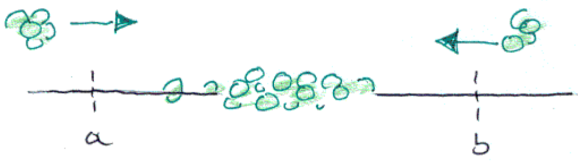
MASS CONSERVATION



RATE OF MASS INCREASE
IN THE INTERVAL

= RATE OF MASS INFLUX
ACROSS THE ENDS

$\forall$ INTERVALS !

$$\frac{\partial}{\partial t}\int_a^b \rho \, dx = \rho_a v_a - \rho_b v_b$$

$$\int_a^b \frac{\partial \rho}{\partial t} \, dx + [\rho v]_a^b = 0$$

$$\int_a^b \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) \right) dx = 0$$

$\forall a < b$

IF F CONTINUOUS !!

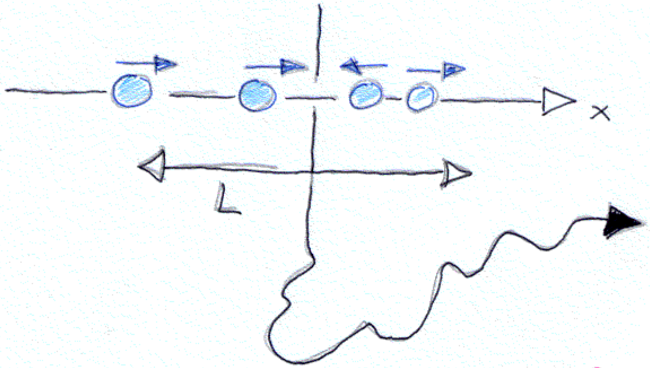
$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0}$$

CONTINUITY EQUATION

ALSO

$$\frac{D\rho}{Dt} + \rho \frac{\partial v}{\partial x} = 0$$

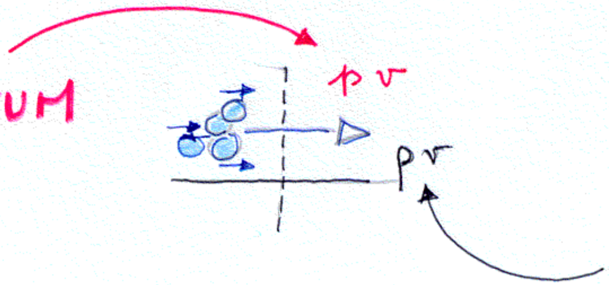
MOMENTUM DENSITY



$$\overbrace{p(x,t)}^{\text{MOMENTUM DENSITY}} \triangleq \underbrace{\rho(x,t)}_{\text{MASS DENSITY}}\, v(x,t)$$

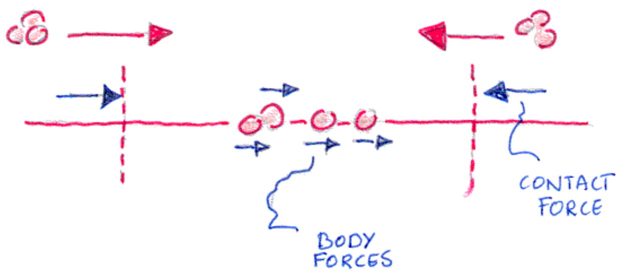
THIS IS IN FACT THE DEFINITION OF THE CONTINUOUS VELOCITY, THE MOMENTUM AND THE MASS DENSITY BEING DEFINED BY THE AVERAGING PROCESS !

MOMENTUM FLUX



MASS FLUX

MOMENTUM CONSERVATION



RATE OF MOMENTUM INCREASE IN THE INTERVAL

= RATE OF MOMENTUM INFLUX ACROSS ENDS

\+ APPLIED BODY FORCES

\+ CONTACT FORCES

$\forall$ INTERVALS !

$$\frac{\partial}{\partial t}\int_a^b \rho \, dx = \rho_A v_A - \rho_B v_B + (-\sigma_A) - (-\sigma_B) + \int_a^b \underbrace{\rho \, \underbrace{g}_{\text{MASS FORCE EX: GRAVITY}}}_{f \text{ BODY FORCE}} \, dx$$

σ

COMPRESSION < 0

TRACTION > 0

$$\int_a^b \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) \, dx = \int_a^b f + \frac{\partial \sigma}{\partial x} \, dx$$

$\forall a < b$

IF CONTINUITY ASSUMPTIONS !

$$\boxed{\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2) = f + \frac{\partial \sigma}{\partial x}}$$

MOMENTUM EQUATION

...... SOME ALGEBRA !

$$\underbrace{\rho \frac{\partial}{\partial t} v + \rho v \frac{\partial v}{\partial x}}_{\rho \frac{Dv}{Dt}} + \underbrace{v \frac{\partial \rho}{\partial t} + v \frac{\partial (\rho v)}{\partial x}}_{v\left(\underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x}}_{=0}\right)} = f + \frac{\partial \sigma}{\partial x}$$

CONTINUITY EQUATION

USUAL FORM OF MOMENTUM EQUATION

$$\rho \frac{Dv}{Dt} = f + \frac{\partial \sigma}{\partial x}$$

NEWTON'S SECOND LAW

$$m a = f$$

CONTINUUM PROBLEM

CONTINUITY EQUATION

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0}$$

MOMENTUM EQUATION (NO BODY FORCE $f = 0$)

$$\boxed{\rho \frac{Dv}{Dt} = \frac{\partial \sigma}{\partial x}}$$

2 EQUATIONS
3 UNKNOWNS ρ v σ

NEED OF A STATE EQUATION

IDEAL GAZ DYNAMICS

$$\boxed{pV = nRT}$$

PRESSURE

VOLUME

MOLES

CONSTANT

TEMPERATURE

$$\rho = \frac{n}{V}$$

$$\sigma = -p + \underbrace{\text{FRICTION TERMS}}_{\text{NEGLECTED HERE}}$$

REVERSIBLE DYNAMICS IS ONLY CONSIDERED

$$T \simeq K\rho^{\gamma-1}$$

$\gamma = 7/5$ FOR AIR

STATE LAW

$$\boxed{\begin{aligned} -\sigma &= \rho R T \\ &= R K \rho^{\gamma} \\ &= \left(\frac{k^2}{\gamma}\right)\rho^{\gamma} \end{aligned}}$$

k, γ CONSTANTS OF THE GAZ

$$k \triangleq \sqrt{\gamma R K}$$

$$\boxed{\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \\ &\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -k^2 \rho^{\gamma-1}\frac{\partial \rho}{\partial x} \end{aligned}}$$

QUASI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

2 UNKNOWNS
2 EQUATIONS

SOLVING THE PROBLEM OF PROPAGATION OF SOUND : DENSITY - VELOCITY WAVES

IDENTIFY A STATE OF REST

$$\left.\begin{aligned} v &= 0 \\ \rho &= \rho^* \end{aligned}\right)$$

"FIXED POINT"

THEN, INTRODUCE PERTUBATIONS

"SMALL PERTURBATIONS"

$$\begin{aligned} v &= \hat{v} \\ \rho &= \rho^* + \hat{\rho} \end{aligned}$$

$$\begin{cases} \dfrac{\partial \hat{\rho}}{\partial t} + \rho_* \dfrac{\partial \hat{v}}{\partial x} = 0 \\ \underbrace{(\rho^* + \hat{\rho})\left(\dfrac{\partial \hat{v}}{\partial t} + \hat{v}\dfrac{\partial \hat{v}}{\partial x}\right)}_{\simeq\, \rho^* \frac{\partial \hat{v}}{\partial t}} = -k^2 \rho_*^{\gamma-1} \underbrace{\left(1 + \dfrac{\hat{\rho}}{\rho_*}\right)^{\gamma-1}}_{\simeq\, 1} \dfrac{\partial \hat{\rho}}{\partial x} \end{cases}$$

$$\begin{cases} \dfrac{\partial \hat{\rho}}{\partial t} + \rho_* \dfrac{\partial \hat{v}}{\partial x} = 0 \\ \rho^* \dfrac{\partial \hat{v}}{\partial t} = -k^2 \rho_*^{\gamma-1} \dfrac{\partial \hat{\rho}}{\partial x} \end{cases}$$

$$\rho^* \frac{\partial^2 \hat{v}}{\partial x \partial t} = -\frac{\partial^2 \hat{\rho}}{\partial t^2}$$

$$\rho^* \frac{\partial^2 \hat{v}}{\partial x \partial t} = -k^2 \rho_*^{\gamma-1} \frac{\partial^2 \rho}{\partial x^2}$$

WAVE EQUATION

$$\boxed{\frac{\partial^2 \hat{\rho}}{\partial t^2} = k^2 \rho_*^{\gamma-1} \frac{\partial^2 \hat{\rho}}{\partial x^2}}$$

WAVE EQUATION

$$\frac{\partial^2 \hat{p}}{\partial t^2} = \underbrace{k^2 \rho_*^{\gamma-1}}_{\triangleq (c^*)^2} \frac{\partial^2 \hat{p}}{\partial x^2}$$

SOLUTION

$$\hat{p}(x,t) = f(x - c^* t) + g(x + c^* t)$$

SPECIFIED BY BOUNDARY AND INITIAL CONDITIONS

LOUD UNIDIRECTIONAL SOUND

WHITOUT ANY LOSS OF GENERALITY,

$$\gamma = 1$$
$$f = 0$$

CONSIDER ONLY SOLUTIONS FOR WHICH

$$p = R(v)$$

AND FIND EXACT SOLUTION

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial}{\partial x}(p v) = 0 \\ p\left(\dfrac{\partial v}{\partial t} + v \dfrac{\partial v}{\partial x}\right) = -k^2 \dfrac{\partial p}{\partial x} \end{cases}$$

$$R' \frac{\partial v}{\partial t} + R' \frac{\partial v}{\partial x} v + R \frac{\partial v}{\partial x} = 0$$

$$\left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}\right) = -\frac{R}{R'} \frac{\partial v}{\partial x}$$

ASSUMPTION $p = R(v)$ IS CONSISTENT ONLY IF THE FACTORS ARE THE SAME!

$$\left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}\right) = -k^2 \frac{R'}{R} \frac{\partial v}{\partial x}$$

$$(R')^2 k^2 = (R)^2$$

$$\boxed{\begin{aligned} R &= \exp(\pm v/k + A) \\ &= \underbrace{\exp(A)}_{\rho_*} \exp(\pm v/k) \\ R' &= \pm \frac{\rho_*}{k} \exp(\pm v/k). \end{aligned}}$$

$$\boxed{\frac{\partial v}{\partial t} + (\pm k + v)\frac{\partial v}{\partial x} = 0.}$$

+ RIGHT TRAVELLING WAVES

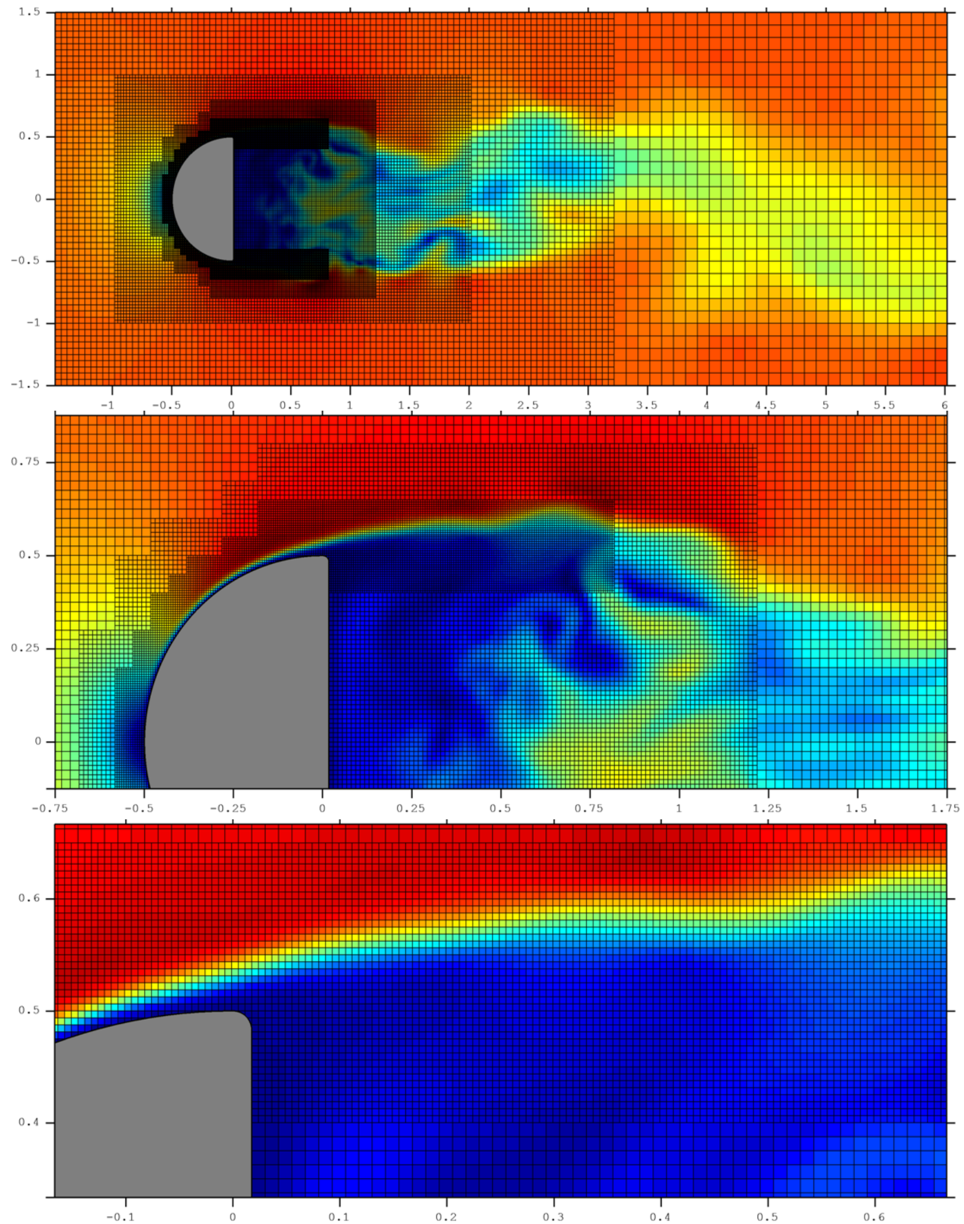
− LEFT TRAVELLING WAVES

NON LINEAR TERM INDUCES SHOCKS TO FORM

SONIC BOOMS

Que des équations ?

Vortex and Boundary Element Methods
Fast multipole solver
Hierarchically refined grid
Large Eddies Simulation
Flow past a hemisphere at Re=3000



Daeninck, Winckelmans, 2006

Nombre de Reynolds

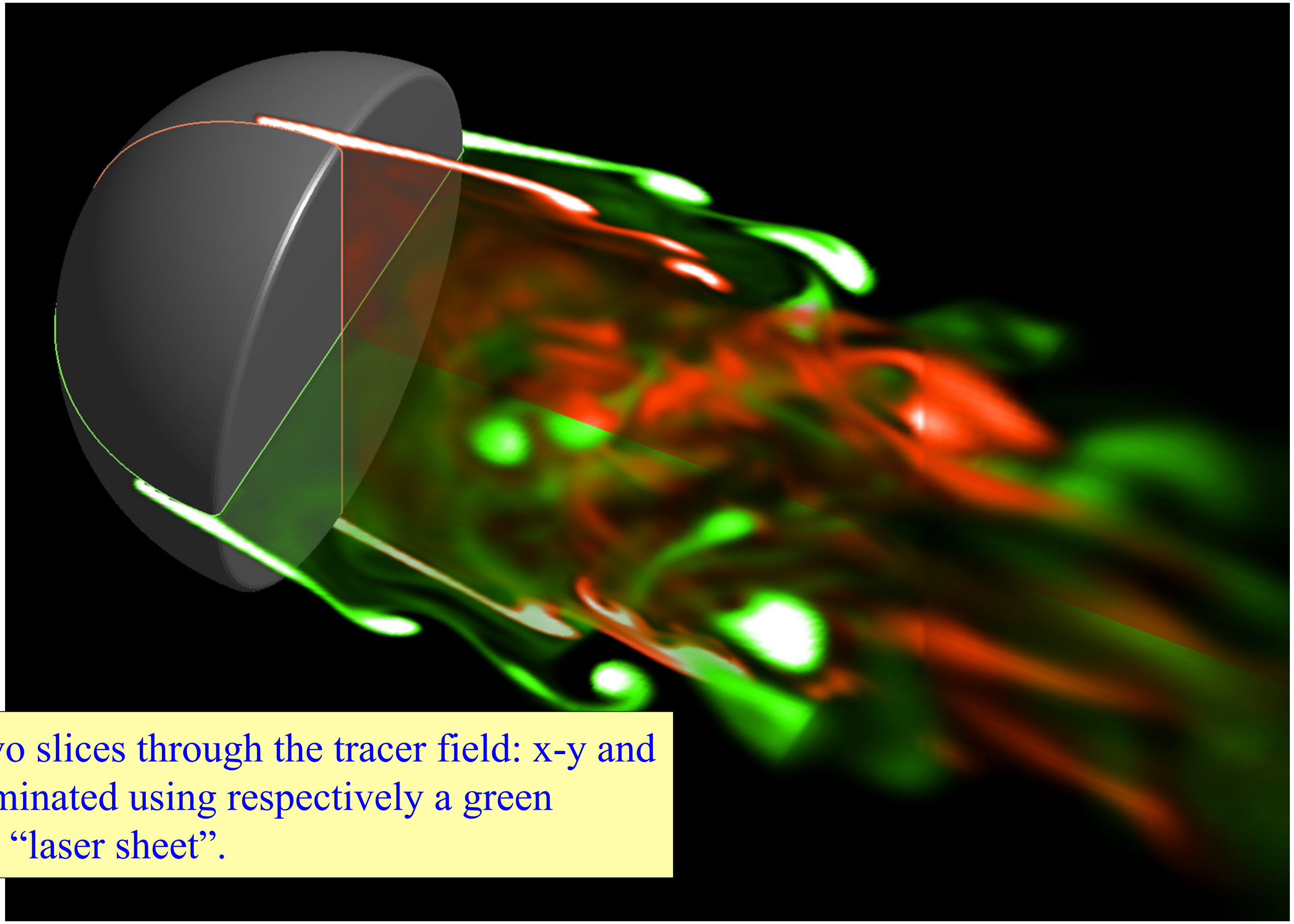
caractérise un écoulement d'un fluide !

$$Re = \frac{\rho_0 u_0 L}{\mu}$$

Born: 23 Aug 1842 in Belfast, Ireland

Died: 21 Feb 1912 in Watchet, Somerset, England

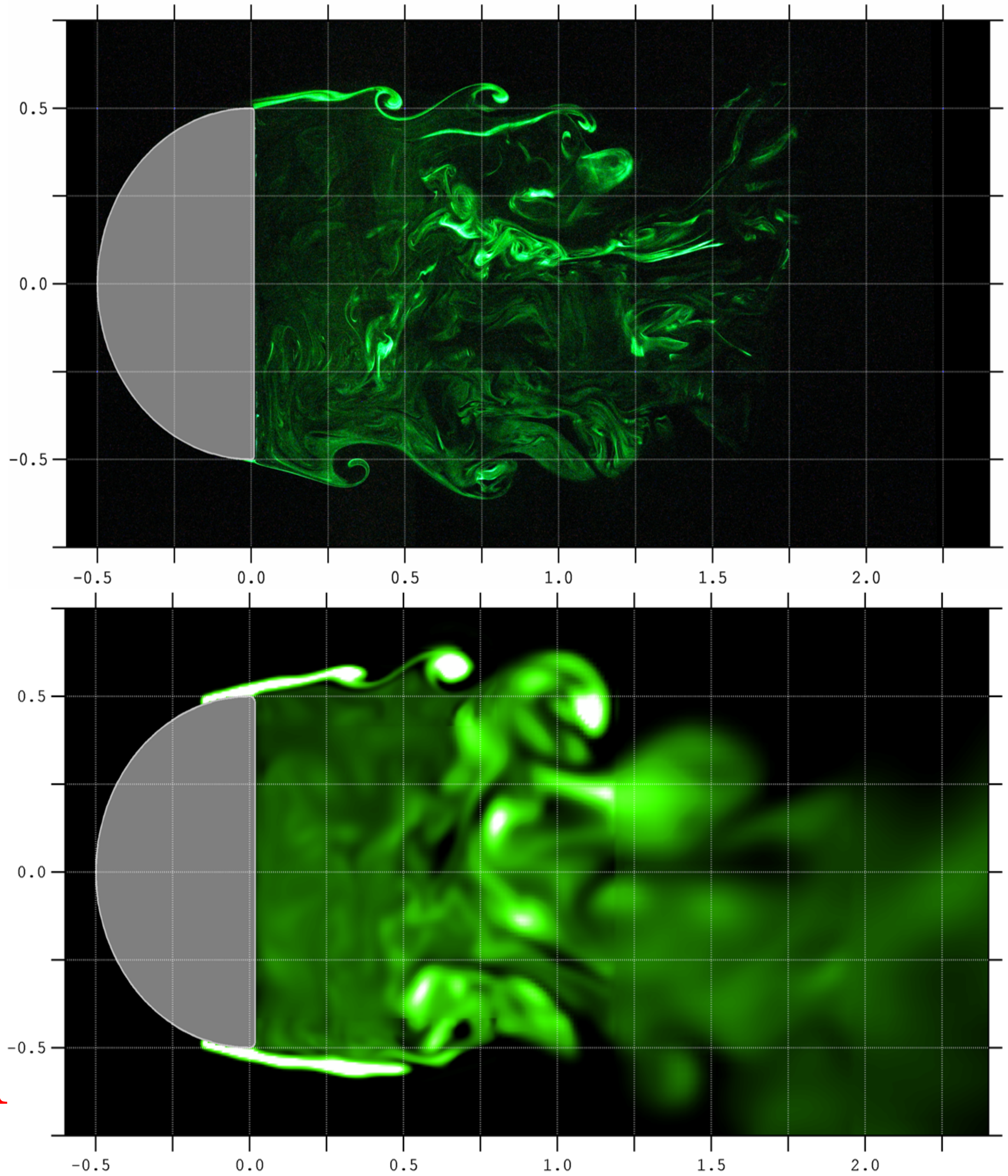
Flow past a hemisphere at Re=3000



Perspective view of two slices through the tracer field: x-y and y-z midplanes are illuminated using respectively a green "laser sheet" and a red "laser sheet".

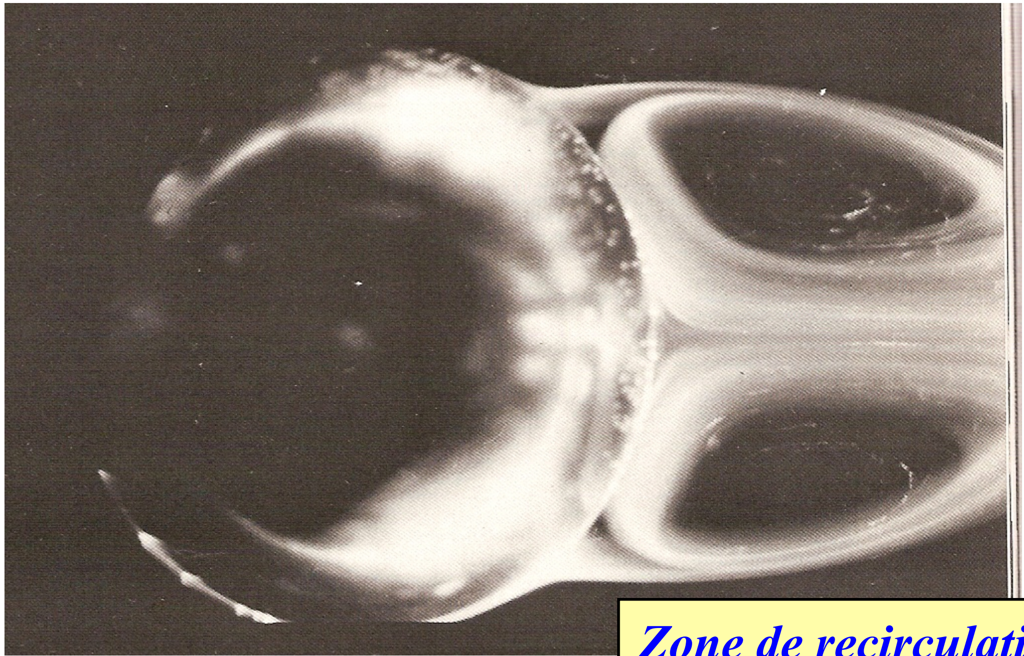
Expérience…

Experimental visualization of the flow past the hemisphere at Re=3,000 in a towing tank. Fluorescein is injected in the boundary layer at the front of the hemisphere.



…et simulation

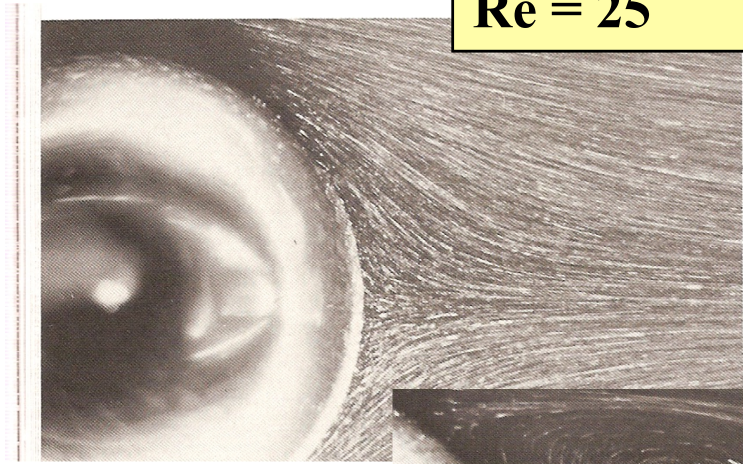
Ecoulement laminaire : Re = 104



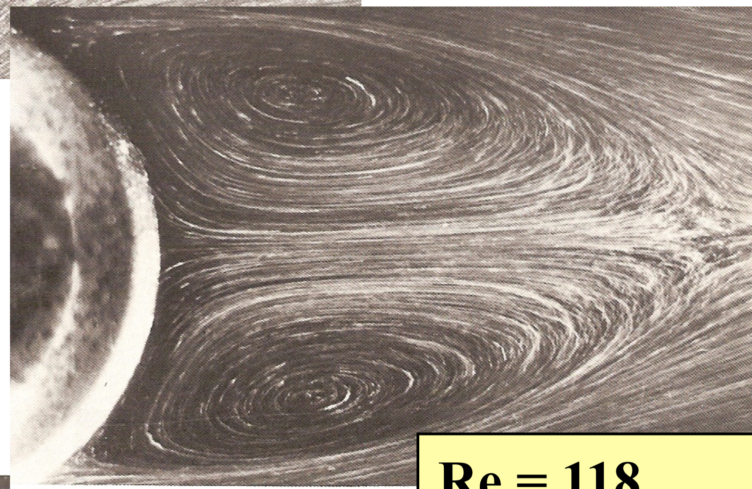
Zone de recirculation derrière une sphère
Re = 104

Taneda 1956
(from An Album of Fluid Motion, Van Dyke)

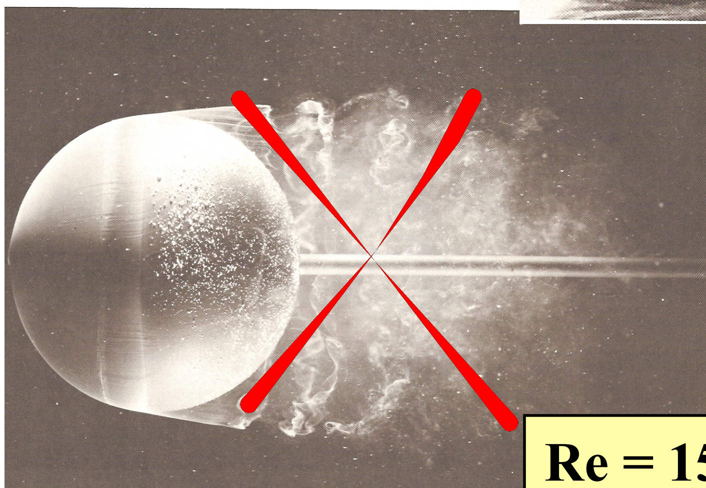
Ecoulement laminaire ?



Re = 25



Re = 118



Re = 15000

(Van Dyke, 1982)