FORMULAIRE

1. Vecteurs et tenseurs

Coordonnées cartésiennes: $x_1,\ x_2,\ x_3$

Vecteurs de la base orthonormée associée: $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$, avec $|\mathbf{e}_1| = |\mathbf{e}_2| = |\mathbf{e}_3| = 1$

Composantes du vecteur **a**: $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$

Norme du vecteur **a**: $|\mathbf{a}| = \left(a_1^2 + a_2^2 + a_3^2\right)^{1/2}$

Produit scalaire des vecteurs **a** et **b**: $\mathbf{a} \bullet \mathbf{b} \ = \ a_1b_1 + a_2b_2 + a_3b_3$

Produit vectoriel des vecteurs **a** et **b**:
$\mathbf{a} \times \mathbf{b} \ = \ (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3$

Double produit vectoriel: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \ = \ (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$

Produits scalaire et vectoriel: $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) \ = \ \mathbf{b} \bullet (\mathbf{c} \times \mathbf{a}) \ = \ \mathbf{c} \bullet (\mathbf{a} \times \mathbf{b})$
$(\mathbf{a} \times \mathbf{b}) \bullet (\mathbf{c} \times \mathbf{d}) \ = \ (\mathbf{a} \bullet \mathbf{c})(\mathbf{b} \bullet \mathbf{d}) - (\mathbf{a} \bullet \mathbf{d})(\mathbf{b} \bullet \mathbf{c})$

Tenseur **A**: $\mathbf{A} = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix}$ (matrice associée au tenseur)

Produits scalaires entre le vecteur **a** et le tenseur **A**:

$$\mathbf{a} \bullet \mathbf{A} \ = \ \sum_{i=1}^{3} \sum_{j=1}^{3} a_i A_{i,j} \mathbf{e}_j \quad \text{ou} \quad \mathbf{A} \bullet \mathbf{a} \ = \ \sum_{i=1}^{3} \sum_{j=1}^{3} A_{i,j} a_j \mathbf{e}_i$$

Produit "tensoriel" entre les vecteurs **a** et **b**: $\mathbf{ab} = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix}$

Produits tensoriel et scalaire: $(\mathbf{a} \bullet \mathbf{b})\mathbf{c} = \mathbf{a} \bullet \mathbf{bc} = \mathbf{a} \bullet (\mathbf{bc})$

Double produit scalaire entre les tenseurs **A** et **B**: $\mathbf{A} : \mathbf{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{i,j} B_{i,j}$

2. Opérateurs différentiels en coordonnées cartésiennes

Divergence du vecteur **a**: $\nabla \bullet \mathbf{a} = \dfrac{\partial a_1}{\partial x_1} + \dfrac{\partial a_2}{\partial x_2} + \dfrac{\partial a_3}{\partial x_3}$

Divergence du tenseur **A**: $\nabla \bullet \mathbf{A} = \sum_{i=1}^{3} \sum_{j=1}^{3} \dfrac{\partial A_{i,j}}{\partial x_j} \mathbf{e}_i$

Divergence du produit tensoriel $\mathbf{ab}$: $\nabla \bullet (\mathbf{ab}) = (\nabla \bullet \mathbf{b})\mathbf{a} + (\nabla \mathbf{a})\mathbf{b} = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(a_i b_j)\mathbf{e}_i$

Gradient du scalaire c: $\nabla c = \frac{\partial c}{\partial x_1}\mathbf{e}_1 + \frac{\partial c}{\partial x_2}\mathbf{e}_2 + \frac{\partial c}{\partial x_3}\mathbf{e}_3$

Gradient du vecteur $\mathbf{a}$: $\nabla \mathbf{a} = \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{pmatrix}$ (matrice associée au tenseur)

Rotationnel du vecteur $\mathbf{a}$: $\nabla \times \mathbf{a} = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathbf{e}_3$

Laplacien du scalaire c: $\nabla^2 c = \frac{\partial^2 c}{\partial x_1^2} + \frac{\partial^2 c}{\partial x_2^2} + \frac{\partial^2 c}{\partial x_3^2}$

Laplacien du vecteur $\mathbf{a}$: $\nabla^2 \mathbf{a} = (\nabla^2 a_1)\mathbf{e}_1 + (\nabla^2 a_2)\mathbf{e}_2 + (\nabla^2 a_3)\mathbf{e}_3 = \nabla(\nabla \bullet \mathbf{a}) - \nabla \times (\nabla \times \mathbf{a})$

Quelques formules utiles:

$$\nabla \times \nabla c = 0$$
$$\nabla \bullet (\nabla \times \mathbf{a}) = 0$$
$$\nabla \bullet (c\mathbf{a}) = \mathbf{a} \bullet \nabla c + c \nabla \bullet \mathbf{a}$$
$$\nabla \times (c\mathbf{a}) = \nabla c \times \mathbf{a} + c \nabla \times \mathbf{a}$$
$$\mathbf{a} \bullet \nabla \mathbf{b} = a_1 \frac{\partial \mathbf{b}}{\partial x_1} + a_2 \frac{\partial \mathbf{b}}{\partial x_2} + a_3 \frac{\partial \mathbf{b}}{\partial x_3} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_j \frac{\partial b_i}{\partial x_j}\mathbf{e}_i$$
$$\nabla(\mathbf{a} \bullet \mathbf{b}) = \mathbf{a} \bullet \nabla \mathbf{b} + \mathbf{b} \bullet \nabla \mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$
$$\nabla \bullet (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \bullet (\nabla \times \mathbf{a}) - \mathbf{a} \bullet (\nabla \times \mathbf{b})$$
$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \bullet \mathbf{b})\mathbf{a} - (\nabla \bullet \mathbf{a})\mathbf{b} + \mathbf{b} \bullet \nabla \mathbf{a} - \mathbf{a} \bullet \nabla \mathbf{b}$$

Vecteur position (en 3 dimensions): $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$

$$\nabla \bullet \mathbf{x} = 3, \quad \nabla \times \mathbf{x} = 0, \quad \nabla \bullet \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{2}{|\mathbf{x}|}$$

3. Intégrales

V: volume limité par la surface S, de normale extérieure unitaire $\mathbf{n}$ ($|\mathbf{n}|=1$); t: temps;
$\mathbf{u}^S$: vitesse des points de la surface S

$$\int_V \nabla \bullet \mathbf{a} \; dV = \int_S \mathbf{a} \bullet \mathbf{n} \; dS$$

$$\int_V \nabla \times \mathbf{a} \; dV = \int_S \mathbf{n} \times \mathbf{a} \; dS$$

$$\int_V \nabla c \; dV = \int_S c\mathbf{n} \; dS$$

$$\frac{d}{dt}\int_V c \; dV = \int_V \frac{\partial c}{\partial t} \; dV + \int_S c\mathbf{u}^S \bullet \mathbf{n} \; dS$$

$$\frac{d}{dt}\int_V \mathbf{a} \; dV = \int_V \frac{\partial \mathbf{a}}{\partial t} \; dV + \int_S \mathbf{a}(\mathbf{u}^S \bullet \mathbf{n}) \; dS$$

4. Coordonnées curvilignes orthogonales

Coordonnées cartésiennes: (x_1, x_2, x_3); vecteurs orthonormés associés: $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$

Coordonnées curvilignes orthogonales: $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$; vecteurs orthonormés associés: $(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)$

Coefficients métriques: (h_1, h_2, h_3), avec $\dfrac{\partial \mathbf{x}}{\partial \tilde{x}_i} = h_i \tilde{\mathbf{e}}_i \quad (i = 1,2,3)$

Composantes d'un vecteur:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \tilde{a}_1\tilde{\mathbf{e}}_1 + \tilde{a}_2\tilde{\mathbf{e}}_2 + \tilde{a}_3\tilde{\mathbf{e}}_3 \;, \quad (a_i = \mathbf{a} \bullet \mathbf{e}_i, \; \tilde{a}_i = \mathbf{a} \bullet \tilde{\mathbf{e}}_i, \; i = 1,2,3)$$

Gradient: $\nabla c = \dfrac{\tilde{\mathbf{e}}_1}{h_1}\dfrac{\partial c}{\partial \tilde{x}_1} + \dfrac{\tilde{\mathbf{e}}_2}{h_2}\dfrac{\partial c}{\partial \tilde{x}_2} + \dfrac{\tilde{\mathbf{e}}_3}{h_3}\dfrac{\partial c}{\partial \tilde{x}_3}$

Divergence: $\nabla \bullet \mathbf{a} = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial \tilde{x}_1}(h_2h_3\tilde{a}_1) + \dfrac{\partial}{\partial \tilde{x}_2}(h_3h_1\tilde{a}_2) + \dfrac{\partial}{\partial \tilde{x}_3}(h_1h_2\tilde{a}_3)\right]$

Rotationnel:

$$\nabla \times \mathbf{a} = \frac{\tilde{\mathbf{e}}_1}{h_2h_3}\left[\frac{\partial(h_3\tilde{a}_3)}{\partial \tilde{x}_2} - \frac{\partial(h_2\tilde{a}_2)}{\partial \tilde{x}_3}\right] + \frac{\tilde{\mathbf{e}}_2}{h_3h_1}\left[\frac{\partial(h_1\tilde{a}_1)}{\partial \tilde{x}_3} - \frac{\partial(h_3\tilde{a}_3)}{\partial \tilde{x}_1}\right] + \frac{\tilde{\mathbf{e}}_3}{h_1h_2}\left[\frac{\partial(h_2\tilde{a}_2)}{\partial \tilde{x}_1} - \frac{\partial(h_1\tilde{a}_1)}{\partial \tilde{x}_2}\right]$$

Laplacien: $\nabla^2 c = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial \tilde{x}_1}\left(\dfrac{h_2h_3}{h_1}\dfrac{\partial c}{\partial \tilde{x}_1}\right) + \dfrac{\partial}{\partial \tilde{x}_2}\left(\dfrac{h_3h_1}{h_2}\dfrac{\partial c}{\partial \tilde{x}_2}\right) + \dfrac{\partial}{\partial \tilde{x}_3}\left(\dfrac{h_1h_2}{h_3}\dfrac{\partial c}{\partial \tilde{x}_3}\right)\right]$

Coordonnées cylindriques:

$$x = r\cos\varphi \;, \quad y = r\sin\varphi \;, \quad z = \xi$$

$$\mathbf{e}_r = \cos\varphi\mathbf{e}_x + \sin\varphi\mathbf{e}_y \;, \quad \mathbf{e}_\varphi = -\sin\varphi\mathbf{e}_x + \cos\varphi\mathbf{e}_y \;, \quad \mathbf{e}_\xi = \mathbf{e}_z$$

$$h_r = 1 \;, \quad h_\varphi = r \;, \quad h_\xi = 1$$

$$\mathbf{a} = a_r\mathbf{e}_r + a_\varphi\mathbf{e}_\varphi + a_\xi\mathbf{e}_\xi$$

$$\nabla c = \mathbf{e}_r\frac{\partial c}{\partial r} + \frac{\mathbf{e}_\varphi}{r}\frac{\partial c}{\partial \varphi} + \mathbf{e}_\xi\frac{\partial c}{\partial \xi}$$

$$\nabla \bullet \mathbf{a} = \frac{1}{r}\frac{\partial(ra_r)}{\partial r} + \frac{1}{r}\frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_\xi}{\partial \xi}$$

$$\nabla \times \mathbf{a} = \mathbf{e}_r\left[\frac{1}{r}\frac{\partial a_\xi}{\partial \varphi} - \frac{\partial a_\varphi}{\partial \xi}\right] + \mathbf{e}_\varphi\left[\frac{\partial a_r}{\partial \xi} - \frac{\partial a_\xi}{\partial r}\right] + \mathbf{e}_\xi\left[\frac{1}{r}\frac{\partial (r a_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial a_r}{\partial \varphi}\right]$$

$$\nabla^2 c = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 c}{\partial \varphi^2} + \frac{\partial^2 c}{\partial \xi^2}$$

Coordonnées sphériques “géographiques” (rayon - longitude - latitude):

$$x = r\cos\varphi\cos\theta \ , \quad y = r\sin\varphi\cos\theta \ , \quad z = r\sin\theta$$

$$\begin{aligned}
\mathbf{e}_r &= \cos\varphi\cos\theta\,\mathbf{e}_x + \sin\varphi\cos\theta\,\mathbf{e}_y + \sin\theta\,\mathbf{e}_z \\
\mathbf{e}_\varphi &= -\sin\varphi\,\mathbf{e}_x + \cos\varphi\,\mathbf{e}_y \\
\mathbf{e}_\theta &= -\cos\varphi\sin\theta\,\mathbf{e}_x + \sin\varphi\sin\theta\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z
\end{aligned}$$

$$h_r = 1 \ , \quad h_\varphi = r\cos\theta \ , \quad h_\theta = r$$

$$\mathbf{a} = a_r\mathbf{e}_r + a_\varphi\mathbf{e}_\varphi + a_\theta\mathbf{e}_\theta$$

$$\nabla c = \mathbf{e}_r\frac{\partial c}{\partial r} + \frac{\mathbf{e}_\varphi}{r\cos\varphi}\frac{\partial c}{\partial \varphi} + \frac{\mathbf{e}_\theta}{r}\frac{\partial c}{\partial \theta}$$

$$\nabla \bullet \mathbf{a} = \frac{1}{r^2}\frac{\partial (r^2 a_r)}{\partial r} + \frac{1}{r\cos\theta}\frac{\partial a_\varphi}{\partial \varphi} + \frac{1}{r\cos\theta}\frac{\partial}{\partial \theta}(\cos\theta\, a_\theta)$$

$$\begin{aligned}
\nabla \times \mathbf{a} = {} & \frac{\mathbf{e}_r}{r\cos\theta}\left[\frac{\partial a_\theta}{\partial \varphi} - \frac{\partial}{\partial \theta}(\cos\theta\, a_\varphi)\right] \\
& + \frac{\mathbf{e}_\varphi}{r}\left[\frac{\partial a_r}{\partial \theta} - \frac{\partial (r a_\theta)}{\partial r}\right] + \frac{\mathbf{e}_\theta}{r}\left[\frac{\partial (r a_\varphi)}{\partial r} - \frac{1}{\cos\theta}\frac{\partial a_r}{\partial \varphi}\right]
\end{aligned}$$

$$\nabla^2 c = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) + \frac{1}{r^2\cos^2\theta}\frac{\partial^2 c}{\partial \varphi^2} + \frac{1}{r^2\cos^2\theta}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial c}{\partial \theta}\right)$$