{\boldsymbol{\tau}^s + \boldsymbol{\tau}^b}{\rho H}.$$

Waves equation
Equal-order discretization !

$P_1 - P_1$

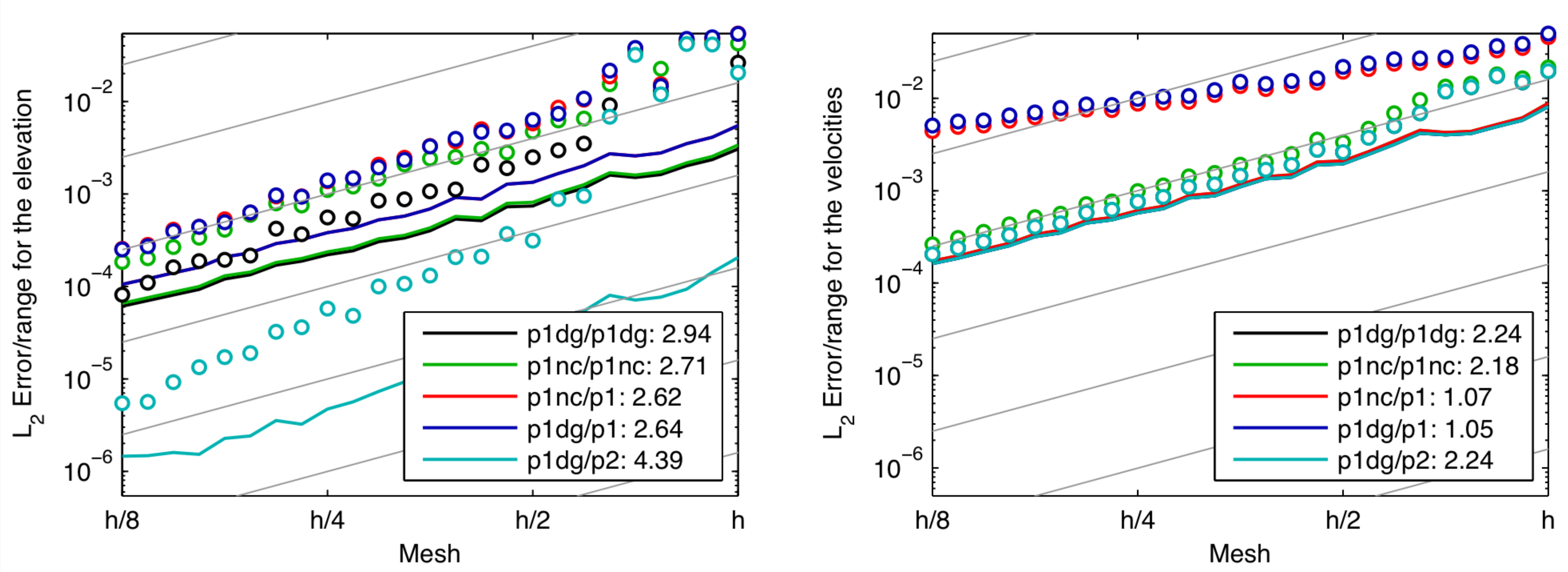
Geostrophy equilibrium
Exactly satisfied ?

$P_1^{DG} - P_2^{DG}$

Stokes problem:
LBB condition occurs !

$P_2 - P_1$

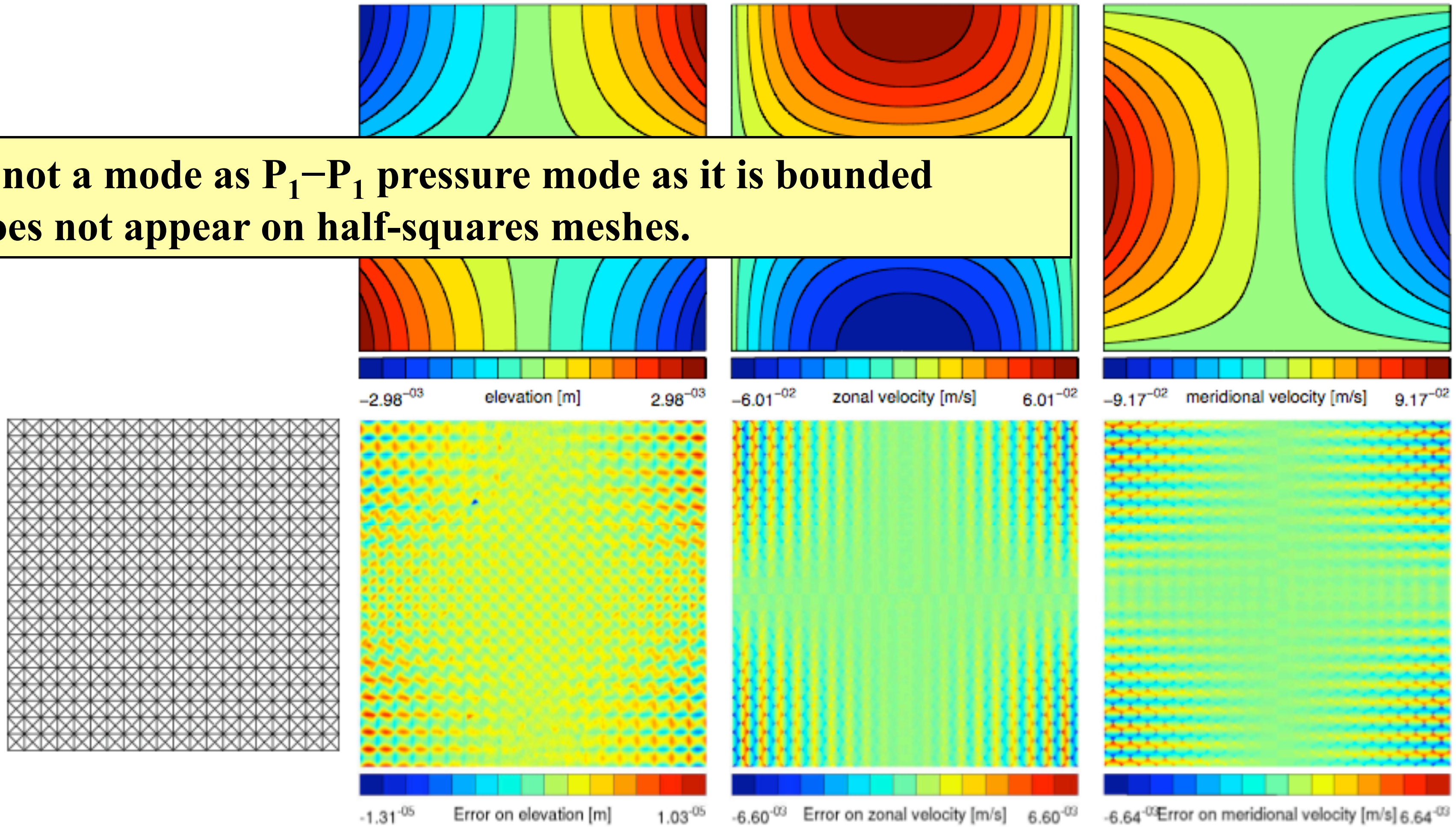
P_1^{NC}-P_1 inviscid computations look pretty nice …



… but exhibit only a first-order convergence!

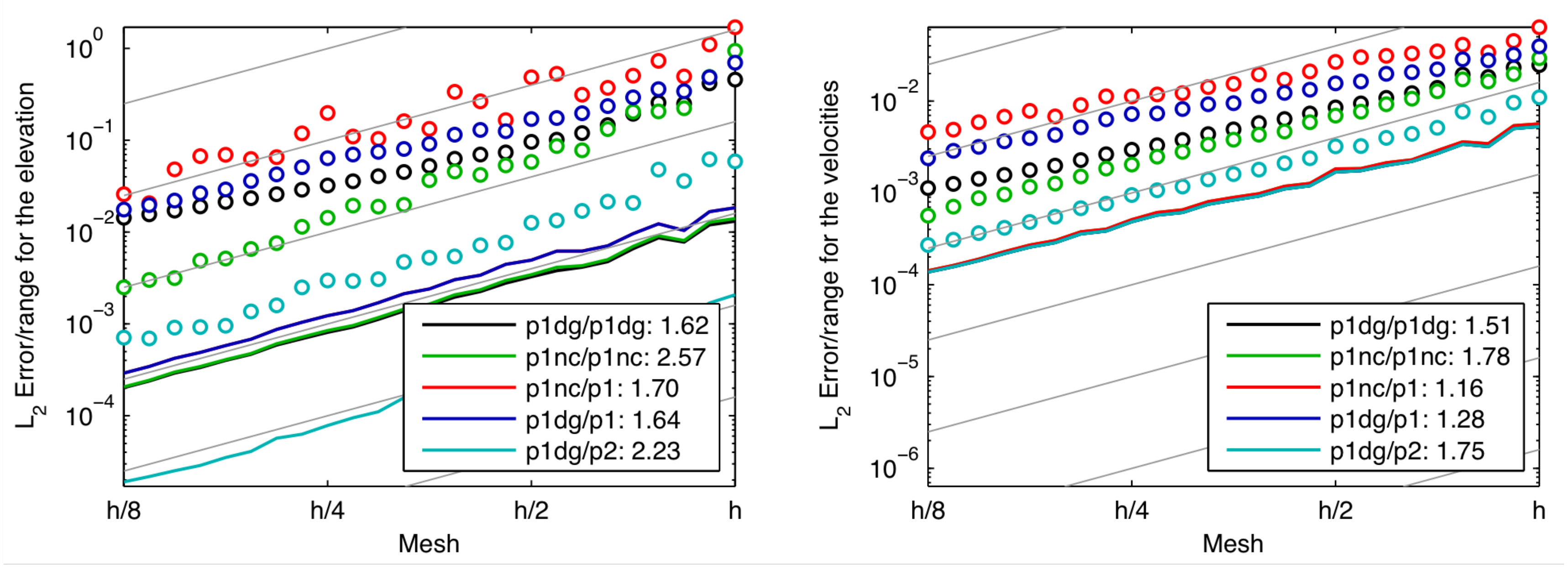
Structured noise is observed !

- it is not a mode as $P_1 - P_1$ pressure mode as it is bounded
- it does not appear on half-squares meshes.



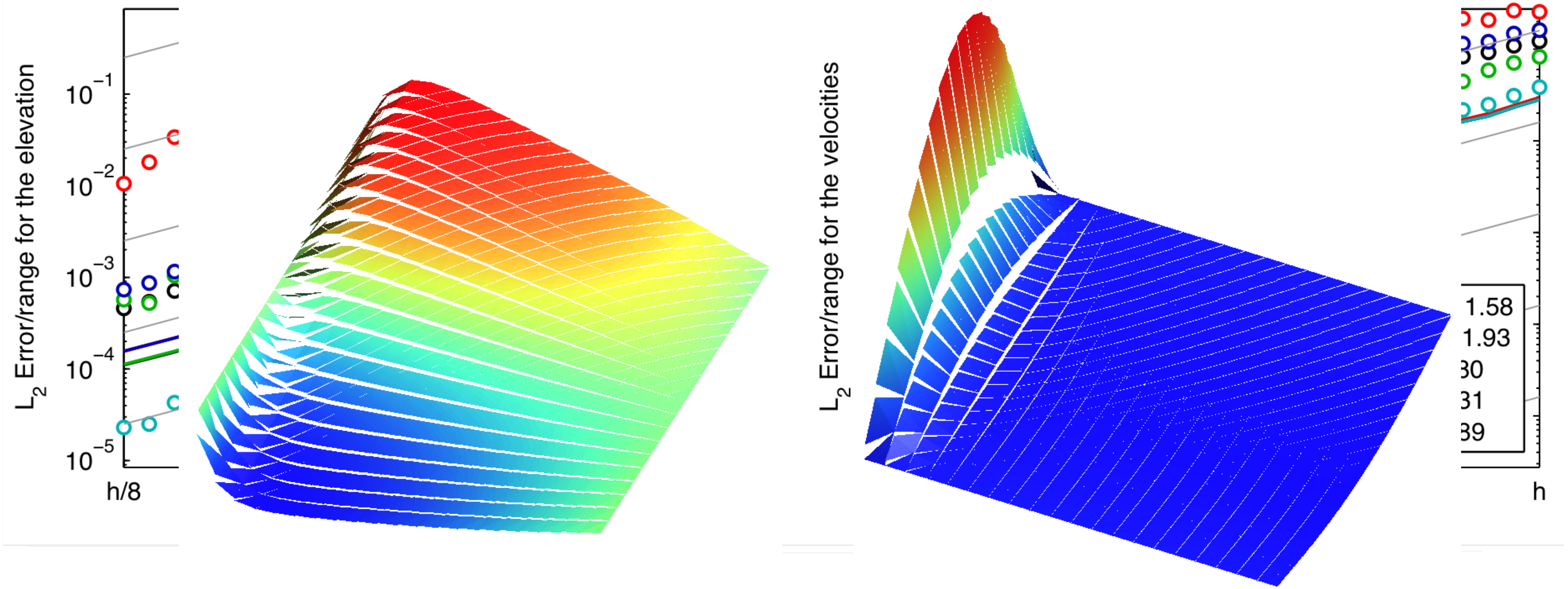
Relative error: 1 % 10% 10%

P_1^{DG}-P_2 wins the accuracy award!



- **Second-order convergence for all benchmarks.**
- **Higher order quadrature rules are required.**
- **Consistency requires to use P_2 tracers !**
- **Efficient iterative solution strategy ?**

Coriolis issue for P_1^{DG}-P_1^{DG}



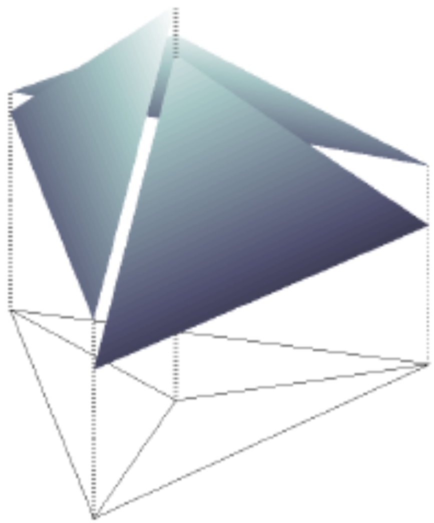
- Half an order of accuracy is lost with Coriolis
- Coriolis term has no corresponding interface term
- Only normal velocity jumps are removed by the Riemann solver
- Tangent velocity jumps amplified by Coriolis term and not damped

Finite Volumes
- **Natural treatment of wave-like terms**
- **Low order on unstructured meshes**

Continous Finite Elements
- **Optimal for second-order terms**
- **High order interpolation spaces**

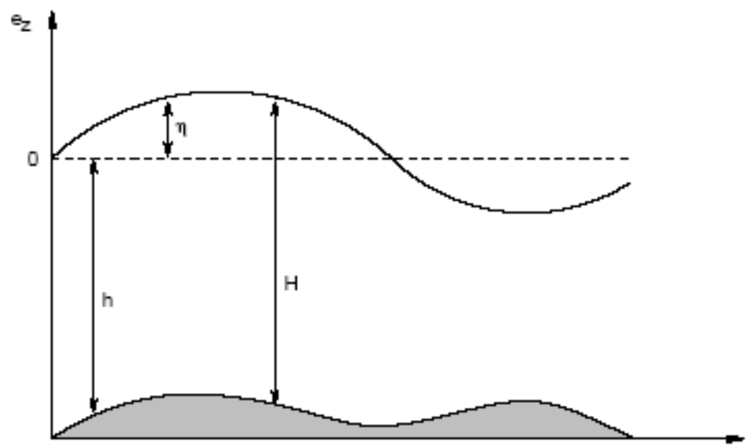
The Galerkin Discontinuous Method

Best of both approaches !
- **Wave terms handled in the finite volume spirit**
- **Second-order terms accurately handled with IP formulation**
- **High order interpolation spaces**



The Shallow Water Equations…

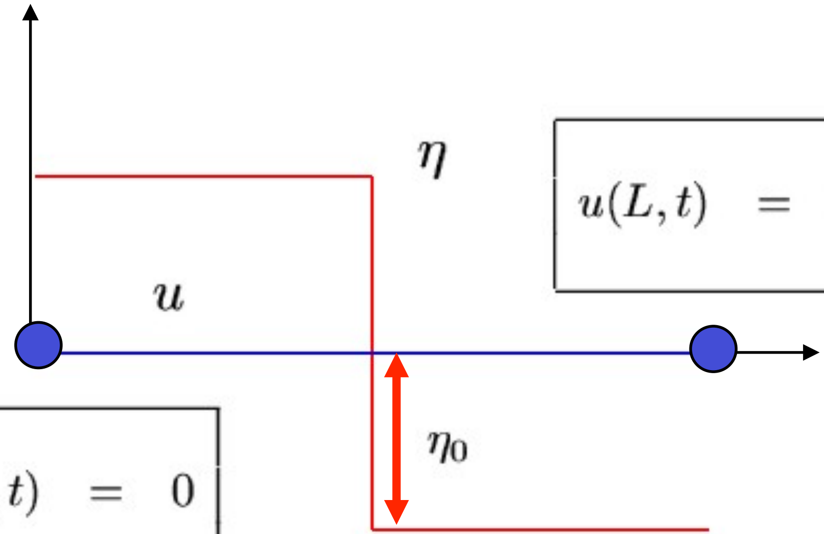
$$\frac{\partial \eta}{\partial t} + H(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0$$
$$\frac{\partial u}{\partial t} - fv + g\frac{\partial \eta}{\partial x} = 0$$
$$\frac{\partial v}{\partial t} + fu + g\frac{\partial \eta}{\partial y} = 0$$



- **Very crude model for geophysical flows, but allows the existence of inertia-gravity waves**
- **Constant depth of the ocean**

A 1D sharp simplified problem in a finite domain

$$
\begin{aligned}
\frac{\partial \eta}{\partial t} + H\,\frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial t} - fv + g\frac{\partial \eta}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + fu &= 0
\end{aligned}
$$



$$u(L,t) = 0$$

$$u(0,t) = 0$$

$$
\begin{aligned}
u(x,0) &= 0 & & \\
v(x,0) &= 0 & & \\
\eta(x,0) &= \eta_0 & & x \in [0, \tfrac{L}{2}[\\
\eta(x,0) &= -\eta_0 & & x \in]\tfrac{L}{2}, L]
\end{aligned}
$$

What is the solution ?

$$\frac{\partial \eta'}{\partial t'} + \frac{\partial u'}{\partial x'} = 0$$

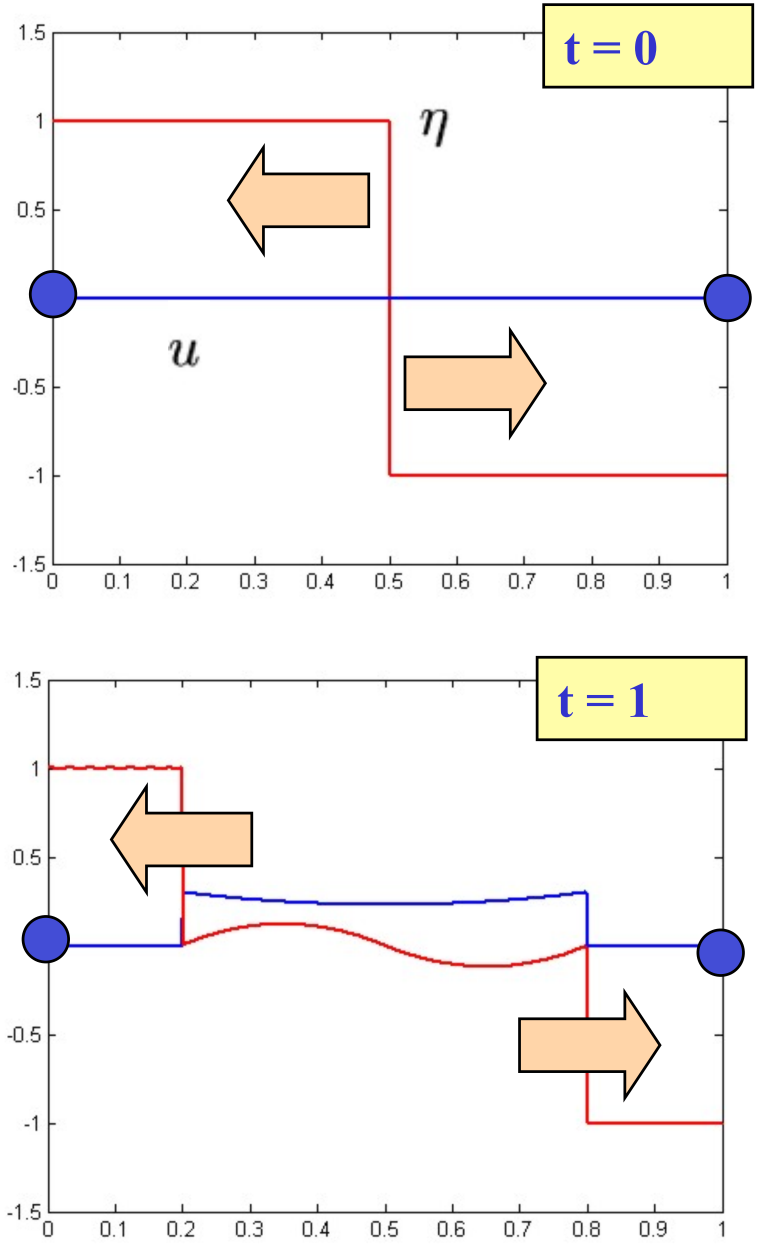
$$\frac{\partial u'}{\partial t'} - v' + \alpha^2 \frac{\partial \eta'}{\partial x'} = 0$$

$$\frac{\partial v'}{\partial t'} + u' = 0$$

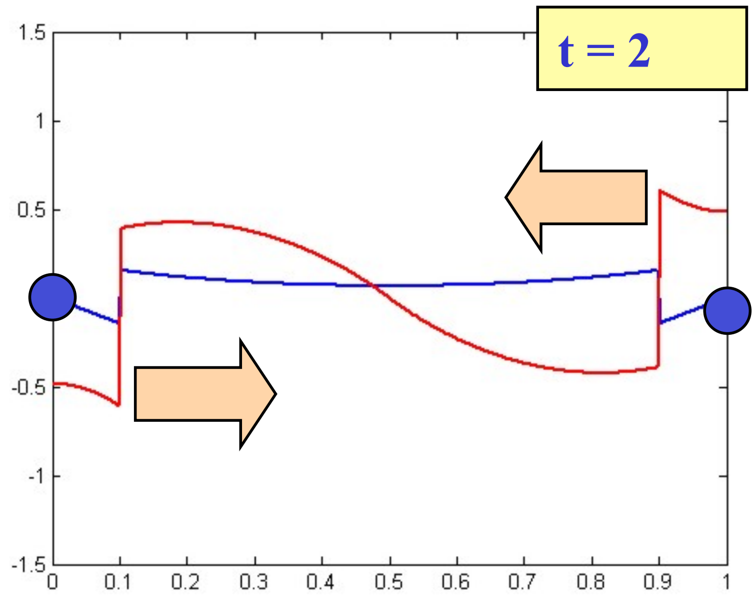
$$x' = \frac{x}{L}, \qquad t' = f\,t, \qquad \eta' = \frac{\eta}{\eta_0}, \qquad \boldsymbol{u}' = \frac{H\boldsymbol{u}}{Lf\eta_0},$$

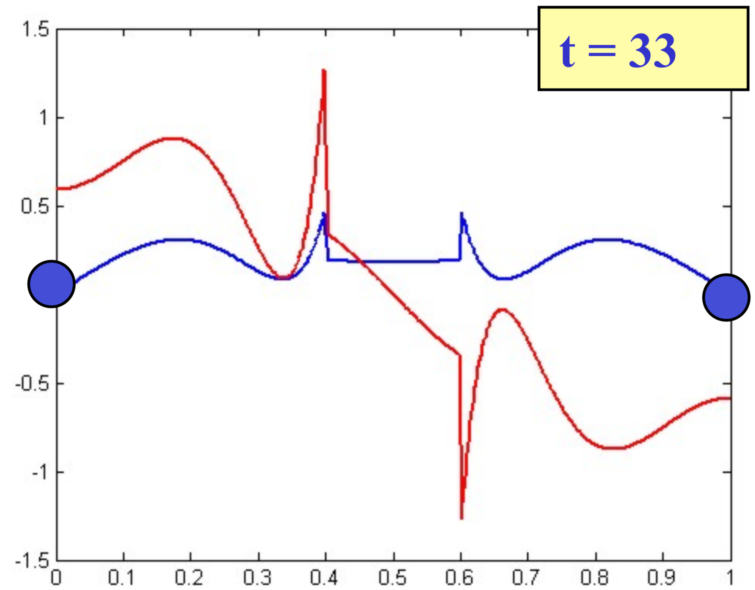
$$\alpha = \boxed{\frac{\sqrt{gH}}{f}}\,\frac{1}{L}$$

Rossby's radius



A more and more complex and interesting solution…

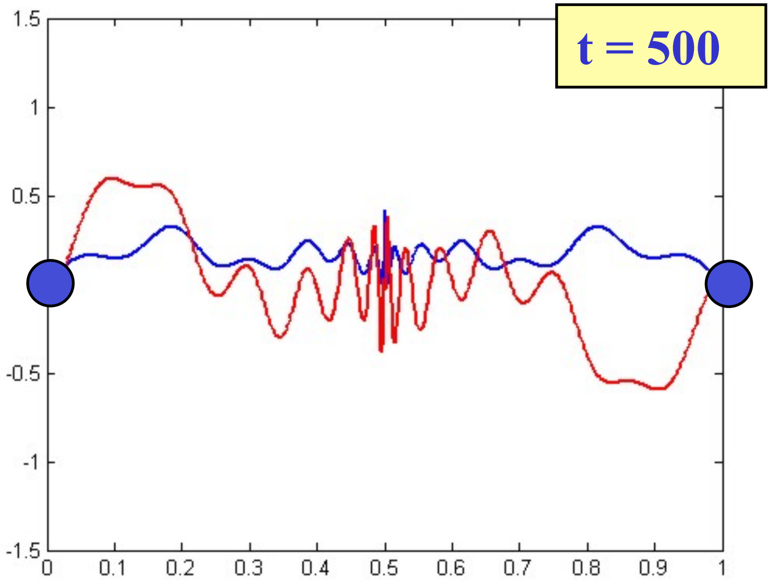




$$\alpha = \frac{\sqrt{gH}}{f}\,\frac{1}{L} = \frac{\sqrt{10}}{10} = 0.3162$$

$$\begin{aligned} f &= 10^{-4}\ 1/s \\ L &= 1000\ Km \\ H &= 100\ m \\ g &= 10\ m/s^2 \end{aligned}$$

What are the equations ?



$$\begin{aligned} \frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} &= 0 \\ \frac{\partial u}{\partial t} - v + \alpha^2 \frac{\partial \eta}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + u &= 0 \end{aligned}$$

$$\frac{\partial^2 u}{\partial t^2} - \underbrace{\frac{\partial v}{\partial t}}_{-u} + \alpha^2 \underbrace{\frac{\partial^2 \eta}{\partial t \partial x}}_{-\frac{\partial^2 u}{\partial x^2}} = 0$$

Helmholtz's Equation
Forced Wave Equation

$$\frac{\partial^2 u}{\partial t^2} + u - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$

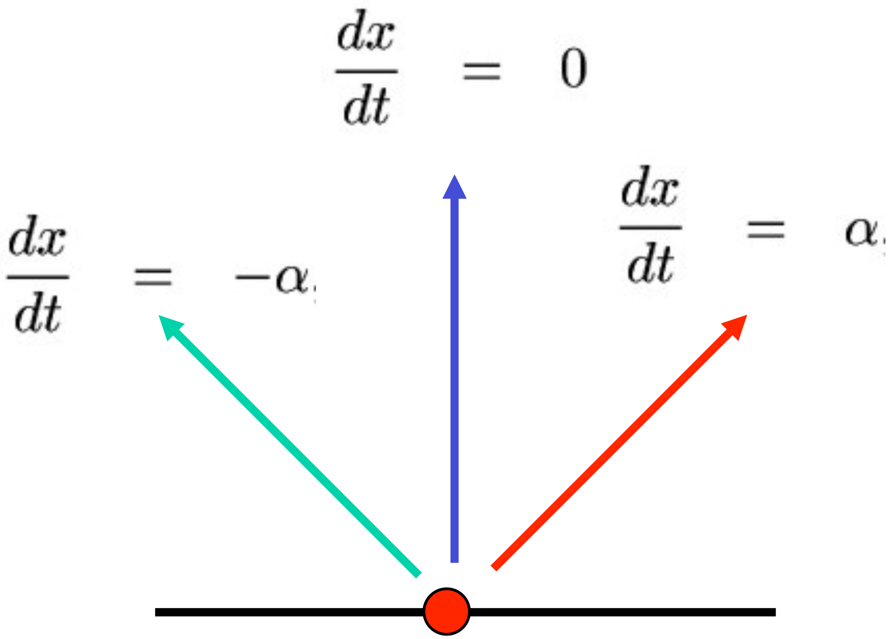
How does information propagate ?

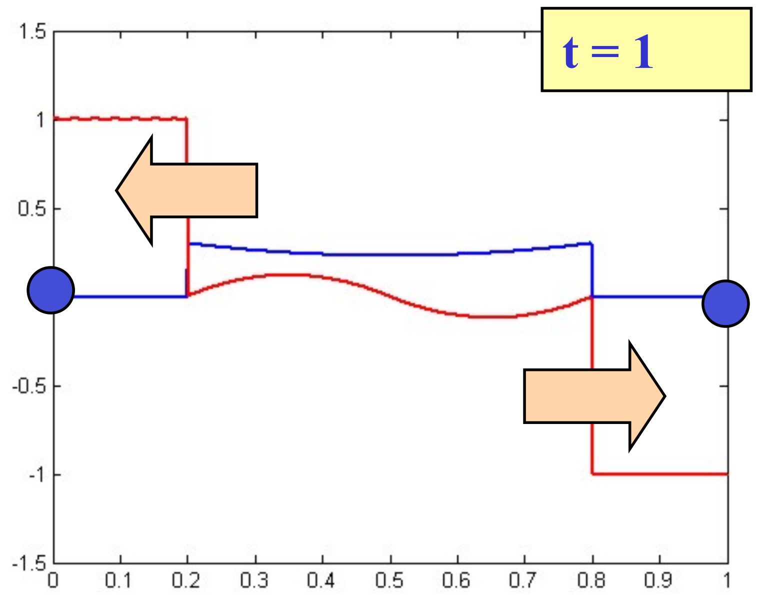
$$\frac{d}{dt}(\alpha\eta - u) = -v \quad \text{on} \quad \frac{dx}{dt} = -\alpha,$$

$$\frac{d}{dt}(\alpha\eta + u) = v \quad \text{on} \quad \frac{dx}{dt} = \alpha,$$

$$\frac{dv}{dt} = -u \quad \text{on} \quad \frac{dx}{dt} = 0,$$

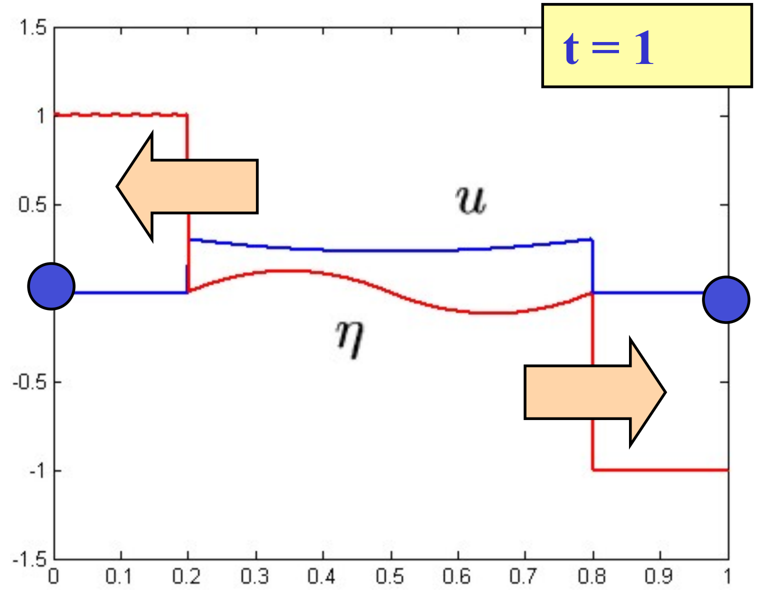
Riemann's Invariants

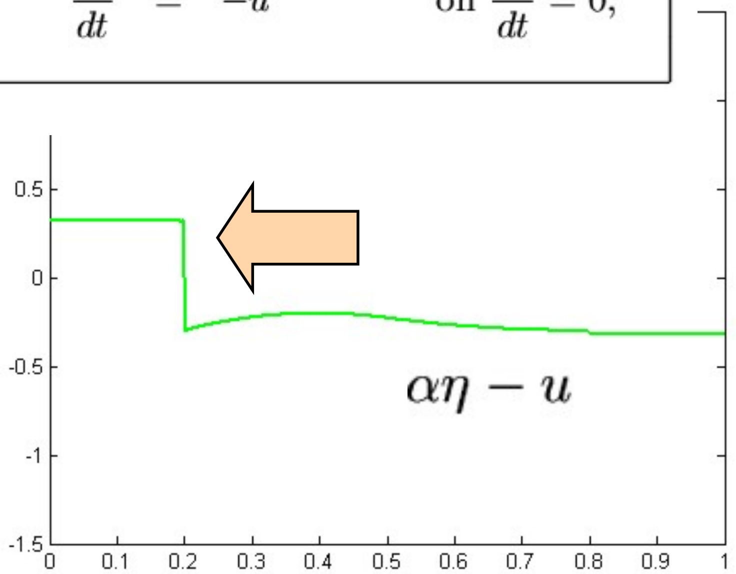


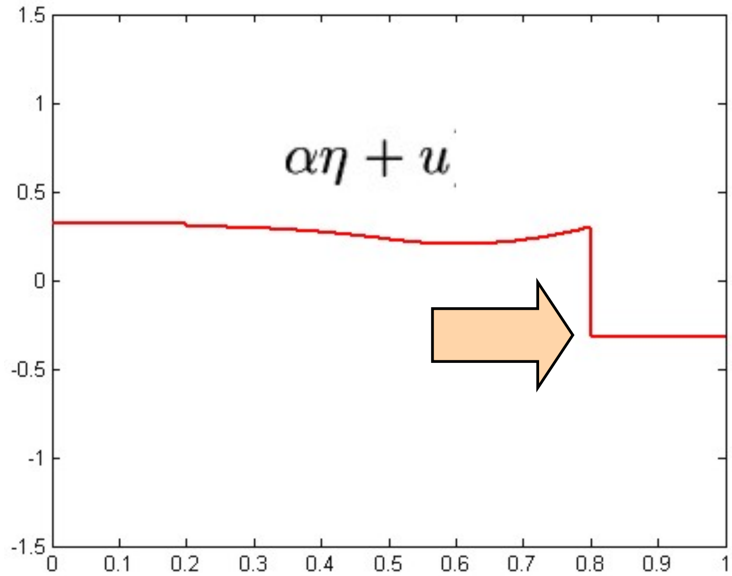


Two distinct waves…

$$
\begin{aligned}
\frac{d}{dt}(\alpha\eta - u) &= -v &\quad \text{on } \frac{dx}{dt} = -\alpha,\\
\frac{d}{dt}(\alpha\eta + u) &= v &\quad \text{on } \frac{dx}{dt} = \alpha,\\
\frac{dv}{dt} &= -u &\quad \text{on } \frac{dx}{dt} = 0,
\end{aligned}
$$

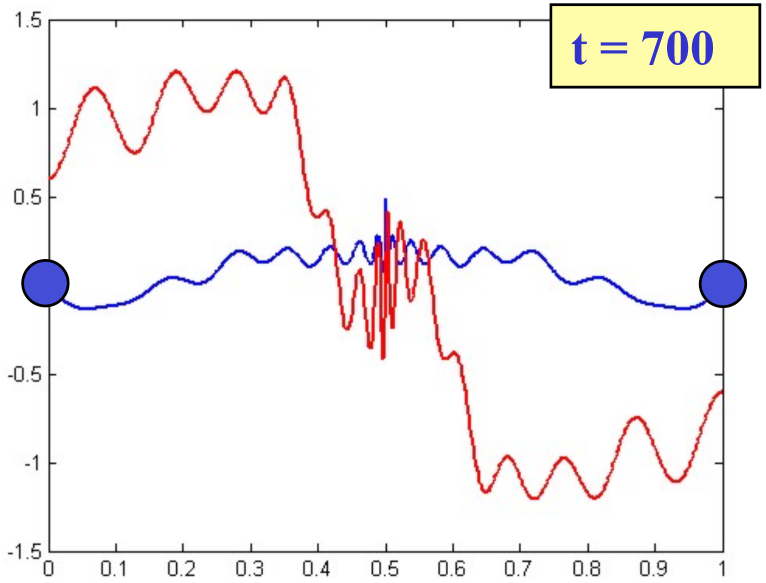






An analytical solution exists !

$$\frac{\partial^2 u}{\partial t^2} + u - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$



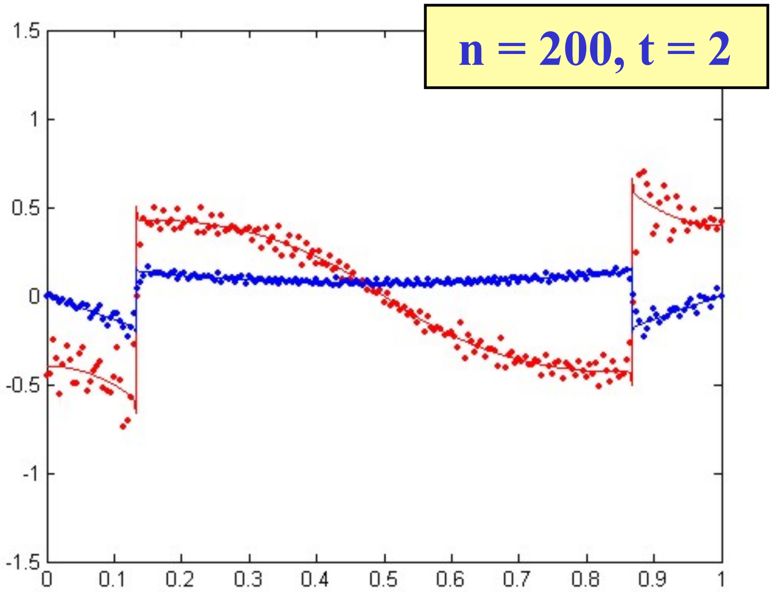
Separation of the Classical Equations with the boundary conditions

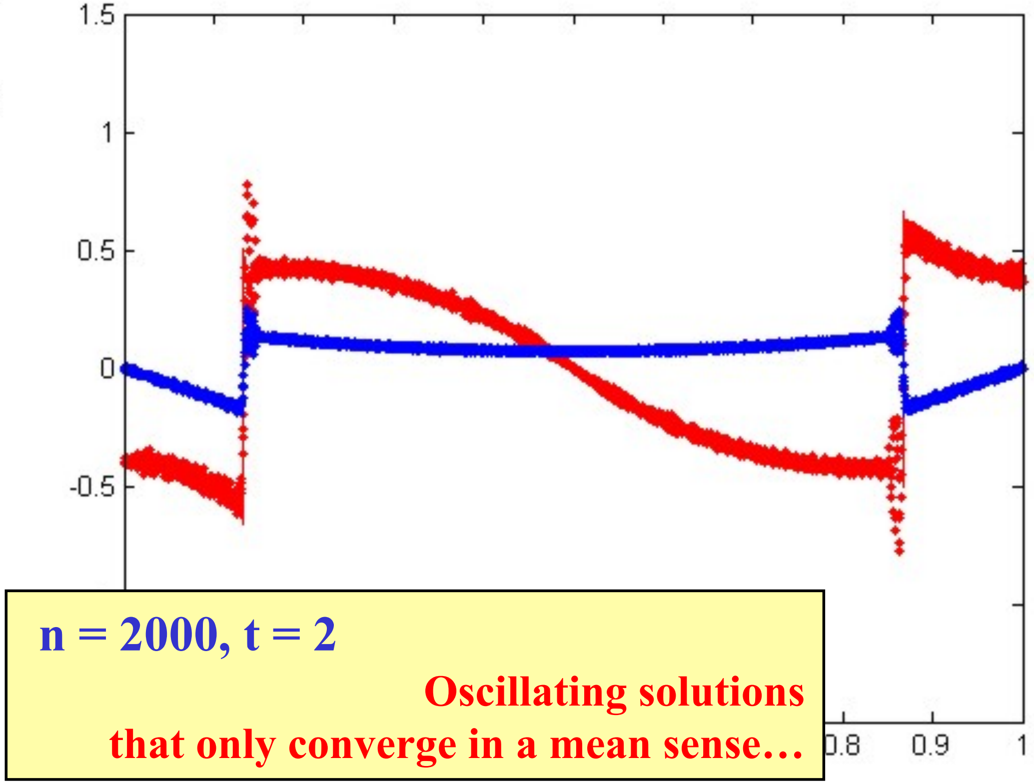
$$u(x,t) = T(t)f(x)$$

$$\frac{T''}{T} = \alpha^2 \frac{f''}{f} - 1$$

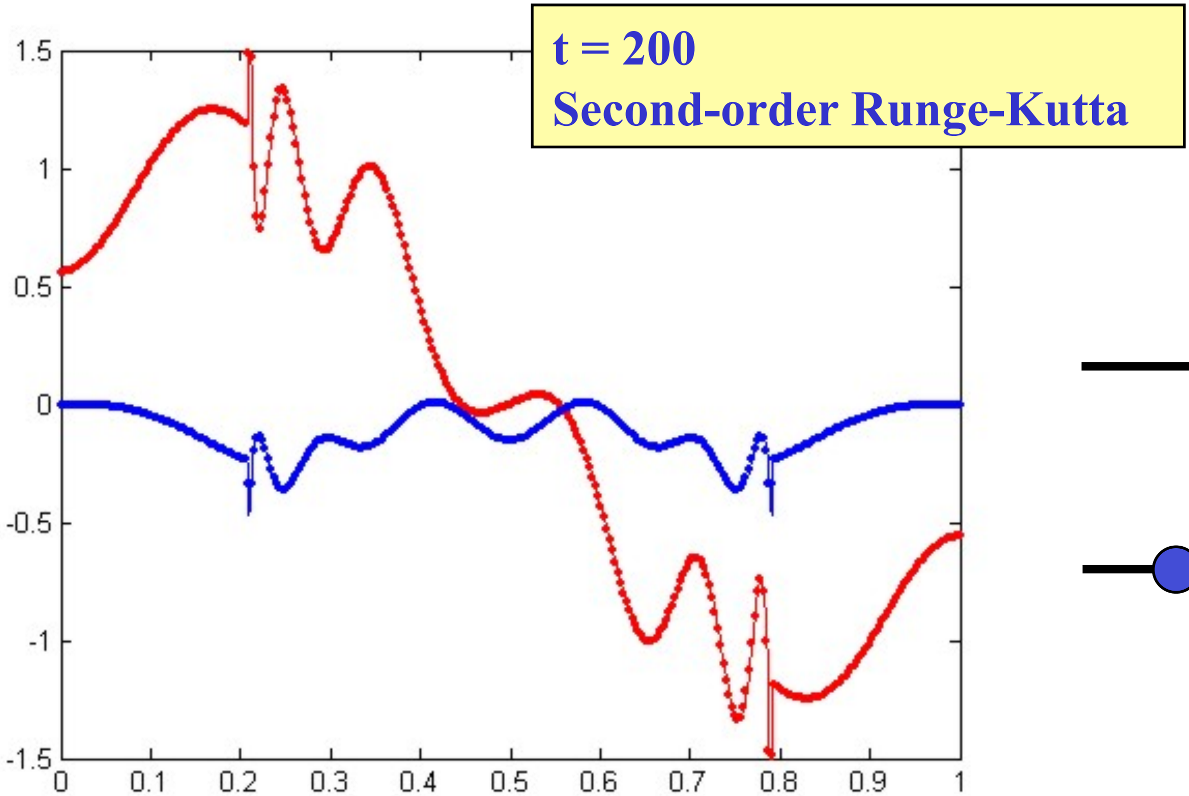
$$\begin{aligned} u(x,t) &= \sum_{i=1}^{\infty} \frac{4\alpha^2(-1)^{i+1}}{\omega_i} \sin(\omega_i t) \sin(k_i x) \\ k_i &= (2i-1)\pi \\ \omega_i &= \sqrt{1+\alpha^2 k_i^2} \end{aligned}$$

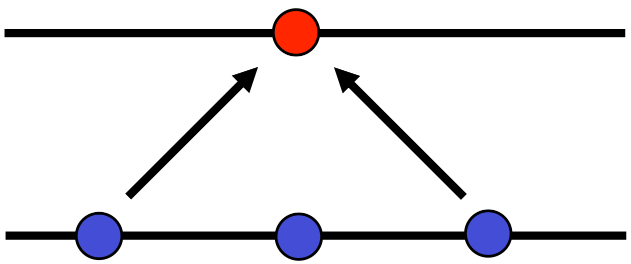
The Continuous Galerkin Method



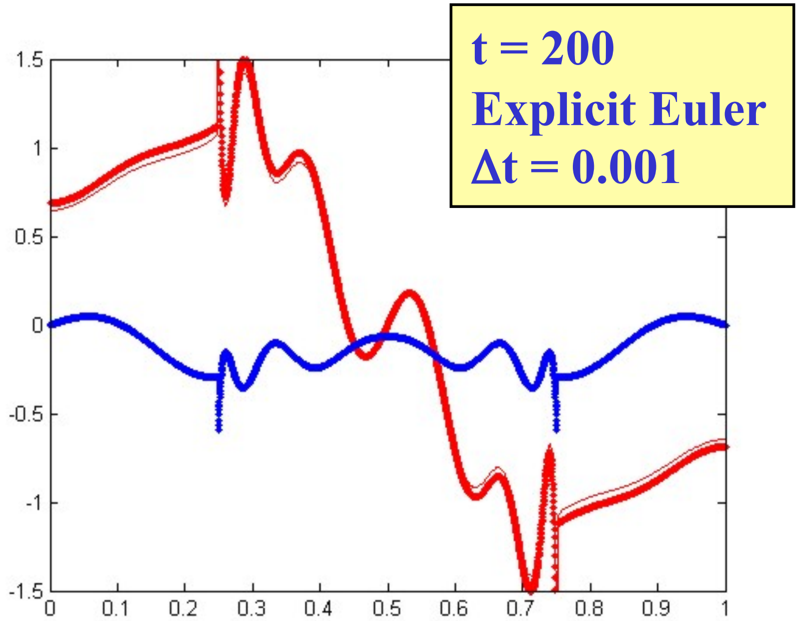


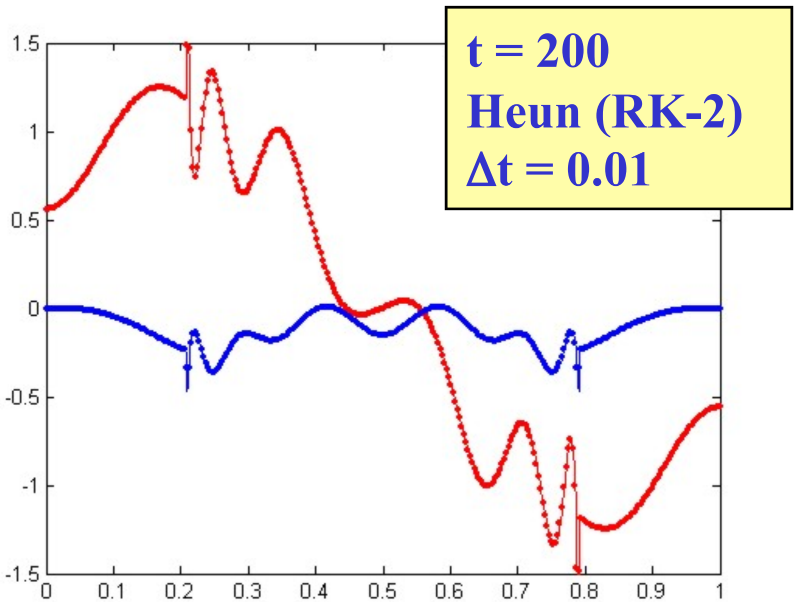
The Optimal Technique : Integrating along characteristics

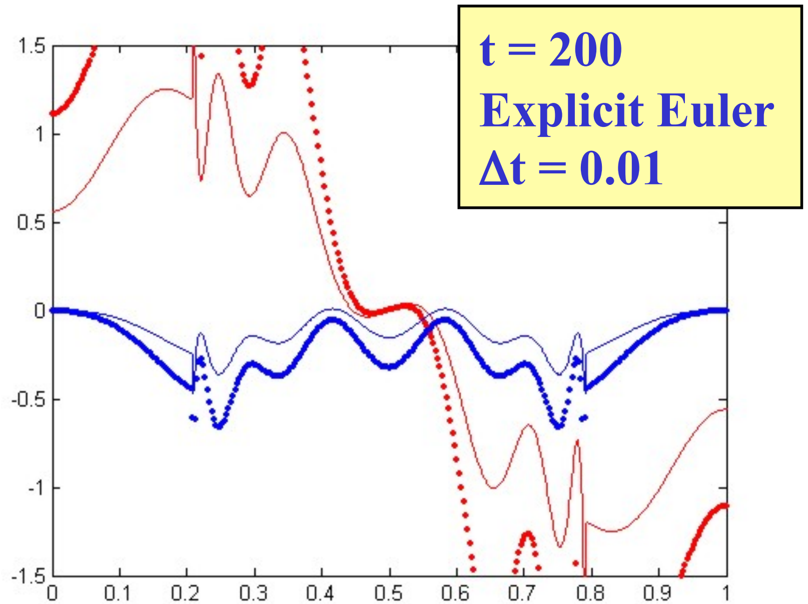




Time integration has to be accurately performed…







The Discontinuous so-called Galerkin Method

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

Find $u^h \in \mathcal{U}^h$ such that

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial u^h}{\partial t} \hat{u}^h + \frac{\partial u^h}{\partial x} \hat{u}^h \right) dx + \sum_{e=1}^{N_E} \Big[a(\hat{u}^h)[u^h] \Big]_{\Omega_e} = 0 \qquad \forall \hat{u}^h \in \hat{\mathcal{U}}^h,$$

Penalty term to enforce weak continuity of the solution

$$a(\hat{u}^h) = \zeta \left(1 - \lambda \, \text{sign}(n)\right) \hat{u}^h$$

Penalty Factor (usually = 1)

Upwinding Factor (usually = 1)

After some tedious algebra…

Find $u^h \in \mathcal{U}^h$ such that

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial u^h}{\partial t} \hat{u}^h + \frac{\partial u^h}{\partial x} \hat{u}^h \right) dx + \sum_{e=1}^{N_E} \Big[\, a(\hat{u}^h)[u^h] \,\Big]_{\Omega_e} \quad = \quad 0 \qquad \forall \hat{u}^h \in \hat{\mathcal{U}}^h,$$

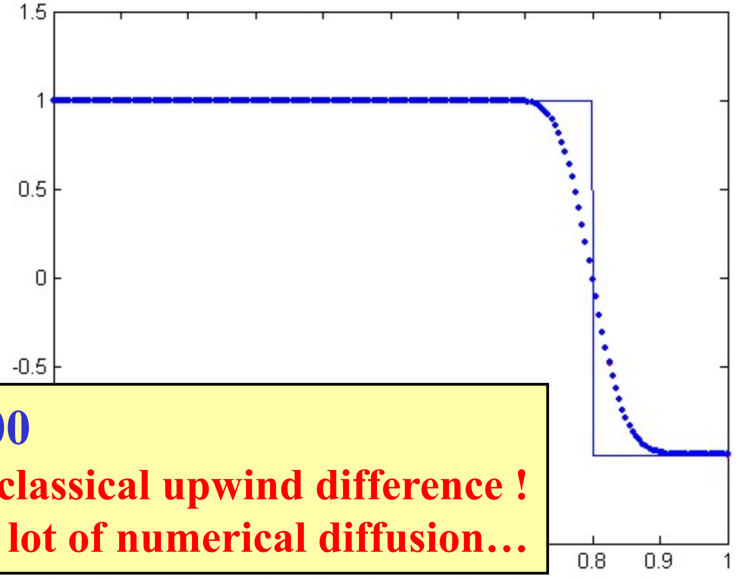
Considering only once integrals along internal segments.

Find $u^h \in \mathcal{U}^h$ such that

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial u^h}{\partial t} \hat{u}^h - u^h \frac{\partial \hat{u}^h}{\partial x} \right) dx$$

$$+ \sum_{i=2}^{N_E} < u^h(X_i) >_\lambda [\hat{u}^h(X_i)] \quad = \quad 0 \qquad \forall \hat{u}^h \in \hat{\mathcal{U}}^h,$$

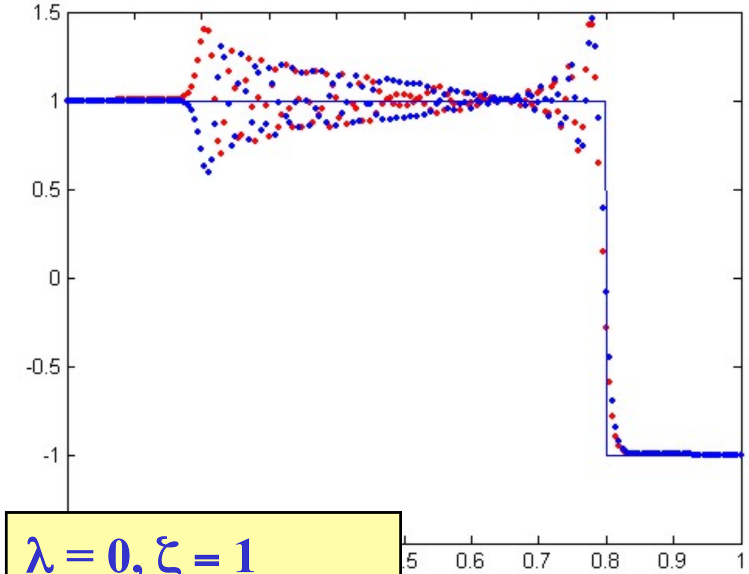
How to impose the continuity constraint ?



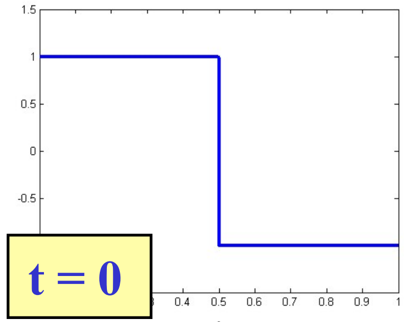
$\lambda = 1, \zeta = 100$

Almost the classical upwind difference !

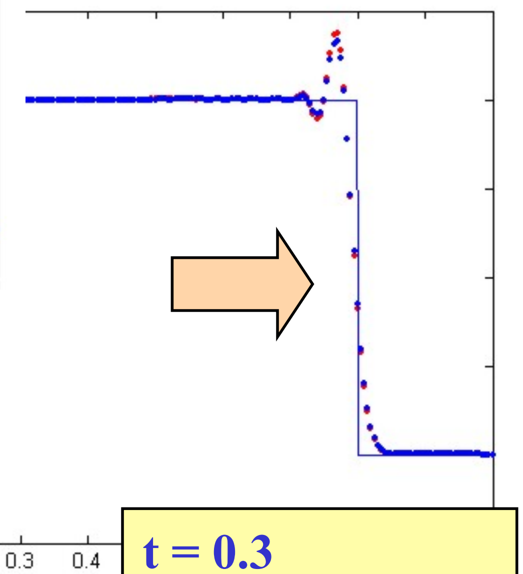
Stable, but a lot of numerical diffusion…



$\lambda = 0, \zeta = 1$



t = 0



t = 0.3

$\lambda = 1, \zeta = 1$

The Discontinuous Galerkin Method

Find $\eta \in \mathcal{E}$ and $(u, v) \in \mathcal{U} \times \mathcal{U}$ such that

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial \eta}{\partial t} \hat{\eta} + \frac{\partial u}{\partial x} \hat{\eta} \right) dx + \sum_{e=1}^{N_E} \Big[a(\hat{\eta})[u] \Big]_{\Omega_e} = 0 \qquad \forall \hat{\eta} \in \mathcal{E},$$

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial u}{\partial t} \hat{u} + v\hat{u} + \alpha^2 \frac{\partial \eta}{\partial x} \hat{u} \right) dx + \sum_{e=1}^{N_E} \Big[b(\hat{u})[\alpha^2 \eta] \Big]_{\Omega_e} = 0 \qquad \forall \hat{u} \in \mathcal{U},$$

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial v}{\partial t} \hat{v} - u\hat{v} \right) dx = 0 \qquad \forall \hat{v} \in \mathcal{U},$$

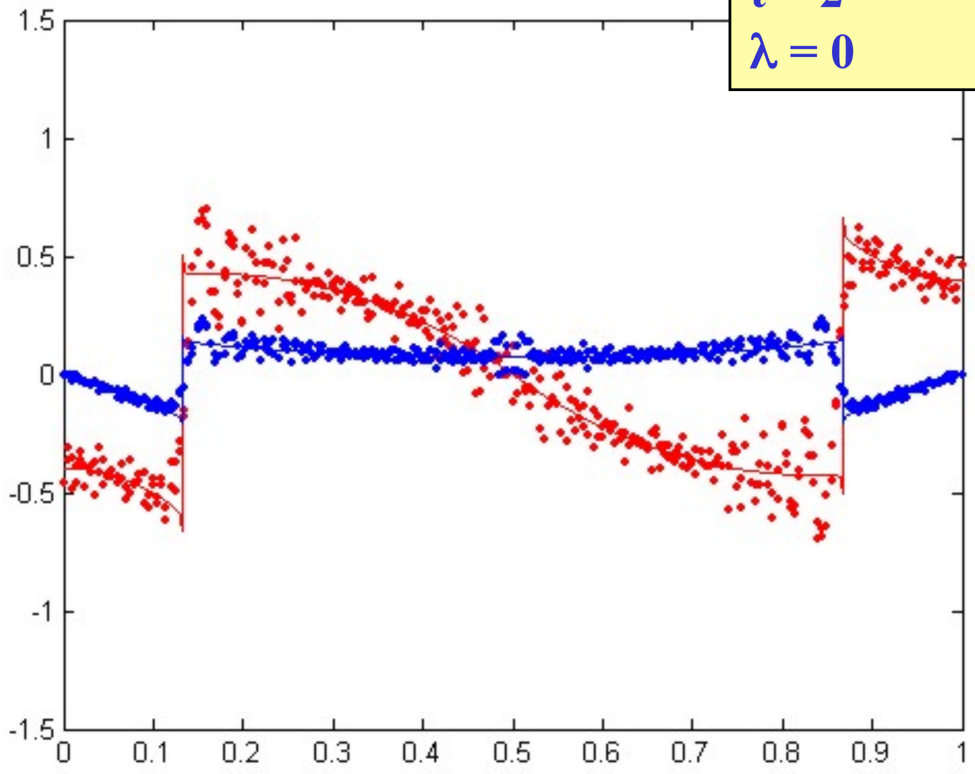
Penalty terms to enforce weak continuity of the solution

$$a(\hat{u}) = \left(1 - \lambda \, \text{sign}(n) \right) \hat{u}$$

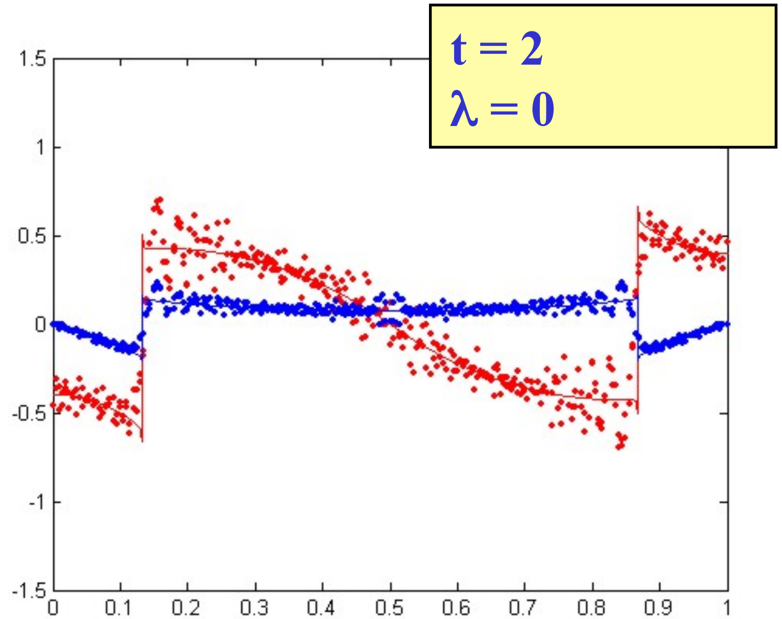
Upwinding Factor
(in fact, the best selection is = 0)

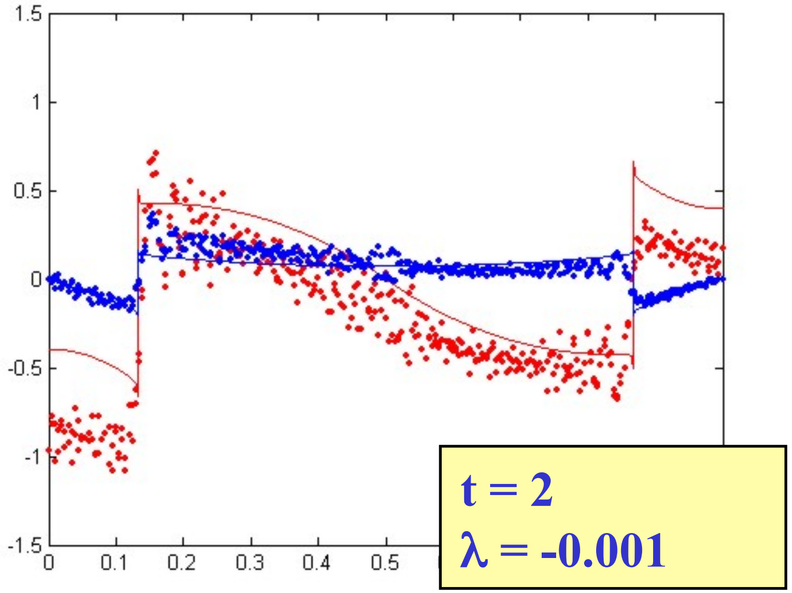
The Discontinuous Galerkin Method

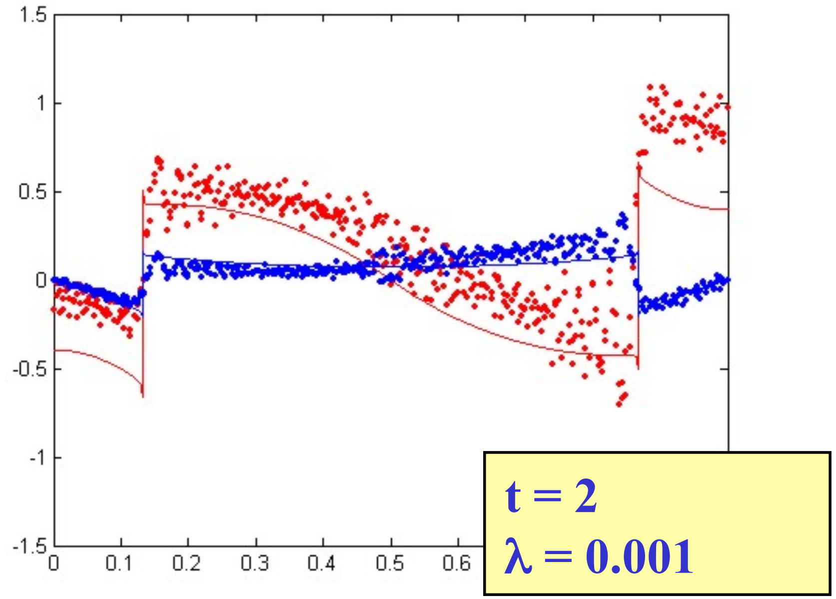
t = 2
λ = 0



How to impose continuity constraint ?







The Discontinuous Riemann-Galerkin Method

Penalty terms to enforce weak continuity of the Riemann's invariants

Find $\eta \in \mathcal{E}$ and $(u, v) \in \mathcal{U} \times \mathcal{U}$ such that

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial \eta}{\partial t} \hat{\eta} + \frac{\partial u}{\partial x} \hat{\eta} \right) dx + \sum_{e=1}^{N_E} \Big[a(\hat{\eta})[u + \alpha \eta] \Big]_{\Omega_e} + \sum_{e=1}^{N_E} \Big[b(\hat{\eta})[u - \alpha \eta] \Big]_{\Omega_e} = 0 \qquad \forall \hat{\eta} \in \mathcal{E},$$

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial u}{\partial t} \hat{u} + v \hat{u} + \alpha^2 \frac{\partial \eta}{\partial x} \hat{u} \right) dx + \sum_{e=1}^{N_E} \Big[a(\hat{u})[\alpha^2 \eta + \alpha u] \Big]_{\Omega_e} + \sum_{e=1}^{N_E} \Big[b(\hat{u})[\alpha^2 \eta - \alpha u] \Big]_{\Omega_e} = 0 \qquad \forall \hat{u} \in \mathcal{U},$$

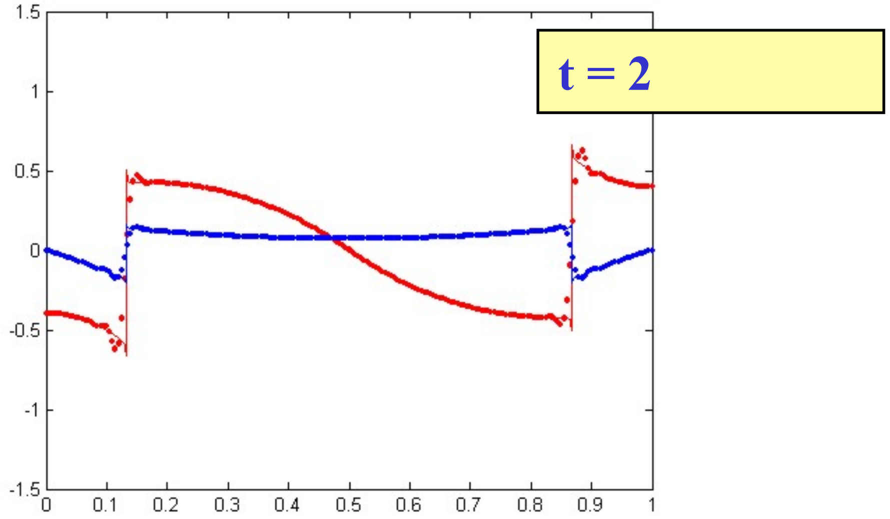
$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial v}{\partial t} \hat{v} - u \hat{v} \right) dx = 0 \qquad \forall \hat{v} \in \mathcal{U},$$

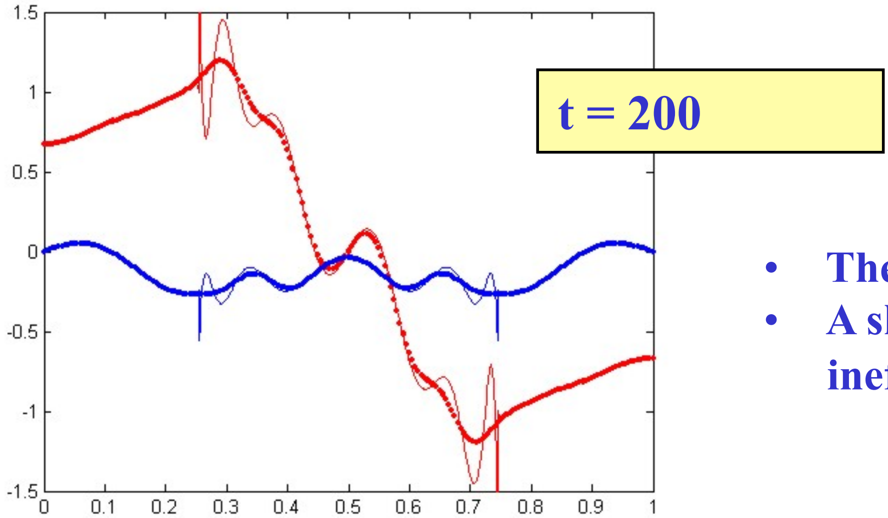
$$a(\hat{u}) = \Big(1 - \lambda \, \text{sign}(n) \Big) \hat{u} \qquad\qquad b(\hat{u}) = \Big(\lambda \, \text{sign}(n) + 1 \Big) \hat{u}$$

Backward Upwinding

Forward Upwinding

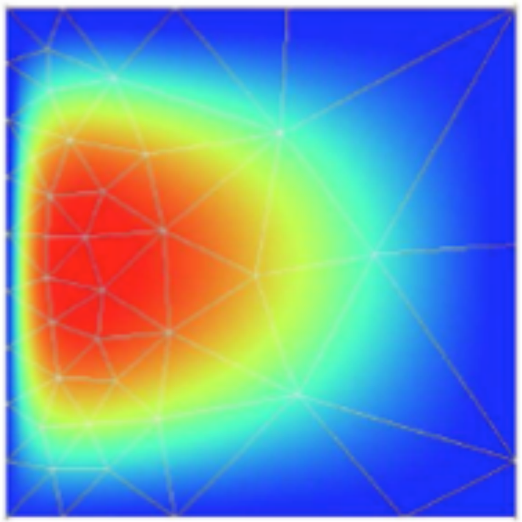
DG Method works !

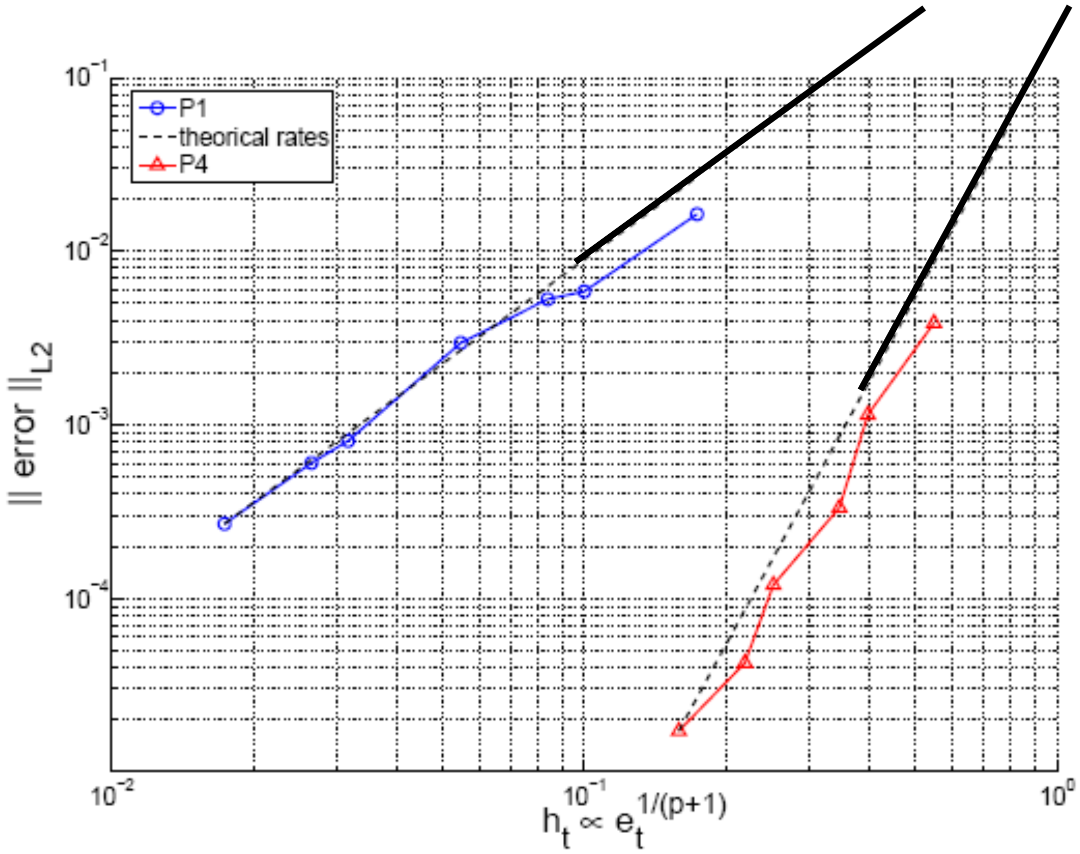




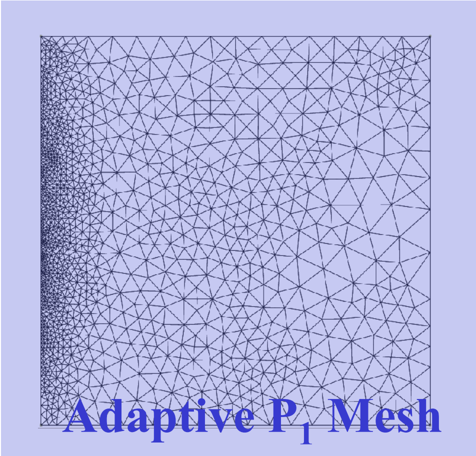
- **The use of a good Riemann solver is mandatory !**
- **A sharp problem is needed to discriminate inefficient or unstable numerical techniques**

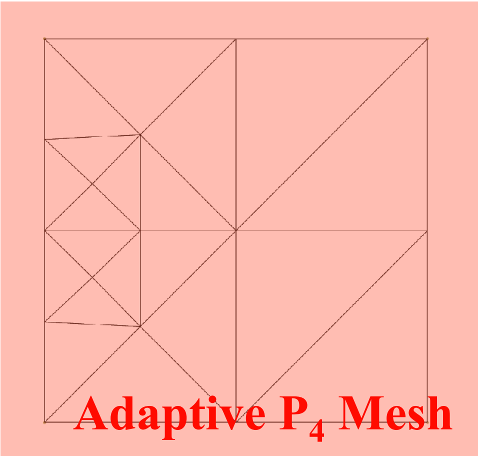
Theoretical rates of convergence are obtained for the analytical Stommel problem

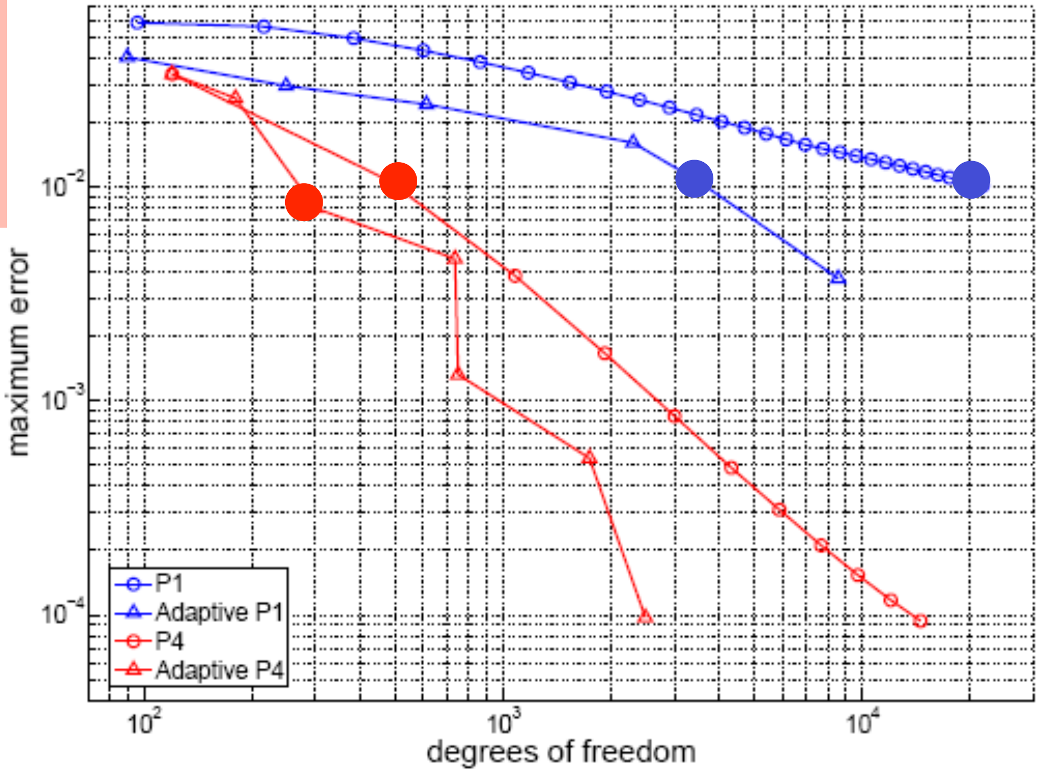




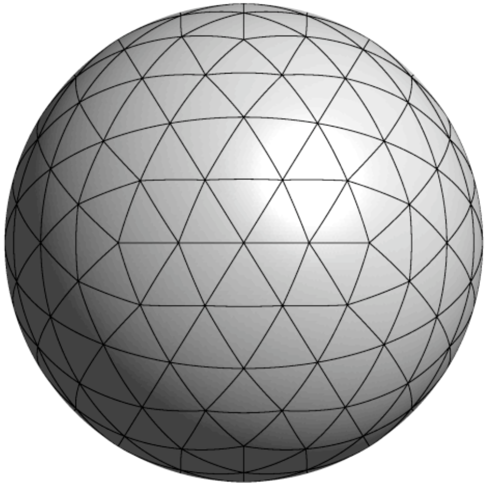
How does it converge ?

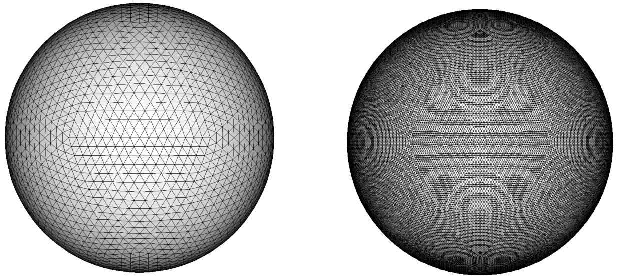


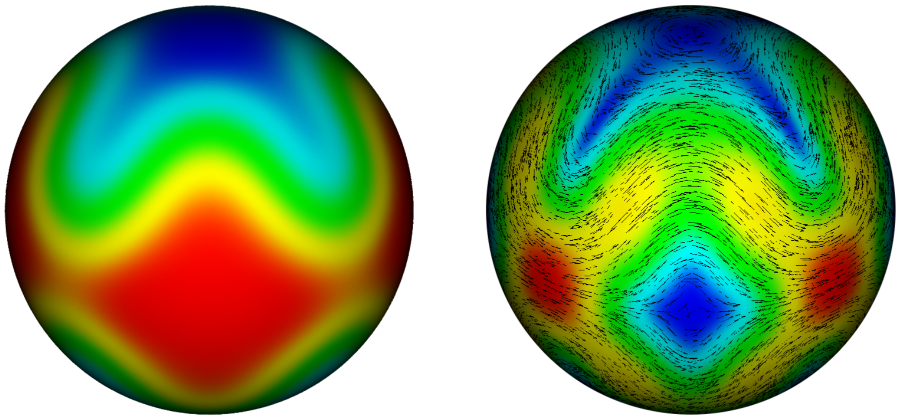




High-order versus low-order meshes

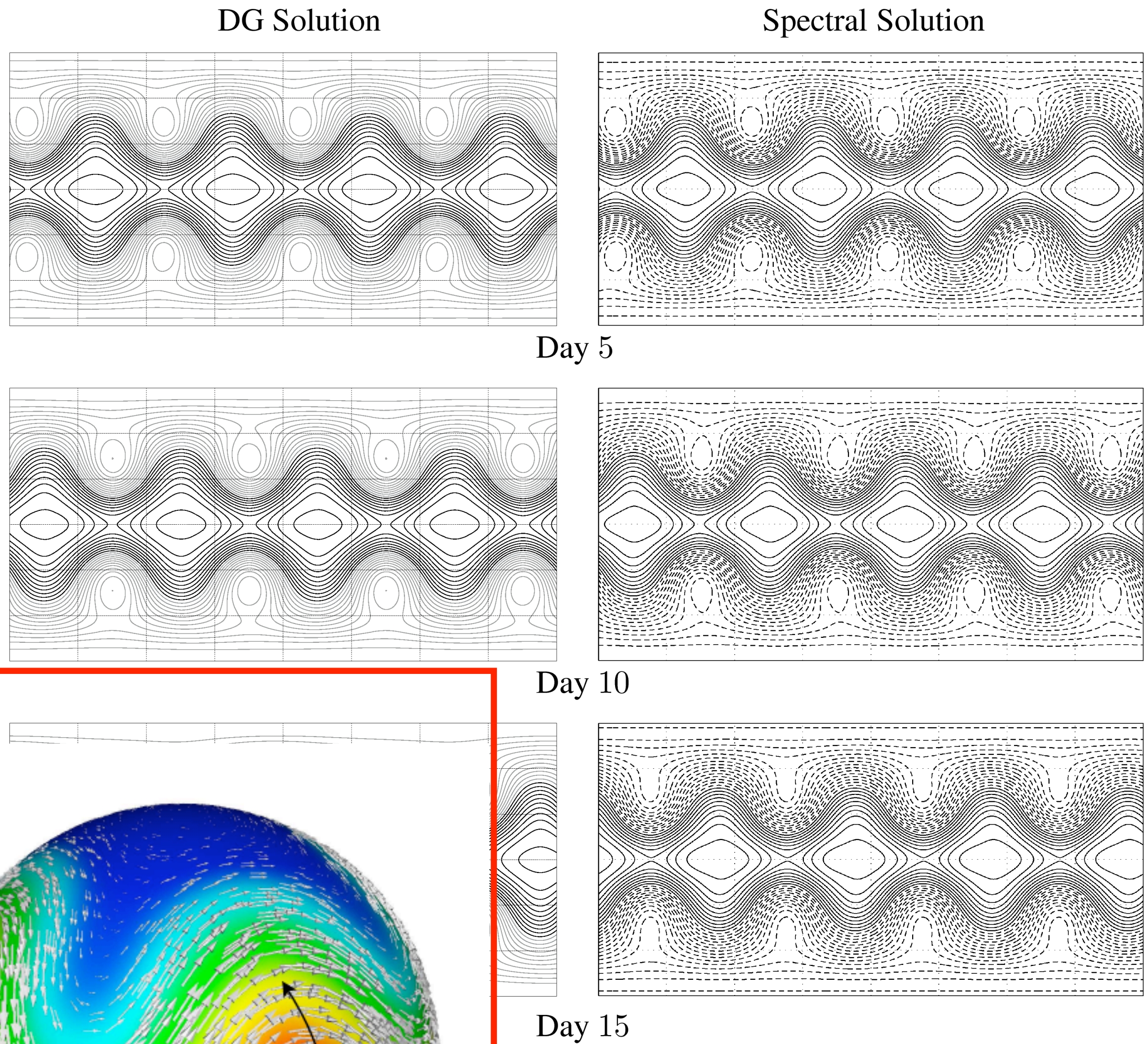






Unsteady balance between pressure term and Coriolis force

Global Rossby-Hauritz waves



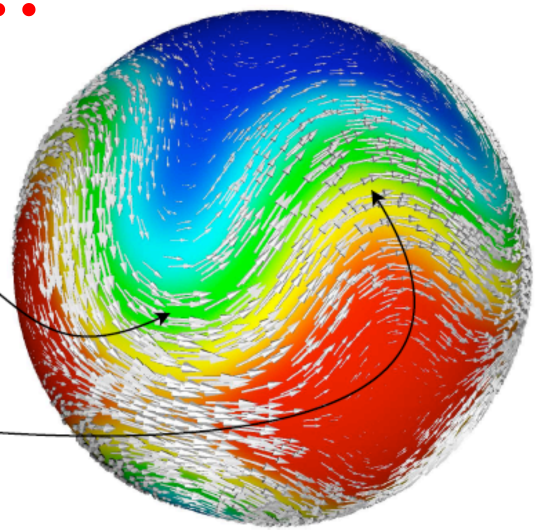
Spectral Transform Method [Jakob-Chien et al. (1995)]

Intuitively...

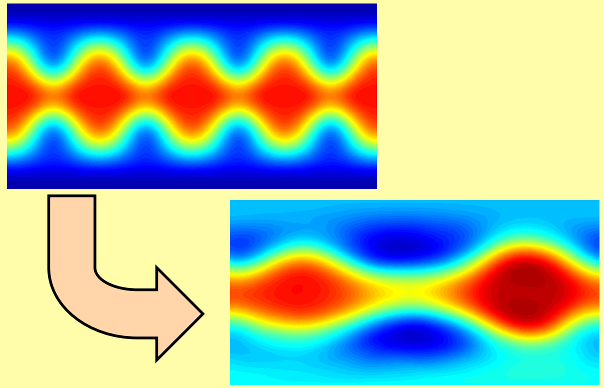
Coriolis vs elevation gradient

Elevation gradient dominating: the flow goes leftward

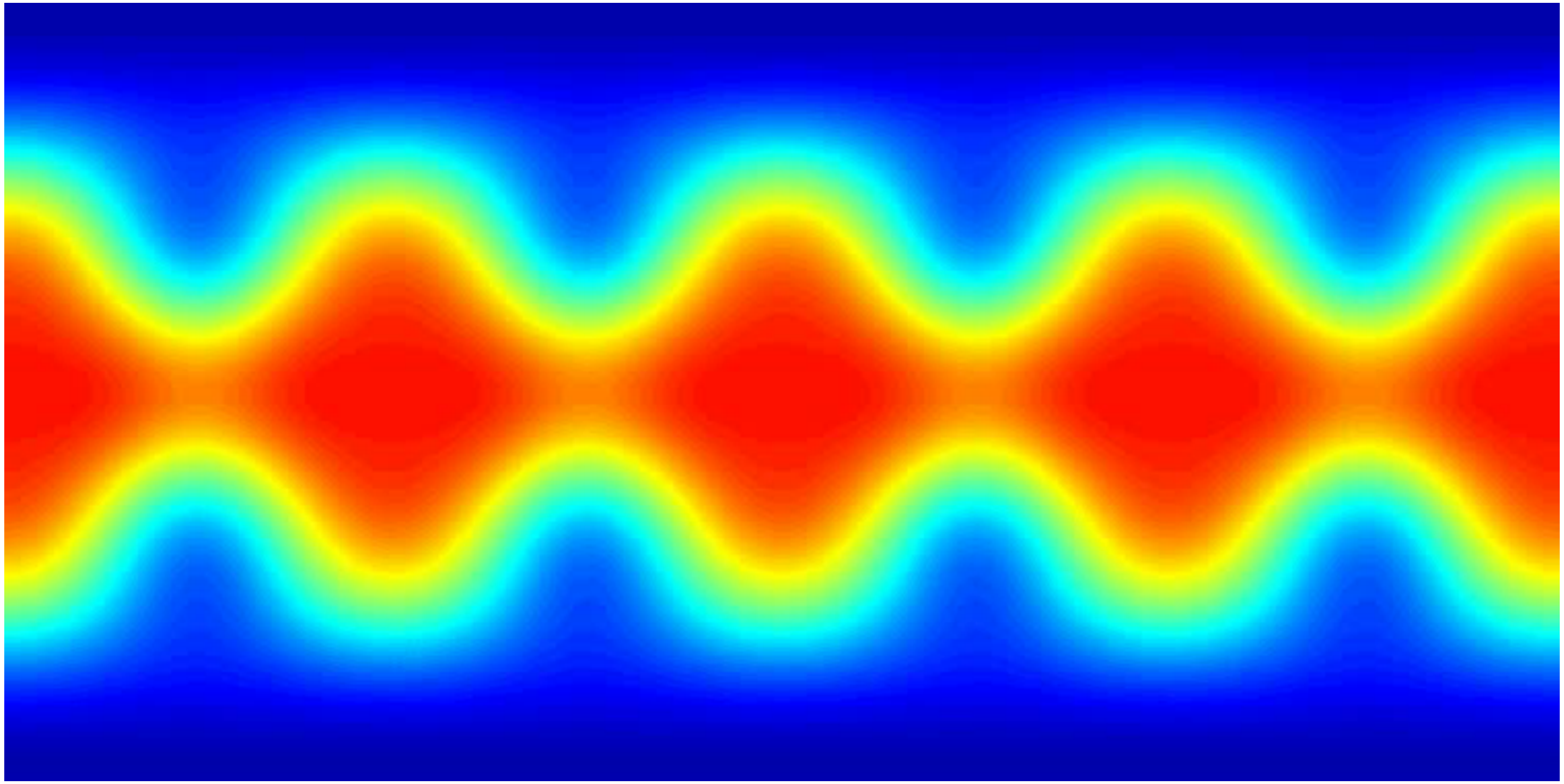
Coriolis term dominating: the flow goes rightward



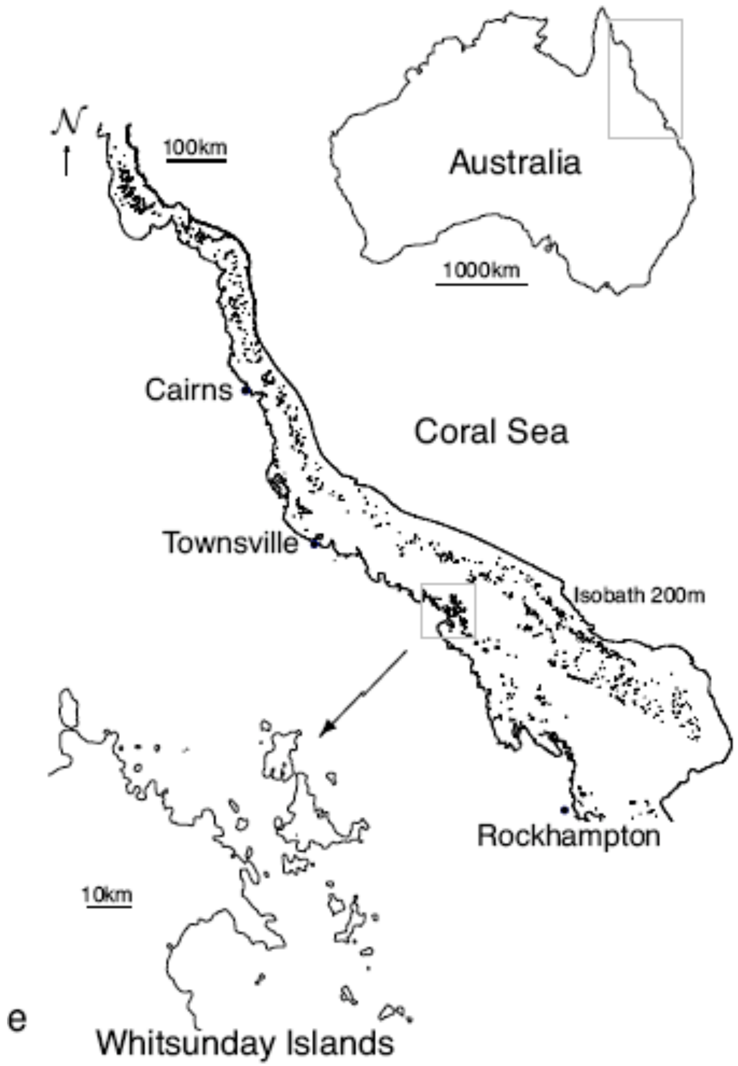
And the flow becomes instable...

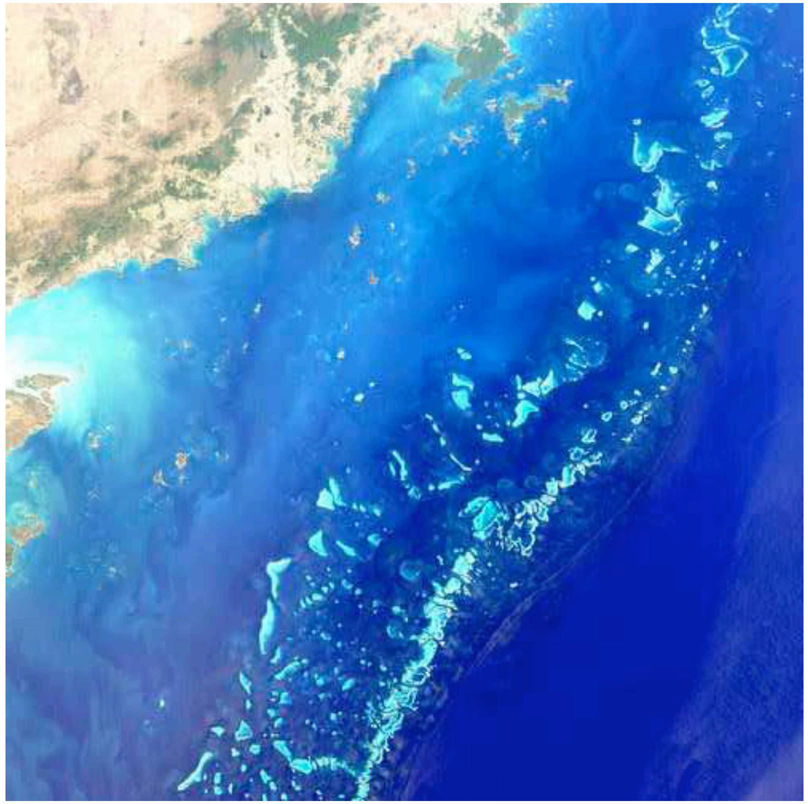


Finally, a pattern characterized by a wave number of two appears!

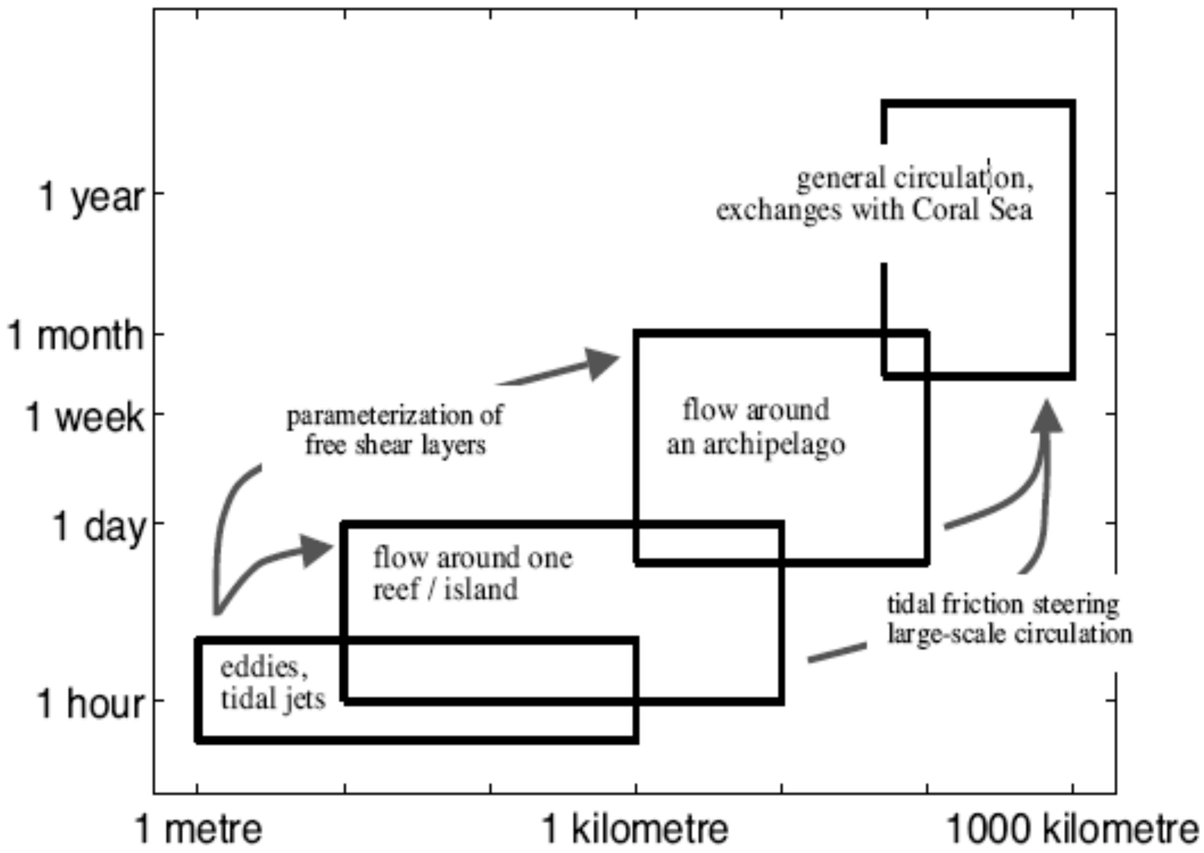


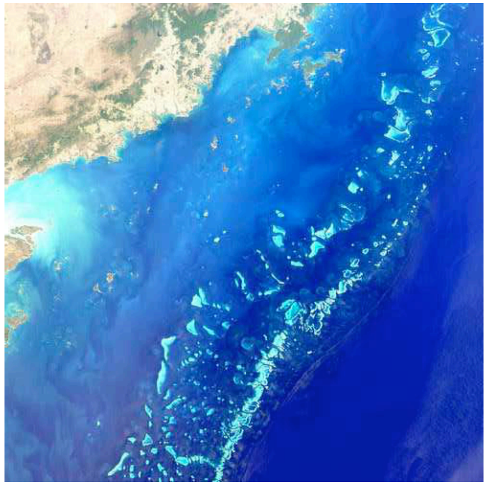
Multi-scale modelling of the Great Barreer Reef (Australia)





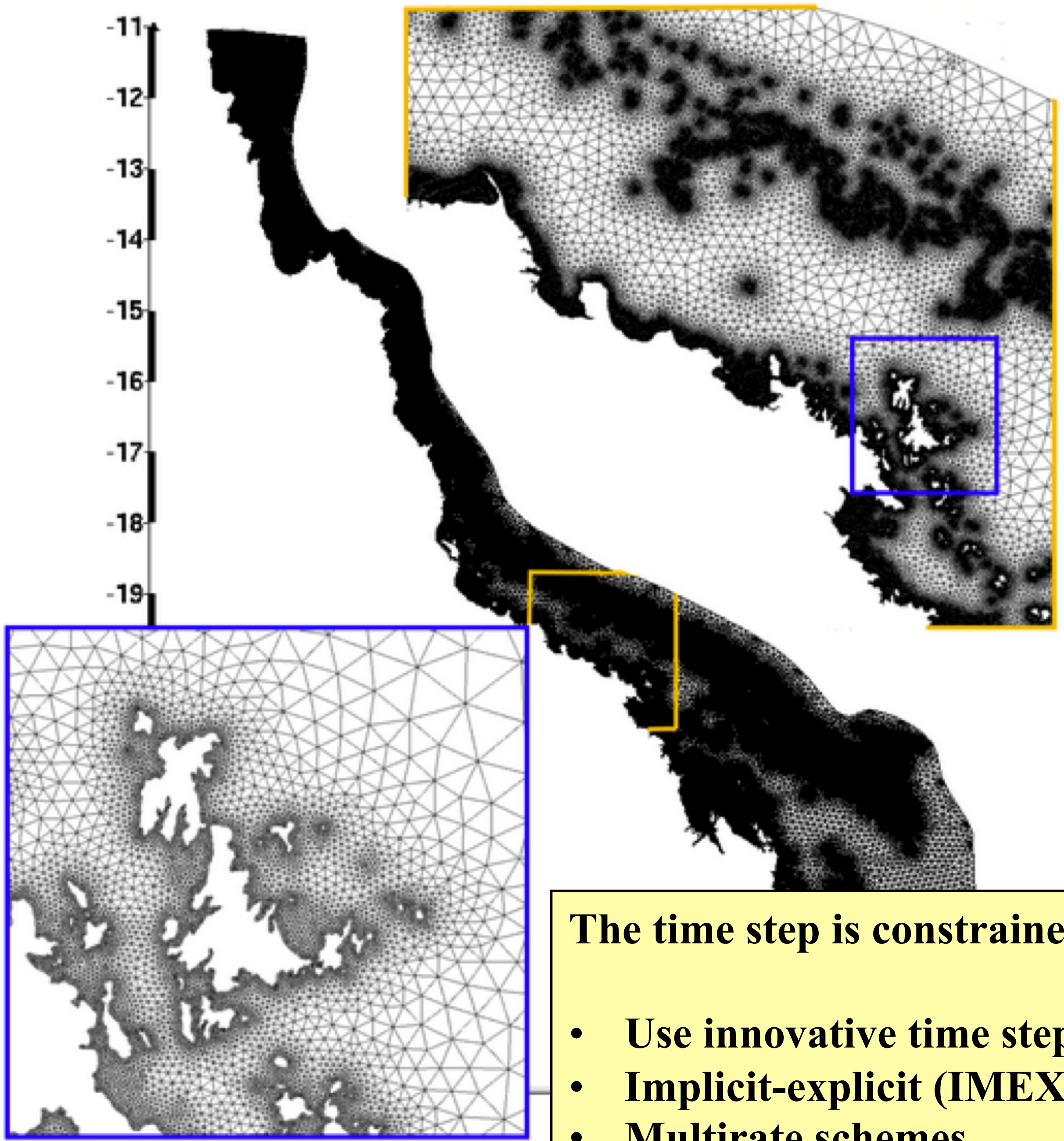
Time-space scales





- ***Forcings :***
  wind, tides, Coral Sea inflow
- ***Wide spectrum of hydrodynamics processes simulated :***
  eddies, tidal jets, sticky waters, general circulation

The time stepping issue



- *890,000 triangles*
- *Smallest element : 7 m*
- *Largest element : 3,300 m*
- *99.9 % > 60m*

The time step is constrained by the smallest element .

- **Use innovative time stepping procedures**
- **Implicit-explicit (IMEX) schemes**
- **Multirate schemes**

Reduce cost by 1000 !
Use high performance computers !

10 Gflops
2 processors

1.759 Pflops
224,162 processors

- **Exploit single precision BLAS/LAPACK for the efficient implementation of the explicit and implicit discontinuous Galerkin methods.**
- **Implement new time-integration procedures adapting the time step to the physical processes.**
- **Introduce multi-level methods for the implicit linear and non-linear solvers with multigrid methods as a preconditioner for stiff, non-linear and non-positive-definite systems.**

Each route could reduce the computational cost by one order of magnitude.

Quotes by (other) famous simulaters

- As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality. **Albert Einstein**
- Everything is vague to a degree you do not realize till you have tried to make it precise. **Bertrand Russell**
- In these matters the only certainty is that nothing is certain. **Pliny the Elder**
- However beautiful the strategy, you should occasionally look at the results. **Sir Winston Churchill**