Differential Geometry: An Introduction

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3D Models

3-D models can be defined using a Boundary Representation. In Gmsh, only four kinds of *model entities* are defined:

1. Model Vertices G_i^0 that are topological entities of dimension 0,

2. Model Edges G_i^1 that are topological entities of dimension 1,

3. Model Faces G_i^2 that are topological entities of dimension 2,

4. Model Regions G_i^3 that are topological entities of dimension 3.

3D Models

Model entities are topological entities, i.e., they only deal with adjacencies in the model

We use a bi-directional data structure for representing the graph of adjacencies. In this representation, a model entity G_i^d of dimension d holds one lists of upward adjacencies $G_j^{d+1}(G_i^d)$, i.e., all its adjacent entities of dimension $d+1$, and one list of downward adjacencies of dimension $d-1$, $G_j^{d-1}(G_i^d)$. Schematically, we have

$$G_i^0 \rightleftharpoons G_i^1 \rightleftharpoons G_i^2 \rightleftharpoons G_i^3.$$

This representation is said to be complete because any model entity is able to build its list of adjacencies of any dimension using local operations, i.e., without having to do a complete traversal of the adjacency graph of the model.

3D Models

Each model entity G_i^d has a shape, a geometry. More precisely, it is a manifold of dimension d that is embedded in 3-D space. (Note that the overall geometric model may itself be non-manifold: Gmsh supports non-manifold features).

The geometry of a model entity depends on the solid modeler kernel for its underlying representation. Solid modelers provide a parametrization of the shapes, i.e., a mapping $\mathbf{x} \in R^d \mapsto \mathbf{p} \in R^3$:

3D models

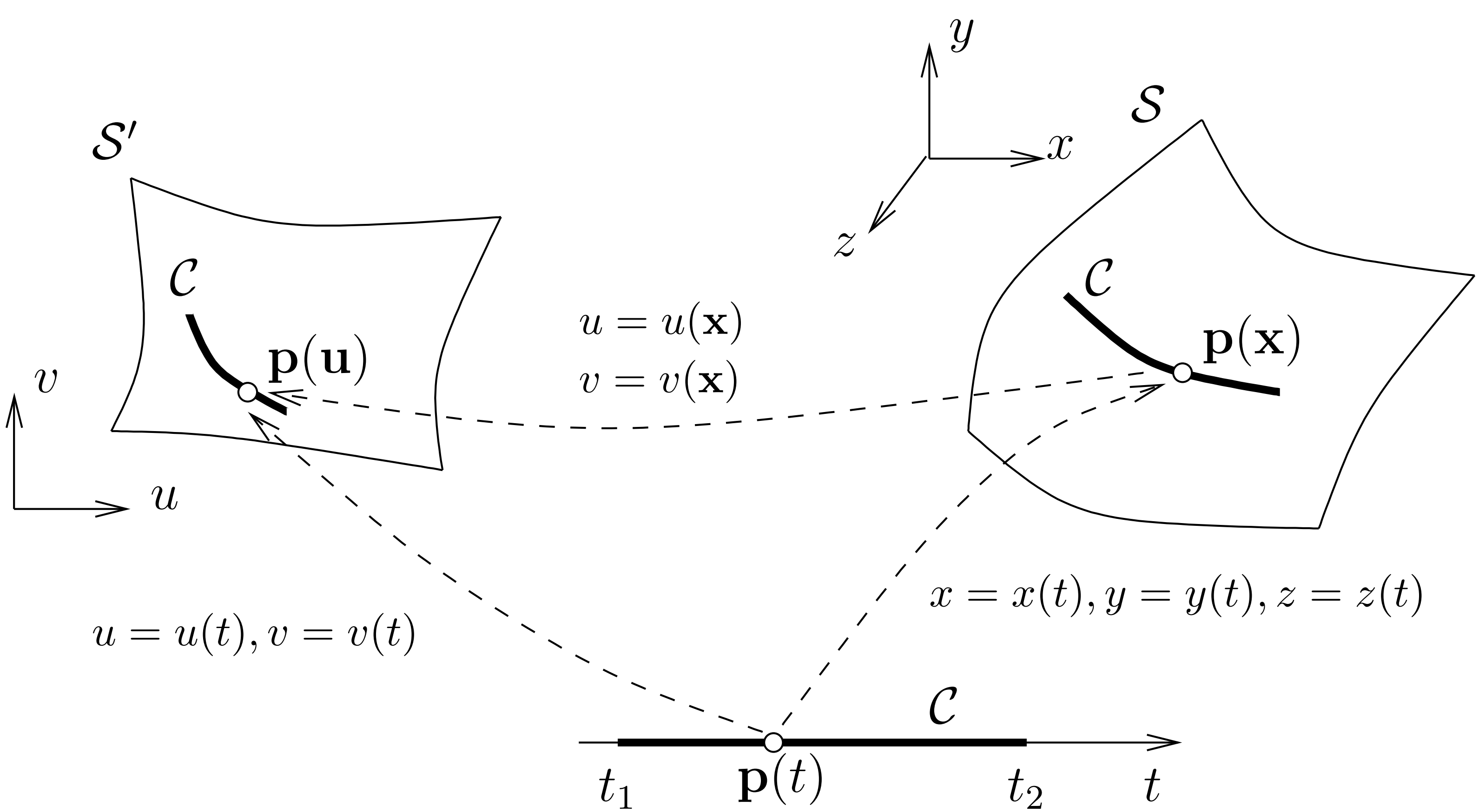
1. The geometry of a model vertex G_i^0 is simply its 3-D location $\mathbf{x}_i = (x_i, y_i, z_i)$.

2. The geometry of a model edge G_i^1 is its underlying curve $\mathcal{C}_i$ with its parametrization $\mathbf{p}(t) \in \mathcal{C}_i$, $t \in [t_1, t_2]$.

3. The geometry of a model face G_i^2 is its underlying surface $\mathcal{S}_i$ with its parametrization $\mathbf{p}(u, v) \in \mathcal{S}_i$. Note that, for any curve $\mathcal{C}_j$ that is on a surface $\mathcal{S}_i$, mesh generation procedures require the ability to reparametrize any point $\mathbf{p}(t) \in \mathcal{C}_j$ on the surface $\mathcal{S}_i$, i.e., to compute the mapping $u = u(t)$ and $v = v(t)$. Gmsh either uses a brute force algorithm to compute the direct mapping $x = x(t)$, $y = y(t)$ and $z = z(t)$ and its inverse $u = u(x, y, z)$ and $v = v(x, y, z)$, or, when the underlying CAD system provides it,

the direct reparametrization of a point on a model face (i.e., a function that directly computes $u = u(t)$ and $v = v(t)$).

4. The geometry associated to a model region is R^3.



Point $\mathbf{p}$ located on the curve $\mathcal{C}$ that is itself embedded in a 2D surface $\mathcal{S}'$ (left) or a 3D surface $\mathcal{S}$ (right).

3D models

```
class GEdge : public GEntity{
  // bi-directional data structure
  GVertex *v1, *v2;
  std::list<GFace*> faces;
public:
  // pure virtual functions that have to be overloaded
  // for every solid modeler
  virtual std::pair<double> parRange() = 0;
  virtual Point3 value(double t) = 0;
  virtual Vector3 tangent(double t) = 0;
  virtual Point2 reparam(modelFace *mf, double t, int dir) = 0;
  virtual bool isSeam(modelFace *mf) = 0;
  // other functions of the class are non pure virtual
  // ...
};
```

3D models

```
class GFace : public modelEntity{
  // bi-directional data structure
  GRegion *r1, *r2;
  std::list<GEdge*> edges;
public:
  // pure virtual functions that have to be overloaded
  // for every solid modeler
  virtual std::pair<double> parRange(int dir) const = 0;
  virtual Point3 value(double u, double v) const = 0;
  virtual std::pair<Vector3> tangent( double u, double v) const = 0;
  virtual double curvature(double u, double v) const;
  // other functions of the class are non pure virtual
  // ...
};
```

3D models

Each model entity G_i^d has a shape, a geometry. More precisely, it is a manifold of dimension d that is embedded in 3-D space.

Solid modelers usually provide a parametrization of the shapes, i.e., a mapping $\mathbf{x} \in \mathcal{R}^d \mapsto \mathbf{x} \in \mathcal{R}^3$. The geometry of a model vertex G_i^0 is simply its 3-D location $\mathbf{x} = (x_1, x_2, x_3)$.

The geometry of a model edge G_i^1 is its underlying curve $\mathcal{C}_i$ with its parametrization

$$t \in [t_1, t_2] \mapsto \mathbf{x}(t) \in \mathcal{R}^3. \quad (1)$$

3D models

The geometry of a model face G_i^2 is its underlying surface $\mathcal{S}_i$ with its parametrization

$$(u, v) \in \mathcal{R}^2 \mapsto \mathbf{x}(u, v) \in \mathcal{R}^3.$$

The geometry associated to a model region is R^3.

If a curve is included within a surface, it is usually drawn on the parameter plane (u, v) of the surface:

$$t \in [t_1, t_2] \mapsto (u, v) \in \mathcal{R}^2 \mapsto \mathbf{x}\left(u(t), v(t)\right) \in \mathcal{R}^3. \tag{2}$$

Differential Geometry of Curves

A parameterization of a curve is a bijective mapping from a 1D domain to the 3D curve. The mapping can be defined as follows:

$$t \in \mathcal{R}^1 \mapsto \mathbf{x}(t) \in \mathcal{R}^3. \qquad (3)$$

Consider a segment of curve $\mathcal{C}$ defined by a range of parameter $t \in [t_a, t_b]$, $t_a \geq t_1$, $t_b \leq t_2$. The length of that segment can be computed as

$$\int_{\mathcal{C}} dl$$

with $dl = \sqrt{dx_1^2 + dx_2^2 + dx_3^2}$. Using $\mathcal{C}$'s parametrization (1), we have

$$\begin{aligned} \int_{\mathcal{C}} \sqrt{dx_1^2 + dx_2^2 + dx_3^2} &= \int_{t_a}^{t_b} \sqrt{x_{1,t}^2 + x_{2,t}^2 + x_{3,t}^2}\, dt \\ &= \int_{t_a}^{t_b} \left\| \mathbf{x}_{,t} \right\| dt \end{aligned}$$

Differential Geometry of Curves

This can be easily extended to the computation of integral quantities over model edges:

$$\int_{\mathcal{C}} f(x_1, x_2, x_3) dl \;=\; \int_{t_a}^{t_b} f(x_1(t), x_2(t), x_3(t)) \left\| \mathbf{x}_{,t} \right\| dt \tag{4}$$

The curvilinear abscissa $l(t)$ of a point $\mathbf{x}(t)$ of curve $\mathcal{C}$, is the length of the segment defined by parameter range $[t_1, t]$, i.e. the length of the curve from the origin $\mathbf{x}(t_1)$ to $\mathbf{x}(t)$:

$$l(t) = \int_{t_1}^{t} \left\| \mathbf{x}_{,t} \right\| dt \tag{5}$$

We have seen before that $dl = \left\| \mathbf{x}_{,t} \right\| dt$.

A parametrization of $\mathcal{C}$ is said to be regular if $\left\| \mathbf{x}_{,t} \right\| \neq 0$. For regular parametrizations, the unit tangent vector is defined as

$$\mathbf{t}(t) = \frac{\mathbf{x}_{,t}}{\left\| \mathbf{x}_{,t} \right\|} = \frac{d\mathbf{x}}{dl}.$$

Differential Geometry of Curves

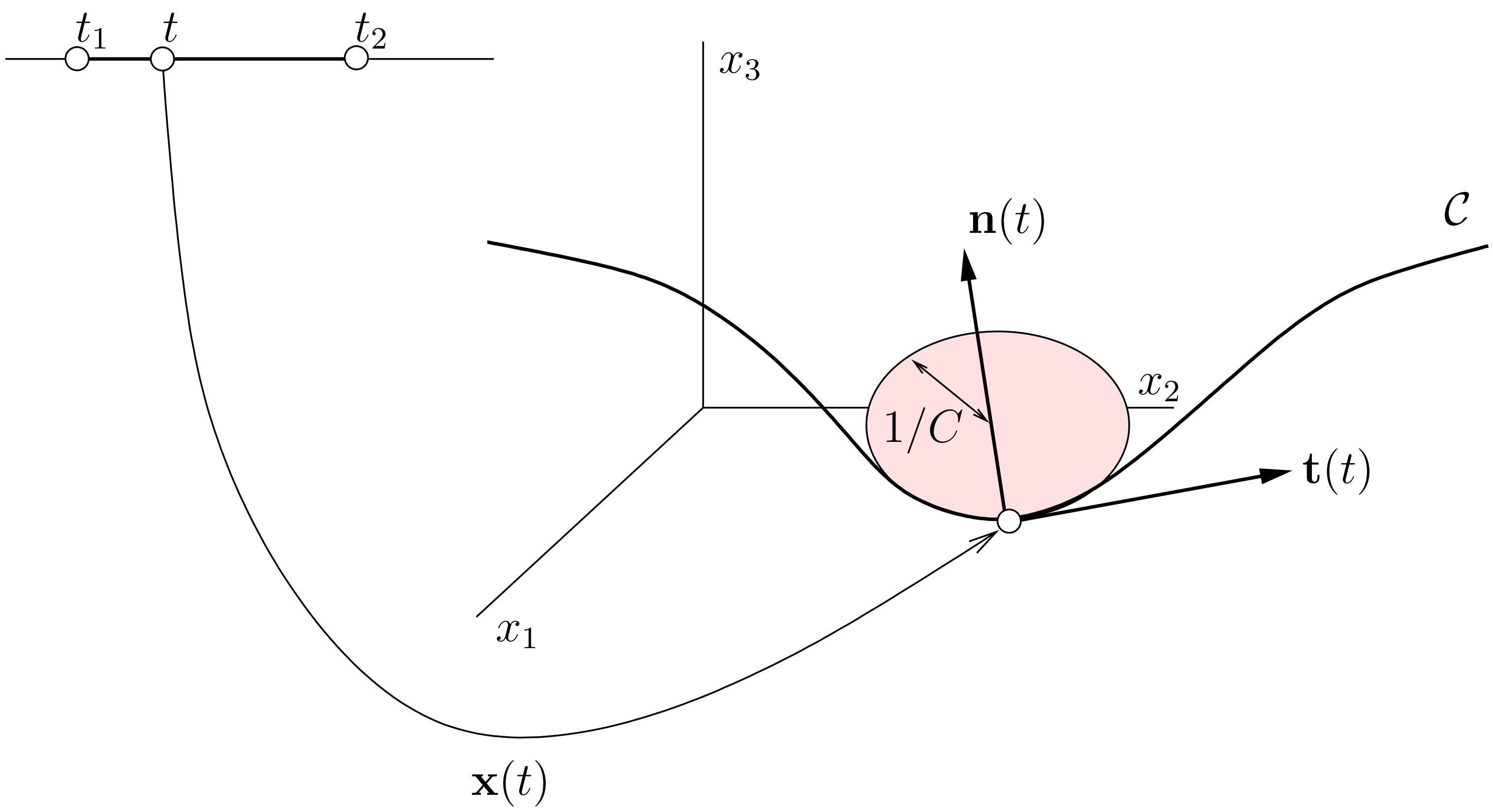
The normal plane at point $\mathbf{x}(t)$ is the plane that contains $\mathbf{x}(t)$ and that has $\mathbf{t}(t)$ as normal vector (see Figure). The curvature of the curve at a point $\mathbf{x}$ can be defined as the amplitude of the variations of the unit tangent $\mathbf{t}$ along the curve. The vector $\mathbf{t}_{,l}$ is obviously orthogonal to $\mathbf{t}$ because $\mathbf{t}$'s amplitude is one along l. Recalling that

$$\frac{d}{dt}\frac{1}{\|\mathbf{x}\|} = -\frac{\mathbf{x}_{,t} \cdot \mathbf{x}}{\|\mathbf{x}\|^3}$$

we have

$$\begin{aligned}
\mathbf{t}_{,l} &= \frac{1}{\left\|\mathbf{x}_{,t}\right\|}\mathbf{t}_{,t} \\
&= \frac{1}{\left\|\mathbf{x}_{,t}\right\|}\left(\frac{\mathbf{x}_{,tt}}{\left\|\mathbf{x}_{,t}\right\|} - \mathbf{x}_{,t}\frac{\mathbf{x}_{,t} \cdot \mathbf{x}_{,tt}}{\|\mathbf{x}_{,t}\|^3}\right) \\
&= \frac{1}{\left\|\mathbf{x}_{,t}\right\|^3}\left(\mathbf{x}_{,tt}\left\|\mathbf{x}_{,t}\right\| - \mathbf{x}_{,t}\frac{\mathbf{x}_{,t} \cdot \mathbf{x}_{,tt}}{\|\mathbf{x}_{,t}\|}\right).
\end{aligned}$$

Differential Geometry of Curves



A curve $\mathcal{C}$ defined by the mapping $\mathbf{x}(t)$.

Differential Geometry of Curves

Clearly, $\mathbf{t}_{,l} \cdot \mathbf{t} = 0$. Because we have defined the curvature as the amplitude of the variations of the unit tangent $\mathbf{t}$ along the curve, we call rewrite

$$\mathbf{t}_{,l} = \|\mathbf{t}_{,l}\| \frac{\mathbf{t}_{,l}}{\|\mathbf{t}_{,l}\|} = C\mathbf{n}$$

with $\mathbf{n}$ a unit normal vector orthogonal to $\mathbf{t}$ and C the curvature that is the norm of $\mathbf{t}_l$. Remembering that

$$\|\mathbf{a}\times\mathbf{b}\|^2 = \|a\|^2\|b\|^2 \sin^2(\mathbf{a},\mathbf{b}) = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 \left(1 - \frac{(\mathbf{a}\cdot\mathbf{b})^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2}\right) = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2$$

it is easy to see that

$$C^2 = \frac{1}{\left\|\mathbf{x}_{,t}\right\|^6}\left(\|\mathbf{x}_{,t}\|^2\|\mathbf{x}_{,tt}\|^2 - (\mathbf{x}_{,tt}\cdot\mathbf{x}_{,t})^2\right) = \frac{1}{\left\|\mathbf{x}_{,t}\right\|^6}\|\mathbf{x}_t \times \mathbf{x}_{,tt}\|^2$$

and we get the formula

$$C = \frac{\|\mathbf{x}_t \times \mathbf{x}_{,tt}\|}{\left\|\mathbf{x}_{,t}\right\|^3}. \qquad (6)$$

Differential Geometry of Curves

Consider the function $y = f(x)$ in the (x, y) plane. Its parametric representation is $x(t) = t$ and $y(t) = f(t)$ or $\mathbf{x}(t) = \{t, f(t)\}$. The tangent vector is $\mathbf{x}_{,t} = \{1, f'(t)\}$ and $\mathbf{x}_{,tt} = \{0, f''(t)\}$. We have therefore the classical formula

$$C = \frac{|f''(t)|}{(1 + f'(t)^2)^{3/2}}. \tag{7}$$

It is possible to define a local system of coordinates at any point $\mathbf{x}$ of the curve

$$(\mathbf{t}, \mathbf{n}, \mathbf{b})$$

with $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. This system of coordinates is usually called the Frenet frame. The osculating plane of the curve at point $\mathbf{x}$ can be defined as the plane containing $\mathbf{x}$ and normal to $\mathbf{b}$. The curvature $C(t)$ is the inverse of the radius of the osculating circle at point $\mathbf{x}$ i.e. the circle which most closely approximates the curve near $\mathbf{x}$:

$$R(t) = \frac{1}{C(t)}.$$

This gives an interesting intuitive interpretation of the curvature.

Differential Geometry of Curves

The tangent, normal, and binormal unit vectors, often called $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$, or collectively the Frenet frame are defined as follows:

- $\mathbf{t}$ is the unit vector tangent to the curve,
- $\mathbf{n}$ is the derivative of $\mathbf{t}$ with respect to the arclength parameter of the curve, divided by its length.
- $\mathbf{b}$ is the cross product of $\mathbf{t}$ and $\mathbf{n}$.

The Frenet formulas are

$$\mathbf{t}_{,l} = C\mathbf{n},$$

$$\mathbf{n}_{,l} = (\mathbf{b} \times \mathbf{t})_{,l} = -C\mathbf{t} + T\mathbf{b},$$

$$\mathbf{b}_{,l} = (\mathbf{t} \times \mathbf{n})_{,l} = \mathbf{t} \times \mathbf{n}_{,l} = -T\mathbf{n}.$$

Here, T is the torsion of curve.

Differential Geometry of Curves

Consider the helix

$$\mathbf{x}(l) = \frac{1}{\sqrt{2}}(\cos l, \sin l, l).$$

Whe have

$$\mathbf{t} = \frac{1}{\sqrt{2}}(-\sin l, \cos l, 1), \;\; \|t\| = 1.$$

Frenet formulas give

$$\mathbf{t}_{,l} = -\frac{1}{\sqrt{2}}(\cos l, \sin l, 0) = C\mathbf{n}.$$

$$\mathbf{n}_{,l} = -C\mathbf{t} + T\mathbf{b} = (\sin l, -\cos l, 0).$$

which means that

$$T\mathbf{b} = (\sin l, -\cos l, 0) + \frac{1}{2}(-\sin l, \cos l, 1) = \frac{1}{\sqrt{2}} \underbrace{\frac{1}{\sqrt{2}}(\sin l, -\cos l, 1)}_{\mathbf{b}}.$$

and the helix has a constant torsion.

Differential Geometry of Curves

Consider the Taylor expansion of $\mathbf{x}(l)$ around $l = 0$. We have

$$\mathbf{x}(l) = \mathbf{x}(0) + \mathbf{x}_{,l}(0)l + \mathbf{x}_{,ll}(0)\frac{l^2}{2} + \mathbf{x}_{,lll}(0)\frac{l^3}{6} + \mathcal{O}(l^4).$$

Using Frenet, we have $\mathbf{x}_{,l} = \mathbf{t}$, $\mathbf{x}_{,ll} = C\mathbf{n}$ and $\mathbf{x}_{,lll} = C_{,l}\mathbf{n} + C(-C\mathbf{t} + T\mathbf{b})$,

$$\mathbf{x}(l) \simeq \mathbf{x}(0) + \mathbf{t}l + C\mathbf{n}\frac{l^2}{2} + (-C^2\mathbf{t} + C_{,l}\mathbf{n} + CT\mathbf{b})\frac{l^3}{6}.$$

or

$$\mathbf{x}(l) \simeq \mathbf{x}(0) + \mathbf{t}\left(l - \frac{C^2 l^3}{6}\right) + \mathbf{n}\left(\frac{Cl^2}{2} + \frac{C_{,l} l^3}{6}\right) + \mathbf{b}\left(\frac{CTl^3}{6}\right).$$

Differential Geometry of Curves

The osculating plane is the plane containing $\mathbf{t}$ and $\mathbf{n}$. The projection of the curve onto this plane has the following expansion

$$\mathbf{x}(l) = \mathbf{x}(0) + \mathbf{t}l + \mathbf{n}\frac{Cl^2}{2} + \mathcal{O}(l^3)$$

that is a parabola that has the same curvature as the curve itself on $l = 0$. This gives a very well known geometrical interpretation of the curvature.

The rectifying plane is the plane containing $\mathbf{n}$ and $\mathbf{b}$.

The normal plane is the plane containing $\mathbf{t}$ and $\mathbf{b}$.

Differential Geometry of Surfaces

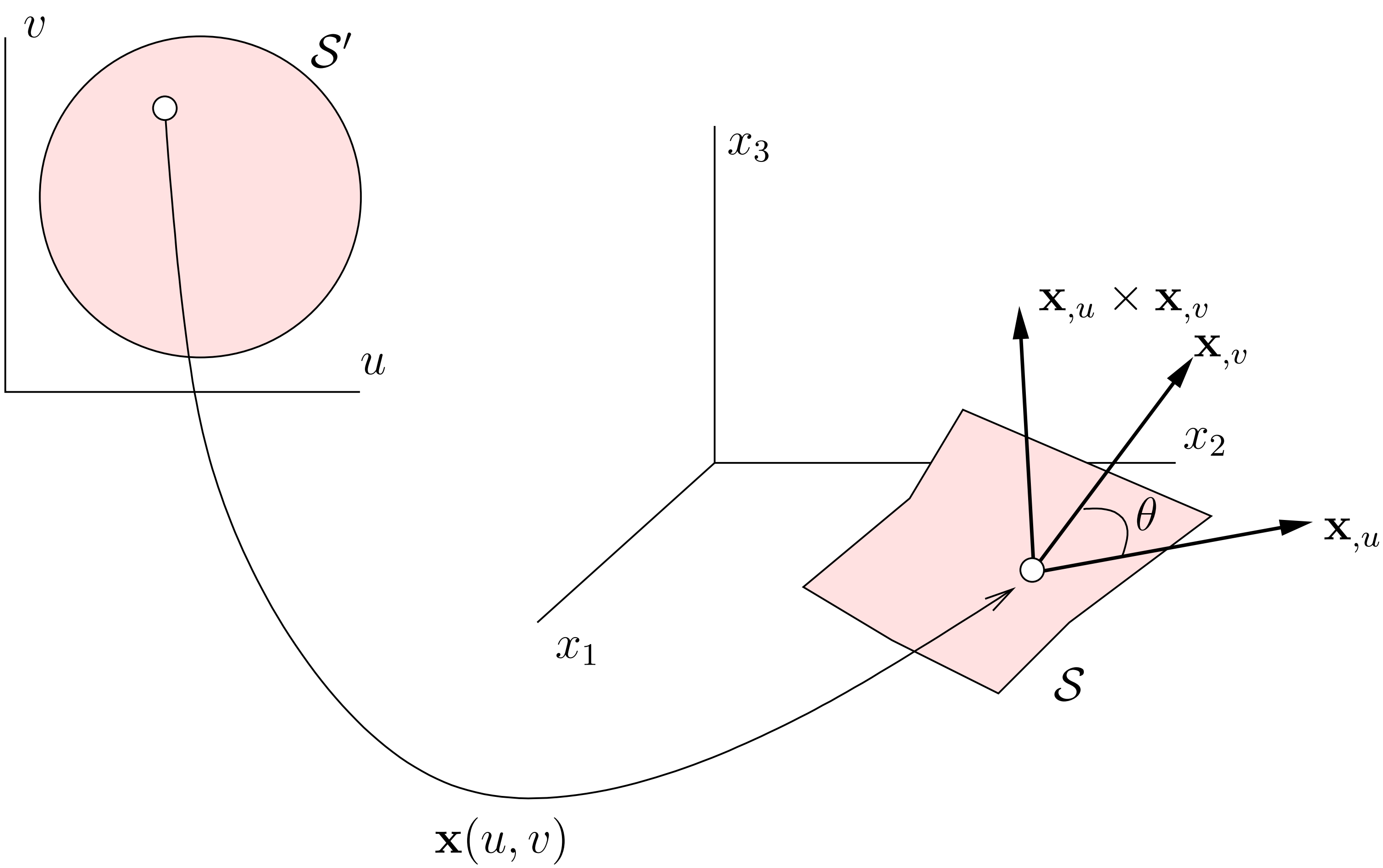
A parameterization of a surface is a bijective mapping from a suitable two-dimensional domain $\mathcal{S}'$ to the 3D surface $\mathcal{S}$ (see Figure). In CAD modelers, surfaces have explicit parametrizations i.e. their parametrization is given explicitly as a continuous and differentiable function:

$$(u, v) \in \mathcal{S}' \subset \mathcal{R}^2 \mapsto \mathbf{x}(u, v) \in \mathcal{S} \subset \mathcal{R}^3. \tag{8}$$

Such a parametrization exists if the two surfaces $\mathcal{S}$ and $\mathcal{S}'$ have the same topology, i.e are both zero genus surfaces ($G = 0$) and have at least one boundary ($N_B \geq 1$)*.

*For example, a sphere has $G = 0$ and $N_B = 0$ and a torus has $G = 1$ and $N_B = 0$.

Differential Geometry of Surfaces



A surface $\mathcal{S}$ defined by the mapping $\mathbf{x}(u, v)$.

Differential Geometry of Surfaces

Given the parametrization $\mathbf{x}(\mathbf{u})$, let us compute fundamental properties of surfaces such as lengths, angles, areas and curvatures. Consider a curve that is included in surface $\mathcal{S}$. It is easy to extend the integration formula (4) as:

$$\begin{aligned}
\int_{\mathcal{C}} f(x,y,z) & \sqrt{dx^2 + dy^2 + dz^2} \\
&= \int_{\mathcal{C}} f(x,y,z)\sqrt{\|\mathbf{x}_{,u}\|^2\, du^2 + 2\ \mathbf{x}_{,u}\cdot\mathbf{x}_{,v} du\ dv + \|\mathbf{x}_{,v}\|^2\, dv^2} \\
&= \int_{\mathcal{C}} f(x,y,z)\sqrt{\begin{bmatrix} du \\ dv \end{bmatrix}^T \begin{bmatrix} \mathbf{x}_{,u}\cdot\mathbf{x}_{,u} & \mathbf{x}_{,u}\cdot\mathbf{x}_{,v} \\ \mathbf{x}_{,v}\cdot\mathbf{x}_{,u} & \mathbf{x}_{,v}\cdot\mathbf{x}_{,v} \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}} \\
&= \int_{\mathcal{C}} f(x,y,z)\sqrt{\mathbf{du}^T\mathbf{M}\ \mathbf{du}}. \qquad (9)
\end{aligned}$$

Differential Geometry of Surfaces

In (10), the matrix $\mathbf{M}$ is called the **metric tensor** or first fundamental form. It is defined as follows:

$$\mathbf{M} = \mathbf{x}_{,\mathbf{u}}^T \mathbf{x}_{,\mathbf{u}} = \begin{bmatrix} \mathbf{x}_{,u} \cdot \mathbf{x}_{,u} & \mathbf{x}_{,u} \cdot \mathbf{x}_{,v} \\ \mathbf{x}_{,v} \cdot \mathbf{x}_{,u} & \mathbf{x}_{,v} \cdot \mathbf{x}_{,v} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \quad (10)$$

The metric tensor is a symmetric definite positive second order tensor that varies smoothly over the manifold and that has two real eigenvalues.

We have just seen that the tensor metric enables to measure curve lengths drawn in the parametric plane. It also allows to generalizes many familiar properties of the dot product of vectors in Euclidean space. In particular, it allows to compute the angle between two tangent vectors to the surface as well as areas. Any tangent vector at a point of the parametric surface can be written in the form

$$\mathbf{t} = a\mathbf{x}_{,u} + b\mathbf{x}_{,v}$$

with $a, b \in \mathcal{R}$.

Differential Geometry of Surfaces

Let us consider two tangent vectors

$$\mathbf{t}_1 = a_1\mathbf{x}_{,u} + b_1\mathbf{x}_{,v} \quad \text{and} \quad \mathbf{t}_2 = a_2\mathbf{x}_{,u} + b_2\mathbf{x}_{,v}.$$

Coordinates $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ are called covariant coordinates of $\mathbf{t}_1$ and $\mathbf{t}_2$. We have

$$\mathbf{t}_1 \cdot \mathbf{t}_2 = a_1a_2\mathbf{x}_{,u} \cdot \mathbf{x}_{,u} + (a_1b_2 + a_2b_1)\mathbf{x}_{,u} \cdot \mathbf{x}_{,v} + b_1b_2\mathbf{x}_{,v} \cdot \mathbf{x}_{,v} = \mathbf{a}^T\mathbf{M}\mathbf{b},$$

which gives us the angle θ between the two tangent vectors (see Fig.):

$$\theta = \arccos\left(\frac{\mathbf{a}^T\mathbf{M}\mathbf{b}}{\mathbf{t}_1\mathbf{t}_2}\right) \tag{11}$$

Consider a small rectangle $du\ dv$ at a point on the parametric plane. Its area on the 3D surface is given by:

$$s = \|\mathbf{x}_{,u}du \times \mathbf{x}_{,v}dv\| = du\ dv\sqrt{\det \mathbf{M}}. \tag{12}$$

Note again that the value of the area only depends on the tensor metric $\mathbf{M}$.

The last important fundamental quantity related to the shape of surfaces is the curvature. Let us first define the unit surface normal $\mathbf{n}$ that is orthogonal to both tangent vectors:

$$\mathbf{n} = \frac{\mathbf{x}_{,u} \times \mathbf{x}_{,v}}{\|\mathbf{x}_{,u} \times \mathbf{x}_{,v}\|} = \frac{\mathbf{x}_{,u} \times \mathbf{x}_{,v}}{\sqrt{\det \mathbf{M}}} = \frac{\mathbf{x}_{,u} \times \mathbf{x}_{,v}}{\sqrt{EG - F^2}}. \qquad (13)$$

The three vectors $(\mathbf{x}_{,u}, \mathbf{x}_{,v}, \mathbf{n})$ form at each point of the surface a local system of coordinates usually called the *local frame*. It is easy to orthonormalize the local frame, i.e. by choosing

$$\mathbf{t}_1 = \frac{\mathbf{x}_{,u}}{\|\mathbf{x}_{,u}\|}, \quad \mathbf{t}_2 = \mathbf{n} \times \mathbf{t}_1$$

so that vectors $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{n})$ form an orthonormal system of coordinates usually called the *Darboux frame*.

Differential Geometry of Surfaces

To study the curvature of the surface at a point $\mathbf{x}$, one can examine the variations of the unit normal $\mathbf{n}$ around $\mathbf{x}$. In particular, one can derivate $\mathbf{n}$ in the direction specified by the tangent vectors at $\mathbf{x}$: this is called the Weingarten map. Note that $\mathbf{n}_{,u}$ and $\mathbf{n}_{,v}$ are both tangent vectors because $\mathbf{n}$ is a unit vector:

$$\frac{\partial}{\partial u}\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{v}_{,u}}{\|\mathbf{v}\|} - \frac{\mathbf{v}}{\|\mathbf{v}\|^3}(\mathbf{v} \cdot \mathbf{v}_{,u})$$

Applying that formula to Equation (13) (an after tedious calculations), we obtain the Weingarten equations that express the derivatives of the normal to a surface using derivatives of the position vector $\mathbf{x}$

$$\mathbf{n}_{,u} = \frac{MF - LG}{EG - F^2}\mathbf{x}_{,u} + \frac{LF - ME}{EG - F^2}\mathbf{x}_{,v}$$

$$\mathbf{n}_{,v} = \frac{NF - MG}{EG - F^2}\mathbf{x}_{,u} + \frac{MF - NE}{EG - F^2}\mathbf{x}_{,v}$$

where the scalars e, f and g are defined as:

$$L = \mathbf{n} \cdot \mathbf{x}_{,uu}, \quad M = \mathbf{n} \cdot \mathbf{x}_{,uv} \quad \text{and} \quad N = \mathbf{n} \cdot \mathbf{x}_{,vv}.$$

Differential Geometry of Surfaces

Those scalars define the **curvature tensor** or second fundamental tensor of the surface denoted $\mathbf{M}_2$:

$$\mathbf{M}_2 = \begin{bmatrix} L & M \\ M & N \end{bmatrix} \tag{14}$$

The curvature tensor is a tensor, and like for any tensor, some invariants can be defined that are independent of the system of coordinates. In other words, invariants are intrinsic measures of the curvature, i.e., their value depends only on how distances are measured on the surface, not on the way it is embedded in space. The principal curvatures κ_1 and κ_2 are the two eigenvalues of $\mathbf{M}_2$. They are the largest and smallest values of the curvature at a point. The mean curvature $\bar{\kappa} = \kappa_1 + \kappa_2$ is proportional to the trace of the curvature tensor and $\bar{\kappa}$ is therefore an invariant of the tensor. It is easy to see that

$$\bar{\kappa} = \nabla \cdot \mathbf{n} = \frac{1}{2}\mathrm{trace}\left(\frac{\mathbf{M}_2}{\mathbf{M}}\right) = \frac{LG - 2MF + NE}{2\left(EG - F^2\right)}. \tag{15}$$

Differential Geometry of Surfaces

The Gaussian curvature κ of a point on a surface is the square root of the product of the principal curvatures, κ_1 and κ_2, of the given point. Its value can be computed as

$$\kappa = \frac{\det \mathbf{M}_2}{\det \mathbf{M}} = \frac{LN - M2}{EG - F^2}. \qquad (16)$$

Example

Consider the following change of (orthogonal) coordinates:

$$\mathbf{q}(q_1, q_2, q_3) \to \mathbf{x} = (x_1(\mathbf{q}), x_2(\mathbf{q}), x_3(\mathbf{q})).$$

Every point of R^3 $\mathbf{x} = (x_1, x_2, x_3)$ has its counterpart $\mathbf{q}$. We define

$$h_i = \left\| \frac{\partial \mathbf{x}}{\partial q_i} \right\| \tag{17}$$

and the three orthonormal unit vector of the local basis

$$\mathbf{e}_{q_i} = \frac{\partial \mathbf{x}}{\partial q_i} \frac{1}{h_i}. \tag{18}$$

Gradient

Considérons une fonction $f(x_1, x_2, x_3)$ continuement dérivable. On consdère la fonction composée $f(x_1(\boldsymbol{q}), x_2(\boldsymbol{q}), x_3(\boldsymbol{q}))$ Soit

$$\nabla f = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \boldsymbol{e}_i$$

le gradient exprimé en coordonnées cartésiennes. On cherche ses composantes dans la base locale i.e

$$\nabla f = \sum_{i=1}^{3} f_i \boldsymbol{e}_{q_i}.$$

On calcule les différentielles

$$dx_i = \sum_{j=1}^{3} \frac{\partial x_i}{\partial q_j} dq_j \quad \text{et} \quad df = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} dx_i.$$

On a donc

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} dx_i = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \sum_{j=1}^{3} \frac{\partial x_i}{\partial q_j} dq_j = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial q_j} dq_j = \sum_{j=1}^{3} \frac{\partial f}{\partial q_j} dq_j.$$

Gradient

Si on pose $\boldsymbol{dx} = (dx_1, dx_2, dx_3)$, on a $\nabla f \cdot \boldsymbol{dx} = df$. Or

$$\boldsymbol{dx} = \sum_{i=1}^{3} \frac{\partial \boldsymbol{x}}{\partial q_i} dq_i = \sum_{i=1}^{3} h_i dq_i \boldsymbol{e}_{q_i}.$$

Le produit scalaire $\nabla f \cdot \boldsymbol{dx} = df$ vaut

$$df = \left(\sum_{i=1}^{3} f_i \boldsymbol{e}_{q_i} \right) \cdot \left(\sum_{i=1}^{3} h_i dq_i \boldsymbol{e}_{q_i} \right).$$

On a supposé être en présence de coordonnées orthogonales, on a donc

$$df = \sum_{i=1}^{3} f_i h_i dq_i.$$

Or, $df = \sum_{j=1}^{3} \frac{\partial f}{\partial q_j} dq_j$ ce qui implique que

$$\frac{\partial f}{\partial q_j} = f_i h_i.$$

Gradient

Les composantes du gradient dans la base locale sont donc

$$f_i = \frac{1}{h_i}\frac{\partial f}{\partial q_j}.$$

On a donc le résultat remarquable

$$\boxed{\nabla f = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial f}{\partial q_j} \boldsymbol{e}_{q_j}. \qquad (19)}$$

Divergence

Rappelons que l'opérateur Laplacien est la divergence du gradient. Soit un vecteur

$$\boldsymbol{A} = A_1\boldsymbol{e}_{q_1} + A_2\boldsymbol{e}_{q_2} + A_3\boldsymbol{e}_{q_3}$$

exprimé dans la base locale. On a *

$$\nabla \cdot \boldsymbol{A} = \sum_{i=1}^{3} A_i \nabla \cdot \boldsymbol{e}_{q_1} + \sum_{i=1}^{3} \nabla A_i \boldsymbol{e}_{q_1}$$

Calculons tout d'abord

$$\begin{aligned} \nabla \cdot \boldsymbol{e}_{q_1} &= \nabla \cdot (\boldsymbol{e}_{q_2} \times \boldsymbol{e}_{q_3}) \\ &= \nabla \cdot (h_2 h_3 \nabla q_2 \times \nabla q_3) \\ &= \nabla(h_2 h_3) \cdot (\nabla q_2 \times \nabla q_3) + h_2 h_3 \nabla \cdot (\nabla q_2 \times \nabla q_3) \\ &= \nabla(h_2 h_3) \cdot (\nabla q_2 \times \nabla q_3) + h_2 h_3 (\nabla \times \nabla q_2 \cdot \nabla q_3 - \nabla q_2 \cdot \nabla \times \nabla q_3) \\ &= \nabla(h_2 h_3) \cdot (\nabla q_2 \times \nabla q_3). \end{aligned} \tag{20}$$

$^*\nabla \cdot (\boldsymbol{A} b) = (\nabla \cdot \boldsymbol{A}) b + (\nabla b) \cdot \boldsymbol{A}$.

Divergence

Dans (20), on a utilisé l'identité

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{B} - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$$

et le fait que rotationnel $\nabla\times$ d'un gradient est nul. On a donc

$$\begin{aligned}
\nabla \cdot \boldsymbol{e}_{q_1} &= \nabla(h_2 h_3) \cdot (\nabla q_2 \times \nabla q_3) \\
&= \frac{1}{h_2 h_3} \nabla(h_2 h_3) \cdot (\boldsymbol{e}_{q_2} \times \boldsymbol{e}_{q_3}) \\
&= \frac{1}{h_2 h_3} \nabla(h_2 h_3) \cdot \boldsymbol{e}_{q_1} \\
&= \frac{1}{h_2 h_3} \left(\sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial(h_2 h_3)}{\partial q_i} \boldsymbol{e}_{q_i} \right) \cdot \boldsymbol{e}_{q_1} \\
&= \frac{1}{h_1 h_2 h_3} \frac{\partial(h_2 h_3)}{\partial q_1}
\end{aligned}$$

Divergence

et donc

$$\begin{aligned}\nabla\cdot(A_1\boldsymbol{e}_{q_1}) &= A_1\nabla\cdot\boldsymbol{e}_{q_1}+\nabla A_1\cdot\boldsymbol{e}_{q_1}\\ &= A_1\frac{1}{h_1h_2h_3}\frac{\partial(h_2h_3)}{\partial q_1}+\left(\sum_{i=1}^{3}\frac{1}{h_i}\frac{\partial A_1}{\partial q_i}\boldsymbol{e}_{q_i}\right)\cdot\boldsymbol{e}_{q_1}\\ &= \frac{1}{h_1h_2h_3}\frac{\partial(A_1h_2h_3)}{\partial q_1}. \qquad (21)\end{aligned}$$

On obtient donc la forme finale de la divergence exprimée en fonction des coordonées dans la base locale et des facteurs h_i

$$\nabla\cdot\boldsymbol{A}=\frac{1}{h_1h_2h_3}\left(\frac{\partial(A_1h_2h_3)}{\partial q_1}+\frac{\partial(h_1A_2h_3)}{\partial q_2}+\frac{\partial(h_1h_1A_3)}{\partial q_3}\right). \quad (22)$$

Laplacien

Soit f une fonction deux fois différentiable. On trouve

$$\begin{aligned}\nabla^2 f &= \nabla \cdot \nabla f \\ &= \nabla \cdot \left(\sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial f}{\partial q_i} \boldsymbol{e}_{q_i} \right) \\ &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial f}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial q_3} \right) \right).\end{aligned}$$

Coordonnées polaires

Dérivons maintenant les opérateurs différentiels défini plus haut dans divers systèmes de coordonnées. Commençons par les coordonnées polaires :

$$x(r,\theta) = r\cos\theta \quad , \quad y(r,\theta) = r\sin\theta.$$

On calcule tout d'abord la matrice jacobienne $J = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}}$ avec $J_{ij} = \frac{\partial x_i}{\partial q_j}$. On a

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{pmatrix}}_{J} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \tag{23}$$

Il est en général plus facile d'obtenir J^{-1}. En effet,

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix}}_{J^{-1}} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}. \tag{24}$$

Coordonnées polaires

On calcule donc

$$J^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

ce qui donne

$$J = \begin{pmatrix} \cos\theta & -\sin\theta/r \\ \sin\theta & \cos\theta/r \end{pmatrix}.$$

Il est donc facile de voir que

$$\nabla u = \begin{pmatrix} \cos\theta\frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial u}{\partial\theta} \\ \sin\theta\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial u}{\partial\theta} \end{pmatrix} = \frac{\partial u}{\partial r}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \frac{1}{r}\frac{\partial u}{\partial\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}. \quad (25)$$

Utilisons maintenant les formules développées plus haut. On commence par calculer les $\boldsymbol{e}_{q_i}$ et les h_i par les formules (17) et (18) :

$$h_1 = 1, \quad h_2 = r, \quad \boldsymbol{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{et} \quad \boldsymbol{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Coordonnées polaires

La formule (19) donne

$$\nabla u = \frac{\partial u}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\boldsymbol{e}_\theta. \qquad (26)$$

ce qui est exactement (25).

La divergence d'un vecteur $\boldsymbol{A}$ en coordonnées polaires est calculée en utilisant (22)

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\left(\frac{\partial (A_r r)}{\partial r} + \frac{\partial A_\theta}{\partial \theta}\right).$$

Coordonnées polaires

Le laplacien en coordonnées polaires est calculé en utilisant (23) :

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

Coordonnées sphériques

Les coordonnées sphériques sont donnés par :

$x(r,\theta,\phi) = r\sin\theta\cos\phi \quad , \quad y(r,\theta,\phi) = r\sin\theta\sin\phi \quad , \quad z(r,\theta,\phi) = r\cos\theta.$

On a

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta,$$

$$\boldsymbol{e}_r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \quad , \quad \boldsymbol{e}_\theta = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} \quad \text{et} \quad \boldsymbol{e}_\phi = \begin{pmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{pmatrix}.$$

Coordonnées sphériques

La formule (19) donne

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi. \qquad (27)$$

La divergence d'un vecteur $\boldsymbol{A}$ en coordonnées sphériques est calculée en utilisant (22)

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2\sin\theta}\left(\frac{\partial(A_r r^2 \sin\theta)}{\partial r} + \frac{\partial(r\sin\theta A_\theta)}{\partial \theta} + \frac{\partial(r A_\phi)}{\partial \phi}\right).$$

Coordonnées sphériques

Le laplacien en coordonnées sphériques est calculé en utilisant (23) :

$$\boxed{\nabla^2 f = \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \phi}\right)\right).}$$

Vibrations d’une membrane rectangulaire

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