LOCALLY COMPACT GROUPS WHOSE ERGODIC OR MINIMAL ACTIONS ARE ALL FREE

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Abstract. We construct locally compact groups with no non-trivial Invariant Random Subgroups and no non-trivial Uniformly Recurrent Subgroups.

1. Introduction

Let G be a locally compact group. We denote by $\mathrm{Sub}(G)$ the space of closed subgroups of G, endowed with the Chabauty topology [Cha50]. The space $\mathrm{Sub}(G)$ is compact, and G acts continuously by conjugation on $\mathrm{Sub}(G)$. Fixed points for this action are precisely the closed normal subgroups of G.

An Invariant Random Subgroup (IRS) of G is a probability measure on $\mathrm{Sub}(G)$ which is invariant under the action of G [AGV14]. For instance any closed subgroup H of finite covolume in G (e.g. a lattice) gives rise to an IRS by considering the push-forward of an invariant probability measure on G/H by the map $G/H \to \mathrm{Sub}(G)$, $gH \mapsto gHg^{-1}$. The space $\mathrm{IRS}(G)$ is closed for the weak-$*$ topology in the space $\mathcal{P}(\mathrm{Sub}(G))$ of probability measures on $\mathrm{Sub}(G)$, and hence is a compact space. Viewing lattices inside in this compact space led to recent developments, see e.g. [ABB$^+$17, Gel15a]. More generally if G acts on a probability space (X, μ) by measure preserving transformations, then the push-forward of μ through the stabilizer map $X \to \mathrm{Sub}(G)$ is an IRS of G. In fact, every IRS arises in this way [AGV14, ABB$^+$17].

The notion of IRS admits a natural topological counterpart, namely closed invariant subspaces of $\mathrm{Sub}(G)$. We will denote by $\mathcal{F}(\mathrm{Sub}(G))$ the compact space of closed subsets of $\mathrm{Sub}(G)$, and by $\mathcal{F}(\mathrm{Sub}(G))^G$ the subspace of closed invariant subsets. Among these, an important role is played by minimal ones. A minimal closed invariant subset of $\mathrm{Sub}(G)$ is called a Uniformly Recurrent Subgroup (URS) [GW15]. For instance if H is a closed cocompact subgroup of G, then the conjugacy class of H is closed in $\mathrm{Sub}(G)$, and is therefore a URS. More generally any minimal action of G on a compact space gives rise to a URS, called the stabilizer URS of the action [GW15]. The converse of this statement is also true, namely that every URS arises as the stabilizer URS of some minimal action, see [MBT17] (for finitely generated groups, this was also established in [Ele17]).

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In this article, we are interested in the following problems:

Question 1. Are there locally compact groups such that every IRS is a convex combination of δ_1 and δ_G ?

Question 2. Are there locally compact groups with no URS other than $\{\{1\}\}$ and $\{G\}$?

We will say that G has no non-trivial IRS's if every IRS is a convex combination of δ_1 and δ_G, and that G has no non-trivial URS's if $\{\{1\}\}$ and $\{G\}$ are the only URS's. Equivalently, G has no non-trivial IRS's if and only if every non-trivial ergodic probability measure preserving action of G is essentially free; and G has no non-trivial URS's if and only if every non-trivial minimal action on a compact space is topologically free, i.e. there is a dense G_δ set of points with trivial stabilizer.

Groups having no non-trivial IRS's or no non-trivial URS's are known to exist among *discrete groups*, and our focus in this paper will be about non-discrete groups, for which Questions 1 and 2 are open. For IRS's this question appears in [ABB+16], and is discussed in more details in [Gel15b]. Since any closed normal subgroup is a fixed point for the action of G on $\mathrm{Sub}(G)$, any potential candidate for having no non-trivial IRS's or no non-trivial URS's must be topologically simple. A group with no non-trivial IRS's should also fail to admit any lattice. Examples of compactly generated simple groups not containing any lattice have been exhibited in [BCGM12, LB16], but it is not known whether these families of groups contain instances with no non-trivial IRS's. Note that all the groups from [BCGM12] and [LB16] do admit non-trivial URS's (coming from their action on the boundary of the tree associated to them).

The goal of this article is to provide a simple construction of non-discrete groups answering Questions 1 and 2 simultaneously. The examples that we give are not compactly generated, and both questions remain open for non-discrete compactly generated groups (see the discussion below).

Groups of piecewise affine homeomorphisms. We denote by $\mathrm{PL}(\mathbb{Q}_p)$ the group of piecewise affine homeomorphisms of the local field $\mathbb{Q}_p$. These are homeomorphisms g such that there exists partition of $\mathbb{Q}_p$ into compact open subsets $(\mathcal{U}_i)_{i\geq 0}$ such that for all $i \geq 0$ the action of g on $\mathcal{U}_i$ is given by a map of the form $x \mapsto ax + b$, with $a \in \mathbb{Q}_p^*$ and $b \in \mathbb{Q}_p$.

Recall that two locally compact groups are locally isomorphic if they have isomorphic open subgroups.

Theorem 1.1. *For every countable product of finite groups U, there exists a totally disconnected locally compact group $G \leq \mathrm{PL}(\mathbb{Q}_p)$ that is locally isomorphic to U and such that:*

(i) the only IRS's of G are convex combinations of δ_1 and δ_G;
(ii) G has no URS other than $\{\{1\}\}$ and $\{G\}$.

We refer to §4.2 for an explicit description of groups G as in Theorem 1.1. All the examples given there share a common dense countable subgroup Λ, such that the groups G are Sclichting completions of Λ [Sch80]. For the proof of the absence of IRS's in G, we invoke results of Dudko–Medynets [DM14] implying that the *discrete group* Λ has no non-trivial IRS's. Using simple approximation arguments, we then deduce the absence of IRS's for the non-discrete group G. The proof of the absence of URS's follows the same strategy, by appealing to results from [LBMB16] concerning discrete groups. However we point out that in general results on IRS's or URS's of a countable group do *not* pass to its completions. This is for instance illustrated by the example of $\mathrm{PSL}(3, \mathbb{Q})$, which has no non-trivial IRS's (see Section 5), but whose completions $\mathrm{PSL}(3, \mathbb{Q}_p)$ do have non-trivial IRS's; for instance lattices. The main point of this paper is to provide a construction for which results on IRS's and URS's of discrete groups can be profitably used in order to study IRS's and URS's of non-discrete groups.

This construction admits some variations. In Section 5 we discuss two of them, arising as completions of the finitary alternating group $\mathrm{Alt}_f(\mathbb{Z})$, and of the group of infinite matrices $\mathrm{SL}(\infty, \mathbb{Q})$. These provide other examples answering (separately) Questions 1 and 2.

About compact generation. It is inherent to our argument that the examples of groups from Theorem 1.1 that we construct are not compactly generated. While candidates to answer Question 1 for compactly generated groups are available [BCGM12, LB16], the situation is rather different for URS's, and we do not know any example of a non-discrete compactly generated group for which we could conjecture the absence of non-trivial URS's. Such a group should be totally disconnected and topologically simple, i.e. it should belong to the class $\mathcal{S}$ studied by Caprace–Reid–Willis [CRW17a, CRW17b]. We refer to Appendix A from [CRW17b] for a list of currently known sources of examples of groups in $\mathcal{S}$, and to [Cap16] for a survey about recent developments.

An intermediate problem is whether there exist groups without non-trivial URS's which are non-elementary in the sense of Wesolek [Wes15].

Organization. The article is organized as follows. In the next section we introduce notation and recall basic facts about the Chabauty topology. In Section 3 we consider an approximation process of a locally compact group, and explain how the study of IRS's and URS's of the ambient group can be reduced to the study of certain IRS's and URS's of the approximating subgroups. We apply these results in Section 4, and construct the groups from Theorem 1.1. Finally in Section 5 we give additional examples of non-discrete groups having no non-tivial IRS's or URS's.

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2. Preliminaries

In this article all locally compact groups G are assumed to be second countable. We denote $\mathrm{Sub}(G)$ the set of closed subgroups of G, endowed with the Chabauty topology. Recall that a pre-basis of open sets for this topology is given by sets of the form

$$\mathcal{U}_V = \{H \in \mathrm{Sub}(G) \colon H \cap V \neq \varnothing\} \quad \mathcal{O}_C = \{H \in \mathrm{Sub}(G) \colon H \cap C = \varnothing\},$$

where V is a relatively compact open subset of G, and C a compact subset of G. The space $\mathrm{Sub}(G)$ is compact and metrisable (as we assume G second countable). The convergence in $\mathrm{Sub}(G)$ can be characterized as follows: a sequence (H_n) converges to H in $\mathrm{Sub}(G)$ if and only if every $h \in H$ is the limit of a sequence (h_n) where $h_n \in H_n$ for every n, and conversely every cluster point of such a sequence belongs to H.

The following facts are well-known (see e.g. [Sch71] for proofs).

Lemma 2.1. *Let G be a locally compact group.*

(i) Let H be a closed subgroup of G. Then the inclusion $\mathrm{Sub}(H) \to \mathrm{Sub}(G)$ is continuous.
(ii) Let N be a closed normal subgroup of G, and $\pi : G \to G/N$ the canonical projection. Then the map $\mathrm{Sub}(G/N) \to \mathrm{Sub}(G)$, $H \mapsto \pi^{-1}(H)$, is continuous. If moreover N is compact, $\mathrm{Sub}(G) \to \mathrm{Sub}(G/N)$, $H \mapsto \pi(H)$, is also continuous.
(iii) If $O \leq G$ is open, the intersection map $\mathrm{Sub}(G) \to \mathrm{Sub}(G)$, $H \mapsto H \cap O$, is continuous.
(iv) If (H_n) is a decreasing sequence of closed subgroups, then (H_n) converges to $\cap H_n$.
(v) If (H_n) is increasing, then (H_n) converges to $\overline{\cup H_n}$.

Let X be a compact metrisable space. A map $\varphi \colon X \to \mathrm{Sub}(G)$ is **upper semicontinuous** if for every sequence $(x_n) \subset X$ converging to a limit $x \in X$, every cluster point K of $(\varphi(x_n))$ in $\mathrm{Sub}(G)$ verifies $K \leq \varphi(x)$. We recall the following classical fact.

Lemma 2.2. *An upper semicontinuous map $\varphi \colon X \to \mathrm{Sub}(G)$ is measurable for the Borel σ-algebras on X and on $\mathrm{Sub}(G)$.*

Proof. It is enough to show that $\varphi^{-1}(\mathcal{U}_V)$ and $\varphi^{-1}(\mathcal{O}_C)$ are measurable whenever $V \subset G$ is open and $C \subset G$ is compact. First, we claim that $\varphi^{-1}(\mathcal{O}_C)$ is actually open, or equivalently that $\varphi^{-1}(\mathcal{O}_C^c) = \{x \in X \colon \varphi(x) \cap C \neq \varnothing\}$ is closed. Let $(x_n) \subset \varphi^{-1}(\mathcal{O}_C^c)$ be a sequence converging to a $x \in X$. Choose a sequence $g_n \in \varphi(x_n) \cap C$, and upon passing to a subsequence, (g_n) converges to $g \in C$. Then every cluster point K of $\varphi(x_n)$ contains g, and by upper semicontinuity we get that $g \in \varphi(x)$, proving that $x \in \varphi^{-1}(\mathcal{O}_C^c)$.

Now let $V \subset G$ be an open set. Since G is locally compact and second countable we may write V as the countable union of compact subsets C_n, and we have $\mathcal{U}_V = \bigcup \mathcal{O}_{C_n}^c$, showing that $\varphi^{-1}(\mathcal{U}_V)$ is also Borel. □

If X is a compact space we denote by $\mathcal{P}(X)$ the convex space of all probability measures on X, endowed with the weak-* topology, and by $\mathcal{F}(X)$ the set of closed subsets of X, endowed with the Hausdorff topology. Recall that $\mathcal{P}(X)$ and $\mathcal{F}(X)$ are metrizable whenever X is metrizable.

Statement (i) of Lemma 2.1 implies that whenever $H \leq G$ is a closed subgroup, the inclusion $\mathrm{Sub}(H) \subset \mathrm{Sub}(G)$ induces closed inclusions $\mathcal{P}(\mathrm{Sub}(H)) \subset \mathcal{P}(\mathrm{Sub}(G))$ and $\mathcal{F}(\mathrm{Sub}(H)) \subset \mathcal{F}(\mathrm{Sub}(G))$. In the sequel we will always use these identifications without further mention.

The conjugation action of G on $\mathrm{Sub}(G)$ induces continuous actions on $\mathcal{P}(\mathrm{Sub}(G))$ and $\mathcal{F}(\mathrm{Sub}(G))$. We denote $\mathrm{IRS}(G) \subset \mathcal{P}(\mathrm{Sub}(G))$ the space of G-invariant probability measures. It is a closed subspace of $\mathcal{P}(\mathrm{Sub}(G))$, and hence a compact space for the induced topology. Similarly we let $\mathcal{F}(\mathrm{Sub}(G))^G \subset \mathcal{F}(\mathrm{Sub}(G))$ be the set of G-invariant closed subsets of $\mathrm{Sub}(G)$. Again $\mathcal{F}(\mathrm{Sub}(G))^G$ is compact for the induced topology. We denote by $\mathrm{URS}(G) \subset \mathcal{F}(\mathrm{Sub}(G))^G$ the uniformly recurrent subgroups, i.e. minimal G-invariant closed subsets of $\mathrm{Sub}(G)$. Note that in general $\mathrm{URS}(G)$ is *not* closed in $\mathcal{F}(\mathrm{Sub}(G))$.

We will say for short that a group G has **no non-trivial IRS's** if the only IRS's of G are convex combinations of δ_1 and δ_G, and that G has **no non-trivial URS's** if its only URS's are $\{1\}$ and G.

3. Approximations in the Chabauty space

3.1. **Bi-approximations.** The approximation process considered in Sections 4 and 5 consists of a locally compact group G which can be written as an ascending union of open subgroups G_n, and which contains a decreasing sequence (U_n) such that U_n is a compact open normal subgroup of G_n. In this section we work in a slightly more general setting (in which U_n needs not be open nor normal in G_n), and explain how the study of IRS's and URS's of the group G can be reduced to the study of IRS's and URS's of G_n "relatively to" the coset space G_n/U_n (in a sense made precise below).

Definition 3.1. Let G be a locally compact group. A **bi-approximation** for G is a sequence (G_n, U_n) of pairs of closed subgroups $U_n \leq G_n \leq G$ such that:

(i) (G_n) is an increasing sequence of open subgroups such that $G = \cup G_n$;
(ii) (U_n) converges to $\{1\}$ in $\mathrm{Sub}(G)$, and $\cup U_n$ is relatively compact in G.

Remark 3.2. When the sequence of compact subgroups (U_n) is decreasing and $\cap U_n = 1$; then the second condition of Definition 3.1 is satisfied (see Lemma 2.1).

Yet another way to formulate the condition on the sequence (U_n) is is the following:

Lemma 3.3. *Let (U_n) be sequence of compact subgroups that is is relatively compact in G. The following are equivalent:*

(i) (U_n) converges to $\{1\}$ in $\mathrm{Sub}(G)$;
(ii) every sequence (u_n) with $u_n \in U_n$ converges to 1 in G.

Proof. If (U_n) converges to $\{1\}$ and $u_n \in U_n$ for all n, then any cluster point of (u_n) should belong to the limit of (U_n), and hence is trivial. Therefore the relatively compact sequence (u_n) admits 1 as unique cluster point, and hence converges to 1. The converse implication is clear. □

Before moving further, let us indicate that we do not suppose (G_n) to be strictly increasing in Definition 3.1, so that all results of this section hold true in the extreme case $G_n = G$

for all n. Similarly the U_n are not necessarily pairwise distinct; and $U_n = \{1\}$ for all n is also allowed.

3.2. **Truncation and saturation maps.** We will need the following terminology.

Definition 3.4. Let G be a group and $U \le G$ be a subgroup. For a subgroup $H \le G$, the U-**saturation** of H in G is the subgroup

$$[H]_U^G = \bigcap_{g \in G} HgUg^{-1}. \tag{1}$$

In other words, $[H]_U^G$ is the largest subgroup of G which acts on G/U with the same orbits as H. In particular $[[H]_U^G]_U^G = [H]_U^G$. We will say that H is U-**saturated** if $[H]_U^G = H$ (note that this depends not only on U but also on the ambient group G). When U is clear from the context, we will simply say that H is saturated.

If U is normal in G, then $[H]_U^G = HU$, and the set of saturated subgroups consists of preimages of subgroups of the quotient G/U. This motivates the terminology.

Remark 3.5. Let $\mathcal{P}$ be a partition of the coset space G/U and denote $G_{\mathcal{P}}$ the subgroup of G that preserves individually each subset of the partition $\mathcal{P}$. Then $G_{\mathcal{P}}$ is saturated, and every saturated subgroup is of this form.

Lemma 3.6. *Let G be a locally compact group, and $U \le G$ a compact subgroup. Then $[H]_U^G$ is closed in G for every closed subgroup $H \le G$.*

Proof. For every $g \in G$, the set $HgUg^{-1}$ is the product of a closed subgoup and a compact subgroup, and hence is a closed subset of G. It follows that $[H]_U^G = \bigcap_g HgUg^{-1}$ remains closed. ☐

Definition 3.7. When U is a compact subgroup of G, the map $\mathrm{Sub}(G) \to \mathrm{Sub}(G)$, $H \mapsto [H]_U^G$, will be called the **saturation map** relative to G/U.

The next lemma says that the saturation map $H \mapsto [H]_U^G$ is well behaved with respect to the Chabauty topology.

Lemma 3.8. *Let G be a locally compact group, and $U \le G$ a compact subgroup. The following hold:*

(i) the saturation map $H \mapsto [H]_U^G$ is equivariant.
(ii) For every H_n converging to H and every cluster point K of $[H_n]_U^G$, we have $H \le K \le [H]_U^G$. In particular $H \mapsto [H]_U^G$ is upper semicontinuous, and measurable.
(iii) If U is moreover assumed to be normal, then $H \mapsto [H]_U^G$ is continuous.

Proof. We denote by φ the map from the statement.

(i). The fact that φ is equivariant is apparent from the formula (1).

(ii). Let (n_k) such that $\varphi(H_{n_k})$ converges to K. Since $H_{n_k} \le \varphi(H_{n_k})$ for all k and H_{n_k} converges to H, we have $H \le K$. Moreover for every $g \in G$, the subset $H_{n_k}gUg^{-1}$ converges to $HgUg^{-1}$ in the Hausdorff topology on all closed subsets of G. Since $\varphi(H_{n_k}) \subset H_{n_k}gUg^{-1}$, it follows that K is contained in $HgUg^{-1}$ for all $g \in G$, and therefore K is contained in $\varphi(H)$. This shows upper semicontinuity, and measurability follows (Lemma 2.2).

(iii) follows from Lemma 2.1. ☐

The image of the saturation map $[\,\cdot\,]_U^G$ is exactly the set of saturated subgroups. We warn the reader that it is not a closed subset of $\mathrm{Sub}(G)$ in general. This is illustrated by the following example, which shows simultaneously that the map $[\,\cdot\,]_U^G$ is not continuous in general.

Example 3.9. Suppose that we have $G = H \rtimes U$, where U is a non-trivial compact group such that for every $u \neq 1$, the set of $[h, u]$, when h varies in H, is the entire H. If H is not isolated in $\mathrm{Sub}(H)$, then the set of saturated subgroups is not closed in $\mathrm{Sub}(G)$, and $[\,\cdot\,]_U^G$ is not continuous.

Indeed, let (H_n) be a sequence of proper subgroups of H converging to H, and let $\gamma \in [H_n]_U^G$. Since $\gamma \in H_n U$, one may write $\gamma = h_n u_0$. Additionally for every $h \in H$, there exists $k_n \in H_n$ and $u \in U$ such that $k_n \gamma = huh^{-1}$, i.e. $k_n h_n u_0 = [h, u]u$. Necessarily we have $u = u_0$, and $k_n h_n = [h, u_0]$. Therefore $[H, u_0]$ lies in a proper subgroup of H, which forces u_0 to be trivial by our assumption. So we deduce $\gamma = h_n$ and finally $[H_n]_U^G = H_n$. Therefore H_n is saturated for all n, while $[H]_U^G = G$ shows that the limit H is not saturated. A concrete example is given by $G = \mathbb{Z}[1/2] \rtimes \{\pm 1\}$ and $H_n = \frac{1}{2^n}\mathbb{Z}$ for $n \geq 1$.

Definition 3.10. Let (G_n, U_n) be a bi-approximation of G. The maps

$$\lambda_n \colon \mathrm{Sub}(G) \to \mathrm{Sub}(G),\ H \mapsto [H \cap G_n]_{U_n}^{G_n}$$

will be called the **trunc-saturation** maps associated (G_n, U_n).

For future reference we summarize the following basic properties:

Proposition 3.11. *For every n, the map λ_n is Borel-measurable, upper semicontinuous, G_n-equivariant (but not necessarily G-equivariant), and takes values in the set of closed U_n-saturated subgroups of G_n. Moreover λ_n is continuous if $U_n \trianglelefteq G_n$.*

Proof. Note that the map $H \mapsto H \cap G_n$ must be continuous since G_n is open in G (Lemma 2.1 (iii)), so all the properties follow from Lemmas 3.6 and 3.8. ☐

The following fact will be used repeatedly.

Lemma 3.12. *Let (G_n, U_n) be a bi-approximation of G, and let $\lambda_n \colon \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ be the associated trunc-saturation maps. Then the sequence of maps (λ_n) converges uniformly, as $n \to \infty$, to the identity map $\mathrm{Sub}(G) \to \mathrm{Sub}(G)$.*

Proof. Since $\mathrm{Sub}(G)$ is metrisable, the statement can be equivalently formulated as follows: for every sequence H_n converging to H in $\mathrm{Sub}(G)$, the sequence $\lambda_n(H_n)$ also converges to H. Let us show that this holds.

Let $h \in H$. Since H_n tends to H, we can choose a sequence $h_n \in H_n$ that converges to h. If n is large enough, we can assume $h \in G_n$ hence also $h_n \in G_n$ for n sufficiently large. It follows that eventally $h_n \in H_n \cap G_n$, which is contained in $\lambda_n(H_n)$ and therefore

h belongs to any cluster point of $(\lambda_n(H_n))$. Conversely let $h_n \in \lambda_n(H_n)$ be a sequence converging to some $h \in G$. Since $\lambda_n(H) \subset HU_n$ for all n and all $H \in \mathrm{Sub}(G)$, we can write $h_n = h'_n u_n, h'_n \in H_n, u_n \in U_n$, and we have $u_n \to 1$ (Lemma 3.3). It follows that also $h'_n \in H_n$ converges to h. Therefore $h \in H$. □

3.3. **Bi-approximations and probability measures on** $\mathrm{Sub}(G)$**.** Assume (G_n, U_n) is a bi-approximation of G, and denote λ_n the associated trunc-saturation maps. Measurability and equivariance of λ_n (Proposition 3.11) imply that λ_n induces a map $\lambda_n^* \colon \mathcal{P}(\mathrm{Sub}(G)) \to \mathcal{P}(\mathrm{Sub}(G))$ by pushforward of measures, which remains G_n-equivariant (but not necessarily G-equivariant).

Definition 3.13. Let G be a locally compact group and $U \leq G$ be a compact subgroup. We say that $\mu \in \mathrm{IRS}(G)$ is U-**saturated** if μ-almost every $H \in \mathrm{Sub}(G)$ is U-saturated.

Remark 3.14. When U is normal in G, an IRS of G is U-saturated if and only if it is the pushforward of an IRS of G/U under the natural map $\mathrm{Sub}(G/U) \to \mathrm{Sub}(G)$, and the set of U-saturated IRS's is naturally identified with $\mathrm{IRS}(G/U)$.

Proposition 3.15. *Let (G_n, U_n) be a bi-approximation of G. Then the sequence of maps*

$$\lambda_n^* \colon \mathcal{P}(\mathrm{Sub}(G)) \to \mathcal{P}(\mathrm{Sub}(G))$$

converges uniformly, as $n \to \infty$, to the identity of $\mathcal{P}(\mathrm{Sub}(G))$. In particular every $\mu \in \mathrm{IRS}(G)$ is the limit of a sequence $\nu_n \in \mathrm{IRS}(G_n)$, where each ν_n is U_n-saturated.

Proof. Arguing as in the proof of Lemma 3.12, it is enough to show that for every sequence $\mu_n \in \mathcal{P}(\mathrm{Sub}(G))$ converging to some limit $\mu \in \mathcal{P}(\mathrm{Sub}(G))$, the sequence $\lambda_n^*(\mu_n)$ still converges to μ. To this end, let $f \colon \mathrm{Sub}(G) \to \mathbb{R}$ be a continuous function, and observe that

$$\left|\int f d\lambda_n^*(\mu_n) - \int f d\mu\right| = \left|\int f \circ \lambda_n d\mu_n - \int f d\mu\right| \leq \left|\int f d\mu_n - \int f d\mu\right| + \int |f \circ \lambda_n - f| d\mu_n.$$

The first term of the last sum tends to 0 because $\mu_n \to \mu$. To bound the second term, observe that $f \circ \lambda_n$ tends to f uniformly, because f is continuous (hence uniformly continuous) and λ_n tends to the identity uniformly by Lemma 3.12. Therefore $\int |f \circ \lambda_n - f| d\mu_n \leq \sup |f \circ \lambda_n - f| \to 0$. Finally the last sentence follows by taking $\nu_n = \lambda_n^*(\mu)$. □

Theorem 3.16. *If (G_n, U_n) is a bi-approximation of G, the following are equivalent:*

(i) the group G has no non-trivial IRS;
(ii) for every sequence $\mu_n \in \mathrm{IRS}(G_n)$ of U_n-saturated IRS's, every cluster point of (μ_n) in $\mathcal{P}(\mathrm{Sub}(G))$ is a convex combination of δ_1 and δ_G.

Proof. The implication (i)⇒(ii) follows from the observation that every cluster point of μ_n is G_n-invariant for arbitrarily high n and therefore is an IRS of G. (ii)⇒(i) follows from the last statement of Proposition 3.15. □

In the particular case when U_n is normal in G_n, we obtain:

Corollary 3.17. *If (G_n, U_n) is a bi-approximation of G such that each U_n is normal in G_n, and the group G_n/U_n has no non-trivial IRS's for all $n \geq 1$, then G has no non-trivial IRS's.*

Proof. Under the assumption, every $\mu_n \in \mathrm{IRS}(G_n, G_n/U_n)$ is a convex combination of δ_{U_n} and δ_{G_n}. Since $U_n \to \{1\}$ and $G_n \to G$ in $\mathrm{Sub}(G)$, it follows that every cluster point of such measures is a convex combination of δ_1 and δ_G. The statement then follows from Theorem 3.16. □

3.4. **Bi-approximations and closed subsets of** $\mathrm{Sub}(G)$**.** We now analyse the topological setting.

Definition 3.18. Let G be a locally compact group and let $U \leq G$ be a compact subgroup. Let $\mathcal{H} \in \mathcal{F}(\mathrm{Sub}(G))^G$. We say that $\mathcal{H}$ is U-**saturated** if the set of U-saturated subgroups is dense in $\mathcal{H}$.

Remarks 3.19. (i) If $\mathcal{H}$ is U-saturated, it automatically follows that the set of U-saturated subgroups of $\mathcal{H}$ is a dense G_δ subset of $\mathcal{H}$. This is because the saturation map $H \mapsto [H]_U^G$, being upper semi-continuous, is continuous on a dense G_δ subset of $\mathcal{H}$ (see [Cho48, p. 95, Th. 1], and recall that $\mathcal{H}$ is metrisable here). Since $[\,\cdot\,]_U^G$ coincides with the identity on a dense subset, it must be the identity on every continuity point.

(ii) If U is normal, then $\mathcal{H} \in \mathcal{F}(\mathrm{Sub}(G))^G$ is saturated if and only if it consists of saturated subgroups. This is because in this case the map $H \mapsto [H]_{G/U}^G$ is continuous by Lemma 3.8. In particular the set of U-saturated $\mathcal{H} \in \mathcal{F}(\mathrm{Sub}(G))^G$ is in bijection with $\mathcal{F}(\mathrm{Sub}(G/U))^{G/U}$.

Definition 3.20. Let (G_n, U_n) be a bi-approximation of a locally compact group G, with trunc-saturation maps $\lambda_n \colon \mathrm{Sub}(G) \to \mathrm{Sub}(G)$. We define a sequence of G_n-equivariant maps $\bar{\lambda}_n \colon \mathcal{F}(\mathrm{Sub}(G)) \to \mathcal{F}(\mathrm{Sub}(G))$ by

$$\bar{\lambda}_n(\mathcal{H}) = \overline{\{\lambda_n(H) \,:\, H \in \mathcal{H}\}}.$$

We will need the following lemma.

Lemma 3.21. *Let $n \geq 1$. Let $\mathcal{H} \in \mathcal{F}(\mathrm{Sub}(G))$. Then for every $K \in \bar{\lambda}_n(\mathcal{H})$, there exists $H \in \mathcal{H}$ such that $H \cap G_n \leq K \leq \lambda_n(H)$.*

Proof. Let $(H_k) \subset \mathcal{H}$ such that $(\lambda_n(H_k))$ converges to K as $k \to \infty$ (note that n is fixed). Upon taking a subsequence we may assume that (H_k) converges to some $H \in \mathcal{H}$. By continuity we have $H_k \cap G_n \to H \cap G_n$ (Lemma 2.1), and the statement then follows from Lemma 3.8. □

The following is the analogue of Proposition 3.15 for closed subspaces of $\mathrm{Sub}(G)$ rather than probability measures on $\mathrm{Sub}(G)$.

Proposition 3.22. *Let G be a locally compact group, and let (G_n, U_n) be a bi-approximation of G. Then the sequence of maps*

$$\bar{\lambda}_n \colon \mathcal{F}(\mathrm{Sub}(G)) \to \mathcal{F}(\mathrm{Sub}(G))$$

converges uniformly, as $n \to \infty$, to the identity of $\mathcal{F}(\mathrm{Sub}(G))$. In particular, every $\mathcal{H} \in \mathcal{F}(\mathrm{Sub}(G))^G$ is the limit of a sequence $\mathcal{K}_n \in \mathcal{F}(\mathrm{Sub}(G_n))^{G_n}$, where each $\mathcal{K}_n$ is U_n-saturated.

Proof. Arguing as in the proof of Lemma 3.12, we have to show that for every sequence $\mathcal{H}_n \in \mathcal{F}(\mathrm{Sub}(G))$ converging to a limit $\mathcal{H}$, the sequence $\bar{\lambda}_n(\mathcal{H}_n)$ also converges to $\mathcal{H}$ as $n \to \infty$.

Let $H \in \mathcal{H}$, and choose a sequence $H_n \in \mathcal{H}_n$ that converges to H. By Lemma 3.12 the sequence $(\lambda_n(H_n))$ also converges to $\mathcal{H}$. Since $\lambda_n(H_n) \in \overline{\lambda}_n(\mathcal{H}_n)$ and $H \in \mathcal{H}$ was arbitrary, this implies that $\mathcal{H}$ is contained in every cluster point of $\overline{\lambda}_n(\mathcal{H}_n)$.

To show the converse, let $K_n \in \overline{\lambda}_n(\mathcal{H}_n)$ be a sequence converging to some $K \in \mathrm{Sub}(G)$. By Lemma 3.21, for every n there exists $H_n \in \mathcal{H}_n$ such that $H_n \cap G_n \leq K_n \leq \lambda_n(H_n)$. Up to taking a subsequence we may assume that H_n converges to a limit H, which belongs to $\mathcal{H}$ since $\mathcal{H}_n \to \mathcal{H}$. It is easy to see that $H_n \cap G_n$ also tends to H. Moreover, applying again Lemma 3.12 we get that $\lambda_n(H_n) \to H$. Since $H_n \cap G_n \leq K_n \leq \lambda_n(H_n)$, it follows that $K_n \to H$ as well. Hence $K = H \in \mathcal{H}$. This shows that every cluster point of $(\mathcal{H}_n)$ is contained in $\mathcal{H}$, and therefore $\mathcal{H}_n$ converges to $\mathcal{H}$. □

Theorem 3.23. *Given a bi-approximation (G_n, U_n) of a locally compact group G, the following are equivalent:*

(i) the group G has no non-trivial URS's;
(ii) for every sequence $\mathcal{H}_n \in \mathcal{F}(\mathrm{Sub}(G_n))^{G_n}$ where each $\mathcal{H}_n$ is U_n-saturated, every cluster point of $(\mathcal{H}_n)$ in $\mathcal{F}(\mathrm{Sub}(G))$ contains $\{1\}$ or G;

In particular, if U_n is normal in G_n and G_n/U_n has no non-trivial URS's for every n, then G has no non-trivial URS's.

Proof. Every cluster point of $(\mathcal{H}_n)$ with $\mathcal{H}_n \in \mathcal{F}(\mathrm{Sub}(G_n))^{G_n}$ must be G-invariant since $G = \cup G_n$, and hence contains a URS by Zorn's lemma. So (i) implies (ii), and the converse follows from Proposition 3.22. □

4. Groups of piecewise affine homeomorphisms

The goal of this section is to apply the results of Section 3 in order to prove Theorem 1.1.

Definition 4.1. A homeomorphism g of $\mathbb{Q}_p$ is **piecewise affine** if there exists a (possibly infinite) partition of $\mathbb{Q}_p$ into clopen subsets $(\mathcal{U}_i)_{i\geq 0}$ such that for all $i \geq 0$ the action of g on $\mathcal{U}_i$ is given by a map of the form $x \mapsto a_i x + b_i$, with $a_i \in \mathbb{Q}_p^*$ and $b_i \in \mathbb{Q}_p$.

We denote by $\mathrm{PL}(\mathbb{Q}_p)$ the group of piecewise affine homeomorphisms of $\mathbb{Q}_p$, and by $\mathrm{PL_c}(\mathbb{Q}_p)$ the (normal) subgroup of $\mathrm{PL}(\mathbb{Q}_p)$ consisting of compactly supported elements.

4.1. **Discrete groups of piecewise affine homeomorphisms.**

Before proving Theorem 1.1, we need some intermediate results about certain countable subgroups of $\mathrm{PL_c}(\mathbb{Q}_p)$. The purpose of this paragraph is to establish them, by combining results from [Nek13, DM14, LBMB16].

Definition 4.2. We denote Λ_p the group of homeomorphisms of $\mathbb{Z}_p$ acting piecewise by maps of the form $x \mapsto ax + b$, with $a \in p^{\mathbb{Z}}$ and $b \in \mathbb{Z}[1/p]$.

The groups Λ_p are member of a larger family of groups studied in [Nek04, Nek13] (with different terminology). To any self-similar group G acting on a rooted tree, Nekrashevych associates a group $\mathcal{V}_G$ of homeomorphisms of the boundary of the tree, which contains a copy of the Higman–Thompson group [Nek13, Def. 3.2]. In the present situation the rooted tree is a p-ary tree, whose boundary is identified with $\mathbb{Z}_p$ via $\{0,\dots,p-1\}^{\mathbb{N}} \xrightarrow{\sim} \mathbb{Z}_p$, $(a_n) \mapsto \sum a_n p^n$. If $A_p \leq \Lambda_p$ is the cyclic subgroup generated by $\alpha_p : x \mapsto x+1$, the action of A_p on $\mathbb{Z}_p$ extends to a self-similar action on the p-ary tree, called the adding machine in [Nek13, Ex. 3.2]. The copy of the Higman–Thompson group V_p inside Λ_p consists of all elements that locally preserve the lexicographic order of $\{0,\dots,p-1\}^{\mathbb{N}}$, and Λ_p is generated by V_p together with A_p.

In particular Theorems 4.7 and 4.8 in [Nek13] imply the following:

Proposition 4.3. *The group Λ_p' is simple, and Λ_p/Λ_p' is isomorphic to $\mathbb{Z}$ if $p=2$ and to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ for odd p.*

More precisely, the map $x \mapsto (x \mod V_p', 0)$ for $x \in V_p$, and $\alpha_p \mapsto (0,1)$, extends to a morphism of Λ_p, and induces an isomorphism $\Lambda_p/\Lambda_p' \simeq V_p/V_p' \times \mathbb{Z}$.

Dudko and Medynets exhibited sufficient dynamical conditions on a countable group of homeomorphisms ensuring the absence of non-trivial IRS's, and applied their results to the family of Higman-Thompson groups [DM14]. Here we apply their results to the group Λ_p'. The arguments are essentially the same as in [DM14], so we only give a sketch of proof, and refer to [DM14, Sec. 3] for details.

Proposition 4.4. *Λ_p' has no non-trivial IRS's.*

Proof. Let us consider the subgroup $\Delta \leq \Lambda_p'$ consisting of elements of Λ_p' acting trivially on a neighbourhood of 0. One can check that Δ can be written as an increasing union of subgroups all isomorphic to Λ_p', so that Δ is also a simple group (Proposition 4.3). Since the action of Δ on $\mathbb{Z}_p \setminus \{0\}$ is easily seen to be compressible [DM14, Def. 2.5], it follows from [DM14, Th. 2.9] that Δ has no proper character. Now applying [DM14, Th. 2.11] to the group Λ_p' and the subgroup Δ (the verification of the assumption on non-trivial conjugacy classes is easy, and we leave it), we obtain that the group Λ_p' has no proper character either, and hence no non-trivial IRS's. □

The next result is based on an application of [LBMB16, Cor. 3.12]. We will use the following notation: for a group G acting on a space X and a point $x \in X$, we will write G_x^0 for the subgroup of G consisting of elements acting trivially on a neighbourhood of x.

Recall that any minimal action of a locally compact group G on a compact space gives rise to a URS of G, called the stabilizer URS of the action [GW15, Prop. 1.2].

Proposition 4.5. *The stabilizer URS of Λ_p' associated to the action on $\mathbb{Z}_p$ is exactly the collection of subgroups $(\Lambda_p')_x^0$, when x varies in $\mathbb{Z}_p$; and it is the only non-trivial URS of Λ_p'.*

Proof. Given $\gamma \in \Lambda_p'$ and $x \in \mathbb{Z}_p$ a fixed point of γ, either x is an isolated fixed point, or γ acts trivially on a neighbourhood of x (because a non-identity affine map has either 0 or 1 fixed point in a given clopen subset of $\mathbb{Z}_p$). Therefore the subgroups in the stabilizer URS are exactly the $(\Lambda_p')_x^0$ according to [LBMB16, Prop. 2.10].

To show that this is the only non-trivial URS, we check that the conditions of [LBMB16, Cor. 3.12] are satisfied. Consider a coset $x+p^n\mathbb{Z}_p \subset \mathbb{Z}_p$. The subgroup H of Λ_p consisting of elements supported in the coset $x+p^n\mathbb{Z}_p$ is isomorphic to Λ_p. It follows from the description of the abelianisation map $\Lambda_p \to \Lambda_p/\Lambda_p'$ given in Theorem 4.8 in [Nek13] that its derived subgroup $H' \simeq \Lambda_p'$ coincides with the subgroup of Λ_p' consisting of elements supported in the coset $x + p^n\mathbb{Z}_p$. We conclude the latter is isomorphic to Λ_p', and hence is simple by Proposition 4.3. Moreover point stabilizers for the action on $\mathbb{Z}_p$ are maximal subgroups of Λ_p' (for instance because Λ_p' acts n-transitively for all n on each of its orbits in $\mathbb{Z}_p$). Hence we can apply [LBMB16, Cor. 3.12] to the action of Λ_p' on $\mathbb{Z}_p$, which gives the conclusion. □

4.2. **Locally compact groups of piecewise affine homeomorphisms.**

Definition 4.6. Let Γ_p be the subgroup of $\mathrm{PL_c}(\mathbb{Q}_p)$ consisting of elements such that the coefficients a_i, b_i from Definition 4.1 satisfy $a_i \in p^{\mathbb{Z}}$ and $b_i \in \mathbb{Z}[1/p]$ for all i. In other words, Γ_p is the set of g such that there exist disjoint compact open subsets $\mathcal{U}_1, \ldots, \mathcal{U}_n \subset \mathbb{Q}_p$ such that g is supported in the union, and g acts on $\mathcal{U}_i$ by $x \mapsto a_i x + b_i$ with $a_i \in p^{\mathbb{Z}}$ and $b_i \in \mathbb{Z}[1/p]$.

For $n \geq 0$, we denote $\mathcal{X}_n$ the set of p-adic numbers with valuation equal to $-(n+1)$, i.e. the complement of $p^{-n}\mathbb{Z}_p$ in $p^{-(n+1)}\mathbb{Z}_p$.

Definition 4.7. Given a family $\mathcal{F} = (F_n)$ of finite subgroups $F_n \leq \Gamma_p$ such that F_n is supported inside $\mathcal{X}_n$, we denote by $G_{\mathcal{F}} \leq \mathrm{PL}(\mathbb{Q}_p)$ the homeomorphisms g such that:

(i) there exists $N \geq 0$ such that g preserves $p^{-N}\mathbb{Z}_p$, and the restriction of g to $p^{-N}\mathbb{Z}_p$ is piecewise $ax + b$, with $a \in p^{\mathbb{Z}}$ and $b \in \mathbb{Z}[1/p]$;
(ii) the restriction of g to $\mathcal{X}_n$ coincides with an element of F_n for all $n \geq N$.

Note that Γ_p is a subgroup of $G_{\mathcal{F}}$, and Γ_p is actually the group of compactly supported elements of $G_{\mathcal{F}}$. Note that since the F_n have disjoint support, we can naturally view the product $\prod F_n$ as a subgroup of $\mathrm{PL}(\mathbb{Q}_p)$, and it follows from Definition 4.7 that $G_{\mathcal{F}}$ is generated by Γ_p and $\prod F_n$.

We endow $G_{\mathcal{F}}$ with the topology for which the sets $(\mathcal{V}_N(g))_{N\geq 1}$, with $\mathcal{V}_N(g) = g\prod_{n\geq N} F_n$, form a basis of neighbourhoods of $g \in G_{\mathcal{F}}$. Since every $g \in G_{\mathcal{F}}$ normalizes all but finitely many F_n, this indeed defines a group topology.

Theorem 4.8. *Assume that $F_n \leq \Gamma_p'$ for all $n \geq 0$. Then the (open, normal) subgroup of $G_{\mathcal{F}}$ generated by Γ_p' and $\prod F_n$ has no non-trivial URS's and no non-trivial IRS's.*

Remark 4.9. If all the groups F_n are perfect, then the subgroup generated by Γ_p' and $\prod F_n$ coincides with $G_{\mathcal{F}}'$. However in general $G_{\mathcal{F}}'$ is not closed in $G_{\mathcal{F}}$, and the subgroup generated by Γ_p' and $\prod F_n$ is the closure of $G_{\mathcal{F}}'$.

Remark 4.10. It is easy to see that any finite alternating group can be realized as a group F_n as in Definition 4.7. Consequently the same holds for any finite group, so that Theorem 1.1 follows from Theorem 4.8.

Proof of Theorem 4.8. We denote by G the subgroup of $G_{\mathcal{F}}$ generated by Γ'_p and $\prod F_n$. For $n \geq 0$, we let $\Gamma_{p,n}$ be the subgroup of Γ_p supported in $p^{-n}\mathbb{Z}_p$. Observe that all $\Gamma_{p,n}$ are isomorphic to each other (actually conjugated in $\mathrm{PL}(\mathbb{Q}_p)$); and isomorphic to the group Λ_p from Definition 4.2. Note also that Γ_p is the increasing union of all $\Gamma_{p,n}$; and hence Γ'_p is also the increasing union of $\Gamma'_{p,n}$.

Let G_n be the subgroup of G generated by $\Gamma'_{p,n}$ and $U_n = \prod_{k \geq n} F_k$. Note that since $\Gamma'_{p,n}$ and U_n have disjoint support, the subgroup G_n splits as a direct product $\Gamma'_{p,n} \times U_n$.

Lemma 4.11. *(G_n, U_n) is a bi-approximation of the group G.*

Proof. That G_n is open in G is clear. We have to argue that the sequence (G_n) is increasing. The description of $\Gamma'_{p,n+1}$ inside $\Gamma_{p,n+1}$ given in Theorem 4.8 from [Nek13] implies the equality $\Gamma_{p,n+1} \cap \Gamma'_p = \Gamma'_{p,n+1}$. Since $F_n \leq \Gamma'_p$ by our assumption and $F_n \leq \Gamma_{p,n+1}$ by definition, we deduce that F_n lies inside $\Gamma'_{p,n+1}$. Therefore G_{n+1} contains both F_n and U_{n+1}, and hence U_n. Since G_{n+1} also contains $\Gamma'_{p,n}$, we have verified the inclusion $G_n \leq G_{n+1}$. Note that $\cup G_n$ contains both Γ'_p and $\prod F_n$, and hence is equal to G. The condition on U_n is also satisfied because (U_n) is decreasing and $\cap U_n = 1$ (see Remark 3.2). □

Since U_n is a normal subgroup of G_n and the discrete quotient G_n/U_n is isomorphic to Λ_p, G_n/U_n has no non-trivial IRS's by Proposition 4.4. We can therefore apply Corollary 3.17 to the bi-approximation (G_n, U_n) of G, and deduce that G has no non-trivial IRS's.

We now have to prove the absence of non-trivial URS's. Recall that since U_n is normal in G_n, the subgroups of G_n which are U_n-saturated are the subgroups containing U_n. For $n \geq 1$, let $\mathcal{H}_n \in \mathcal{F}(\mathrm{Sub}(G_n))^{G_n}$ be a closed G_n-invariant U_n-saturated subset. Let $\mathcal{K}_n \in \mathcal{F}(\mathrm{Sub}(\Gamma'_{p,n+1}))^{\Gamma'_{p,n+1}}$ such that $\mathcal{H}_n$ is the preimage of $\mathcal{K}_n$ under the quotient map. The subset $\mathcal{K}_n$ must contain a URS of $\Gamma'_{p,n+1}$, so we deduce from Proposition 4.5 that $\mathcal{K}_n$ contains at least one of $\{1\}$, or $\Gamma'_{p,n+1}$, or the stabilizer URS associated to the action of $\Gamma'_{p,n+1}$ on $p^{-n}\mathbb{Z}_p$. In terms of $\mathcal{H}_n$, this implies that $\mathcal{H}_n$ contains U_n, or G_n, or $(G_n)^0_{x_n}$ with $x_n = p^{-n}$ (recall that the notation G^0_x has been defined before Proposition 4.5).

Lemma 4.12. *Let $x_n \in \mathbb{Q}_p$ such that $v_p(x_n) \to -\infty$. Then $(G_n)^0_{x_n} \to G$ in $\mathrm{Sub}(G)$.*

Proof. Write $k_n = v_p(x_n)$. Given $g \in G$, we define an element g_n by declaring that g_n acts on $p^{k_n+1}\mathbb{Z}_p$ like g, and trivially elsewhere. Since g preserves $p^{k_n+1}\mathbb{Z}_p$ eventually, the element g_n is well defined, $g_n \in (G_n)^0_{x_n}$ and $gg_n^{-1} \in \prod_{k \geq -k_n-1} F_k$. Therefore (g_n) converges to g, and the lemma is proved. □

Now assume that $(\mathcal{H}_{n_k})$ is a sequence of closed G_{n_k}-invariant subsets saturated relatively to G_{n_k}/U_{n_k}, such that $(\mathcal{H}_{n_k})$ converges to $\mathcal{H}$ in $\mathcal{F}(\mathrm{Sub}(G))$. If I_1, I_2, I_3 are respectively the set of integers k such that $U_{n_k} \in \mathcal{H}_{n_k}$, $G_{n_k} \in \mathcal{H}_{n_k}$ and $(G_{n_k})^0_{x_{n_k}} \in \mathcal{H}_{n_k}$; then according to the previous paragraph at least one of I_1, I_2, I_3 is infinite. Since $U_n \to \{1\}$, $G_n \to G$ and $(G_n)^0_{x_n} \to G$ by Lemma 4.12, we deduce that $\mathcal{H}$ must contain at least one of $\{1\}$ or G. Therefore we are in position to apply Theorem 3.23, which shows that G has no non-trivial URS's. □

5. More examples

5.1. **Locally elliptic groups.** Our goal in this paragraph is to provide other (and maybe more tractable) examples of non-discrete groups with no non-trivial URS's. These are based on a variation of a construction independently due to Willis [Wil07, Sec. 3] and Akin–Glasner–Weiss [AGW08, Sec. 4], and also considered by Caprace–Cornulier in [CC14]. As in [Wil07, AGW08], the groups will be defined as groups of permutations which are prescribed on certain blocks; the difference here being that we force the fixed point set of every permutation to be infinite.

The construction goes as follows. Let k_n be a strictly increasing sequence of natural numbers, with $k_0 = 0$. For all $n \geq 0$, choose a finite group $D_n \leq \mathrm{Alt}(\{k_n, \ldots, k_{n+1}-1\})$, and see the compact group $\prod_{n\geq 0} D_n$ as a permutation group on $\mathbb{Z}$ acting on $\mathbb{Z}_{\geq 0}$ on each interval $[k_n, k_{n+1}-1]$ via D_n, and acting trivially on $\mathbb{Z}_{<0}$. Consider the group G of permutations of $\mathbb{Z}$ generated by $\mathrm{Alt}_f(\mathbb{Z})$ (the group of finitary permutations of $\mathbb{Z}$) and $\prod D_n$. Equivalently, G can be described as the group of permutations moving only finitely many negative integers, and acting on all but finitely many intervals $[k_n, k_{n+1} - 1]$ like an element of D_n. Since every finitary permutation centralizes all but finitely many D_n, the group G carries a locally compact group topology for which the inclusion $\prod D_n \hookrightarrow G$ is a homeomorphism onto its image.

Theorem 5.1. *For every sequence (D_n), the following hold:*

(i) G has no non-trivial URS's;
(ii) G has no proper subgroup of finite covolume.

Recall that a if H is a closed subgroup of a locally compact group G, we say that H has finite covolume if G/H carries a G-invariant probability measure.

Before giving the proof of Theorem 5.1, let us make several observations:

Remark 5.2. (i) In contrast with the groups considered in Section 4, the group G does admit non-trivial IRS's, that arise from random invariant partitions of $\mathbb{Z}$ (as in the case of the alternating group [Ver12]).

(ii) A common feature between these groups and the ones from Section 4 is that every single element normalizes a compact open subgroup. Equivalently, Willis' scale function is identically equal to one [Wil94]. However an important difference is that here G is locally elliptic (i.e. every finite subset generates a relatively compact subgroup), while the groups from Definition 4.7 have plenty of infinite finitely generated discrete subgroups.

As in the previous section, the proof of the absence of URS's in G will appeal to auxiliary results about discrete groups, namely:

Proposition 5.3. *For an infinite set Ω, the group $\mathrm{Alt}_f(\Omega)$ does not admit non-trivial URS's.*

Here $\mathrm{Alt}_f(\Omega)$ is the group of finitely supported alternating permutations of Ω. Proposition 5.3 was proved by Thomas and Tucker-Drob in [TTD16] (for Ω countable), where it is deduced from Vershik's classification of the IRS's of this group [Ver12]. A closely related and more precise result goes back to Sehgal–Zalesskii [SZ93], who characterized the subgroups of $\mathrm{Alt}_f(\Omega)$ whose conjugacy class avoids an open neighbourhood of $\{1\}$ in the Chabauty space $\mathrm{Sub}(\mathrm{Alt}_f(\Omega))$. See the equivalence (ii) $\Leftrightarrow$ (iii) from Theorem 1 in [SZ93] (from which Proposition 5.3 easily follows).

Proof of Theorem 5.1. For $n \geq 1$, consider the open subgroup

$$G_n = \langle \mathrm{Alt}_f(\mathbb{Z}_{<k_n}) \cup \prod D_j \rangle,$$

which consists of elements of G acting on $[k_i, k_{i+1} - 1]$ like an element of D_i for all $i \geq n$. The sequence (G_n) is increasing and ascends to G, and if we write $U_n = \prod_{j \geq n} D_j \leq G_n$, then (U_n) is decreasing and $\cap U_n = 1$. So (G_n, U_n) is a bi-approximation of G. Note that U_n is normal in G_n and that $G_n \cong \mathrm{Alt}_f(\mathbb{Z}_{<k_n}) \times U_n$, so that the quotient G_n/U_n has no non-trivial URS's by Proposition 5.3. Therefore statement (i) follows by applying Theorem 3.23.

For (ii), let $H \leq G$ be a closed subgroup of finite covolume. Since G_n is open in G for all $n \geq 1$, $H \cap G_n$ has finite covolume in G_n. The subgroup U_n being compact, we deduce that the projection of $H \cap G_n$ to G_n/U_n is a closed subgroup of finite covolume. But $G_n/U_n = \mathrm{Alt}_f(\mathbb{Z}_{<k_n})$ is a discrete infinite simple group, and hence has no proper finite index subgroups. Therefore $H \cap G_n$ surjects onto G_n/U_n, and since (G_n) ascends to G and (U_n) decreases, we actually have $G = HU_n$ for arbitrary n. This shows that H is dense in G. Since H is also closed by assumption, we must have $H = G$. □

5.2. **Groups of infinite matrices.** In this paragraph we describe additional examples of non-discrete groups without IRS's. The construction is in spirit very close to the one carried out in [Wil07, Prop. 3.5].

Let $\mathbb{Q}^{(\mathbb{N})}$ be the infinite dimensional vector space over $\mathbb{Q}$ with basis $(e_i)_{i \in \mathbb{N}}$. Let $\mathrm{SL}(\infty, \mathbb{Q}) = \lim_d \mathrm{SL}(d, \mathbb{Q})$ be the group of linear isomorphisms of $\mathbb{Q}^{(\mathbb{N})}$ that fix for all but finitely many e_i, and with determinant one. Let (d_n) be a strictly increasing sequence of natural numbers, and for convenience we will assume that d_n is odd for infinitely many n. Write $k_n = d_{n+1} - d_n$. We choose a finite subgroup $D_n \leq \mathrm{SL}(k_n, \mathbb{Q})$ for all n, and view the group $\prod_{n \geq 0} D_n$ as a group of block-diagonal linear transformations of $\mathbb{Q}^{(\mathbb{N})}$, where each D_n acts on the subspace spanned by $\{e_{d_n+1}, \ldots, e_{d_{n+1}}\}$ and fixes all other elements of the basis. Finally we consider the group $G = \langle \mathrm{SL}(\infty, \mathbb{Q}) \cup \prod_{n \geq 0} D_n \rangle$, equipped with the topology for which the inclusion of $\prod D_n$ is continuous and open.

Proposition 5.4. *For every choice of (D_n), the group G has no non-trivial IRS's.*

A result of Kirillov [Kir65] (extended by Peterson and Thom in [PT16]) says that the group $\mathrm{PSL}(d, \mathbb{Q})$ has no non-trivial characters for $d \geq 3$, and therefore no non-trivial IRS's [DM14, Th. 2.11], [PT16, Th. 3.2]:

Proposition 5.5. *For $d \geq 3$, the group $\mathrm{PSL}(d, \mathbb{Q})$ has no non-trivial IRS's.*

The proof of Proposition 5.4 will again follow the same scheme, which consists in applying Corollary 3.17 to a suitable bi-approximation of G.

Proof of Proposition 5.4. If we denote by G_n the subgroup of G generated by $\mathrm{SL}(d_n, \mathbb{Q})$ and $\prod_{j\geq 0} D_j$, and by $U_n = \prod_{j\geq n} D_j \leq G_n$, we easily verify that (G_n, U_n) forms a bi-approximation of the group G. Moreover U_n is normal in G_n and $G_n/U_n \cong \mathrm{SL}(d_n, \mathbb{Q})$. Since d_n is odd for infinitely many n, up to taking a subsequence, G_n/U_n has no nontrivial IRS's by Proposition 5.5. Therefore the conclusion of Proposition 5.4 follows by applying Corollary 3.17. □

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