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par

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Géométrie des groupes localement compacts. Arbres. Action !

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Résumé

Dans le Chapitre 1 nous étudions les groupes localement compacts lacunaires hyperboliques. Nous caractérisons les groupes ayant un cône asymptotique qui est un arbre réel et dont l'action naturelle est focale. Nous étudions également la structure des groupes lacunaires hyperboliques, et montrons que dans le cas unimodulaire les sous-groupes ne satisfont pas de loi. Nous appliquons au Chapitre 2 les résultats précédents pour résoudre le problème de l'existence de points de coupure dans un cône asymptotique dans le cas des groupes de Lie connexes.

Dans le Chapitre 3 nous montrons que le groupe de Neretin est compactement présenté et donnons une borne supérieure sur sa fonction de Dehn. Nous étudions également les propriétés métriques du groupe de Neretin, et prouvons que certains sous-groupes remarquables sont quasi-isométriquement plongés.

Nous étudions dans le Chapitre 4 une famille de groupes agissant sur un arbre, et dont l'action locale est prescrite par un groupe de permutations. Nous montrons entre autres que ces groupes ont la propriété (PW), et exhibons des groupes simples au sein de cette famille.

Dans le Chapitre 5 nous introduisons l'éventail des relations d'un groupe de type fini, qui est l'ensembles des longueurs des relations non engendrées par des relations plus courtes. Nous établissons un lien entre la simple connexité d'un cône asymptotique et l'éventail des relations du groupe, et donnons une grande classe de groupes dont l'éventail des relations est aussi grand que possible.

Abstract

In Chapter 1 we investigate the class of locally compact lacunary hyperbolic groups. We characterize locally compact groups having one asymptotic cone that is a real tree and whose natural isometric action is focal. We also study the structure of lacunary hyperbolic groups, and prove that in the unimodular case subgroups cannot satisfy a law. We apply the previous results in Chapter 2 to solve the problem of the existence of cut-points in asymptotic cones for connected Lie groups.

In Chapter 3 we prove that Neretin's group is compactly presented and give an upper bound on its Dehn function. We also study metric properties of Neretin's group, and prove that some remarkable subgroups are quasi-isometrically embedded.

In Chapter 4 we study a family of groups acting on a tree, and whose local action is prescribed by some permutation group. We prove among other things that these groups have property (PW), and exhibit some simple groups in this family.

In Chapter 5 we introduce the relation range of a finitely generated group, which is the set of lengths of relations that are not generated by relations of smaller length. We establish a link between simple connectedness of asymptotic cones and the relation range of the group, and give a large class of groups having a relation range as large as possible.

Introduction

Cette thèse s'articule essentiellement autour de deux axes de recherche, représentant mes centres d'intérêts mathématiques jusqu'à aujourd'hui. Le premier a pour sujet la notion de cône asymptotique d'un groupe, et le second concerne l'étude de groupes agissant sur un arbre ou sur son bord, et dont l'action locale satisfait une condition de rigidité.

Groupes. La notion principale au coeur de cette thèse est la notion de groupe. Les groupes trouvent leur origine entre la fin du 18^e^ et le début du 19^e^ siècle, avec les travaux de Lagrange et Gauss, puis de Galois sur la résolution d'équations polynomiales. Ils apparaissent depuis dans de nombreux domaines mathématiques : en géométrie, topologie, théorie des nombres, théorie ergodique, et bien d'autres encore.

La fin du 20^e^ siècle voit naître la théorie géométrique des groupes avec entre autres les travaux de Mostow, Milnor, Stallings, Abels, Gromov, qui soulignent l'importance de l'apport de la géométrie à l'étude des objets algébriques que sont les groupes. L'idée sous-jacente à la géométrie à grande échelle, et plus généralement à la géométrie grossière, est d'étudier des objets en les observant « de très loin ». Cette approche consiste à négliger toute propriété locale, pour ne retenir que les configurations géométriques globales.

Si G est un groupe et S un sous-ensemble générateur, on peut munir G de la métrique des mots d_S associée à S. Si de plus G est muni d'une topologie de groupe localement compact et si S est supposé compact, alors la classe de quasi-isométrie de (G, d_S) ne dépend pas du choix de S. Cette observation fondamentale signifie que tout groupe localement compact compactement engendré possède une géométrie à grande échelle intrinsèque, et en particulier tout invariant de quasi-isométrie est un invariant du groupe lui-même.

Ce point de vue a été largement adopté et a conduit à des avancées spectaculaires dans le cadre des groupes discrets, l'exemple le plus frappant étant surement la théorie des groupes hyperboliques de type fini. A l'opposé, si G est un groupe de Lie réel connexe, la manière traditionnelle de munir G d'une distance est de considérer sur G une métrique riemannienne invariante à gauche. Associer à G une métrique des mots provenant d'un voisinage compact de l'identité peut dans un premier temps sembler brutal du point de vue de la géométrie riemannienne classique, mais se révèle utile dans l'étude des propriétés asymptotiques de G. En outre, les espaces métriques obtenus en munissant G d'une métrique riemannienne ou d'une métrique des mots sont quasi-isométriques.

Voir les groupes localement compacts compactement engendrés comme des objets géométriques à part entière, est un point de vue qui, comparativement au cas des groupes discrets et des groupes de Lie, a été nettement moins adopté.

Arbres. La seconde notion centrale de cette thèse est la notion d'arbre. Les arbres sont des espaces métriques très simples, satisfaisant la propriété agréable qu'étant donnés deux points dans un arbre, il existe un unique chemin injectif menant de l'un à l'autre.

Les arbres ayant un caractère continu sont appelés arbres réels. Ceux-ci apparaissent notamment en géométrie, souvent comme limite d'espaces hyperboliques, en topologie dans l'étude des variétés hyperboliques, en théorie des groupes ou encore en probabilités. Dans le Chapitre 1, nous étudions les groupes localement compacts compactement engendrés G admettant un cône asymptotique qui est un arbre réel. Un tel cône asymptotique est naturellement doté d'une action transitive par isométries, et le point de vue adopté ici est de relier les caractéristiques de cette action aux propriétés du groupe G.

Les arbres ayant un caractère discret sont appelés arbres simpliciaux. Ceux-ci sont des objets centraux en théorie des groupes, notamment via la théorie de Bass-Serre. Dans les Chapitres 3 et 4, nous étudions des groupes agissant sur un arbre simplicial ou sur son bord (ou, de manière informelle, sur un « voisinage » de ce bord), et dont l'action locale satisfait une condition de rigidité.

Cônes asymptotiques

Le théorème de croissance polynomiale de Gromov, qui caractérise les groupes de type finis virtuellement nilpotents comme étant ceux dont la croissance est polynomiale [Gro81], est sans doute l'un des résultats les plus frappant de la théorie géométrique des groupes. L'une des étapes de la preuve de Gromov consiste à construire, étant donné un groupe à croissance polynomiale Γ, un Γ-espace obtenu comme limite de copies de Γ avec métriques échelonnées. Cette construction a ensuite été généralisée par van den Dries et Wilkie, et nous rappelons maintenant sa définition.

Si (X, d) est un espace métrique non vide et $e \in X$, et si $\mathbf{s} = (s_n)$ est une suite de nombres réels positifs tendant vers l'infini, on note $\mathrm{Precone}(X, d, \mathbf{s})$ l'ensemble des suites (x_n) telles que $d(x_n, e) = O(s_n)$. Etant donné un ultrafiltre non-principal ω, on peut munir $\mathrm{Precone}(X, d, \mathbf{s})$ de la pseudo-distance $d_\omega((x_n), (y_n)) = \lim^\omega d(x_n, y_n)/s_n$, et le cône asymptotique $\mathrm{Cone}^\omega(X, d, \mathbf{s})$ de X associé aux paramètres $\mathbf{s}$ et ω est par définition l'espace métrique obtenu en identifiant dans $\mathrm{Precone}(X, d, \mathbf{s})$ les points à distance nulle. Lorsque (X, d) est un groupe localement compact muni de la métrique des mots d_S associée à un sous-ensemble compact générateur S, l'ensemble $\mathrm{Precone}(G, d_S, \mathbf{s})$ ne dépend pas du choix de S, et nous le notons $\mathrm{Precone}(G, \mathbf{s})$. Le cône asymptotique de G associée à la suite $\mathbf{s}$ et à l'ultrafiltre non principal ω est noté $\mathrm{Cone}^\omega(G, \mathbf{s})$. Notons que l'action du groupe sur lui-même se prolonge naturellement en une action isométrique et transitive du groupe $\mathrm{Precone}(G, \mathbf{s})$ sur l'espace $\mathrm{Cone}^\omega(G, \mathbf{s})$.

Pourquoi les cônes asymptotiques ?

L'un des intérêts de la notion de cône asymptotique est que, étant donnés $\mathbf{s}$ et ω fixés, le cône asymptotique $\mathrm{Cone}^\omega(G, \mathbf{s})$ est un invariant topologique (et même bi-Lipschitz) de la classe de quasi-isométrie de G. Les propriétés métriques ou topologiques des cônes asymptotiques d'un espace métrique X reflètent en un certain sens les propriétés géométriques de X, comme le montrent les résultats suivants :

(i) un groupe de type fini G est virtuellement nilpotent si et seulement si tous ses cônes asymptotiques sont localement compacts [Gro81, Dru02] ;

(ii) un espace métrique géodésique est hyperbolique si et seulement si tous ses cônes asymptotiques sont des arbres réels [Gro93, Dru02] ;

(iii) si G est un groupe localement compact compactement engendré dont tous les cônes asymptotiques sont simplement connexes, alors G est compactement présenté et a une fonction de Dehn polynomialement bornée [Gro93].

Notons que par un théorème dû à Pansu [Pan83], dans la situation (i) tous les cônes asymptotiques de G sont isométriques à un groupe de Lie « Carnot-gradué » canoniquement associé à G. Ce résultat a été étendu par Breuillard [Bre14] au cadre des groupes localement compacts à croissance polynomiale. Mentionnons également que Sapir a récemment démontré qu'un résultat d'Hrushovski entraîne que l'implication indirecte de (i) reste vraie sous l'hypothèse que G admet *un* cône asymptotique localement compact [Sap13].

La notion de cône asymptotique a également été utilisée pour établir des résultats de rigidité concernant entre autres les immeubles euclidiens et espaces symétriques de rang supérieur [KL97], les résaux non uniformes dans les groupes de Lie semi-simples de rang supérieur [Dru00], ou les groupes relativement hyperboliques [DS05].

Groupes lacunaires hyperboliques

Le premier exemple de groupe de type fini admettant des cônes asymptotiques non homéomorphes fut construit par Thomas et Velickovic [TV00]. L'idée de leur construction est de considérer un groupe défini par deux générateurs a, b et une suite de relateurs $w_n(a, b)$ satisfaisant une condition de petite simplification, et dont la longueur s_n croît très vite. Le groupe ainsi obtenu est tel que $\mathrm{Cone}^\omega(G, \mathbf{s})$ est non simplement connexe pour tout ultrafiltre ω, alors que les cônes asymptotiques correspondant à des suites d'échelons s'intercalant entre les valeurs de $\mathbf{s} = (s_n)$ sont des arbres réels.

L'étude générale des groupes de type fini ayant un cône asymptotique qui est un arbre réel a été systématisée par Olshanskii, Osin et Sapir [OOS09], qui ont baptisé ces groupes *lacunaires hyperboliques*. Les groupes lacunaires hyperboliques peuvent être caractérisés comme limites directes de groupes hyperboliques G_n avec morphismes surjectifs $\alpha_n : G_n \twoheadrightarrow G_{n+1}$, tels que la constante d'hyperbolicité de G_n est petite devant le rayon d'injectivité de α_n [OOS09]. Les travaux d'Olshanskii, Osin et Sapir montrent également qu'un groupe lacunaire hyperbolique peut avoir des propriétés très éloignées de celles des groupes hyperboliques : il existe des groupes lacunaires hyperboliques non virtuellement cycliques qui sont élémentairement moyennables, ou qui ont un centre infini, ou qui sont de torsion [OOS09].

La définition de groupe lacunaire hyperbolique s'étend immédiatement au cas d'un groupe G localement compact compactement engendré. Si ω et $\mathbf{s}$ sont tels que $\mathrm{Cone}^\omega(G, \mathbf{s})$ est un arbre réel, alors le groupe $\mathrm{Precone}(G, \mathbf{s})$ hérite immédiatement d'une action (transitive) sur un arbre réel. Rappelons que si Γ est un groupe agissant transitivement sur un arbre réel X non réduit à un point ou à une droite, alors soit l'action est de type général, i.e. Γ contient deux isométries hyperboliques sans bout commun, soit Γ fixe un unique point dans le bord de X. Le théorème suivant donne une caractérisation des groupes lacunaires hyperboliques pour lesquels cette dernière situation a lieu. Nous renvoyons au Chapitre 1 pour la définition d'automorphisme compactant.

Théorème. *Soit G un groupe localement compact compactement engendré. Supposons que G admet un cône asymptotique* $\mathrm{Cone}^\omega(G, \mathbf{s})$ *qui est un arbre réel, et tel que le groupe* $\mathrm{Precone}(G, \mathbf{s})$ *fixe un unique point dans le bord de* $\mathrm{Cone}^\omega(G, \mathbf{s})$. *Alors G admet une décomposition en produit semi-direct $H \rtimes \mathbb{Z}$ ou $H \rtimes \mathbb{R}$, où l'élément $1 \in \mathbb{Z}$ ou $\mathbb{R}$ induit un automorphisme compactant de H.*

Ce théorème permet entre autres de retrouver un résultat de Caprace, Cornulier, Monod et Tessera [CCMT] affirmant que tout groupe hyperbolique fixant un unique point dans son bord est de la forme $H \rtimes \mathbb{Z}$ ou $H \rtimes \mathbb{R}$ avec action compactante.

Rappelons que tout groupe topologique est naturellement une extension d'un groupe connexe par un groupe totalement discontinu, si bien que la résolution d'un problème peut dans certains cas consister à traiter séparément les cas connexes et totalement discontinus, et étudier comment recoller ces morceaux pour obtenir une solution générale. D'après la résolution du cinquième problème de Hilbert [MZ55], tout groupe connexe localement compact admet un sous-groupe compact normal tel que le quotient soit un groupe de Lie. Ce résultat permet très souvent de réduire un problème de géométrie grossière du cas d'un groupe connexe au cas d'un groupe de Lie connexe. Au sein de la classe des groupes localement compacts compactement engendrés, les groupes totalement discontinus (ou plus généralement les groupes admettant des sous-groupes compacts ouverts) jouissent de propriétés supplémentaires. Par exemple, par une construction d'Abels, ceux-ci agissent géométriquement sur un graphe connexe localement fini.

Mis à part le cas bien compris d'un groupe admettant un cône asymptotique réduit à un point ou à une droite, il découle du théorème ci-dessus que pour tous $G, \omega, \mathbf{s}$ tels que $\mathrm{Cone}^\omega(G, \mathbf{s})$ est un arbre réel, alors soit le groupe G est un groupe hyperbolique, soit l'action de $\mathrm{Precone}(G, \mathbf{s})$ sur $\mathrm{Cone}^\omega(G, \mathbf{s})$ est de type général. En utilisant des arguments géométriques au niveau de l'action de $\mathrm{Precone}(G, \mathbf{s})$ sur $\mathrm{Cone}^\omega(G, \mathbf{s})$, nous montrons que dans cette dernière situation la composante connexe du neutre dans G est toujours un sous-groupe compact ou cocompact ; et déduisons que tout groupe lacunaire hyperbolique est soit hyperbolique, ou bien admet des sous-groupes compacts ouverts. Ce résultat nous permet d'étendre la caractérisation des groupes lacunaires hyperboliques discrets d'Olshanskii, Osin et Sapir au cadre des groupes localement compacts.

Théorème. *Soit G un groupe localement compact compactement engendré. Alors G est lacunaire hyperbolique si et seulement si*

(a) soit G est hyperbolique ; ou

(b) il existe un groupe localement compact hyperbolique G_0 agissant géométriquement sur un arbre localement fini, et une suite croissante de sous-groupes discrets normaux N_n de G_0, dont la réunion N est discrète et telle que G est isomorphe à G_0/N ; et si S est un sous-ensemble compact engendrant G_0 et

$$\rho_n = \min\{|g|_S \,:\, g \in N_{n+1} \backslash N_n\},$$

alors G_0/N_n est δ_n-hyperbolique avec $\delta_n = o(\rho_n)$.

Il est clair que tout sous-groupe quasi-isométriquement plongé dans un groupe lacunaire hyperbolique est lui-même lacunaire hyperbolique. En revanche, sans hypothèse de non-distortion, il est a priori difficile de donner des propriétés significatives des sous-groupes des groupes lacunaires hyperboliques. Nous démontrons le résultat suivant dans la Section 1.6, à laquelle nous renvoyons pour la définition de satisfaire une loi.

Théorème. *Soit G un groupe lacunaire hyperbolique unimodulaire. Si H est un sous-groupe compactement engendré ayant une croissance relative exponentielle dans G, et n'admettant pas $\mathbb{Z}$ comme sous-groupe discret co-compact, alors H ne satisfait pas de loi.*

En particulier ce théorème répond à une question d'Olshanskii, Osin et Sapir posée dans le cas discret dans [OOS09]. Il est aisé de vérifier que l'hypothèse d'unimodularité est nécessaire, et notons que d'après un exemple construit dans [OOS09], l'hypothèse de croissance relative exponentielle est également nécessaire. L'idée de la preuve de ce théorème est d'exhiber des groupes libres, qui constituent une obstruction évidente au fait de satisfaire une loi. On ne peut pas espérer trouver de tels groupes dans H, par exemple car H peut être de torsion ou être moyennable. En revanche, nous montrons que si $\mathrm{Cone}^{\omega}(G, \mathbf{s})$ est un arbre réel, alors le sous-groupe de $\mathrm{Precone}(G, \mathbf{s})$ constitué des suites à valeurs dans H contient toujours des groupes libres. Nous y parvenons en étudiant l'action de ce groupe sur l'arbre réel $\mathrm{Cone}^{\omega}(G, \mathbf{s})$.

Points de coupure asymptotiques

Si X est un espace métrique géodésique, un point $x \in X$ est appelé point de coupure si l'espace $X \setminus \{x\}$ n'est plus connexe. Les arbres réels sont des exemples d'espaces métriques ayant des points de coupure. Pour un espace métrique géodésique, la propriété d'avoir des points de coupure dans ses cônes asymptotiques peut être vue comme une forme très faible d'hyperbolicité. La classe des groupes de type fini admettant des points de coupure dans tout cône asymptotique contient par exemple les groupes relativement hyperboliques [DS05], ou les groupes modulaires de surfaces épointées [Beh06].

La propriété de n'avoir de point de coupure dans aucun cône asymptotique peut être caractérisée par une propriété géométrique de l'espace appelée divergence. Grossièrement, la divergence estime le coût de relier deux points d'un espace métrique en évitant une grande boule entre ces deux points. Nous renvoyons à la Section 2.2 pour une définition précise de la notion de divergence, et le lien avec l'existence de points de coupure asymptotiques. Notons qu'il existe des groupes de type fini admettant un cône asymptotique ayant des points de coupure, et un cône asymptotique sans point de coupure.

Drutu, Mozes et Sapir [DMS10] conjecturent qu'un réseau dans un groupe de Lie semi-simple de rang supérieur n'admet de point de coupure dans aucun cône asymptotique, et prouvent cette conjecture dans le cas des réseaux de $\mathbb{Q}$-rang un et dans le cas du groupe spécial linéaire à coefficients dans l'anneau des $\mathcal{S}$-entiers d'un corps de nombres. Dans la Section 2.3, nous prouvons que ce dernier résultat est également vrai en caractéristique positive.

Théorème. *Soit q une puissance d'un nombre premier, soit $\mathcal{S}$ un ensemble de valuations d'une extension finie du corps $\mathbb{F}_q(t)$ des fractions rationnelles à coefficients dans le corps fini $\mathbb{F}_q$, et soit $\mathcal{O}_\mathcal{S}$ l'anneau des $\mathcal{S}$-entiers associé. Pour tout $n \geq 3$, le groupe $\mathrm{SL}_n(\mathcal{O}_\mathcal{S})$ a une divergence linéaire.*

De manière équivalente, le groupe $\mathrm{SL}_n(\mathcal{O}_\mathcal{S})$ n'admet de point de coupure dans aucun cône asymptotique.

Rappelons que le groupe SOL est le groupe de Lie de dimension trois $\mathbb{R}^2 \rtimes \mathbb{R}$, où l'action de $t \in \mathbb{R}$ sur $\mathbb{R}^2$ est donnée par la matrice $\mathrm{diag}(e^t, e^{-t})$. C'est un exemple caractéristique de groupe n'admettant de point de coupure dans aucun cône asymptotique : tous ses cônes asymptotiques sont homéomorphes à $\{(x, y) \in \mathbb{T} \times \mathbb{T} : b(x) + b(y) = 0\}$, où $\mathbb{T}$ est un arbre réel homogène de degré continu et b est une fonction de Busemann sur $\mathbb{T}$ [Cor08, Section 9] ; et il est facile de vérifier que cet espace métrique n'admet pas de point de coupure.

Dans la Section 2.5, nous résolvons le problème de l'existence de points de coupure dans un cône asymptotique dans le cas des groupes de Lie réels et des groupes algébriques p-adiques. Nous montrons que ceux-ci sont soit hyperboliques, ou bien n'admettent pas de point de coupure dans aucun cône asymptotique.

Théorème. *Soit G un groupe localement compact compactement engendré. Supposons que G est soit un groupe de Lie réel connexe, soit un groupe algébrique linéaire sur un corps local non archimédien de caractéristique nulle. Si G admet des points de coupure dans l'un de ses cônes asymptotiques, alors G est en fait hyperbolique.*

Nous déduisons ce résultat d'un énoncé plus général (Corollary 2.34) dans la Section 2.5, à laquelle nous renvoyons pour plus de détails.

Croissance des relations

Dans le Chapitre 5, nous associons à un groupe de type fini G un ensemble d'entiers naturels, qui est l'ensemble des longueurs des relations entre des générateurs de G qui ne sont pas engendrées par des relations de longueur plus petite. On vérifie que cet ensemble ne dépend pas du choix d'un système de générateurs fini, modulo la relation d'équivalence $\sim$ sur $\mathcal{P}(\mathbb{N})$ « être à distance de Hausdorff multiplicative bornée ». Cet ensemble est appelé *éventail des relations* de G, et noté $\mathcal{R}(G)$. On dit que G est densément présenté si $\mathcal{R}(G) \sim \mathbb{N}$, et lacunairement présenté sinon.

Nous prouvons que $\mathcal{R}(G)$ est invariant par passage à un sous-groupe d'indice fini, et par quotient par un sous-groupe normal fini. L'éventail des relations se comporte également de manière agréable par rapport aux opérations de produit libre et produit direct. Nous donnons dans la Section 5.2 une classe de groupes densément présentés, obtenus en itérant un épimorphisme non-injectif dans un groupe non Hopfien. La classe des groupes densément présentés contient les exemples standards de groupes métabéliens de présentation infinie que sont $\mathbb{Z} \wr \mathbb{Z}$ et $\mathbb{Z}[1/n]^2 \rtimes \mathbb{Z}$ pour $n \geq 2$, où l'action de $\mathbb{Z}$ sur $\mathbb{Z}[1/n]^2$ est définie par la multiplication par n sur le premier facteur et n^{-1} sur le second. Nous montrons également que le groupe de Grigorchuk est densément présenté.

Rappelons que par un résultat de Gromov, si un groupe de type fini G a tous ses cônes asymptotiques simplement connexes, alors G est de présentation finie, autrement dit l'éventail des relations de G est fini. On ne peut bien sûr pas espérer la même conclusion si on suppose juste la simple connexite au niveau d'un seul cône asymptotique, par exemple parce que tout groupe lacunaire hyperbolique non hyperbolique est de présentation infinie. En revanche, nous prouvons que la simple connexité d'un cône asymptotique impose tout de même une restriction sur l'éventail des relations du groupe.

Théorème. *Soit G un groupe de type fini admettant un cône asymptotique simplement connexe. Alors G est lacunairement présenté.*

Ce résultat implique par exemple que tout groupe de type fini lacunaire hyperbolique est lacunairement présenté. En particulier il est impossible de construire des groupes lacunaires hyperboliques en itérant à l'infini un épimorphisme non injectif sur un groupe de type fini.

Groupes agissant (presque) sur un arbre

Le second axe majeur de cette thèse est d'étudier certains groupes agissant sur un arbre simplicial ou sur son bord, et dont l'action locale satisfait une condition de rigidité. Les deux familles de groupes que nous étudions fournissent des exemples de groupes localement compacts, totalement discontinus, non discrets, compactement engendrés et simples. Cette classe de groupes joue un rôle majeur dans l'étude générale de la structure des groupes localement compacts compactement engendrés [CM11, CRW13, CRW14].

Groupe de presqu'automorphismes d'un arbre

Pour tout entier $d \geq 3$, notons T_d l'arbre simplicial régulier de degré d. Muni de la topologie compacte-ouverte, le groupe $\mathrm{Aut}(T_d)$ des automorphismes de T_d est un groupe localement compact totalement discontinu. L'action du groupe $\mathrm{Aut}(T_d)$ sur T_d est propre, continue et cocompacte, si bien que $\mathrm{Aut}(T_d)$ et T_d sont quasi-isométriques.

Le bord à l'infini ∂T_d de l'arbre T_d peut être pensé comme l'espace des directions à l'infini dans T_d. Il s'agit d'un espace métrique compact, et l'action d'un élément de $\mathrm{Aut}(T_d)$ sur l'arbre induit un homéomorphisme de son bord. La notion de presqu'automorphisme de l'arbre T_d est en quelque sorte une relaxation de la notion d'automorphisme. Un presqu'automorphisme n'agit pas sur l'arbre T_d, mais sur son bord à l'infini : c'est par définition une transformation bicontinue du bord de T_d qui est un automorphisme d'arbre par morceaux. Cela signifie que localement l'action d'un presqu'automorphisme sur ∂T_d provient de l'action sur ∂T_d d'un automorphisme de T_d. Le groupe $\mathrm{AAut}(T_d)$ des presqu'automorphismes de l'arbre T_d admet une topologie de groupe localement compacte faisant de $\mathrm{Aut}(T_d)$ un sous-groupe ouvert.

Le groupe $\mathrm{AAut}(T_d)$ a été introduit par Neretin comme un analogue combinatoire, du point de vue de la théorie des représentations, du groupe des difféomorphismes du cercle [Ner92]. Lorsque $d = p + 1$ avec p premier, le bord de T_{p+1} s'identifie naturellement avec la droite projective $\mathbb{P}^1(\mathbb{Q}_p)$, et le groupe $\mathrm{AAut}(T_{p+1})$ contient le groupe des difféomorphismes localement analytiques de $\mathbb{P}^1(\mathbb{Q}_p)$ [Ner92]. Motivé par la simplicité du groupe des difféomorphismes du cercle de classe $\mathcal{C}^\infty$ préservant l'orientation, Kapoudjian a montré que le groupe $\mathrm{AAut}(T_d)$ est simple [Kap99]. Récemment Bader, Caprace, Gelander et Mozes ont prouvé que le groupe $\mathrm{AAut}(T_d)$ n'admet pas de réseaux [BCGM12], fournissant le premier exemple de groupe simple (compactement engendré) ne possédant pas de réseaux.

Les groupes de Thompson, communément notés F, T et V, sont des groupes de type fini apparaissant dans divers domaines mathématiques, et étant au centre de nombreux problèmes de géométrie des groupes. Nous renvoyons à [CFP96] pour plus de détails sur ces groupes. Higman étendit la définition du groupe V en une famille de groupes $V_{d,k}$, ceux-ci étant tous infinis de présentation finie, et ayant un sous-groupe simple d'indice au plus deux [Hig74]. On peut vérifier que pour tout $d \geq 2$, le groupe $\mathrm{AAut}(T_{d+1})$ contient une copie du groupe $V_{d,2}$ comme sous-groupe dense. Ce fait est par exemple utilisé de manière cruciale dans [Kap99]. Nous précisons le lien entre les groupes $\mathrm{AAut}(T_{d+1})$ et $V_{d,2}$ dans la Section 3.2, en montrant que $\mathrm{AAut}(T_{d+1})$ s'identifie au complété de Schlichting du groupe $V_{d,2}$ relativement à un sous-groupe commensuré localement fini.

Présentation compacte et fonction de Dehn. On peut vérifier que pour tout $d \geq 2$, le groupe de presqu'automorphismes $\mathrm{AAut}(T_{d+1})$ est engendré par un système fini de générateurs de $V_{d,2}$, auquel on ajoute un sous-groupe compact de $\mathrm{Aut}(T_{d+1})$. De fait, le groupe $\mathrm{AAut}(T_{d+1})$ est compactement engendré.

La notion de présentation compacte est plus forte que la notion de génération compacte. Un groupe localement compact G est dit compactement présenté s'il admet un sous-ensemble générateur compact tel que G admet une présentation, en tant que groupe abstrait, ayant S pour ensemble de générateurs et un ensemble de relateurs de longueur bornée. Lorsque le groupe G est discret, cela revient à dire que G est de présentation finie. On vérifie que cette définition ne dépend pas du choix du système de générateurs compact S.

L'un des intérêts majeurs de cette notion a priori algébrique est qu'il s'agit d'une propriété *géométrique*. En effet, le fait d'être compactement présenté s'interprête en termes de simple connexité grossière du graphe de Cayley de G associé à un sous-ensemble générateur compact, et en particulier être compactement présenté est invariant par quasi-isométries. La classe des groupes compactement présentés contient par exemple les groupes abéliens et nilpotents, les groupes connexes, et les groupes Gromov-hyperboliques.

L'une des raisons à l'origine de l'intérêt des théoriciens des groupes pour les groupes de Thompson et leurs généralisations, est que ceux-ci jouissent à la fois de propriétés de simplicité et de finitude. Alors que la simplicité de $\mathrm{AAut}(T_d)$ a été obtenue dans [Kap99], nous démontrons le théorème suivant dans la Section 3.2.

Théorème. *Pour tout $d \geq 3$, le groupe $\mathrm{AAut}(T_d)$ est compactement présenté.*

Nous montrons en fait ce résultat pour de nombreux groupes agissant sur le bord de l'arbre T_d, et dont l'action locale est prescrite par un groupe régulièrement branché. Nous renvoyons à la Section 3.2 pour plus de détails.

La fonction de Dehn δ_G d'un groupe localement compact compactement présenté G est un invariant du groupe G, aux aspects à la fois géométrique et combinatoire. D'un point de vue géométrique, $\delta_G(n)$ est l'aire maximale d'un lacet dans G de longueur au plus n. En d'autres termes, la fonction δ_G est la meilleure fonction isopérimétrique du groupe G. D'un point de vue combinatoire, la fonction de Dehn fournit une estimation quantitative du fait que G est compactement présenté : $\delta_G(n)$ est le supremum pour toutes les relations w dans G de longueur au plus n, du nombre minimal de relateurs nécessaires pour réduire w au mot trivial.

Rappelons sa définition précise. Si G est un groupe localement compact compactement présenté et S un sous-ensemble compact générateur, il existe un entier $k \geq 1$ tel que G admette une présentation $\langle S \mid R_k \rangle$, où R_k est l'ensemble des relations dans G (i.e. des mots en les lettres de S représentant l'élément trivial du groupe G) de longueur au plus k. L'aire $a(w)$ d'une relation w est le plus petit entier m tel que w admet une décomposition dans le groupe libre F_S en un produit de m conjugués d'éléments de R_k. On définit la fonction de Dehn de G par

$$\delta_G(n) = \sup \{a(w) : w \text{ relation de longueur au plus } n\}.$$

Cette fonction dépend du choix de S et k, mais son comportement asymptotique n'en dépend pas, et est en fait un invariant géométrique du groupe G.

Avoir un petite fonction de Dehn a des conséquences géométriques remarquables. Par exemple, un groupe est hyperbolique si et seulement si il a une fonction de Dehn linéaire [Gro87]. De plus, tout groupe ayant une fonction de Dehn sous-quadratique a en fait une fonction de Dehn linéaire [Gro87, Bow91]. Le comportement asymptotique de la fonction de Dehn d'un groupe est intimement lié aux propriétés topologiques de ses cônes asymptotiques. Par un theorème de Gromov, si un groupe localement compact compactement engendré G a tous ses cônes asymptotiques simplement connexes, alors le groupe G est compactement présenté et a une fonction de Dehn polynomialement bornée [Gro93]. Ce résultat admet une réciproque partielle dûe à Papasoglu, qui affirme que si G a une fonction de Dehn quadratique, alors tous les cônes asymptotiques de G sont simplement connexes.

Nous prouvons la majoration suivante sur la fonction de Dehn du groupe $\mathrm{AAut}(T_{d+1})$ dans la Section 3.2.

Théorème. *Pour tout $d \geq 2$, le groupe $\mathrm{AAut}(T_{d+1})$ a une fonction de Dehn asymptotiquement bornée par celle du groupe $V_{d,2}$.*

Ce résultat est en fait obtenu pour une famille de groupes plus générale définie à la fin de la Section 3.1, nous renvoyons le lecteur au Théorème 3.17.

Il n'est pas difficile de voir que le groupe $\mathrm{AAut}(T_{d+1})$ n'est pas hyperbolique, et par conséquent sa fonction de Dehn est au moins quadratique. Lorsque $d = 2$, le groupe $V_{2,2}$ est isomorphe au groupe de Thompson V. Tandis qu'il est connu que la fonction de Dehn du groupe de Thompson F est quadratique, on ne sait pas s'il en est de même pour le groupe de Thompson V. Cependant, en invoquant un résultat de Guba [Gub06], on peut déduire du théorème précédent le résultat suivant.

Corollaire. *Le groupe de presqu'automorphismes d'un arbre régulier trivalent a une fonction de Dehn polynomialement bornée ($\preccurlyeq n^{11}$).*

Une question qui découle naturellement du théorème ci-dessus est de se demander si la fonction de Dehn du groupe $\mathrm{AAut}(T_{d+1})$ est strictement plus petite que celle de $V_{d,2}$. Nous adressons cette question et en discutons certains aspects à la fin de cette thèse.

Diagrammes et propriétés métriques. Par définition les éléments du groupe $\mathrm{AAut}(T_d)$ sont définis par leur action au bord de l'arbre T_d, et on peut les penser comme des transformations à l'infini. Cependant, le fait que l'action locale provienne de l'action d'un automorphisme de l'arbre permet de penser qu'un élément du groupe $\mathrm{AAut}(T_d)$ agit sur un « voisinage » du bord de l'arbre. Cette approche a l'avantage de permettre de représenter les éléments du groupe $\mathrm{AAut}(T_d)$ par une donnée combinatoire se situant maintenant dans l'arbre, et non plus dans son bord. Cette idée est bien connue et a été massivement utilisée dans le cas des groupes de Thompson et certaines de leurs généralisations. La notion de diagramme associé à un presqu'automorphisme que nous définissons est liée, mais est en générale différente de celle utilisée pour les groupes de Thompson.

L'un des intérêts de notre construction est que cette notion fournit une fonction de longueur $\mathcal{C}$ sur le groupe $\mathrm{AAut}(T_d)$, et nous permet d'obtenir une pseudo-métrique sur $\mathrm{AAut}(T_d)$ ayant les propriétés agréables d'être propre et invariante à gauche. De manière générale, il est naturel de se demander si une telle pseudo-métrique sur un groupe localement compact compactement engendré est quasi-isométrique à la métrique des mots. Nous répondons dans ce cas par la négative à cette question, en prouvant le résultat suivant.

Proposition. *Pour tout $d \geq 3$ et tout sous-ensemble générateur compact S de $\mathrm{AAut}(T_d)$, il existe une constante $c > 0$ telle que pour tout $g \in \mathrm{AAut}(T_d)$, nous avons*

$$c^{-1}\mathcal{C}(g) \leq |g|_S \leq c\mathcal{C}(g)\log(1+\mathcal{C}(g)).$$

De plus les fonctions de longueur $\mathcal{C}$ et $|\cdot|_S$ ne sont pas asymptotiquement équivalentes.

Ce résultat peut en quelque sorte être vu comme une version non discrète d'un énoncé similaire sur le groupe de Thompson V dû à Birget. Alors que la preuve de la seconde partie de l'énoncé consiste dans le cas du groupe V en un argument de dénombrement sur des ensembles finis, notre preuve repose sur une estimation de la croissance de la mesure de Haar d'une famille de sous-groupes compacts ouverts du groupe $\mathrm{AAut}(T_d)$.

La question de comparer la métrique des mots dans $\mathrm{AAut}(T_d)$ et la métrique fournie par la notion de diagramme est aussi justifiée par le fait que cette dernière apparaît comme la métrique induite sur une orbite de $\mathrm{AAut}(T_d)$ dans un complexe cubique CAT(0) sur lequel ce groupe agit. Bien que la construction d'un complexe cubique CAT(0) muni d'une action propre du groupe $\mathrm{AAut}(T_d)$ n'apparaisse pas clairement dans la littérature, celle-ci semble être connue et l'idée de sa construction est similaire à la construction de Farley dans le cas des groupes de Thompson [Far03] (voir aussi [Nav02]). Nous rappelons cette construction en adoptant le point de vue des actions commensurantes dans la Section 3.3. Le résultat précédent peut donc être interprété comme une étude de la distortion de l'application orbitale du groupe $\mathrm{AAut}(T_d)$ dans un complexe cubique CAT(0), et affirme en particulier que cette application n'est pas un plongement quasi-isométrique. Ce résultat apparait comme une motivation naturelle à la question de savoir si le groupe $\mathrm{AAut}(T_d)$ peut se plonger quasi-isométriquement dans un espace métrique CAT(0), que nous adressons à la fin de cette thèse.

La construction de diagrammes représentant un presqu'automorphisme et l'étude de la métrique qui en découle est également motivée par le fait que pour certains sous-groupes remarquables de $\mathrm{AAut}(T_d)$, celle-ci est comparable à la métrique des mots. De manière générale, si G est un groupe localement compact compactement engendré et H un sous-groupe compactement engendré, alors H peut être muni d'une part de sa métrique des mots, et d'autre part de la métrique induite par la métrique des mots de G. On dit que H est quasi-isométriquement plongé, ou non distordu dans G, si ces deux métrique sont comparables. Nous prouvons :

Proposition. *Pour tout $d \geq 2$,*

(a) il existe un sous-groupe discret de $\mathrm{AAut}(T_{d+1})$ isomorphe au groupe de Thompson F_d et quasi-isométriquement plongé dans $\mathrm{AAut}(T_{d+1})$;

(b) le groupe $\mathrm{Aut}(T_{d+1})$ des automorphismes de l'arbre T_{d+1} est quasi-isométriquement plongé dans $\mathrm{AAut}(T_{d+1})$.

Prescrire l'action locale presque partout

La seconde classe de groupes auxquels nous nous intéressons consiste en des groupes agissant cette fois sur l'arbre T_d lui même, et dont l'action locale satisafait une condition de rigité presque partout. Nous remercions chaleuresement les auteurs de [BCGM12] d'avoir attiré notre attention sur la définition de ces groupes.

Diverses propriétés. Rappelons tout d'abord la définition du groupe « universel » $U(F)$ défini par Burger et Mozes. Fixons un coloriage des arêtes de T_d par les entiers $1, \ldots, d$, tel que des arêtes provenant d'un même sommet ont des couleurs deux à deux distinctes. Par définition, un automorphisme g de T_d envoie l'ensemble des arêtes émanant d'un sommet v

sur l'ensemble des arêtes émanant de $g(v)$, et par conséquent induit pour chaque sommet v une bijection $\sigma(g,v) \in \mathrm{Sym}(d)$. Étant donné un groupe de permutations $F \leq \mathrm{Sym}(d)$, le groupe $U(F)$ est le sous-groupe de $\mathrm{Aut}(T_d)$ dont l'action locale est prescrite par F, c'est-à-dire l'ensemble des automorphismes g de l'arbre T_d tels que $\sigma(g,v) \in F$ pour tout sommet v. Il s'agit d'un sous-groupe fermé de $\mathrm{Aut}(T_d)$ agissant transitivement sur l'ensemble des sommets de T_d, et dont la classe de conjugaison dans $\mathrm{Aut}(T_d)$ ne dépend pas du choix du coloriage des arêtes.

Nous considérons une famille de groupes $G(F)$ définis en relaxant la condition de rigidité locale de Burger et Mozes en un nombre fini de sommets. Plus précisément, $G(F)$ est constitué des éléments $g \in \mathrm{Aut}(T_d)$ tels que l'ensemble des sommets v pour lesquels $\sigma(g,v) \notin F$ est fini. Clairement, le groupe $U(F)$ est un sous-groupe de $G(F)$. Le groupe $G(F)$ n'est en général pas fermé dans $\mathrm{Aut}(T_d)$, et on peut décrire explicitement son adhérence (voir Corollary 4.6). On vérifie qu'il existe une topologie de groupe sur $G(F)$ faisant de l'inclusion $U(F) \hookrightarrow G(F)$ une application ouverte et continue, et telle que les stabilisateurs de sommets dans $G(F)$ sont ouverts et localement elliptiques. Muni de cette topologie, $G(F)$ est un groupe localement compact totalement discontinu compactement engendré.

Les groupes $G(F)$ ont des propriétés communes avec le groupe de Neretin $\mathrm{AAut}(T_d)$. Par exemple, le fait que le stabilisateur d'une arête dans $G(F)$ est ouvert et localement elliptique peut être vu comme un analogue du fait que le sous-groupe de $\mathrm{AAut}(T_d)$ préservant la mesure visuelle dans ∂T_d issue de cette arête, est ouvert et localement elliptique. Cependant, les groupes $G(F)$ sont tout de même beaucoup plus rigides que $\mathrm{AAut}(T_d)$, dans le sens où ceux-ci respectent la structure d'arbre de T_d. Cette rigidité supplémenatire impose par exemple que les groupes $G(F)$ ont une dimension asymptotique égale à un (Corollary 4.23), alors que $\mathrm{AAut}(T_d)$ a une dimension asymptotique infinie. Ainsi, les groupes $G(F)$ aparaissent comme des groupes « intermédiaires » entre les groupes $U(F)$ et $\mathrm{AAut}(T_d)$.

La plus grande flexibilité dont jouit l'action du groupe $G(F)$ sur l'arbre T_d par rapport au groupe $U(F)$ permet par exemple de construire des groupes simples de type fini. En particulier le théorème suivant fournit des exemples de groupes de type fini, simples, et de dimension asymptotique égale à un. L'existence de tels groupes n'était pas connue jusqu'à aujourd'hui.

Théorème. *Soit $d \geq 3$, et soit $F \leq \mathrm{Sym}(d)$ un groupe de permutation supposé simplement transitif. Soit également F' un sous-groupe de* $\mathrm{Sym}(d)$ *contenant F, dont les stabilisateurs de points sont parfaits et engendrent F'. Alors le groupe $G(F) \cap U(F')$ est un groupe simple de type fini, et de dimension asymptotique égale à un.*

L'un des ingrédients majeurs de la preuve de ce résultat est un critère de simplicité qui apparait comme un avatar du théorème de Tits [Tit70] (voir Theorem 4.15). Nous renvoyons à la Section 4.1 pour plus de détails.

Une question naturelle est de se demander si le groupe $G(F)$ ou certains de ses sous-groupes remarquables sont compactement présentés. Nous répondons à cette question dans la Section 4.2 pour une classe relativement large de sous-groupes, incluant le groupe $G(F)$ lui même (voir Proposition 4.26). Ce résultat implique en particulier que le groupe $G(F)$ n'est pas compactement présenté dès que $F \leq \mathrm{Sym}(d)$ est un sous-groupe propre transitif. Dans le cas où $F \leq \mathrm{Sym}(d)$ est un sous-groupe propre 2-transitif, nous donnons également une seconde preuve complètement indépendante de ce résultat, reposant sur une version topologique du théorème de scindement de Bieri-Strebel.

Diagrammes et action commensurante. Exploitant l'analogie entre la famille de groupes $G(F)$ et le groupe des presqu'automorphismes de l'arbre T_d, nous définissons une notion de diagramme pour représenter et manipuler les éléments du groupe $G(F)$. Nous renvoyons à la Section 4.3 pour plus de détails sur cette construction. L'intérêt de cette approche provient en partie du fait que, quand F est supposé transitif, la taille des diagrammes en question fournit une estimation quasi-isométrique à la métrique des mots dans le groupe $G(F)$. Notons que les constantes en jeu sont complètement explicites et linéaires en d.

Proposition. *Soit $d \geq 3$ et soit $F \leq \mathrm{Sym}(d)$ supposé transitif. Alors il existe un sous-ensemble (explicite) compact générateur S de $G(F)$ tel que, si $\mathcal{N}(g)$ désigne la taille du diagramme représentant l'élément $g \in G(F)$, alors*

$$\mathcal{N}(g) \leq |g|_S \leq (3d-2)\mathcal{N}(g) + 3d + 2$$

pour tout $g \in G(F)$.

Nous exploitons aussi l'idée sous-jacente à notre construction de diagrammes pour construire un ensemble muni d'une action commensurante de $G(F)$. Rappelons que si G est un groupe et X un G-ensemble, un sous-ensemble $A \subset X$ est dit commensuré si $\#(gA \triangle A)$ est fini pour tout $g \in G$.

Théorème. *Soit $d \geq 3$ et soit $F \leq \mathrm{Sym}(d)$ supposé transitif. Alors il existe un ensemble X muni d'une action de $G(F)$ et un sous-ensemble $A \subset X$ tels que $\#(gA \triangle A) = 2\mathcal{N}(g)$ pour tout $g \in G(F)$.*

Par un principe général, nous déduisons de ce théorème qu'il existe un complexe X cubique $\mathrm{CAT}(0)$ sur lequel $G(F)$ agit proprement, et un sommet $x_0 \in X$ tel que $d_{\ell^1}(gx_0, x_0) = 2\mathcal{N}(g)$ pour tout $g \in G(F)$. Ainsi le résultat ci-dessus affirmant que la quantité $\mathcal{N}(g)$ est quasi-isométrique à la métrique des mots dans le groupe $G(F)$ s'interprète géométriquement : il signifie que l'application orbitale de $G(F)$ dans ce complexe cubique $G(F) \to X$, $g \mapsto gx_0$, est un plongement quasi-isométrique.

Contents

Chapter 1

Lacunary hyperbolic groups

In this chapter we investigate the class of locally compact lacunary hyperbolic groups. We prove that if a locally compact compactly generated group G admits one asymptotic cone that is a real tree and whose natural transitive isometric action is focal, then G must be a focal hyperbolic group (Theorem 1.1). We also prove several results for locally compact lacunary hyperbolic groups, and extend the characterization of finitely generated lacunary hyperbolic groups of [OOS09] to the setting of locally compact groups. We moreover answer a question of Olshanskii, Osin and Sapir about subgroups of lacunary hyperbolic groups.

1.1 Introduction

1.1.1 Locally compact hyperbolic groups

If G is a locally compact group and S a compact generating subset, then G can be equipped with the word metric associated to S. A locally compact compactly generated group is hyperbolic if it admits some compact generating subset such that the associated word metric is Gromov-hyperbolic. By [CCMT, Corollary 2.6], this is equivalent to asking that the group acts continuously, properly and cocompactly by isometries on some proper geodesic hyperbolic metric space. Examples of non-discrete hyperbolic groups include semisimple real Lie groups of rank one, or the full automorphism group of a semi-regular locally finite tree. We freely use the shorthand *hyperbolic LC-group* for locally compact compactly generated hyperbolic group.

Finitely generated hyperbolic groups have received much attention over the last twenty-five years, and their study led to a rich and powerful theory. On the other hand, hyperbolic LC-groups have not been studied to the same extent, and this disparity leads to the natural problem of discussing the similarities and differences between the discrete and non-discrete setting. One positive result in this vein is the extension of Bowditch's topological characterization of discrete hyperbolic groups, as those finitely generated groups that act properly and cocompactly on the space of distinct triples of a compact metrizable space, to the setting of locally compact groups [CD14]. However it turns out that some hyperbolic LC-groups exhibit some completely opposite behavior to what happens for discrete hyperbolic groups: while a non-virtually cyclic finitely generated hyperbolic group always contains a non-abelian free group, some hyperbolic LC-groups are non-elementary hyperbolic and amenable. It follows from the work of Caprace, Cornulier, Monod and Tessera that those can be characterized in terms of the dynamics of the action of the group on its boundary, and that they coincide with the class of mapping tori of compacting automorphisms (see Theorem 1.2).

1.1.2 Lacunary hyperbolic groups

The definition of asymptotic cones of a metric space makes sense for a locally compact compactly generated group G. Let $\mathbf{s} = (s_n)$ be a sequence of positive real numbers tending to infinity, and ω a non-principal ultrafilter. We denote by $\mathrm{Precone}(G, \mathbf{s})$ the set of sequences (g_n) in G such that there exists some constant $C > 0$ so that the word length of g_n is at most Cs_n for every $n \geq 1$; and equip it with the pseudo-metric $d_\omega((g_n),(h_n)) = \lim^\omega d_S(g_n, h_n)/s_n$. It inherits a group structure by component-wise multiplication, and the asymptotic cone $\mathrm{Cone}^\omega(G, \mathbf{s})$ of G associated to the parameters $\mathbf{s}, \omega$ is the homogeneous space $\mathrm{Precone}(G, \mathbf{s}) \,/\, \mathrm{Sublin}^\omega(G, \mathbf{s})$, where $\mathrm{Sublin}^\omega(G, \mathbf{s})$ is the subgroup of sequences at distance d_ω zero from the identity. The group $\mathrm{Precone}(G, \mathbf{s})$ can be viewed as a large picture of the group G, and the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathrm{Cone}^\omega(G, \mathbf{s})$ is inherited from the action of G on itself. Asymptotic cones capture the large-scale geometry of the word metric on G. In some sense, the metric space $\mathrm{Cone}^\omega(G, \mathbf{s})$ reflects the properties of the group G that are visible at scale $\mathbf{s}$.

For example if G is a hyperbolic LC-group, then all its asymptotic cones are real trees. Interestingly enough, thanks to a result of Gromov [Gro93, Dru02], one can characterize hyperbolicity in terms of asymptotic cones: a locally compact compactly generated group is hyperbolic if and only if all its asymptotic cones are real trees. However there exist finitely generated non-hyperbolic groups with some asymptotic cone a real tree. The first example appeared in [TV00], where small cancellation theory is used to construct a finitely generated group with one asymptotic cone a real tree, and one asymptotic cone that is not simply connected. The systematic study of the class of finitely generated groups with one asymptotic cone a real tree, called *lacunary hyperbolic groups*, was then initiated in [OOS09]. Olshanskii, Osin and Sapir characterized finitely generated lacunary hyperbolic groups as direct limits of sequences of finitely generated hyperbolic groups satisfying some conditions on the hyperbolicity constants and injectivity radii [OOS09, Theorem 3.3]. They also proved that the class of finitely generated lacunary hyperbolic groups contains examples of groups that are very far from being hyperbolic: a non-virtually cyclic lacunary hyperbolic group can have all its proper subgroups cyclic, can have an infinite center or can be elementary amenable.

Following [OOS09], we call a locally compact compactly generated group *lacunary hyperbolic* if one of its asymptotic cones is a real tree. For example if X is a proper geodesic metric space with a cobounded isometric group action, and if X has one asymptotic cone that is a real tree, then the full isometry group $G = \mathrm{Isom}(X)$ is a locally compact lacunary hyperbolic group, which has a priori no reason to be discrete.

By construction any asymptotic cone $\mathrm{Cone}^{\omega}(G, \mathbf{s})$ of a locally compact compactly generated group G comes equipped with a natural isometric action of the group $\mathrm{Precone}(G, \mathbf{s})$. So in particular if G admits one asymptotic cone $\mathrm{Cone}^{\omega}(G, \mathbf{s})$ that is a real tree, then we have a transitive action by isometries of the group $\mathrm{Precone}(G, \mathbf{s})$ on a real tree. Recall that isometric group actions on real trees are classified as follows: if the the translation length is trivial then there is a fixed point or a fixed end, and otherwise either there is an invariant line, a unique fixed end or two hyperbolic isometries without common endpoint. It turns out that when G is a hyperbolic LC-group, then for every choice of parameters $\mathbf{s}$ and ω, the asymptotic cone $\mathrm{Cone}^{\omega}(G, \mathbf{s})$ is a real tree, and the type of the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathrm{Cone}^{\omega}(G, \mathbf{s})$ is inherited from the type of the G-action on itself. Recall that a hyperbolic LC-group G is called focal if its action on ∂G has a unique fixed point. In particular when G is a focal hyperbolic group, then for every scaling sequence $\mathbf{s}$ and non-principal ultrafilter ω, the asymptotic cone $\mathrm{Cone}^{\omega}(G, \mathbf{s})$ is a real tree and the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathrm{Cone}^{\omega}(G, \mathbf{s})$ fixes a unique boundary point. This naturally leads to the question as to whether this phenomenon may appear when considering non-hyperbolic groups. Our first result shows that this is not the case. More precisely, we prove the following statement (see Theorem 1.21).

Theorem 1.1. *Let G be a locally compact compactly generated group. Assume that G admits* one *asymptotic cone* $\mathrm{Cone}^{\omega}(G, \mathbf{s})$ *that is a real tree and such that the group* $\mathrm{Precone}(G, \mathbf{s})$ *fixes a unique end of* $\mathrm{Cone}^{\omega}(G, \mathbf{s})$*. Then $G = H \rtimes \mathbb{Z}$ or $H \rtimes \mathbb{R}$, where the element $1 \in \mathbb{Z}$ or $\mathbb{R}$ induces a compacting automorphism of H.*

Recall that an automorphism $\alpha \in \mathrm{Aut}(H)$ of a locally compact group H is said to be compacting if there exists a compact subset $V \subset H$ such that for every $h \in H$, for n large enough $\alpha^n(h) \in V$. In particular we recover and strengthen the implication $(i) \Rightarrow (iii)$ of the following theorem of Caprace, Cornulier, Monod and Tessera.

Theorem 1.2. *[CCMT, Theorem 7.3] If G is a locally compact compactly generated group, then the following statements are equivalent:*

(i) G is a focal hyperbolic group;

(ii) G is amenable and non-elementary hyperbolic;

(iii) G is a semidirect product $H \rtimes \mathbb{Z}$ or $H \rtimes \mathbb{R}$, where the element $1 \in \mathbb{Z}$ or $\mathbb{R}$ induces a compacting automorphism of the non-compact group H.

Our method is different from that of [CCMT]: indeed the latter makes a crucial use of amenability, and the fact that quasi-characters on amenable groups are characters, while we only use geometric arguments at the level of the real tree arising as an asymptotic cone.

We call a locally compact compactly generated group G *lacunary hyperbolic of general type* if it admits one asymptotic cone $\mathrm{Cone}^{\omega}(G, \mathbf{s})$ that is a real tree and such that the action of $\mathrm{Precone}(G, \mathbf{s})$ has two hyperbolic isometries without common endpoint. Drutu and Sapir proved that any non-virtually cyclic finitely generated lacunary hyperbolic group is of general type (see the end of the proof of Theorem 6.12 in [DS05]). In the locally compact setting, it will follow from Theorem 1.1 that any lacunary hyperbolic group that is neither an elementary nor a focal hyperbolic group, is lacunary hyperbolic of general type (see Theorem 1.30).

It is often the case in topological group theory that a given problem can be reduced to the case of connected groups and totally disconnected groups, by using the fact that any topological group decomposes as an extension with connected kernel and totally disconnected quotient. For instance if one wants to study the large scale geometry of a given class of compactly generated groups (say that is stable by modding out by a compact normal subgroup and passing to a cocompact normal subgroup), then this can be reduced to the study of connected and totally disconnected groups as soon as the identity component of a group in this class is either compact or cocompact. It is worth pointing out that this process cannot be applied in generality for hyperbolic LC-groups, because it may happen that the unit component of a hyperbolic LC-group is neither compact nor cocompact. A typical example is $(\mathbb{Q}_p \times \mathbb{R}) \rtimes \mathbb{Z}$, where the automorphism of $\mathbb{Q}_p \times \mathbb{R}$ is the multiplication by (p, p^{-1}). However, apart from focal groups, it is true that the identity component of a hyperbolic LC-group is either compact or cocompact [CCMT, Proposition 5.10]. Here we will extend this result to the setting of lacunary hyperbolic groups in Theorem 1.32.

As a consequence, we will be able to deduce that a locally compact lacunary hyperbolic group is either hyperbolic or admits a compact open subgroup. Compactly generated groups with compact open subgroups are generally more tractable than compactly generated locally compact groups. For example they act geometrically on a locally finite connected graph thanks to a construction due to Abels recalled in Proposition 1.11. Most importantly for our purpose, the fact that any finitely generated group is a quotient of a finitely generated free group, admits a topological extension to the class of compactly generated groups with compact open subgroups (see Proposition 1.12). This will allow us to extend the characterization of finitely generated lacunary hyperbolic groups of Olshanskii, Osin and Sapir to the locally compact setting (see also Theorem 1.38).

Theorem 1.3. *Let G be a locally compact, compactly generated group. Then G is lacunary hyperbolic if and only if*

(a) either G is hyperbolic; or

(b) there exists a hyperbolic LC-group G_0 acting geometrically on a locally finite tree, and an increasing sequence of discrete normal subgroups N_n of G_0, whose discrete union N is such that G is isomorphic to G_0/N; and if S is a compact generating set of G_0 and

$$\rho_n = \min\{|g|_S \,:\, g \in N_{n+1} \backslash N_n\},$$

then G_0/N_n is δ_n-hyperbolic with $\delta_n = o(\rho_n)$.

1.1.3 Subgroups of lacunary hyperbolic groups

In [OOS09], the authors initiated the study of subgroups of finitely generated lacunary hyperbolic groups. They proved for example that any finitely presented subgroup of a lacunary hyperbolic group is a subgroup of a hyperbolic group, or that a subgroup of bounded torsion of a lacunary hyperbolic group cannot have relative exponential growth. This prohibits Baumslag-Solitar groups, free Burnside groups with sufficiently large exponent or lamplighter groups from occurring as subgroups of a finitely generated lacunary hyperbolic group [OOS09, Corollary 3.21]. These groups are examples of groups satisfying a law, and the authors ask whether it is possible that a non-virtually cyclic finitely generated group of relative exponential growth in a finitely generated lacunary hyperbolic group satisfies a law.

Let G be a compactly generated group and $\mathbf{s}$ a scaling sequence. For every subgroup $H \leq G$, the set of H-valued sequences of $\mathrm{Precone}(G, \mathbf{s})$ is a subgroup of $\mathrm{Precone}(G, \mathbf{s})$, which will be denoted $\mathrm{Precone}_G(H, \mathbf{s})$. In particular when $\mathrm{Cone}^\omega(G, \mathbf{s})$ is a real tree, we have an isometric action of the group $\mathrm{Precone}_G(H, \mathbf{s})$ on the real tree $\mathrm{Cone}^\omega(G, \mathbf{s})$, and one might wonder what is the type of this action in terms of the subgroup H. In Section 1.6 we carry out a careful study of the possible type of the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathrm{Cone}^\omega(G, \mathbf{s})$, which leads to the following result.

Proposition 1.4. *Let G be a unimodular lacunary hyperbolic group. If $H \leq G$ is a compactly generated subgroup of relative exponential growth in G not having $\mathbb{Z}$ as a discrete cocompact subgroup, then H cannot satisfy a law.*

In particular when specified to the setting of discrete groups, Proposition 1.4 answers Question 7.2 in [OOS09]. This prohibits for example finitely generated solvable groups from appearing as subgroups of finitely generated lacunary hyperbolic groups (see Corollary 1.47).

1.2 Preliminaries

1.2.1 Asymptotic cones

We start this section by recalling the definition of asymptotic cones. Let ω be a non-principal ultrafilter, i.e. a finitely additive probability measure on $\mathbb{N}$ taking values in $\{0, 1\}$ and vanishing on singletons. A statement $\mathrm{P}(n)$ is said to hold ω-almost surely if the set of integers n such that $\mathrm{P}(n)$ holds has measure 1. For any bounded function $f : \mathbb{N} \to \mathbb{R}$, there exists a unique real number ℓ such that for every $\varepsilon > 0$, we have $f(n) \in [\ell - \varepsilon, \ell + \varepsilon]$ ω-almost surely. The number ℓ is called the limit of f along ω, and we denote $\ell = \lim^\omega f(n)$.

Consider a non-empty metric space (X, d), a base point $e \in X$, and a scaling sequence $\mathbf{s} = (s_n)$, i.e. a sequence of positive real numbers tending to infinity. A sequence (x_n) of elements of X is said to be $\mathbf{s}$-linear if there exists a constant $C > 0$ so that $d(x_n, e) \leq Cs_n$ for all $n \geq 1$. We denote by $\mathrm{Precone}(X, d, \mathbf{s})$ the set of $\mathbf{s}$-linear sequences. If ω is a non-principal ultrafilter, the formula $d_\omega(x, y) = \lim^\omega d(x_n, y_n)/s_n$ makes $\mathrm{Precone}(X, d, \mathbf{s})$ a pseudometric space, i.e. d_ω satisfies the triangle inequality, is symmetric and vanishes on the diagonal. The asymptotic cone $\mathrm{Cone}^\omega(X, d, \mathbf{s})$ of (X, d) relative to the scaling sequence $\mathbf{s}$ and the non-principal ultrafilter ω, is defined by identifying elements of $\mathrm{Precone}(X, d, \mathbf{s})$ at distance d_ω zero. More precisely, $\mathrm{Cone}^\omega(X, d, \mathbf{s})$ is the set of equivalence classes of $\mathbf{s}$-linear sequences, where $x, y \in \mathrm{Precone}(X, d, \mathbf{s})$ are equivalent if $d_\omega(x, y) = 0$. We will denote by $(x_n)^\omega$ the class of the $\mathbf{s}$-linear sequence (x_n).

If two metric spaces X, Y are quasi-isometric, then their asymptotic cones corresponding to the same parameters $\mathbf{s}$ andω are bi-Lipschitz homeomorphic.

Now if G is a locally compact compactly generated group, it can be viewed as a metric space when endowed with the word metric d_S associated to some compact generating subset S. Since word metrics associated to different compact generating sets are bi-Lipschitz equivalent, $\mathrm{Precone}(G, d_S, \mathbf{s})$ does not depend on the choice of S and will be denoted by $\mathrm{Precone}(G, \mathbf{s})$. It inherits a group structure by component-wise multiplication. For any non-principal ultrafilter ω, the set of $\mathbf{s}$-linear sequences that are at distance d_ω zero from the constant sequence (e) is a subgroup of $\mathrm{Precone}(G, \mathbf{s})$, denoted by $\mathrm{Sublin}^\omega(G, \mathbf{s})$. The asymptotic cone $\mathrm{Cone}^\omega(G, d_S, \mathbf{s})$ is by definition the space of left cosets

$$\mathrm{Cone}^\omega(G, d_S, \mathbf{s}) = \mathrm{Precone}(G, \mathbf{s}) \,/\, \mathrm{Sublin}^\omega(G, \mathbf{s}),$$

endowed with the metric $d_\omega((g_n)^\omega, (h_n)^\omega) = \lim^\omega d_S(g_n, h_n)/s_n$. By construction the group $\mathrm{Precone}(G, \mathbf{s})$ acts transitively by isometries on $\mathrm{Cone}^\omega(G, d_S, \mathbf{s})$. Note that as a set, $\mathrm{Cone}^\omega(G, d_S, \mathbf{s})$ does not depend on S. Moreover if S_1, S_2 are two compact generating sets, then the identity map is a bi-Lipschitz homeomorphism between $\mathrm{Cone}^\omega(G, d_{S_1}, \mathbf{s})$ and $\mathrm{Cone}^\omega(G, d_{S_2}, \mathbf{s})$. We will denote by $\mathrm{Cone}^\omega(G, \mathbf{s})$ the corresponding class of metric spaces up to bi-Lipschitz homeomorphism.

If H is a subgroup of a locally compact compactly generated group G, then for every scaling sequence $\mathbf{s}$, we will denote by $\mathrm{Precone}_G(H, \mathbf{s})$ the subgroup of $\mathrm{Precone}(G, \mathbf{s})$ consisting of H-valued sequences. Remark that if H is a normal subgroup of G then $\mathrm{Precone}_G(H, \mathbf{s})$ is normal in $\mathrm{Precone}(G, \mathbf{s})$, and if H satisfies a law then $\mathrm{Precone}_G(H, \mathbf{s})$ satisfies the same law. These two simple observations will be used repeatedly throughout this chapter.

1.2.2 Isometric actions on hyperbolic spaces and real trees

Isometric actions on hyperbolic metric spaces and hyperbolic groups. Let X be a geodesic δ-hyperbolic metric space, and $x \in X$ a base-point. Recall that it means that X is a geodesic metric space such that any side of any geodesic triangle is contained in the δ-neighbourhood of the union of the two other sides. We define the Gromov product relative to x by the formula $2(y, z)_x = d(y, x) + d(z, x) - d(y, z)$. A sequence (y_n) of points in X is called Cauchy-Gromov if $(y_n, y_m)_x \to \infty$ as $m, n \to \infty$. The relation on the set of Cauchy-Gromov sequences defined by $(y_n) \sim (z_n)$ if $(y_n, z_n)_x \to \infty$ as $n \to \infty$, is an equivalence relation, and the boundary ∂X of the hyperbolic metric space X is by definition the set of equivalence classes of Cauchy-Gromov sequences.

Recall that if φ is an isometry of X, then the quantity $d(\varphi^n x, x)/n$ always converges to some real number $l(\varphi) \geq 0$ as $n \to \infty$. When $l(\varphi) = 0$, the isometry φ is called *elliptic* if it has bounded orbits, and *parabolic* otherwise. When $l(\varphi) > 0$, the isometry φ is called *hyperbolic*. The limit set of φ, also called the set of endpoints of φ, is the subset of ∂X of Cauchy-Gromov sequences defined along an orbit of φ. It is empty if φ is elliptic, a singleton if φ is parabolic and has cardinality two if φ is hyperbolic.

Now let Γ be a group acting by isometries on X. Gromov's classification [Gro87], which is summarized in Figure 1.1, says that exactly one of the following happens:

1. orbits are bounded, and the action of Γ on X is said to be *bounded*;
2. orbits are unbounded and Γ does not contain any hyperbolic element, in which case the action is said to be *horocyclic*;
3. Γ has a hyperbolic element and any two hyperbolic elements share the same endpoints. Such an action is termed *lineal*;
4. Γ has a hyperbolic element, the action is not lineal and any two hyperbolic elements share an endpoint. In this situation we say that the action is *focal*;
5. there exist two hyperbolic elements not sharing any endpoint. Such an action is said to be of *general type*.

Now recall that a locally compact compactly generated group G is called hyperbolic if its Cayley graph is hyperbolic for some (any) compact generating subset S. The type of G is defined as the type of the action of G on its Cayley graph. Since horocyclic isometric actions are always distorted (see for example Proposition 3.2 in [CCMT]), hyperbolic LC-groups are never horocyclic. It is easily seen that a hyperbolic LC-group is bounded if and only if it is compact, and hyperbolic LC-groups that are lineal are exactly the locally compact compactly generated groups with two ends. These two types of hyperbolic LC-groups are usually gathered under the term of *elementary* hyperbolic groups.

When dealing with discrete groups, it is a classical result that a finitely generated non-elementary hyperbolic group is of general type. On the other hand, focal hyperbolic groups do exist in the realm of non-discrete locally compact groups. Examples include some connected Lie groups (e.g. $\mathbb{R}^{n-1} \rtimes \mathbb{R}$, $n \geq 2$, which admits a free and transitive isometric action on the n-dimensional hyperbolic space $\mathbb{H}^n$ fixing a boundary point), or the stabilizer of an end in the automorphism group of a semi-regular locally finite tree. Beyond the connected and totally disconnected cases, a simple example of a focal hyperbolic group is $(\mathbb{Q}_p \times \mathbb{R}) \rtimes \mathbb{Z}$, where the element $1 \in \mathbb{Z}$ acts by multiplication by p on $\mathbb{Q}_p$ and by p^{-1} on $\mathbb{R}$. Caprace, Cornulier, Monod and Tessera characterized focal hyperbolic groups as those hyperbolic LC-groups

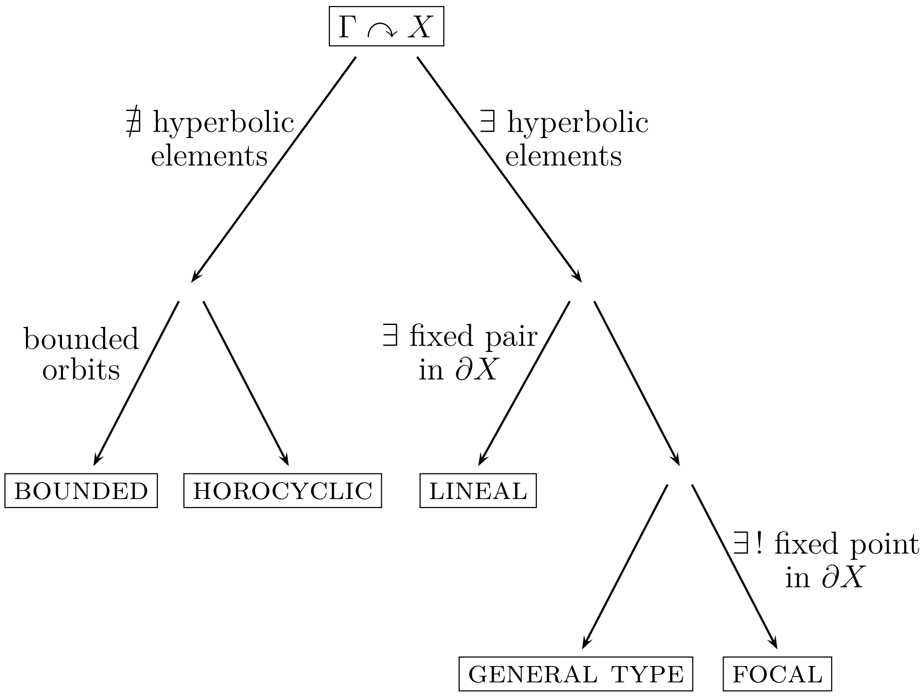


Figure 1.1 – Types of actions on hyperbolic spaces

that are non-elementary and amenable, and gave a precise description of the structure of these groups (see Theorem 7.3 in [CCMT]).

Actions on real trees. We now recall some basic facts about real trees and isometric group actions on these. A metric space is a real tree if it is geodesic and 0-hyperbolic, or equivalently if any two points are connected by a unique topological arc. If T is a real tree, a non-empty subset $T' \subset T$ is called a subtree if it is connected, which is equivalent to saying that T' is convex. We insist on the fact that by definition a subtree is necessarily non-empty. A point $x \in T$ is said to be a branching point if $T \setminus \{x\}$ has at least three connected components, and the branching cardinality of x is the cardinality of the set of connected components of $T \setminus \{x\}$.

If φ is an isometry of a real tree T, then the translation length of φ is defined as

$$\|\varphi\| = \inf_{x \in T} d(\varphi x, x),$$

and the characteristic set Min_φ of φ is the set of points where the translation length is attained. The following proposition, a proof of which can be consulted in [CM87], shows that the dynamics of an individual isometry of a real tree is easily understood.

Proposition 1.5. *The characteristic set* Min_φ *is a closed subtree of* T *which* *is invariant by φ. If $\|g\| = 0$ then φ is elliptic and Min_φ is the set of fixed points of φ; and if $\|\varphi\| > 0$ then Min_φ is a line isometric to $\mathbb{R}$, called the axis of φ, along which φ translates by $\|\varphi\|$.*

If Γ is a group acting by isometries on a real tree T, an invariant subtree T' is called minimal if it does not contain any proper invariant subtree. When this holds we also say that the action of Γ on T' is minimal, or that Γ acts minimally on T'. Since a real tree is a hyperbolic metric space, the classification of isometric group actions on hyperbolic spaces recalled in the previous paragraph holds, and the five possible types of actions may occur for groups acting on real trees. However if the action of Γ on T is minimal, then this action cannot be bounded unless T is reduced to a point, is never horocyclic, and is lineal if and only if T is isometric to the real line.

The following lemma is standard, see Proposition 3.1 in [CM87].

Lemma 1.6. *Suppose that Γ is a group acting on a real tree. If Γ contains some hyperbolic element, then the union of the axes of the hyperbolic elements of Γ is an invariant subtree contained in any other invariant subtree.*

A simple but useful consequence is the following result.

Lemma 1.7. *Let Γ be a group acting minimally on a real tree T, and let $\Lambda \lhd \Gamma$ be a normal subgroup containing some hyperbolic element. Then the action of Λ on T is minimal as well, and every point of T lies on the axis of some hyperbolic element of Λ.*

Proof. Let T' be the union of the axes of the hyperbolic elements of Λ, which is a minimal Λ-invariant subtree by the previous lemma. To prove the statement, it is enough to prove that $T' = T$. But this is clear because the condition that Λ is a normal subgroup of Γ implies that T' is also a Γ-invariant subtree, and by minimality of the action of Γ on T, one must have $T' = T$. □

1.2.3 Locally compact groups

We now aim to recall some structural results about locally compact compactly generated groups that will be needed later. As it is often the case, we will deal separately with connected and totally disconnected groups.

Connected locally compact groups. The material of this paragraph is classical. It is an illustration of how the solution of Hilbert's fifth problem can be used to derive results about connected locally compact groups from the study of connected Lie groups.

Proposition 1.8. *Every connected locally compact group has a unique maximal compact normal subgroup, called the compact radical, and the corresponding quotient is a connected Lie group.*

Proof. Let G be a connected locally compact group. By Theorem 4.6 of [MZ55], there exists a compact normal subgroup K of G such that G/K is a connected Lie group. Since on the one hand having a unique maximal compact normal subgroup is preserved by group extension with compact kernel, and on the other hand any connected Lie group has a unique maximal compact normal subgroup, the conclusion follows. $\square$

If G is a topological group, we denote by G° the connected component of the identity. It is a closed characteristic subgroup of G, and the quotient G/G°, endowed with the quotient topology, is a totally disconnected group.

Corollary 1.9. *Every locally compact group G has a compact subgroup K that is characteristic and contained in G°, such that the quotient G°/K is a connected Lie group without non-trivial compact normal subgroup.*

Proof. Take K the compact radical of G°. Being characteristic in the characteristic subgroup G°, it is characteristic in G. $\square$

The following result will be used in Section 2.5.

Corollary 1.10. *Every connected-by-compact locally compact group is quasi-isometric to a compactly generated solvable group.*

Proof. Clearly it is enough to prove the result for a connected locally compact group G. Modding out by the compact radical of G, we may assume by Proposition 1.8 that G is a connected Lie group, and the result now follows from the classical fact that any connected Lie group has a (possibly non-connected) cocompact solvable Lie subgroup. $\square$

Locally compact groups with compact open subgroups. Recall that if G is a locally compact totally disconnected group, then according to van Dantzig's theorem, compact open subgroups of G exist and form a basis of identity neighbourhoods. In this paragraph we will deal with the slightly more general class of groups, namely the class of groups G having compact open subgroups. Note that by van Dantzig's theorem, this is equivalent to saying that G is a locally compact group with a compact identity component.

The following result, originally due to Abels, associates a connected locally finite graph to any compactly generated locally compact group with a compact open subgroup.

Proposition 1.11. *Let G be a compactly generated locally compact group having a compact open subgroup. Then there exists a connected locally finite graph X on which G acts by automorphisms, transitively and with compact open stabilizers on the set of vertices.*

Recall that the construction consists in choosing a compact open subgroup K, and a compact generating subset S of G that is bi-invariant under the action of K. We take G/K as vertex set for the graph X, and two different cosets g_1K and g_2K are adjacent if there exists $s \in S^{\pm 1}$ such that $g_2 = g_1 s$. The resulting graph is connected and locally finite. The action of G on X is vertex-transitive, and the stabilizer of the base-vertex is the compact open subgroup K. The graph X is called the Cayley-Abels graph of G associated to the compact open subgroup K and compact generating subset S. For example when G is the affine p-adic group $\mathbb{Q}_p \rtimes \mathbb{Q}_p^\times$, its Cayley-Abels graph associated to $K = \mathbb{Z}_p \rtimes \mathbb{Z}_p^\times$ and $S = K \cup K(0,p)$ is a regular tree $\mathbb{T}_{p+1}$ of degree $p+1$. Another example is the p-adic SOL group $\mathbb{Q}_p^2 \rtimes \mathbb{Q}_p^\times$, whose Cayley-Abels graph with respect to some appropriately chosen parameters is the Diestel-Leader graph $\mathrm{DL}(p,p)$, namely the subset of $\mathbb{T}_{p+1} \times \mathbb{T}_{p+1}$ defined by the equation $b(x) + b(y) = 0$, where b is a Busemann function on $\mathbb{T}_{p+1}$.

In some sense, the following result is a topological analogue of the fact that any finitely generated group is a quotient of a finitely generated free group. The result is not new (see for example [CH15, Proposition 8.A.18]), but the proof we give here is different from the one in [CH15].

Proposition 1.12. *Let G be a compactly generated locally compact group having a compact open subgroup. Then there exists a compactly generated locally compact group G_0 acting on a locally finite tree, transitively and with compact open stabilizers on the set of vertices; and an open epimorphism $\pi : G_0 \twoheadrightarrow G$ with discrete kernel.*

Proof. Let K be a compact open subgroup of G, and S a K-bi-invariant compact symmetric generating subset of G containing the identity. Note that this implies that $K \subset S$. We let $R_{K,S}$ be the set of words of the form $s_1 s_2 k^{-1}$, with $s_1, s_2 \in S$ and $k \in K$, when the relation $s_1 s_2 = k$ holds in the group G. We denote by G_0 the group defined by the abstract presentation $G_0 = \langle S \mid R_{K,S} \rangle$. Note that by construction, the group G_0 comes equipped with a natural morphism $\pi : G_0 \to G$, which is onto since S is a generating subset of G.

We claim that G_0 admits a commensurated subgroup isomorphic to the subgroup K of G. Indeed, let K_0 be the subgroup of G_0 generated by $K \subset S$ (here K is seen as a subset of the abstract generating set S). To prove that K_0 is isomorphic to K, it is enough to prove that K_0 intersects trivially the kernel of π. But this is clear, because by construction all the relations in

G of the form $k_1k_2 = k_3$ are already satisfied in G_0, so the map π induces an isomorphism between K_0 and the subgroup K of G. Now it remains to prove that K_0 is commensurated in G_0. Since by definition S generates G_0, it is enough to prove that the subset S commensurates K_0 in G_0. Being compact and open in G, the subgroup K is commensurated in G. Therefore for every $s \in S$ there exists a finite index subgroup $K^{(s)} \leq K$ such that $sK^{(s)}s^{-1} \leq K$. This can be rephrased by saying that for every $k^{(s)} \in K^{(s)}$, there exists $k \in K$ such that $sk^{(s)}s^{-1} = k$. But now using twice the set or relators $R_{K,S}$, it is not hard to check that these relations hold in G_0 as well, which implies that the subgroup K_0 is commensurated in G_0. This finishes the proof of the claim.

Now if we equip K_0 with the pullback topology under the restriction of the map $\pi_{/K_0} : K_0 \xrightarrow{\sim} K$, we obtain a group topology on G_0 turning K_0 into a compact open subgroup (see Lemma 3.13). Note that by construction the epimorphism $\pi : G_0 \twoheadrightarrow G$ is open and has a discrete kernel (because the latter intersects trivially the open subgroup K_0).

To end the proof of the proposition, we need to construct a locally finite tree on which G_0 acts with the desired properties. Let us consider the Cayley-Abels graph X of G_0 associated to K_0 and S. The action of G_0 on X is transitive and with compact open stabilizers on the set of vertices, so the only thing that needs to be checked is that X is a tree, i.e. X does not have non-trivial loops. To every loop in X can be associated a word $s_1 \cdots s_n$ so that the relation $s_1 \cdots s_n k = 1$ holds in G_0 for some $k \in K$. This means that in the free group over the set S, we have a decomposition of the form

$$s_1 \cdots s_n k = \prod_{i=1}^{N} w_i \left(s_{i,1}s_{i,2}k_i^{-1}\right) w_i^{-1},$$

with $s_{i,1}s_{i,2}k_i^{-1} \in R_{K,S}$. Now remark that in X, any loop indexed by a word of the form $s_{i,1}s_{i,2}k_i^{-1} \in R_{K,S}$ is nothing but a simple backtrack, and it follows that we have a decomposition of our original loop as a sequence of backtracks. This implies that X is a tree and finishes the proof. □

1.3 Preliminary results on asymptotic cones

This section gathers a few lemmas that will be used in the sequel. As we have seen earlier, any asymptotic cone of a locally compact compactly generated group comes equipped with a natural isometric group action. The next lemma describes to what extent this data varies for instance when modding out by a compact normal subgroup or passing to a cocompact normal subgroup. We point out that in the second statement, the assumption that $\pi(G)$ is normal in Q is essential (think of $\mathbb{R} \rtimes \mathbb{R}$ inside $\mathrm{SL}_2(\mathbb{R})$).

Lemma 1.13. *Consider a proper homomorphism with cocompact image $\pi : G \to Q$ between locally compact compactly generated groups. Then for every scaling sequence* $\mathbf{s}$ *and non-principal ultrafilter* ω*, the induced map at the level of asymptotic cones* $\tilde{\pi} : \mathrm{Cone}^\omega(G, \mathbf{s}) \to \mathrm{Cone}^\omega(Q, \mathbf{s})$ *is a bi-Lipschitz homeomorphism.*

If we assume in addition that $\mathrm{Cone}^\omega(G, \mathbf{s})$ *(and hence* $\mathrm{Cone}^\omega(Q, \mathbf{s})$*) is a real tree and that* $\pi(G)$ *is normal in* Q*, then the actions of* $\mathrm{Precone}(G, \mathbf{s})$ *on* $\mathrm{Cone}^\omega(G, \mathbf{s})$ *and of* $\mathrm{Precone}(Q, \mathbf{s})$ *on* $\mathrm{Cone}^\omega(Q, \mathbf{s})$ *have the same type.*

Proof. Since the homomorphism π has compact kernel and cocompact image, it is a quasi-isometry. Therefore the map $\tilde{\pi}$ defined by $\tilde{\pi}((g_n)^\omega) = (\pi(g_n))^\omega$ is a bi-Lipschitz homeomorphism, which is equivariant under the actions of $\mathrm{Precone}(G, \mathbf{s})$.

It follows that $\mathrm{Cone}^\omega(Q, \mathbf{s})$ is a real tree if and only if $\mathrm{Cone}^\omega(G, \mathbf{s})$ is a real tree. When this is so and when $\pi(G)$ is supposed to be normal in Q, if $\mathrm{Precone}(G, \mathbf{s})$ stabilizes some finite subset in the boundary of $\mathrm{Cone}^\omega(G, \mathbf{s})$, then the same holds for the group $\mathrm{Precone}(Q, \mathbf{s})$. The converse implication being clear, the proof is complete. □

Recall that a metric space (X, d) is coarsely connected if there exists a constant $c > 0$ such that for any $x, y \in X$, there exists a sequence of points $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_i, x_{i+1}) \leq c$ for every $i = 0, \ldots, n-1$.

Lemma 1.14. *Let* (X, d) *be a coarsely connected non-empty metric space. If* (X, d) *is unbounded, then so are all its asymptotic cones.*

Proof. Let $e \in X$ be a base point, $\mathbf{s}$ a scaling sequence and ω a non-principal ultrafilter. We prove the stronger statement that for every $\ell > 0$, there exists a point in $\mathrm{Cone}^\omega(X, d, \mathbf{s})$ at distance exactly ℓ from the point $(e)^\omega$.

Since (X, d) is unbounded, for every $n \geq 1$ there is a point $x_n \in X$ at distance at least ℓs_n from the base point e. Now by coarse connectedness, x_n can be chosen to be at distance at most $\ell s_n + c$ from e, where $c > 0$ is the constant from the definition of coarse connectedness. By construction, the sequence (x_n) defines a point $(x_n)^\omega \in \mathrm{Cone}^\omega(X, d, \mathbf{s})$ that is at distance ℓ to the point $(e)^\omega$. □

Lemma 1.15. *Let* G *be a compactly generated locally compact group, and* H *a closed compactly generated subgroup of* G*. Then for any asymptotic cone of* G*, the following statements are equivalent:*

(i) H *is compact;*

(ii) $\mathrm{Precone}_G(H, \mathbf{s})$ *fixes the point* $(e)^\omega \in \mathrm{Cone}^\omega(G, \mathbf{s})$*;*

(iii) $\mathrm{Precone}_G(H, \mathbf{s})$ *has a bounded orbit in* $\mathrm{Cone}^\omega(G, \mathbf{s})$*.*

Proof. The implications $i) \Rightarrow ii) \Rightarrow iii)$ are trivial. Let us prove $iii) \Rightarrow i)$ by proving the contrapositive statement.

Since H is a closed compactly generated subgroup of G, the metric space (H, d_G) is coarsely connected [CH15, Proposition 4.B.8]. So if H is assumed not to be compact, it follows from Lemma 1.14 that none of the asymptotic cones of (H, d_G) are bounded. But the asymptotic cone of H with the induced metric from G can be naturally identified with the orbit under $\mathrm{Precone}_G(H, \mathbf{s})$ of the point $(e)^\omega \in \mathrm{Cone}^\omega(G, \mathbf{s})$. So it follows that $\mathrm{Precone}_G(H, \mathbf{s})$ has one unbounded orbit, and since the action is isometric, every orbit must be unbounded. $\square$

Remark 1.16. We illustrate the failure of Lemma 1.15 when H is not compactly generated. Let $G = \mathbb{F}_p(\!(t)\!) \rtimes_t \mathbb{Z}$, where $\mathbb{F}_p(\!(t)\!)$ is the field of Laurent series over some finite field $\mathbb{F}_p$, and let H be the subgroup generated by $(t^{-\alpha_n}, 0)$, $n \geq 1$, where $\alpha_n = 2^{2^n}$. Then for any scaling sequence $\mathbf{s}$ such that $\alpha_n << s_n << \alpha_{n+1}$ (take for example $s_n = 2^{3 \cdot 2^{n-1}}$) and for any non-principal ultrafilter ω, the group $\mathrm{Precone}_G(H, \mathbf{s})$ fixes the point $(e)^\omega \in \mathrm{Cone}^\omega(G, \mathbf{s})$, whereas H is clearly not compact.

Lemma 1.17. *Let G be a compactly generated locally compact group, and let N be a closed normal subgroup of G. Assume that N is not cocompact in G. Then for every asymptotic cone $\mathrm{Cone}^\omega(G, \mathbf{s})$, there exists a bi-Lipschitz ray $\gamma : [0, +\infty[\to \mathrm{Cone}^\omega(G, \mathbf{s})$ such that for every $t \geq 0$,*

$$d_\omega\left(\gamma(t), \mathcal{C}_N\right) \geq ct$$

for some constant $c > 0$, where $\mathcal{C}_N$ is the orbit of the point $(e)^\omega$ under $\mathrm{Precone}_G(N, \mathbf{s})$.

Proof. Since the group G/N is non-compact, it has an infinite quasi-geodesic ray, that can be lifted to a quasi-geodesic ray $\rho : [0, +\infty[\to G$ such that for every $t \geq 0$, $d_G(\rho(t), N) \geq ct$ for some constant c. Now we easily check that for every non-principal ultrafilter ω and scaling sequence $\mathbf{s}$, the ω-limit of the quasi-geodesic ray ρ in $\mathrm{Cone}^\omega(G, \mathbf{s})$ is a bi-Lipschitz ray satisfying the required property. $\square$

Corollary 1.18. *Let G be a compactly generated locally compact group, and let N be a closed normal subgroup of G. If for some parameters $\omega, \mathbf{s}$ the action of $\mathrm{Precone}_G(N, \mathbf{s})$ on $\mathrm{Cone}^\omega(G, \mathbf{s})$ is cobounded, then N is cocompact in G.*

When G is a compactly generated group with an asymptotic cone $\mathrm{Cone}^\omega(G, \mathbf{s})$ that is a real tree and such that the action of $\mathrm{Precone}(G, \mathbf{s})$ is of general type, the five types of actions on real trees may happen for the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathrm{Cone}^\omega(G, \mathbf{s})$, where H is a subgroup of G. However the situation is more restrictive under the additional assumption that H is a normal subgroup.

Lemma 1.19. *Let G be a locally compact compactly generated group. Assume that G admits an asymptotic cone* $\mathrm{Cone}^\omega(G,\mathbf{s})$ *that is a real tree and such that the action of* $\mathrm{Precone}(G,\mathbf{s})$ *is of general type. Then for any normal subgroup N of G, the action of* $\mathrm{Precone}_G(N,\mathbf{s})$ *on* $\mathrm{Cone}^\omega(G,\mathbf{s})$ *is either bounded or of general type.*

Proof. If the group $\mathrm{Precone}_G(N,\mathbf{s})$ preserves a finite subset in the boundary of $\mathrm{Cone}^\omega(G,\mathbf{s})$, then this finite subset is also preserved by $\mathrm{Precone}(G,\mathbf{s})$ because $\mathrm{Precone}_G(N,\mathbf{s})$ is normal in $\mathrm{Precone}(G,\mathbf{s})$. By assumption this does not happen, so it follows that the action of $\mathrm{Precone}_G(N,\mathbf{s})$ on $\mathrm{Cone}^\omega(G,\mathbf{s})$ is either bounded or of general type. $\square$

We point out that it may happen that the group $\mathrm{Precone}_G(N,\mathbf{s})$ fixes the point $(e)^\omega \in \mathrm{Cone}^\omega(G,\mathbf{s})$ even if N is non-compact. Indeed, if G is a non-virtually cyclic finitely generated lacunary hyperbolic group with an infinite center Z (such groups have been constructed in [OOS09]), then the action of the abelian group $\mathrm{Precone}_G(Z,\mathbf{s})$ cannot be of general type, and therefore must have a fixed point.

Let (X,d) be a non-empty metric space, and let $x_0 \in X$. Recall that an isometry φ of X is hyperbolic if the limit as $n \to \infty$ of $d(\varphi^n x_0, x_0)/n$ is positive. If $G \leq \mathrm{Isom}(X)$ is a subgroup of the isometry group of X, we can endow G with the pseudo-metric $d_{x_0}(g,h) = d(gx_0, hx_0)$. Note that for every scaling sequence $\mathbf{s}$ and non-principal ultrafilter ω, the group $\mathrm{Precone}(G, d_{x_0}, \mathbf{s})$ admits a natural action on the asymptotic cone $\mathrm{Cone}^\omega(X,d,\mathbf{s})$.

The following lemma says that if X is a geodesic hyperbolic metric space and $G \leq \mathrm{Isom}(X)$, in many cases the type of the action of $\mathrm{Precone}(G, d_{x_0}, \mathbf{s})$ on $\mathrm{Cone}^\omega(X,d,\mathbf{s})$ is the same as the type of the action of G on X. Note that both situations of statement (b) may happen (see Remark 1.16).

Lemma 1.20. *Let X be a geodesic hyperbolic metric space, and $x_0 \in X$. If G is a subgroup of the isometry group of X, then:*

(a) if the action of G on X is either bounded, lineal, focal or of general type, then for every asymptotic cone of X, the action of $\mathrm{Precone}(G, d_{x_0}, \mathbf{s})$ *on* $\mathrm{Cone}^\omega(X,d,\mathbf{s})$ *has the same type;*

(b) if the action of G on X is horocyclic, then the action of $\mathrm{Precone}(G, d_{x_0}, \mathbf{s})$ *on* $\mathrm{Cone}^\omega(X,d,\mathbf{s})$ *is either bounded or horocylic.*

Proof. We start by making the easy observation that if $g \in G$ is a hyperbolic element, then for every asymptotic cone $\mathrm{Cone}^\omega(X,d,\mathbf{s})$ of X, the element $(g^{s_n}) \in \mathrm{Precone}(G, d_{x_0}, \mathbf{s})$ is a hyperbolic isometry of $\mathrm{Cone}^\omega(X,d,\mathbf{s})$, and the axis of (g^{s_n}) is the asymptotic cone of any geodesic line in X between the two endpoints of g.

(a). The statement is obvious for bounded and lineal actions, and follows from the previous observation for actions of general type. Let us give the proof in the case when the action of G on X is focal. Let $\gamma \in G$ be a hyperbolic element. Since any two hyperbolic elements of G share an endpoint, upon changing γ into its inverse, we may assume that $(\gamma^k x_0) \in \partial X$ is the unique boundary point that is fixed by G. This implies (see for instance [GdlH90, Chap.7 Cor.3]) that there exists some constant $c > 0$ such that for every $g \in G$, we have $d(g\gamma^k x_0, \gamma^k x_0) \leq cd(gx_0, x_0)$ for every integer $k \geq 1$. It follows that for every element $(g_n) \in \mathrm{Precone}(G, d_{x_0}, \mathbf{s})$, there exists some constant $C > 0$ such that $d(g_n\gamma^{\lfloor ts_n \rfloor}x_0, \gamma^{\lfloor ts_n \rfloor}x_0) \leq Cs_n$ for every $t \geq 0$ and $n \geq 1$. This implies that, if we let $\xi : [0, +\infty[\to \mathrm{Cone}^\omega(X, d, \mathbf{s})$ be the ray defined by $\xi(t) = (\gamma^{\lfloor ts_n \rfloor}x_0)^\omega$, in the real tree $\mathrm{Cone}^\omega(X, d, \mathbf{s})$ the distance between $g{\cdot}\xi(t)$ and $\xi(t)$ is uniformly bounded, which means that the two rays $g{\cdot}\xi$ and ξ represent the same end of $\mathrm{Cone}^\omega(X, d, \mathbf{s})$. Combined with the fact that $\mathrm{Precone}(G, d_{x_0}, \mathbf{s})$ contains hyperbolic elements not having the same endpoints (because G already does), this implies that the action of $\mathrm{Precone}(G, d_{x_0}, \mathbf{s})$ on $\mathrm{Cone}^\omega(X, d, \mathbf{s})$ is focal.

(b). We assume that the action of $\mathrm{Precone}(G, d_{x_0}, \mathbf{s})$ on $\mathrm{Cone}^\omega(X, d, \mathbf{s})$ is not bounded, and we prove that it is horocylic. Let $(g_nx_0)^\omega$ be a point of $\mathrm{Cone}^\omega(X, d, \mathbf{s})$ such that $d_\omega((x_0)^\omega, (g_nx_0)^\omega) = \ell > 0$, and let γ_n be a geodesic in X between x_0 and g_nx_0. Call m_n the mid-point of γ_n. Recall that since the action of G on X is horocylic, for every $c > 0$ there exists some constant c' such that the intersection in X between any c-quasi-geodesic and any G-orbit lies in the union of two c-balls. This implies that ω-almost surely the ball or radius $\ell s_n/3$ around m_n in X does intersect the orbit Gx_0. Therefore the mid-point of the unique geodesic in $\mathrm{Cone}^\omega(X, d, \mathbf{s})$ between $(x_0)^\omega$ and $(g_nx_0)^\omega$ is at distance at least $\ell/3$ from any point in the $\mathrm{Precone}(G, d_{x_0}, \mathbf{s})$-orbit of $(x_0)^\omega$. In particular this proves that $\mathrm{Precone}(G, d_{x_0}, \mathbf{s})$ cannot preserve a geodesic line in $\mathrm{Cone}^\omega(X, d, \mathbf{s})$, and therefore does not have any hyperbolic isometry. □

1.4 Focal lacunary hyperbolic groups

This section is devoted to the proof of Theorem 1.1. We call a locally compact compactly generated group *focal lacunary hyperbolic* if it admits one asymptotic cone $\mathrm{Cone}^\omega(G, \mathbf{s})$ that is a real tree, and such that the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathrm{Cone}^\omega(G, \mathbf{s})$ is focal. According to Lemma 1.20, any focal hyperbolic group is a focal lacunary hyperbolic group. The rest of this section will be devoted to the proof of the following converse implication.

Theorem 1.21. *Any focal lacunary hyperbolic group admits a topological semidirect product decomposition $H \rtimes \mathbb{Z}$ or $H \rtimes \mathbb{R}$, where the element $1 \in \mathbb{Z}$ or $\mathbb{R}$ acts on H as a compacting automorphism.*

Recall that an automorphism $\alpha \in \mathrm{Aut}(H)$ of a locally compact group

H is called compacting if there exists a compact subset $V \subset H$, called a pointwise vacuum set for α, such that for every $h \in H$, there exists an integer $n_0 \geq 1$ such that $\alpha^n(h) \in V$ for every $n \geq n_0$. Note that $\alpha \in \mathrm{Aut}(H)$ is compacting if and only if some positive power of α is compacting.

The idea of the proof of Theorem 1.21 is to deduce a contracting dynamics at the level of the group from a focal dynamics at the level of one asymptotic cone. The first step in the argument is to prove that a focal lacunary hyperbolic group is a topological semidirect product $H \rtimes \mathbb{Z}$ or $H \rtimes \mathbb{R}$. This will be achieved in Corollary 1.25.

Recall that if G is a locally compact group endowed with the word metric associated to some compact generating subset, a cyclic subgroup $\langle g \rangle$ is said to be undistorted if the left multiplication by g is a hyperbolic isometry of (G, d_S), i.e. if the limit of $|g^n|_S/n$ is not zero. A sufficient condition for $\langle g \rangle$ to be undistorted is the existence of a continuous homomorphism $f : G \to \mathbb{Z}$ such that $f(g) \neq 0$.

The following result provides a criterion for a normal subgroup of a focal lacunary hyperbolic group to be cocompact.

Lemma 1.22. *Let G be a focal lacunary hyperbolic group, and N a closed normal subgroup containing an undistorted element. Then N is cocompact in G.*

Proof. We denote by $\mathcal{C} = \mathrm{Cone}^\omega(G, \mathbf{s})$ an asymptotic cone of G that is a real tree and such that the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathcal{C}$ is focal. Let $\xi : [0, +\infty[\to \mathcal{C}$ be the ray emanating from $(e)^\omega$ representing the end of $\mathcal{C}$ that is fixed by $\mathrm{Precone}(G, \mathbf{s})$. Since the group N contains an undistorted element, it follows that the group $\mathrm{Precone}_G(N, \mathbf{s})$ acts on $\mathcal{C}$ with a hyperbolic element h, whose translation length will be denoted by ℓ. Without loss of generality, we may assume that $(e)^\omega$ belongs to the axis of h. Indeed, if $(g_n)^\omega$ is a point on the axis of h and if we denote by $g = (g_n)$, then $g^{-1}hg$ is hyperbolic and contains $(e)^\omega$ on its axis. Since $\mathrm{Precone}_G(N, \mathbf{s})$ is normal in $\mathrm{Precone}(G, \mathbf{s})$, the element $g^{-1}hg$ remains in $\mathrm{Precone}_G(N, \mathbf{s})$, and the claim is proved. Now since the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathcal{C}$ is supposed to be focal, the axis of h must contain the entire ray ξ.

Let us now prove that the action of $\mathrm{Precone}_G(N, \mathbf{s})$ on $\mathcal{C}$ is cocompact. According to Corollary 1.18, this finishes the proof of the proposition. Let x be a point of $\mathcal{C}$. We will prove that the $\mathrm{Precone}_G(N, \mathbf{s})$-orbit of x in $\mathcal{C}$ intersects the segment joining $(e)^\omega$ and $\xi(\ell)$. According to Lemma 1.7, there exists some hyperbolic element $\gamma \in \mathrm{Precone}_G(N, \mathbf{s})$ whose axis contains x. But since the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathcal{C}$ is focal, the axis of γ intersects ξ along an infinite ray, and by translating along the axis of γ, there exists some $n \in \mathbb{Z}$ so that $y = \gamma^n x$ belongs to ξ. But now since the axis of h contains the ray ξ and since h translates along its axis by an amount of ℓ, we can find $m \in \mathbb{Z}$ so that $h^m y$ remains in ξ and is at distance at most ℓ from $(e)^\omega$. $\square$

Remark 1.23. Actually the same proof works with the only assumption that N is a closed normal subgroup such that $\mathrm{Precone}_G(N, \mathbf{s})$ acts on $\mathcal{C}$ with a hyperbolic element. This will be used in the proof of Proposition 1.26.

If G is a locally compact group, we denote by $\Delta_G : G \to \mathbb{R}_+^*$ the modular function of G. Recall that Δ_G is a continuous group homomorphism.

The following proposition, which is a crucial step in the argument, consists in obtaining an estimate on the modular function of a focal lacunary hyperbolic group. In the proof, we take advantage of an idea appearing in the end of the proof of Theorem 6.12 in [DS05].

Lemma 1.24. *Let G be focal lacunary hyperbolic group, and $\mathcal{C} = \mathrm{Cone}^\omega(G, \mathbf{s})$ an asymptotic cone of G that is a real tree and such that the action of* $\mathrm{Precone}(G, \mathbf{s})$ *is focal. Let $\xi : [0, +\infty[\to \mathcal{C}$, $\ell \mapsto (\xi_n(\ell))^\omega$, be the geodesic ray emanating from $(e)^\omega$ representing the end of $\mathcal{C}$ that is fixed by* $\mathrm{Precone}(G, \mathbf{s})$. *Then there exist some constants $c > 0$, $\rho > 1$, such that for every $\ell \geq 0$, we have*

$$c\rho^{\ell s_n} \leq \Delta_G(\xi_n(\ell))$$

ω-almost surely.

Proof. First note that for every $\ell \geq 0$, the element $(\xi_n(\ell)) \in \mathrm{Precone}(G, \mathbf{s})$ sends the point $(e)^\omega$ to $\xi(\ell)$ by definition. But since the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathcal{C}$ is supposed to be focal, the image of the geodesic ray ξ by $(\xi_n(\ell))$ eventually coincides with ξ. It follows that $(\xi_n(\ell)) \cdot \xi$ is exactly the infinite subray of ξ emanating from $\xi(\ell)$. In particular for every $k \geq 1$, $(\xi_n(\ell)) \cdot \xi(k\ell) = \xi((k+1)\ell)$, and by a straightforward induction we obtain $\xi(k\ell) = (\xi_n(\ell)^k)^\omega$.

Let S be a compact generating subset of G, and denote by $B_S(r)$ the closed ball of radius $r \geq 0$ around the identity with respect to the word metric associated to S. Let $\ell \geq 0$, and $(g_n) \in \mathrm{Precone}(G, \mathbf{s})$ such that $|g_n| \leq \ell s_n$ for every $n \geq 1$. The image of the point $(e)^\omega$ under such an element (g_n) is at distance at most ℓ from $(e)^\omega$. The action being focal, it follows from this observation that the two rays $(g_n) \cdot \xi$ and ξ intersect along an infinite subray of ξ containing the point $\xi(\ell)$.

Now let us assume for a moment that the element (g_n) is either elliptic or has the fixed end of $\mathcal{C}$ for attractive endpoint. Since the translation length of (g_n) is at most ℓ, it follows from the above observation that $(g_n) \cdot \xi(\ell)$ is at distance at most $\ell/2$ from either $\xi(\ell)$ or $\xi(2\ell)$. This implies that ω-almost surely

$$d_S\left(g_n \xi_n(\ell), \left\{\xi_n(\ell), \xi_n(\ell)^2\right\}\right) \leq \frac{2}{3}\ell s_n,$$

where $d_S(g, \{h, k\})$ is by definition the minimum between $d_S(g, h)$ and $d_S(g, k)$. This inequality can be reformulated by saying that ω-almost surely

$$g_n \in \xi_n(\ell) \cdot B_S(2\ell s_n/3) \cdot \xi_n(\ell)^{-1} \cup \xi_n(\ell)^2 \cdot B_S(2\ell s_n/3) \cdot \xi_n(\ell)^{-1}.$$

Now if the element (g_n) is a hyperbolic isometry having the fixed end of $\mathcal{C}$ for repulsive endpoint, then we can apply the previous argument to (g_n^{-1}).

So we have proved that for every $\ell \geq 0$, ω-almost surely the ball of radius ℓs_n around the identity in G lies inside

$$\xi_n(\ell) \cdot B_S(2\ell s_n/3) \cdot \xi_n(\ell)^{-1} \cup \xi_n(\ell)^2 \cdot B_S(2\ell s_n/3) \cdot \xi_n(\ell)^{-1} \cup \xi_n(\ell) \cdot B_S(2\ell s_n/3) \cdot \xi_n(\ell)^{-2}.$$

Now if we let μ be a left-invariant Haar measure on G, then for every $\ell \geq 0$, ω-almost surely

$$\mu(B_S(\ell s_n)) \leq 2\mu\left(B_S(2\ell s_n/3) \cdot \xi_n(\ell)^{-1}\right) + \mu\left(B_S(2\ell s_n/3) \cdot \xi_n(\ell)^{-2}\right).$$

Dividing by $\mu(B_S(2\ell s_n/3))$, we obtain

$$\frac{\mu(B_S(\ell s_n))}{\mu(B_S(2\ell s_n/3))} \leq 2\Delta_G(\xi_n(\ell)) + \Delta_G(\xi_n(\ell))^2 \leq 3\Delta_G(\xi_n(\ell))^2.$$

We claim that the Haar-measure μ is not right-invariant. Let us argue by contradiction and assume that μ is right-invariant, which implies that the right-hand side of the last inequality is constant equal to 3. Then for every $\ell \geq 0$, ω-almost surely $\mu(B_S(\ell s_n)) \leq 3\mu(B_S(2\ell s_n/3))$. Now since every point of the real tree $\mathcal{C}$ is a branching point, spheres of any given radius in $\mathcal{C}$ are infinite, and it is not hard to see that this establishes a contradiction with the above inequality on the growth function of G. So μ cannot be right-invariant, i.e. G is non-unimodular. In particular the group G has exponential growth, and we easily deduce that the left-hand side of the above inequality is at least $c_1\alpha^{\ell s_n}$ for some constants $c_1 > 0$, $\alpha > 1$, and the conclusion follows with $c = \sqrt{c_1/3}$ and $\rho = \sqrt{\alpha}$. $\square$

Corollary 1.25. *If G is a focal lacunary hyperbolic group, then G admits a topological semidirect product decomposition $H \rtimes \mathbb{Z}$ or $H \rtimes \mathbb{R}$, where H is the kernel of the modular function of G.*

Proof. Let H be the kernel of the modular function $\Delta_G : G \to \mathbb{R}_+^*$. Assume that we have proved that the image of Δ_G is a closed non-trivial subgroup of $\mathbb{R}_+^*$. Then the image of Δ_G is either discrete and infinite cyclic, or topologically isomorphic to $\mathbb{R}$. In the first case we easily have $G = H \rtimes \mathbb{Z}$, and in the other case we use the fact that any quotient homomorphism from a locally compact group to the group $\mathbb{R}$ is split, and deduce that $G = H \rtimes \mathbb{R}$.

So we should prove that the image of Δ_G is closed and non-trivial. According to Lemma 1.24, we can choose some $\xi_n(\ell) = \gamma \in G$ such that

$\Delta_G(\gamma) > 1$. Let us consider the subgroup $N = H \rtimes \langle\gamma\rangle$ of G generated by H and γ. Since H contains the derived subgroup of G, the subgroup N is normal in G, and being the preimage by Δ_G of the discrete subgroup of $\mathbb{R}_+^*$ generated by $\Delta_G(\gamma)$, N is a closed subgroup.

Since $\Delta_G(\gamma) \neq 1$, the cyclic subgroup generated by γ is undistorted in G, and therefore we are in position to apply Lemma 1.22, which implies that N is a cocompact subgroup of G. Hence Δ_G induces a homomorphism from the compact group G/N to $\mathbb{R}_+^*/\Delta_G(N)$, which necessarily has a closed image. Being the preimage in $\mathbb{R}_+^*$ of this closed subgroup, the image of Δ_G is closed. $\square$

So we have proved that any focal lacunary hyperbolic group is either of the form $H \rtimes \mathbb{Z}$ or $H \rtimes \mathbb{R}$. We must now prove that the associated action is compacting. The first step towards this result is the following proposition, which says that a focal lacunary hyperbolic group satisfies in some sense a weak local contracting property.

If H is a subgroup of a compactly generated group G, we denote by $B_{G,H}(r)$ the closed ball of radius $r \geq 0$ in H around the identity, where H is endowed with the induced metric from G.

Proposition 1.26. *Let G be a focal lacunary hyperbolic group. Assume that G admits a topological semidirect product decomposition $G = H \rtimes \langle t_0\rangle$. Then there exist $t \in \{t_0, t_0^{-1}\}$ and infinitely many $N \geq 1$ such that*

$$t^N \cdot B_{G,H}(2N) \cdot t^{-N} \subset B_{G,H}(N).$$

Proof. Let us denote by $\mathcal{C} = \mathrm{Cone}^\omega(G, \mathbf{s})$ an asymptotic cone of G that is a real tree and such that the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathcal{C}$ is focal. Observe that the element $(t_0^{s_n}) \in \mathrm{Precone}(G, \mathbf{s})$ is hyperbolic, and its axis is the image of the map $\mathbb{R} \to \mathrm{Cone}^\omega(G, \mathbf{s})$, $x \mapsto (t_0^{-\lfloor xs_n \rfloor})^\omega$. One of the two ends of this axis must be the end of $\mathcal{C}$ that is fixed by $\mathrm{Precone}(G, \mathbf{s})$, so there is $t \in \{t_0, t_0^{-1}\}$ such that the ray emanating from $(e)^\omega$ representing the fixed end of $\mathcal{C}$ is the image of $\xi : [0, +\infty[\to \mathrm{Cone}^\omega(G, \mathbf{s})$, $x \mapsto (t^{-\lfloor xs_n \rfloor})^\omega$.

We claim that $\mathrm{Precone}_G(H, \mathbf{s})$ cannot fix a point in $\mathcal{C}$. Indeed, if the set of fixed points of $\mathrm{Precone}_G(H, \mathbf{s})$ is not empty, then it is a subtree of $\mathcal{C}$ that is invariant by $\mathrm{Precone}(G, \mathbf{s})$ since $H \lhd G$. But $\mathrm{Precone}(G, \mathbf{s})$ acts transitively on $\mathcal{C}$, so we deduce that the set of fixed points of $\mathrm{Precone}_G(H, \mathbf{s})$ is the entire $\mathcal{C}$. It follows that the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathcal{C}$ is trivial, and this implies that the asymptotic cone $\mathcal{C}$ is a line, which contradicts the fact that the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathcal{C}$ is focal. On the other hand, if $\mathrm{Precone}_G(H, \mathbf{s})$ contains some hyperbolic isometry, then according to Remark 1.23 the conclusion of Lemma 1.22 holds and the subgroup H is cocompact in G, which is a contradiction. So the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathcal{C}$ must be horocyclic. It follows that if (h_n) is a sequence in H such that $|h_n|_S \leq 2\ell s_n$ for every

$n \geq 1$ (which implies that the distance in $\mathrm{Cone}^\omega(G, \mathbf{s})$ between $(e)^\omega$ and $(h_n)^\omega$ is at most 2ℓ), then the element (h_n) fixes $\xi([\ell, +\infty[)$. In particular if $|h_n|_S \leq 2s_n$ for every $n \geq 1$, then (h_n) fixes the point $\xi(1) = (t^{-s_n})^\omega$, and we have

$$\lim{}^\omega \frac{d(h_n t^{-s_n}, t^{-s_n})}{s_n} = \lim{}^\omega \frac{|t^{s_n} h_n t^{-s_n}|_S}{s_n} = 0.$$

So for every $h_n \in B_{G,H}(2s_n)$, ω-almost surely we have $|t^{s_n} h_n t^{-s_n}|_S \leq s_n$, which is equivalent to saying that ω-almost surely $t^{s_n} \cdot B_{G,H}(2s_n) \cdot t^{-s_n} \subset B_{G,H}(s_n)$. $\square$

Corollary 1.27. *Let G be a focal lacunary hyperbolic group with a topological semidirect product decomposition $G = H \rtimes \langle t_0 \rangle$. Then there exist $t \in \{t_0, t_0^{-1}\}$, an integer $n_0 \geq 1$ and a compact symmetric subset $K \subset H$ containing the identity such that:*

i) $\langle K, t^{n_0} \rangle = H \rtimes \langle t^{n_0} \rangle$;

ii) $t^{n_0} \cdot K^2 \cdot t^{-n_0} \subset K$.

Proof. Let t coming from Proposition 1.26, and $N_0 \geq 1$ an integer such that $B_{G,H}(N_0)$ together with t generate the group G. According to Proposition 1.26, there exists $N_1 \geq N_0$ such that

$$t^{N_1} \cdot B_{G,H}(2N_1) \cdot t^{-N_1} \subset B_{G,H}(N_1).$$

If we set

$$K_1 = \bigcup_{i=0}^{N_1 - 1} t^i \cdot B_{G,H}(N_1) \cdot t^{-i},$$

then K_1 is a compact subset of H and by construction conjugating by t sends K_1 into itself because

$$t \cdot K_1 \cdot t^{-1} \subset K_1 \cup t^{N_1} \cdot B_{G,H}(N_1) \cdot t^{-N_1} \subset K_1.$$

In particular the sequence of compact subsets $(t^{-n} \cdot K_1 \cdot t^n)_{n \geq 0}$, is increasing. A fortiori the same holds for the sequence of subgroups $(t^{-n} \cdot \langle K_1 \rangle \cdot t^n)_{n \geq 0}$, and it follows that the subgroup they generate is nothing but their union. But now by assumption K_1 and t generate G, so this increasing union of subgroups is the entire subgroup H. This observation implies in particular that for every $n_0 \geq 1$, the subgroup generated by K_1 and t^{n_0} is equal to $H \rtimes \langle t^{n_0} \rangle$.

Now we let n_0 be an integer satisfying the conclusion of Proposition 1.26 and so that $B_{G,H}(n_0)$ contains K_1, and we check that $K = B_{G,H}(n_0)$ satisfies the conclusion. It follows from the last paragraph that the subgroup generated by K together with t^{n_0} is equal to $H \rtimes \langle t^{n_0} \rangle$ because K contains K_1. Besides it is clear that $K^2 \subset B_{G,H}(2n_0)$, so the inclusion $t^{n_0} \cdot K^2 \cdot t^{-n_0} \subset K$ follows immediately from the conclusion of Proposition 1.26. $\square$

The following result provides a sufficient condition on a group $G = H \rtimes \langle t \rangle$ so that the conjugation by the element t induces a compacting automorphism of the group H.

Proposition 1.28. *Let $G = H \rtimes \langle t \rangle$ be a locally compact group such that there is some compact symmetric subset $K \subset H$ containing the identity so that:*

(a) $S = K \cup \{t\}$ *generates the group* G*;*

(b) $t \cdot K^2 \cdot t^{-1} \subset K$.

Then the automorphism of H induced by the conjugation by t is compacting.

Proof. We check that for every $h \in H$, we have $t^n h t^{-n} \in K$ eventually. The hypotheses imply that H is generated by the increasing union of compact sets $t^{-n} \cdot K \cdot t^n$, so that every element of H lies inside $t^{-n} \cdot K^{2^k} \cdot t^n$ for some integers $n, k \geq 0$. The latter being included in $t^{-n-k} \cdot K \cdot t^{n+k}$ thanks to (b), the proof is complete. $\square$

We are now able to prove the main result of this section.

Proof of Theorem 1.21. Let G be a focal lacunary hyperbolic group. According to Corollary 1.25, the group G admits a topological semidirect product decomposition of the form $H \rtimes_\alpha \mathbb{Z}$ or $H \rtimes_{\alpha(t)} \mathbb{R}$. To conclude we need to prove that the action of α (resp. $\alpha(1)$) on H is compacting. For the sake of simplicity we denote $\alpha(1)$ by α as well.

We claim that upon changing α into its inverse, there is some positive power of α satisfying the hypotheses of Proposition 1.28. In the case when $G = H \rtimes_\alpha \mathbb{Z}$ this follows directly from Corollary 1.27. When $G = H \rtimes_{\alpha(t)} \mathbb{R}$, the subgroup $H \rtimes_{\alpha(1)} \mathbb{Z}$ is normal and cocompact in G, and therefore focal lacunary hyperbolic as well by Lemma 1.13, so that Corollary 1.27 can also be applied.

Consequently Proposition 1.28 implies that some positive power of α is compacting, and it follows that α is compacting as well. $\square$

1.5 Structure of locally compact lacunary hyperbolic groups

1.5.1 Identity component in lacunary hyperbolic groups

Recall that a locally compact compactly generated group G is *lacunary hyperbolic of general type* if it admits one asymptotic cone $\mathrm{Cone}^\omega(G, \mathbf{s})$ that is a real tree and such that the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathrm{Cone}^\omega(G, \mathbf{s})$ is of general type. It turns out that, apart from the case of hyperbolic LC-groups, every lacunary hyperbolic group is of general type. This will be proved in Theorem 1.30 below.

It is proved in [DS05, Proposition 6.1] that if a finitely generated group G has one asymptotic cone that is a line, then G is virtually infinite cyclic. The following lemma is an extension of this result to coarsely connected metric groups. In particular it encompasses the case of a closed compactly generated subgroup H of a locally compact compactly generated group G, where H is endowed with the induced word metric from G.

Lemma 1.29. *Let (Γ, d) be a group equipped with a coarsely connected left-invariant metric. If (Γ, d) admits one asymptotic cone that is quasi-isometric to the real line, then Γ admits an infinite cyclic cobounded subgroup.*

Proof. If $\mathcal{C} = \mathrm{Cone}^\omega(\Gamma, d, \mathbf{s})$ is an asymptotic cone of (Γ, d) that is quasi-isometric to the real line, the action of $\mathrm{Precone}(\Gamma, d, \mathbf{s})$ on $\mathcal{C}$ is lineal. Therefore $\mathrm{Precone}(\Gamma, d, \mathbf{s})$ contains some hyperbolic element $\gamma = (\gamma_n)$, and there exists $\ell > 0$ such that the ℓ-neighbourhood of the $\langle\gamma\rangle$-orbit of the point $(e)^\omega$ is the entire $\mathcal{C}$.

For every $n \geq 1$, we let Γ_n be the subgroup of Γ generated by γ_n. We claim that ω-almost surely, Γ is contained in the $(\ell+1)s_n$-neighbourhood of Γ_n. Let us argue by contradiction and assume that ω-almost surely there exists $x_n \in \Gamma$ such that $d(x_n, \Gamma_n) \geq (\ell+1)s_n$. Since (Γ, d) is coarsely connected, we can assume that $d(x_n, \Gamma_n) \leq (\ell+1)s_n + c$ for some constant $c > 0$. Upon multiplying x_n on the left by an element of Γ_n, we can moreover assume that $d(x_n, \Gamma_n) = d(x_n, e)$, which implies that the sequence (x_n) defines a point $x \in \mathcal{C}$. But by construction, ω-almost surely $d(x_n, \gamma_n^i) \geq (\ell+1)s_n$ for every $i \in \mathbb{Z}$, so the point x is at distance at least $(\ell+1)$ from any point in the $\langle\gamma\rangle$-orbit of the point $(e)^\omega$. Contradiction. ☐

Theorem 1.30. *Let G be a locally compact lacunary hyperbolic group. Then exactly one of the following holds:*

- *(a) G is either an elementary or a focal hyperbolic group;*
- *(b) for every asymptotic cone $\mathrm{Cone}^\omega(G, \mathbf{s})$ that is a real tree, the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathrm{Cone}^\omega(G, \mathbf{s})$ is of general type.*

Proof. Let $\mathcal{C} = \mathrm{Cone}^\omega(G, \mathbf{s})$ be an asymptotic cone of G that is a real tree. By homogeneity $\mathcal{C}$ can be either a point, a line, or such that every point is branching with the same branching cardinality. The case when $\mathcal{C}$ is a point is trivial, as it easily implies that the group G is compact. If $\mathcal{C}$ is a line then G must have an infinite cyclic discrete and cocompact subgroup by Lemma 1.29. So we may assume that $\mathcal{C}$ is neither a point nor a line. This implies that if the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathcal{C}$ is not of general type, then it is focal, and by Theorem 1.21 this implies that G is focal hyperbolic. $\square$

We now aim to establish some structural results about locally compact lacunary hyperbolic groups. Since any topological group naturally lies into an extension with a connected kernel and a totally disconnected quotient, it is natural to wonder what can be said about the identity component of a locally compact lacunary hyperbolic group. Recall that even for hyperbolic LC-groups, it may happen that the identity component is neither compact nor cocompact. Take for example the semidirect product $(\mathbb{R}\times\mathbb{Q}_p)\rtimes\mathbb{Z}$, where the action of $\mathbb{Z}$ is by multiplication by $1/2$ on $\mathbb{R}$ and by p on $\mathbb{Q}_p$. However, if G is a hyperbolic LC-group of general type, it follows from [CCMT, Proposition 5.10] that the identity component of G is either compact or cocompact. We will extend this result to lacunary hyperbolic groups in Theorem 1.32 below.

Recall that if G is a locally compact group, the Braconnier topology is a Hausdorff topology on the group $\mathrm{Aut}(G)$ of topological automorphisms of G. For an introduction to this topology, see for example [CM11, Appendix I].

Lemma 1.31. *Let G be a σ-compact locally compact group, and $N \triangleleft G$ a closed normal subgroup with trivial center and finite outer automorphism group. Assume moreover that the group $\mathrm{Inn}(N)$ of inner automorphisms of N is closed in $\mathrm{Aut}(N)$. Then G has a finite index open subgroup that is topologically isomorphic to the direct product of N with its centralizer in G.*

Proof. If we let C be the centralizer of N in G, we want to prove that the subgroup NC is open in G, has finite index and is topologically the direct product of N and C. Since N is a closed normal subgroup of G, the action of G by conjugation on N yields a continuous map $G \to \mathrm{Aut}(N)$ [HR79, Theorem 26.7]. Being the preimage of the closed finite index subgroup $\mathrm{Inn}(N)$ of $\mathrm{Aut}(N)$ under this map, the subgroup NC is a closed finite index (and hence open) subgroup of G. It follows that NC is a σ-compact locally compact group, and we deduce that the natural epimorphism $N\times C \to NC$ is a quotient morphism between topological groups. Since it is clearly onto, and injective because N has trivial center, it is an isomorphism of topological groups. $\square$

Theorem 1.32. *Let G be a locally compact lacunary hyperbolic group of general type. Then G° is either compact or cocompact in G.*

Proof. According to Corollary 1.9 there exists a compact characteristic subgroup W of G contained in G° such that G°/W is a connected Lie group without non-trivial compact normal subgroup. Now by Lemma 1.13, the group G/W is lacunary hyperbolic of general type as well, so the proof can be reduced to the case when G° is a connected Lie group without non-trivial compact normal subgroup.

Let $\mathcal{C} = \mathrm{Cone}^\omega(G, \mathbf{s})$ be an asymptotic cone of G that is a real tree and such that the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathcal{C}$ is of general type. According to Lemma 1.19, the action of $\mathrm{Precone}_G(G^\circ, \mathbf{s})$ on $\mathcal{C}$ is either bounded or of general type. Since G° is compactly generated, if the action of $\mathrm{Precone}_G(G^\circ, \mathbf{s})$ on $\mathcal{C}$ is bounded then G° is compact by Lemma 1.15. So we may assume that this action is of general type and we will prove that G° is cocompact in G.

We denote by R the non-connected solvable radical of G°, that is its largest normal solvable subgroup. It is a closed, compactly generated subgroup of G°, and being characteristic in the normal subgroup G°, the subgroup R is normal in G. We will prove that R is reduced to the identity. For the same reason as above, the action of $\mathrm{Precone}_G(R, \mathbf{s})$ on $\mathcal{C}$ must be either bounded or of general type. However it cannot be of general type because otherwise $\mathrm{Precone}_G(R, \mathbf{s})$ would contain a non-abelian free subgroup (see Theorem 2.7 in [CM87]), which is clearly impossible because $\mathrm{Precone}_G(R, \mathbf{s})$ is a solvable group. Therefore the action of $\mathrm{Precone}_G(R, \mathbf{s})$ on $\mathcal{C}$ is bounded, and by Lemma 1.15 this implies that R is a compact subgroup. But G° is assumed not to contain any non-trivial compact normal subgroup, so R must be trivial.

It follows that G° is a semisimple Lie group with trivial center, and consequently G° has finite outer automorphism group. So we are in position to apply Lemma 1.31, and we obtain that G admits a finite index open subgroup decomposing as a topological direct product $G' = G^\circ \times Q$. Now since G' has finite index in G, $\mathrm{Cone}^\omega(G', \mathbf{s}) \simeq \mathrm{Cone}^\omega(G^\circ, \mathbf{s}) \times \mathrm{Cone}^\omega(Q, \mathbf{s})$ is a real tree. This implies that either $\mathrm{Cone}^\omega(G^\circ, \mathbf{s})$ or $\mathrm{Cone}^\omega(Q, \mathbf{s})$ is a point, that is either G° or Q is compact. But by assumption G° is not compact so Q must be compact, and the conclusion follows. $\square$

As a consequence of this result, we deduce the following property for locally compact lacunary hyperbolic groups.

Proposition 1.33. *If G is a locally compact lacunary hyperbolic group, then either G is hyperbolic or G has a compact open subgroup.*

Proof. According to Lemma 1.29, if G is not an elementary hyperbolic LC-group, then G must be either focal lacunary hyperbolic or lacunary hyperbolic of general type. If G is focal lacunary hyperbolic then G is focal hyperbolic by Theorem 1.21. Now if G is lacunary hyperbolic of general type, then according to Theorem 1.32 the identity component G° is either compact or cocompact in G. In the latter case G must be hyperbolic (see

Remark 2.31), and in the former G has a compact open subgroup by van Dantzig's theorem. $\square$

1.5.2 Characterization of lacunary hyperbolic groups

Cartan-Hadamard Theorem. This paragraph consists of a recall of a Cartan-Hadamard type theorem due to Gromov, and its application to lacunary hyperbolic groups due to Kapovich and Kleiner, stated for topological groups rather than discrete ones.

Let (X,d) be a non-empty geodesic metric space, $x_0 \in X$ a base point and $c > 0$. A c-loop based at x_0 is a sequence of points $x_0 = x_1, x_2, \ldots, x_n = x_0$ such that $d(x_i, x_{i+1}) \leq c$ for every $i = 1, \ldots, n-1$. Two c-loops are said to be c-elementarily homotopic if one of them can be obtained from the other by inserting a new point, and c-homotopic if they are the extremities of a finite sequence of c-loops such that any two consecutive terms are c-elementarily homotopic. Recall that X is c-large scale simply connected if any c-loop based at x_0 is c-homotopic to the trivial loop.

The following result can be deduced from [BH99, Part III.H Lemma 2.6].

Proposition 1.34. *There exists some universal constant $C > 0$ so that every geodesic δ-hyperbolic metric space is $C\delta$-large scale simply connected.*

The following result appears as a large scale analogue of Cartan-Hadamard Theorem in metric geometry. The idea of this local-global principle goes back to [Gro87], but the version we use here is inspired from Theorem 8.3 of the Appendix of [OOS09] (see also Chapter 8 of [Bow91]).

Theorem 1.35. *There exist some constants $c_1, c_2, c_3 > 0$ such that the following holds: every geodesic, c-large scale simply connected metric space X with the property that there exists some $R \geq c_1 c$ such that every ball in X of radius R is $c_2 R$-hyperbolic; is $c_3 R$-hyperbolic.*

Kapovich and Kleiner [OOS09, Appendix] observed that geodesic metric spaces with one asymptotic cone that is a real tree fulfill the assumption of local hyperbolicity appearing in Theorem 1.35, which yields the following corollary.

Corollary 1.36. *Let (X,d) be a homogeneous, geodesic, c-large-scale simply connected metric space. If X is lacunary hyperbolic then X is hyperbolic.*

Proof. Let $e \in X$ be a base point, $\mathbf{s}$ a scaling sequence and ω a non-principal ultrafilter such that $\mathrm{Cone}^{\omega}(X, d, \mathbf{s})$ is a real tree. Then ω-almost surely, the ball or radius s_n in X around e is δ_n-hyperbolic, with $\delta_n = o(s_n)$. But since X is homogeneous, every ball in X or radius s_n is δ_n-hyperbolic. Now for some large enough n we have $s_n \geq c_1 c$ and $\delta_n / s_n \leq c_2$, so it follows from Theorem 1.35 that X is hyperbolic. $\square$

Since for a locally compact compactly generated group, compact presentability can be characterized in terms of large scale simple connectedness (see for example [CH15, Proposition 8.A.3]), we obtain the following result, which is the topological counterpart of [OOS09, Theorem 8.1] by Kapovich and Kleiner.

Corollary 1.37. *Any compactly presented group that is lacunary hyperbolic is a hyperbolic group.*

Characterization of locally compact lacunary hyperbolic groups. We are now able to generalize to the locally compact setting the structural theorem of Olshanskii, Osin, Sapir [OOS09] for finitely generated lacunary hyperbolic groups.

Theorem 1.38. *Let G be a locally compact, compactly generated group with a compact open subgroup. Then the following assertions are equivalent:*

(i) G is lacunary hyperbolic;

(ii) There exists a scaling sequence $\mathbf{s}$ *such that for every non-principal ultrafilter ω, the asymptotic cone* $\mathrm{Cone}^{\omega}(G, \mathbf{s})$ *is a real tree;*

(iii) There exists a hyperbolic LC-group G_0 acting on a locally finite tree, transitively and with compact open stabilizers on the set of vertices, and an increasing sequence of discrete normal subgroups N_n, whose discrete union N is such that G is topologically isomorphic to G_0/N; and if S is a compact generating set of G_0 and

$$\rho_n = \min\{|g|_S \, : \, g \in N_{n+1} \backslash N_n\},$$

then G_0/N_n is δ_n-hyperbolic with $\delta_n = o(\rho_n)$.

The proof of the implication $(iii) \Rightarrow (ii)$ is similar than in the discrete setting, so we choose not to repeat it here and refer the reader to [OOS09, p.16]. The implication $(ii) \Rightarrow (i)$ being trivial, we only have to prove $(i) \Rightarrow (iii)$.

Proof of $(i) \Rightarrow (iii)$. Let G be a lacunary hyperbolic group with a compact open subgroup. We let G_0 and $\pi : G_0 \to G$ be as in Proposition 1.12. Recall that G_0 is a locally compact compactly generated group acting geometrically on a locally finite tree and π is an open morphism from G_0 onto G with discrete kernel N. Let ω be a non-principal ultrafilter and $\mathbf{s}$ a scaling sequence such that $\mathrm{Cone}^{\omega}(G, \mathbf{s})$ is a real tree. Choose a compact open subgroup K of G_0 intersecting N trivially, and a K-bi-invariant compact generating set S of G_0. For every $k \geq 1$, let N_k be the normal subgroup of G_0 generated by elements of N of word length at most d_k with respect to S, and set $G_k = G_0/N_k$. Note that since $\mathbf{s}$ is an increasing sequence tending to infinity, by construction (N_k) is an increasing sequence of normal subgroups of G_0 whose union is N. This can be rephrased by saying that we have an infinite sequence of locally compact groups and quotient morphisms

$$G_0 \twoheadrightarrow \cdots \twoheadrightarrow G_k \twoheadrightarrow G_{k+1} \twoheadrightarrow \cdots$$

whose direct limit is topologically isomorphic to the group G. Observe that the injectivity radius of the map $G_k \twoheadrightarrow G$ is larger than d_k, and a fortiori the same holds for the injectivity radius of the map $G_k \twoheadrightarrow G_{k+1}$.

For every $k \geq 1$, we push the pair (K, S) in G_k and in G, and we denote by X_k (resp. X) the Cayley-Abels graph of G_k (resp. G) with respect to this compact open subgroup and compact generating set. By abuse of notation, we still denote by K the image of the subgroup K in G_k. To the above sequence of groups and epimorphisms corresponds an infinite sequence of coverings of graphs

$$X_0 \twoheadrightarrow \cdots \twoheadrightarrow X_k \twoheadrightarrow X_{k+1} \twoheadrightarrow \cdots$$

Note that the map $X_k \twoheadrightarrow X$ is injective on the ball $B_{X_k}(K, d_k)$ of radius d_k around the vertex K.

Now since G is quasi-isometric to its Cayley-Abels graph X, their asymptotic cones $\mathrm{Cone}^\omega(X, \mathbf{s})$ and $\mathrm{Cone}^\omega(G, \mathbf{s})$ are bi-Lipschitz homeomorphic. It follows that $\mathrm{Cone}^\omega(X, \mathbf{s})$ is a real tree, and therefore ω-almost surely the ball of radius d_k in X is δ_k-hyperbolic with $\delta_k = o(d_k)$. By the above observation on the injectivity radius of the map $X_k \twoheadrightarrow X$, the same is true in X_k. According to Proposition 1.34, the ball $B_{X_k}(K, d_k)$ is $O(\delta_k)$-large scale simply connected. But by construction of the group G_k, any loop in X_k is built from loops of length at most d_k, so it follows that the entire graph X_k is C_k-large scale simply connected, with $C_k = O(\delta_k)$.

Now let us pick a sequence (Δ_k) such that $\delta_k << \Delta_k << d_k$. If we let c_1, c_2, c_3 be the constants from Theorem 1.35, then ω-almost surely $\Delta_k \geq c_1 C_k$ and $\Delta_k \geq \delta_k/c_2$. So we are in position to apply Theorem 1.35, which implies that ω-almost surely X_k is $c_3\Delta_k$-hyperbolic.

Now as observed earlier, the injectivity radius ρ_k of $G_k \twoheadrightarrow G_{k+1}$ satisfies $\rho_k \geq d_k$. Since $\Delta_k = o(d_k)$, we clearly have $\Delta_k = o(\rho_k)$. It follows that ω almost surely, the graph X_k (and a fortiori the group G_k) is $o(\rho_k)$-hyperbolic, and the conclusion follows. $\square$

The next proposition establishes some stability properties of the class of locally compact lacunary hyperbolic groups. We note that, as observed in [OOS09], the class of finitely generated lacunary hyperbolic groups is not stable under free product.

Proposition 1.39. *The class of locally compact lacunary hyperbolic groups is stable under taking:*

(a) a semidirect product with a compact group;

(b) an HNN-extension over some compact open subgroup;

(c) an amalgamated product with a hyperbolic LC-group over some compact open subgroup.

Proof. The statement (a) is trivial. Let us prove (b). Let G be a lacunary hyperbolic group, K, L two compact open subgroups, $\varphi : K \to L$ a topological isomorphism, and $G' = \mathrm{HNN}(G, K, L, \varphi)$ the corresponding HNN-extension. We want to prove that G' is lacunary hyperbolic. If G is hyperbolic then there is nothing to prove because since K, L are compact, the group G' is hyperbolic as well. Otherwise G has a compact open subgroup by Proposition 1.33, and we let G_0 be a hyperbolic LC-group and (N_n) an increasing sequence of discrete normal subgroups as in Theorem 1.38. There exists an integer $n_0 \geq 1$ such that for every $n \geq n_0$, the group G_n has subgroups isomorphic to K and L, which we still denote by K and L by abuse of notation. Let us form the HNN-extension $G'_n = \mathrm{HNN}(G_n, K, L, \varphi)$. Since G_n is δ_n-hyperbolic and K, L are compact, the group G'_n is δ'_n-hyperbolic. Moreover since K, L have bounded diameter in G_n, we have $\delta'_n = O(\delta_n)$. Now the epimorphism $\alpha_n : G_n \twoheadrightarrow G_{n+1}$ naturally extends to $\alpha'_n : G'_n \twoheadrightarrow G'_{n+1}$ by mapping the stable letter to itself, and the injectivity radius ρ'_n of α'_n is equal to the injectivity radius ρ_n of α_n. Since by assumption $\rho_n << \delta_n$, we have $\rho'_n << \delta'_n$, and the fact that G' is lacunary hyperbolic follows from the implication $(iii) \Rightarrow (i)$ in Theorem 1.38.

The case (c) of an amalgamated product with a hyperbolic LC-group over some compact open subgroup is analogous, and relies on the fact that the amalgamated product of two hyperbolic LC-groups over some compact open subgroup remains hyperbolic, with a control on the hyperbolicity constant in terms of the hyperbolicity constants of the two groups and the diameter of the compact subgroup. □

Example 1.40. Here is a construction providing examples of locally compact lacunary hyperbolic groups with a non-discrete topology. Let Γ be a discrete lacunary hyperbolic group, and G a hyperbolic LC-group with some compact open subgroup U. Let us consider the semidirect product $H = (*_{G/U}\Gamma) \rtimes G$, where G acts on the free product $*_{G/U}\Gamma$ by permuting the factors according to the natural action of G on G/U, and the topology on H is such that the subgroup G is open. Equivalently, H can be defined as the topological amalgamated product of $\Gamma \times U$ with G over the subgroup U. It follows from the statements (a) and (c) of Proposition 1.39 that the group H is lacunary hyperbolic. Note that the group H may be far from discrete, because for example H is non-unimodular as soon as G is.

1.6 Subgroups of lacunary hyperbolic groups

In this section we carry on the investigation started in [OOS09] of groups that may appear as subgroups of lacunary hyperbolic groups.

1.6.1 Quasi-isometrically embedded normal subgroups

It is a classical result that if G is a hyperbolic LC-group, and N a compactly generated quasi-isometrically embedded normal subgroup of G, then N must be either compact or cocompact in G. The following proposition is a generalization of this result to the realm of lacunary hyperbolic groups, which seems to be new even for discrete groups.

Proposition 1.41. *Let G be a locally compact lacunary hyperbolic group, and N a closed normal subgroup of G. Assume that N is compactly generated and quasi-isometrically embedded in G. Then N is either compact or cocompact in G.*

Proof. We let $\mathcal{C} = \mathrm{Cone}^{\omega}(G, \mathbf{s})$ be an asymptotic cone of G that is a real tree, and we denote by $\mathcal{C}_N$ the $\mathrm{Precone}_G(N, \mathbf{s})$-orbit of $(e)^{\omega} \in \mathrm{Cone}^{\omega}(G, \mathbf{s})$. Since N is compactly generated and quasi-isometrically embedded in G, the subset $\mathcal{C}_N$ is a subtree of $\mathcal{C}$ that is clearly invariant by $\mathrm{Precone}_G(N, \mathbf{s})$.

First assume that $\mathrm{Precone}_G(N, \mathbf{s})$ acts on $\mathcal{C}$ with some hyperbolic element. Then we are in position to apply Lemma 1.7, which implies that $\mathcal{C}_N$ must be the entire $\mathcal{C}$. The fact that N is cocompact in G then follows from Corollary 1.18.

We now have to deal with the case when $\mathrm{Precone}_G(N, \mathbf{s})$ does not have any hyperbolic element. We claim that the action of $\mathrm{Precone}_G(N, \mathbf{s})$ on $\mathcal{C}$ cannot be horocyclic. Indeed otherwise the action of $\mathrm{Precone}_G(N, \mathbf{s})$ on the subtree $\mathcal{C}_N$ would be horocylic as well, which is impossible since a transitive isometric action on a real tree cannot be horocyclic. This implies that if $\mathrm{Precone}_G(N, \mathbf{s})$ does not contain any hyperbolic element then $\mathrm{Precone}_G(N, \mathbf{s})$ must have a fixed point, and by Lemma 1.15 this forces the subgroup N to be compact. $\square$

1.6.2 Subgroups satisfying a law

The goal of this paragraph is to exhibit some obstruction for a given group to be a subgroup of a lacunary hyperbolic group.

Recall that if G is a compactly generated group endowed with a compact generating set S, and if H is a subgroup of G, we denote by $B_{G,H}(n)$ the intersection between H and the ball in G of radius $n \geq 1$ around the identity. If μ is a left-invariant Haar measure on G, a measurable subgroup H is said to have relative exponential growth in G if there exists $\rho > 1$ such that $\rho^n \leq \mu(B_{G,H}(e,n))$ for every $n \geq 1$. Note that this condition implies that the subgroup H has positive Haar measure, and hence is open in G. If H_1 is an open subgroup of G, the restriction to H_1 of a Haar measure on G is a Haar measure on H_1, so if H_2 is a subgroup of H_1 of relative exponential growth in H_1, then H_2 has relative exponential growth in G. For example a compactly generated open subgroup of exponential growth has relative exponential growth in the ambient group.

Proposition 1.42. *Let G be a unimodular lacunary hyperbolic group, and $H \leq G$ a subgroup of relative exponential growth in G. If $\mathcal{C} = \mathrm{Cone}^{\omega}(G, \mathbf{s})$ is an asymptotic cone of G that is a real tree, then the action of* $\mathrm{Precone}_G(H, \mathbf{s})$ *on $\mathcal{C}$ cannot have a fixed point or be horocyclic.*

Proof. We shall prove that the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathcal{C}$ cannot be horocyclic. The case of an action with a fixed point can be ruled out with the same kind of arguments, and is actually easier.

Let S be a compact generating set of G, and μ a left-invariant Haar-measure on G. We argue by contradiction and assume that the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathcal{C}$ is horocyclic, and denote by $\xi : [0, +\infty[\to \mathcal{C}$ the ray emanating from $(e)^{\omega}$ representing the end of $\mathcal{C}$ that is fixed by $\mathrm{Precone}_G(H, \mathbf{s})$. Then every element $(h_n) \in \mathrm{Precone}_G(H, \mathbf{s})$ such that $|h_n|_S \leq s_n$ fixes the point $\xi(1/2) = (\xi_n)^{\omega}$, that is

$$\lim^{\omega} \frac{d_S(h_n \xi_n, \xi_n)}{s_n} = \lim^{\omega} \frac{|\xi_n^{-1} h_n \xi_n|_S}{s_n} = 0.$$

This means that for every $\varepsilon > 0$, ω-almost surely the element $\xi_n^{-1} h_n \xi_n$ has length at most εs_n, which is equivalent to saying that h_n belongs to $\xi_n \cdot B_G(e, \varepsilon s_n) \cdot \xi_n^{-1}$. So for every $\varepsilon > 0$, ω-almost surely

$$B_{G,H}(e, s_n) \subset \xi_n \cdot B_G(e, \varepsilon s_n) \cdot \xi_n^{-1}.$$

Combined with the fact that G is unimodular, we obtain that ω-almost surely

$$\mu(B_{G,H}(e, s_n)) \leq \mu\left(\xi_n \cdot B_G(e, \varepsilon s_n) \cdot \xi_n^{-1}\right) = \mu(B_G(e, \varepsilon s_n)) \leq \alpha^{\varepsilon s_n}$$

for some constant $\alpha \geq 1$. This implies that

$$\liminf_{n \to \infty} \frac{\log \mu(B_{G,H}(e, s_n))}{s_n} = 0,$$

which is a contradiction with the fact that H has relative exponential growth in G. $\square$

Let us derive the following consequence of Proposition 1.42, which recovers Theorem 3.18 (c) of [OOS09], and generalizes it to the setting of unimodular locally compact lacunary hyperbolic groups.

Corollary 1.43. *Let G be a unimodular lacunary hyperbolic group, and $H \leq G$ a subgroup of finite exponent. Then H cannot have relative exponential growth in G.*

Proof. For any scaling sequence $\mathbf{s}$, the group $\mathrm{Precone}_G(H, \mathbf{s})$ has finite exponent as well. It follows that for any asymptotic cone $\mathrm{Cone}^\omega(G, \mathbf{s})$ that is a real tree, the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathrm{Cone}^\omega(G, \mathbf{s})$ must have a fixed point or be horocyclic, and H cannot have relative exponential growth in G according to Proposition 1.42. $\square$

We point out that both Corollary 1.43 and Proposition 1.42 fail without the assumption that the group is unimodular. Actually the corresponding statements at the level of groups rather than asymptotic cones already fail for hyperbolic LC-groups of general type. Take for example the amalgamated product of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{F}_p[\![t]\!]$ and $\mathbb{F}_p(\!(t)\!) \rtimes_t \mathbb{Z}$ over the compact open subgroup $\mathbb{F}_p[\![t]\!]$. The resulting group is hyperbolic of general type and non-unimodular. Having relative exponential growth in the open subgroup $\mathbb{F}_p(\!(t)\!) \rtimes \mathbb{Z}$, the finite exponent subgroup $\mathbb{F}_p(\!(t)\!)$ has relative exponential growth in the ambient group. To see why the conclusion of Proposition 1.42 fails, note that the action of $\mathbb{F}_p(\!(t)\!)$ on the quasi-isometrically embedded subgroup $\mathbb{F}_p(\!(t)\!) \rtimes \mathbb{Z}$ is horocylic, so its action on the entire group must be horocyclic as well.

Proposition 1.44. *Let G be a unimodular lacunary hyperbolic group. If H is a subgroup of relative exponential growth in G, and if $\mathcal{C} = \mathrm{Cone}^\omega(G, \mathbf{s})$ is an asymptotic cone of G that is a real tree, then the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathcal{C}$ cannot be focal.*

Proof. The argument will be a slight modification of the beginning of the proof of Lemma 1.24. Assume that $\xi : [0, +\infty[\to \mathcal{C}$ is a geodesic ray starting at $(e)^\omega$ representing and end of $\mathcal{C}$ that is fixed by $\mathrm{Precone}_G(H, \mathbf{s})$. Let us fix some $k \geq 1$, and consider $k+1$ points $\xi(1) = x^{(0)}, x^{(1)}, \dots, x^{(k)} = \xi(2)$ dividing the interval $[\xi(1), \xi(2)]$ into k segments of equal length. For every $(h_n) \in \mathrm{Precone}_G(H, \mathbf{s})$ such that $|h_n|_S \leq s_n$ for every $n \geq 1$, upon changing (h_n) in $(h_n)^{-1}$, there exists some point $x^{(i)}$ such that the distance in $\mathcal{C}$ between $(h_n) \cdot \xi(1)$ and $x^{(i)}$ is at most $1/2k$. This implies that ω-almost surely, the distance in G between $h_n \xi_n(1)$ and $x_n^{(i)}$ is at most s_n/k.

So for every $k \geq 1$, ω-almost surely

$$B_{G,H}(e, s_n) \subset \bigcup_{i=0}^{k} \left(x_n^{(i)} \cdot B_G(e, s_n/k) \cdot \xi_n(1)^{-1} \right)^{\pm 1},$$

and certainly

$$\begin{aligned}
\mu\left(B_{G,H}(e, s_n)\right) &\leq \sum_{i=0}^{k} 2\mu \left(x_n^{(i)} \cdot B_G(e, s_n/k) \cdot \xi_n(1)^{-1} \right) \\
&= 2(k+1)\mu\left(B_G(e, s_n/k)\right) \\
&\leq 2(k+1)\alpha^{s_n/k}
\end{aligned}$$

for some constant $\alpha \geq 1$. Now since H has relative exponential growth in G, we obtain that there exists $\rho > 1$ such that for every $k \geq 1$, ω-almost surely $\rho^{s_n} \leq 2(k+1)\alpha^{s_n/k}$. This implies that $\rho \leq \alpha^{1/k}$ for every $k \geq 1$, which contradicts the fact that $\rho > 1$. □

Proposition 1.45. *Let G be a unimodular lacunary hyperbolic group, and $H \leq G$ a compactly generated subgroup of relative exponential growth in G. Assume that H does not have a cyclic cocompact subgroup. Then for any asymptotic cone $\mathcal{C} = \mathrm{Cone}^{\omega}(G, \mathbf{s})$ of G that is a real tree, the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathcal{C}$ is of general type.*

Proof. We carry out a case-by-case analysis of the possible type of the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathcal{C}$, and prove that other types of actions all lead to a contradiction.

If $\mathrm{Precone}_G(H, \mathbf{s})$ fixes a point in $\mathcal{C}$ then Lemma 1.15 implies that H is compact, which is a contradiction with the fact that H has relative exponential growth. Now assume that the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathcal{C}$ is lineal. Since H is compactly generated, the metric space (H, d_G) is coarsely connected [CH15, Proposition 4.B.8]. So we are in position to apply Lemma 1.29 to obtain that H admits an infinite cyclic cocompact subgroup, which is again a contradiction. Finally, it follows from Proposition 1.42 that the action of $\mathrm{Precone}_G(H, \mathbf{s})$ on $\mathcal{C}$ cannot be horocylic, and according to Proposition 1.44 it cannot be focal either. □

We immediately deduce the following result.

Corollary 1.46. *Let G be a unimodular lacunary hyperbolic group. If $H \leq G$ is a compactly generated subgroup of relative exponential growth in G not having $\mathbb{Z}$ as a discrete cocompact subgroup, then H cannot satisfy a law.*

When specified to finitely generated groups, Corollary 1.46 answers Question 7.2 in [OOS09]. As an example, we deduce the following result.

Corollary 1.47. *Any finitely generated solvable subgroup of a finitely generated lacunary hyperbolic group is virtually cyclic.*

Proof. Let H be a finitely generated solvable group that is a subgroup of a finitely generated lacunary hyperbolic group G. Assume that H has exponential growth. Then H has relative exponential growth in G, and according to Corollary 1.46 the group H must be virtually cyclic, contradiction. Therefore the solvable group H does not have exponential growth, and we deduce that H must be virtually nilpotent [Mil68, Wol68]. In particular H is finitely presented and therefore must be a subgroup of a hyperbolic group [OOS09, Theorem 3.18 (a)], and the conclusion follows from the fact that any finitely generated virtually nilpotent subgroup of a hyperbolic group is virtually cyclic. □

Chapter 2

Asymptotic cut-points

In this chapter we study the existence of cut-points in asymptotic cones for certain locally compact compactly generated groups. In Section 2.3 we show how the machinery developed in [DMS10] to study this problem for certain lattices in semi-simple groups of zero characteristic, applies in positive characteristic as well. In Section 2.5 we use the results from Section 1.4 to characterize connected Lie groups and linear algebraic groups over the p-adics having cut-points in one asymptotic cone.

2.1 Introduction

Recall that in a geodesic metric space X, a point $x \in X$ is a cut-point if $X \setminus \{x\}$ is not connected. Typical examples of geodesic metric spaces with cut-points are real trees. The question of studying the existence of cut-points in asymptotic cones of groups have been shown to be very interesting [DMS10]. In a sense, the property of having cut-points in asymptotic cones can be seen as a very weak form of hyperbolicity. Note that this property is invariant under quasi-isometries.

The divergence is a geometric notion of a metric space, which roughly speaking estimates the cost of going from a point a to another point b while remaining outside a large ball centered at a third point c. See the next subsection for precise definitions. It follows from the work of Stallings [Sta68] that the understanding of the space of ends of the Cayley graph of a finitely generated group yields some significant information about the algebraic structure of the group. Finitely generated groups with no ends are the same as finite groups, and groups with at least two ends are precisely those that split as an HNN-extension or nontrivial amalgam over a finite subgroup. From the geometric viewpoint, being one-ended corresponds to being *connected at infinity*, and the divergence is a quantified version of this connectedness at infinity, estimating how hard it is to connect two given points avoiding a large ball.

The property of having a linear divergence is closely related to the existence of cut-points in asymptotic cones. More precisely, a finitely generated group has a linear divergence if and only if none of its asymptotic cones has cut-points [DMS10].

Examples of finitely generated groups with a linear divergence include any direct product of two infinite groups, non-virtually cyclic groups satisfying a law or non-virtually cyclic groups with central elements of infinite order [DS05]. It is conjectured in [DMS10] that any irreducible lattice in a higher rank semisimple Lie group has a linear divergence. Note that in the case of a cocompact lattice, the conjecture is known to be true because such a lattice is quasi-isometric to the ambient Lie group, so their asymptotic cones (corresponding to the same ultrafilter and scaling constants) are bi-Lipschitz equivalent. Now any asymptotic cone of a semisimple Lie group of $\mathbb{R}$-rank ≥ 2 is known to have the property that any two points belong to a common flat of dimension 2 [KL97], and thus does not have cut-points.

On the other hand, the class of finitely generated groups with cut-points in all their asymptotic cones include relatively hyperbolic groups [DS05, Theorem 1.11], or mapping class groups of punctured surfaces [Beh06, Theorem 7.1]. Actually relatively hyperbolic groups and mapping class groups belong to the class of so-called acylindrically hyperbolic groups, and it is proved in [Sis13] that any acylindrically hyperbolic group has cut-points in all its asymptotic cones.

2.2 Divergence and asymptotic cut-points

In this subsection we provide preliminary material about the notion of divergence. The main result is the equivalence between the linearity of the growth rate of the divergence and the fact that no asymptotic cone admits cut-points. More details can be found in [DMS10].

As usual in the context of studying large-scale geometric properties, we consider functions measuring asymptotic properties of groups modulo a certain equivalence relations.

Definition 2.1. Let $f, g : \mathbb{R}_+ \to \mathbb{R}_+$. We write $f \preccurlyeq g$ if there exists $C > 0$ such that for all $x \in \mathbb{R}_+$,

$$f(x) \leq Cg(Cx + C) + Cx + C.$$

If $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $f \preccurlyeq g$ and $g \preccurlyeq f$, then we write $f \simeq g$, and f, g are said to be $\simeq$-equivalent.

In [DMS10] the authors introduce several definitions of divergence, and prove that they give $\simeq$-equivalent functions. We will focus on the following definition of divergence.

Definition 2.2. Let (X, d) be a geodesic metric space, and let $0 < \delta < 1$ and $\gamma \geq 0$. We define the divergence $\mathrm{div}_\gamma(a, b, c; \delta)$ of a pair of points $a, b \in X$ relative to a point $c \in X$, to be the length of a shortest path in X connecting a and b and avoiding the ball centered at c of radius $\delta d(c, \{a, b\}) - \gamma$. If there is no such path, put $\mathrm{div}_\gamma(a, b, c; \delta) = \infty$. The divergence $\mathrm{div}_\gamma(a, b; \delta)$ of the pair (a, b) is defined as the supremum of the divergences relative to all points $c \in X$, and the divergence function $\mathrm{Div}_\gamma(n; \delta)$ is defined as the supremum of all divergences of pairs (a, b) with $d(a, b) \leq n$.

The following proposition is proved in Lemma 3.4 and Lemma 3.11 of [DMS10].

Proposition 2.3. *Let X be a connected homogeneous locally finite graph with one end (e.g. the Cayley-Abels graph of a locally compact totally disconnected one-ended group). Then there exist $\gamma_0, \delta_0 > 0$ such that for every $\gamma \geq \gamma_0$ and every $\delta \leq \delta_0$, the function $n \mapsto \mathrm{Div}_\gamma(n; \delta)$ takes only finite values, and is independent, up to the equivalence relation $\simeq$, of the parameters γ and δ.*

The following result says that the divergence function is a quasi-isometry invariant. We refer the reader to [DMS10, Lemma 3.2] for a proof.

Proposition 2.4. *Let X and Y be connected homogeneous locally finite graphs with one end. If X and Y are quasi-isometric then they have $\simeq$-equivalent divergence functions.*

Let X be an infinite connected homogeneous graph. As we claimed in the introduction, the topological property for asymptotic cones of having cut-points is closely related to the geometric property for X of having a linear divergence, i.e. a divergence function $\simeq n$. For a proof of this result, see [DMS10, Lemma 3.17].

Proposition 2.5. *All the asymptotic cones of X have no cut-points if and only if there exist γ, δ such that the function $\mathrm{Div}_\gamma(n, \delta)$ is linear.*

2.3 The special linear group over a function ring

The motivation of this subsection comes from the following result of Drutu, Mozes and Sapir.

Theorem 2.6 ([DMS10], Theorem 1.4).
Let G be a semisimple Lie group of $\mathbb{R}$-rank ≥ 2, and let Γ be an irreducible lattice of G which is either of $\mathbb{Q}$-rank one or of the form $\mathrm{SL}_n(\mathcal{O}_\mathcal{S})$ with $n \geq 3$, where $\mathcal{S}$ is a finite set of valuations of a number field containing all the Archimedean ones and $\mathcal{O}_\mathcal{S}$ is corresponding ring of $\mathcal{S}$-integers. Then Γ has a linear divergence.

In [CDG10], Caprace, Dahmani and Guirardel prove that the divergence of twin building lattices is linear. This implies in particular that the group $\mathrm{SL}_n(\mathbb{F}_q[t, t^{-1}])$ has a linear divergence for every $n \geq 2$. Except for this particular case, to the best of our knowledge, nothing has been done before concerning the study of the divergence of arithmetic groups over function fields.

In this subsection we compute the divergence of $\mathrm{SL}_n(\mathcal{O}_\mathcal{S})$, where $\mathcal{S}$ is a finite non-empty set of pairwise non-equivalent valuations of a global function field, and $\mathcal{O}_\mathcal{S}$ is the corresponding ring of $\mathcal{S}$-integers. By global function field we mean a finite extension of the field $\mathbb{F}_q(t)$ of rational functions with coefficients in the finite field $\mathbb{F}_q$. They are the analogues in positive characteristic of number fields, i.e. finite extensions of the field $\mathbb{Q}$ of rational integers.

The analogy between number fields and function fields sometimes extends to groups over these fields, but only up to a certain limit. For instance, an important difference between $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{F}_q[t])$ is that the latter fails to be virtually torsion-free, i.e. does not admit a finite index subgroup without torsion elements. Finiteness properties may also change with the characteristic. The group $\mathrm{SL}_2(\mathbb{Z})$ is finitely presented whereas $\mathrm{SL}_2(\mathbb{F}_q[t])$ is not even finitely generated [Nag59]. In the same way, $\mathrm{SL}_3(\mathbb{Z})$ is finitely presented, which is not the case of $\mathrm{SL}_3(\mathbb{F}_q[t])$ [Beh79]. Since finite presentability can be interpreted in terms of coarse simple connectedness, it means that $\mathrm{SL}_3(\mathbb{Z})$ and $\mathrm{SL}_3(\mathbb{F}_q[t])$ do not behave the same way from the point of view of simple connectedness at infinity. The following result, in contrast, shows that they behave the same way from the point of view of connectedness at infinity.

Theorem 2.7. *For any $n \geq 3$, the finitely generated group $\mathrm{SL}_n(\mathcal{O}_\mathcal{S})$ has a linear divergence.*

Note that Theorem 2.7 also holds for $n = 2$ when $|\mathcal{S}| > 1$ because $\mathrm{SL}_2(\mathcal{O}_\mathcal{S})$ quasi-isometrically embeds into a product of $|\mathcal{S}|$ trees, and the image under this embedding can be identified as the complement of disjoint horoballs. Now it follows from Theorem 5.12 of [DMS10] that such a space has a linear divergence. On the other hand if $|\mathcal{S}| = 1$ then $\mathrm{SL}_2(\mathcal{O}_\mathcal{S})$ is not finitely generated.

Combining Theorem 2.7 and Proposition 2.5, we obtain the following.

Theorem 2.8. *For any $n \geq 3$, the asymptotic cones of $\mathrm{SL}_n(\mathcal{O}_{\mathcal{S}})$ do not have cut-points.*

2.3.1 The case of the polynomial ring $\mathbb{F}_q[t]$

In this subsection we prove that the finitely generated group $\mathrm{SL}_n(\mathbb{F}_q[t])$ has a linear divergence, explicitly constructing a short path joining any two given large matrices and avoiding a large ball around the origin. It turns out that moving in the Cayley graph from a vertex to another corresponds to making column operations in terms of matrices. We will write down the proof only for $n = 3$ but our method applies directly to the general case.

Preliminary material. Throughout this paragraph, we let $\Bbbk$ be the the field $\mathbb{F}_q(t)$ and $\Bbbk_\infty$ be the field of Laurent series $\mathbb{F}_q(\!(t^{-1})\!)$. Recall that $\Bbbk_\infty$ is the completion of $\Bbbk$ with respect to the valuation ν_∞, see Example 2.20. Let us denote by $|\cdot|$ the associated norm. Note that for any polynomial $a \in \mathbb{F}_q[t]$, we have $|a| = q^{\deg a}$. If γ is a matrix with entries in $\mathbb{F}_q[t]$, we denote by

$$\|\gamma\| = \max_{i,j} \{|\gamma_{i,j}|\}$$

its norm as a matrix over $\Bbbk_\infty$.

We now recall some basic results coming from the theory of matrices with elements in a euclidean ring (see Theorem 22.4 of [Mac56] for example). We let $e_{i,j}(r)$ be the elementary unipotent matrix whose (i,j)-entry is r, $i \neq j$.

Theorem 2.9. *Let A be a euclidean ring and $n \geq 2$. Then any element of $\mathrm{SL}_n(A)$ is a product of elementary matrices.* □

Corollary 2.10. *For any $n \geq 3$, the group $\mathrm{SL}_n(\mathbb{F}_q[t])$ is generated by the finite set*

$$S_0 = \bigcup_{i \neq j} \left\{ e_{i,j}(\alpha) \,:\, \alpha \in \mathbb{F}_q^* \right\} \cup \{e_{i,j}(t)\}.$$

Proof. According to Theorem 2.9 it is enough to prove that for any $P \in \mathbb{F}_q[t]$, the matrix $e_{i,j}(P)$ is a product of elements of S_0. Since $\mathbb{F}_q^*$ and t generate $\mathbb{F}_q[t]$ as a ring, this follows, using a straightforward induction, from the identity

$$e_{i,j}(x+y) = e_{i,j}(x)e_{i,j}(y)$$

and the commutator relation

$$e_{i,j}(xy) = [e_{i,k}(x), e_{k,j}(y)]$$

for all $x, y \in \mathbb{F}_q[t]$ and $k \neq i, j$ (using that $n \geq 3$). □

Since all the Cayley graphs of $\mathrm{SL}_3(\mathbb{F}_q[t])$ are quasi-isometric, we can choose a particular generating set S. Its construction goes as follows.

Let us still denote by S_0 the previous finite generating set. We will add some elements to S_0 in order to get a more convenient one.

Denote by S_1 the set of monomial matrices whose non-zero entries are ± 1 (by a monomial matrix we mean a matrix with exactly one non-zero element in each row and each column).

Consider a matrix $A_1 \in \mathrm{SL}_2(\mathbb{F}_q[t])$ with two eigenvalues $\lambda_+, \lambda_- \in \Bbbk_\infty$ such that $|\lambda_+| > 1$ and $|\lambda_-| < 1$, and another matrix A_2 with the same eigenvalues but with different eigen-directions. For example the matrices

$$A_1 = \begin{pmatrix} 1 & t \\ 1 & t+1 \end{pmatrix}$$

and its conjugate

$$A_2 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 1 & t+1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} t+1 & t \\ 1 & 1 \end{pmatrix}$$

would work. To complete our generating set, put

$$S_2 = \bigcup_{i=1}^{2} \left\{ \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A_i}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{A_i}} \\ 0 & & \end{pmatrix} \right\},$$

and take $S = S_0 \cup S_1 \cup S_2$.

We now set some notation and terminology, extensively borrowed from [DMS10].

Definition 2.11. Let $0 < \varepsilon < 1$. An entry $\gamma_{i,j}$ of $\gamma \in \mathrm{SL}_3(\mathbb{F}_q[t])$ is called ε-large if we have

$$\log |\gamma_{i,j}| \geq \varepsilon \log \|\gamma\| .$$

Note that any entry $\gamma_{i,j}$ of $\gamma \in \mathrm{SL}_3(\mathbb{F}_q[t])$ such that $|\gamma_{i,j}| = \|\gamma\|$ is ε-large, but these may not be the only ε-large entries.

Definition 2.12. Let $\delta_1, \delta_2, \delta_3 > 0$. A $(\delta_1, \delta_2, \delta_3)$-external trajectory connecting two elements γ_1, γ_2 of $\mathrm{SL}_3(\mathbb{F}_q[t])$ is a path η in its Cayley graph from γ_1 to γ_2 such that:

i) η remains outside the ball centered at e of radius $\delta_1 \times d_S(e, \{\gamma_1, \gamma_2\}) - \delta_2$;

ii) the length of η is bounded by $\delta_3 \times d_S(\gamma_1, \gamma_2)$.

Two elements γ_1 and γ_2 are said to be $(\delta_1, \delta_2, \delta_3)$-externally connected if there exists a $(\delta_1, \delta_2, \delta_3)$-external trajectory between them, and uniformly externally connected if there exist some constants $\delta_1, \delta_2, \delta_3 > 0$ which do not depend on γ_1 and γ_2, such that γ_1 and γ_2 are $(\delta_1, \delta_2, \delta_3)$-externally connected.

Clearly $\mathrm{SL}_3(\mathbb{F}_q[t])$ has a linear divergence if and only if any two elements are uniformly externally connected.

In the proof, when constructing paths in the Cayley graph of $\mathrm{SL}_3(\mathbb{F}_q[t])$, we will substantially make use of the work of Lubotzky, Mozes and Raghunathan. In [LMR00] they prove that any irreducible lattice in a semisimple Lie group of $\mathbb{R}$-rank ≥ 2, endowed with some word metric associated to a finite generating set, is quasi-isometrically embedded in the ambient Lie group. Their result allows us to estimate the size of a matrix $\gamma \in \mathrm{SL}_3(\mathbb{F}_q[t])$ with respect to the word metric d_S in terms of its norm $\|\gamma\|$.

Proposition 2.13. *There exists $C > 0$ such that for any $\gamma \in \mathrm{SL}_3(\mathbb{F}_q[t])$,*

$$C^{-1}(1 + \log \|\gamma\|) \leq d_S(e, \gamma) \leq C(1 + \log \|\gamma\|). \tag{2.1}$$

Roughly speaking, Proposition 2.13 states that the size of $\gamma \in \mathrm{SL}_3(\mathbb{F}_q[t])$ is the maximum of the degrees of the entries of γ.

Quasi-isometrically embedded lamplighters. While connecting points in the Cayley graph of $\mathrm{SL}_3(\mathbb{F}_q[t])$, we will take advantage of embedded subgroups with nice geometric properties with respect to our problem, i.e. with a linear divergence. For $i = 1, 2$, define

$$\Lambda_i = \begin{pmatrix} 1 & 0 \quad 0 \\ \mathbb{F}_q[t] & \\ \mathbb{F}_q[t] & A_i^{\mathbb{Z}} \end{pmatrix}.$$

Since $\mathbb{F}_q[t]$ is a cocompact lattice in $\Bbbk_\infty$, the group Λ_i is a cocompact lattice in

$$G_i = \begin{pmatrix} 1 & 0 \quad 0 \\ \Bbbk_\infty & \\ \Bbbk_\infty & A_i^{\mathbb{Z}} \end{pmatrix}.$$

Now since A_i is diagonalizable in $\mathrm{SL}_2(\Bbbk_\infty)$, it follows that G_i is conjugated in $\mathrm{SL}_3(\Bbbk_\infty)$ to the group

$$\begin{pmatrix} 1 & 0 & 0 \\ \Bbbk_\infty & \lambda_+^n & 0 \\ \Bbbk_\infty & 0 & \lambda_+^{-n} \end{pmatrix},$$

with $|\lambda_+| > 1$. So the groups Λ_i both are quasi-isometric to $\Bbbk_\infty^2 \rtimes \mathbb{Z}$, where the action of $\mathbb{Z}$ is the multiplication by $(\lambda_+, \lambda_+^{-1})$, and in particular we obtain

Proposition 2.14. *For $i = 1, 2$, all the asymptotic cones of Λ_i are homeomorphic to the hypersurface of equation $b(x) + b(y) = 0$ in the product of two copies of the universal real tree $\mathbb{T}$ with continuum branching everywhere, where b is a Busemann function on $\mathbb{T}$.*

Proof. See Section 9 of [Cor08]. □

Since this metric space has no cut-points (see Section 2.4), Proposition 2.5 tells us that the groups Λ_i have a linear divergence. In order to use this property inside $\mathrm{SL}_3(\mathbb{F}_q[t])$, we need to ensure that these groups are quasi-isometrically embedded. This follows from the fact that the groups Λ_i are quasi-isometric to $\Bbbk_\infty^2 \rtimes \mathbb{Z}$ and from the following standard lemma, which gives an estimate of the word metric in the latter group.

Lemma 2.15. *Let $(\mathbb{K}_i)_{i=1..m}$ be a family of local fields, each one being endowed with a multiplicative norm $|\cdot|_i$. For $i = 1 \dots m$, let $\lambda_i \in \mathbb{K}_i$ be an element of norm different from 1. Let $G = (\bigoplus \mathbb{K}_i) \rtimes \mathbb{Z}$, where the diagonal action of $\mathbb{Z}$ is by multiplication by λ_i on $\mathbb{K}_i$. Then G is compactly generated and for any compact generating set S of G, there exists a constant C such that for any $(x, n) = (x_1, \dots, x_m, n) \in G$,*

$$C^{-1}\left(\log\left(1 + \max_i |x_i|_i\right) + |n|\right) \leq |(x,n)|_S \leq C\left(\log\left(1 + \max_i |x_i|_i\right) + |n|\right).$$

Proof. It is not hard to see that it is enough to prove the result for only one local field $\mathbb{K}$, and that without loss of generality we can assume that the element λ defining the action of $\mathbb{Z}$ has norm strictly smaller than 1. In this situation G is generated by the compact set $S = \mathbb{K}_0 \cup t$, where $\mathbb{K}_0$ denotes the set of elements of $\mathbb{K}$ of norm at most 1, and $t = (0, 1)$. Since different compact generating sets yield bi-Lipschitz equivalent word metrics, it is enough to prove the result for this generating set.

Let us prove the upper bound first. If $(x, n) \in G$ and if we denote by

$$n_0 = \max\left(\lfloor -\log|x| / \log|\lambda| \rfloor + 1, 0\right),$$

then the reader can check that $\lambda^{n_0} x \in \mathbb{K}_0$. Therefore there exists $x_0 \in \mathbb{K}_0$ such that the word $t^{-n_0} x_0 t^{n_0} t^n$ represents the element (x, n) of G, which therefore has length at most $2n_0 + 1 + |n| \leq c(\log(1 + |x|) + |n|)$ for some constant c.

To prove the lower bound, let us consider a word $t^{n_1} x_1 \dots t^{n_k} x_k$ of length at most ℓ representing an element (x, n) of G, where $x_i \in \mathbb{K}_0$ and $k, \sum |n_i| \leq \ell$. Then (x, n) is also represented by the word

$$t^{n_1} x_1 t^{-n_1} t^{n_1+n_2} x_2 t^{-n_1-n_2} \dots t^{n_1+\dots+n_k} x_k t^{-n_1-\dots-n_k} t^{n_1+\dots+n_k},$$

and therefore the equality

$$\sum_{i=1}^{k} \lambda^{n_1+\dots+n_i} x_i = x$$

holds in $\mathbb{K}$, and $n = n_1 + \ldots + n_k$. Consequently

$$|x| \leq \sum_{i=1}^{k} |\lambda^{n_1+\ldots+n_i} x_i| \leq k \max_i |\lambda|^{n_1+\ldots+n_i} \leq \ell |\lambda|^{-\ell} \leq |\lambda|^{-c'\ell}$$

for some constant c'. So now

$$\log(1+|x|) + |n| \leq \log(1+|\lambda|^{-c'\ell}) + \sum |n_i| \leq c''\ell,$$

for some constant c'' and ℓ large enough, which completes the proof. $\square$

Construction of external trajectories. The following lemma will be used repeatedly in this paragraph.

Lemma 2.16. *Let $0 < \varepsilon < 1$ be fixed. For any $\alpha \in \mathrm{SL}_3(\mathbb{F}_q[t])$ having an ε-large entry in its third column and any*

$$\theta = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{F}_q[t]),$$

α and $\alpha\theta$ are uniformly externally connected.

Proof. Let $\alpha \in \mathrm{SL}_3(\mathbb{F}_q[t])$ having an ε-large entry $\alpha_{j,3}$ in its third column. First note that we have a control of the size of $\alpha\theta$ in the sense that $\alpha\theta$ is bounded away from the identity. Indeed, since the third column of $\alpha\theta$ is the third column of α,

$$\log \|\alpha\theta\| \geq \log |\alpha_{j,3}| \geq \varepsilon \log \|\alpha\| .$$

Now according to (2.1),

$$\begin{aligned} d_S(e, \alpha\theta) &\geq C^{-1}(1 + \log \|\alpha\theta\|) \\ &\geq C^{-1}(1 + \varepsilon \log \|\alpha\|) \\ &\geq C^{-1}(1 + \varepsilon(C^{-1} d_S(e,\alpha) - 1)) \\ &= \varepsilon C^{-2} d_S(e,\alpha) + C^{-1}(1-\varepsilon). \end{aligned}$$

Since Λ_i is quasi-isometrically embedded in $\mathrm{SL}_3(\mathbb{F}_q[t])$, for every

$$\theta = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{F}_q[t])$$

there exists a short word $s_1^{(i)} \ldots s_n^{(i)}$ representing θ, where all the $s_k^{(i)}$ belong to

$$\left\{ \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ 0 & A_i \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix} \right\},$$

with $\alpha, \beta \in \mathbb{F}_q^*$. It gives us two short trajectories η_1, η_2 from α to $\alpha\theta$. To conclude that α and $\alpha\theta$ are uniformly externally connected, it is enough to prove that one of these two trajectories does not get too close to the identity.

Following [DMS10], for $0 < c < 1$ and $i = 1, 2$, we define the non-contracting cone of the matrix A_i in $\Bbbk_\infty^2$,

$$\mathrm{NC}_c(A_i) = \left\{ v \in \Bbbk_\infty^2 \ : \ \|vA\| > c\,\|v\| \ \ \forall A \in \langle A_i \rangle \right\} \cup \{0\}\,.$$

Since A_1 and A_2 have distinct eigen-directions, it is easily checked that we can choose $c_0 > 0$ small enough such that

$$\mathrm{NC}_{c_0}(A_1) \cup \mathrm{NC}_{c_0}(A_2) = \Bbbk_\infty^2,$$

and therefore we can find i such that $(\alpha_{j,2}, \alpha_{j,3}) \in \mathrm{NC}_{c_0}(A_i)$, where $\alpha_{j,3}$ still denotes an ε-large entry of α.

Note that in each trajectory η_i, only the elements $s_k^{(i)}$ belonging to S_2 can affect the second or the third column. It follows from this observation, and from the choice of i such that $(\alpha_{j,2}, \alpha_{j,3}) \in \mathrm{NC}_{c_0}(A_i)$, that the same computation we made at the beginning of the proof yields that any point in the trajectory η_i is at distance at least

$$\varepsilon C^{-2} d_S(e, \alpha) + C^{-1}(1 - \varepsilon + \log c_0)$$

from the identity, which concludes the proof. □

As in the proof in [DMS10] for the case of $\mathrm{SL}_3(\mathbb{Z})$, the strategy for uniformly externally connecting two given points will consist in first moving each of them to another point of the same size, lying in a particular subgroup. This is achieved by the following lemma.

Lemma 2.17. *There exist $c_1, c_2 > 0$ such that for any $\alpha \in \mathrm{SL}_3(\mathbb{F}_q[t])$, there is an element*

$$\alpha' = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{F}_q[t])$$

satisfying

$$c_1 \le \frac{1 + d_S(e, \alpha)}{1 + d_S(e, \alpha')} \le c_2,$$

and such that α and α' are uniformly externally connected.

Then using the fact that the subgroup

$$\Lambda_1' = \begin{pmatrix} \begin{matrix} & A_1 & \\ 0 & & 0 \end{matrix} & \begin{matrix} x \\ y \\ 1 \end{matrix} \end{pmatrix}$$

(which is isomorphic to Λ_1) has a linear divergence and is quasi-isometrically embedded inside $\mathrm{SL}_3(\mathbb{F}_q[t])$, we obtain that any two points in $\mathrm{SL}_3(\mathbb{F}_q[t])$ are uniformly externally connected, i.e. $\mathrm{SL}_3(\mathbb{F}_q[t])$ has a linear divergence.

Proof of Lemma 2.17. We are going to proceed in a finite number of steps to construct a path in the Cayley graph connecting α to a suitable α'. In each step, we can check that α_{i+1} and α_i satisfy

$$c_1^{(i)} \leq \frac{1 + d_S(e, \alpha_i)}{1 + d_S(e, \alpha_{i+1})} \leq c_2^{(i)}$$

for some constants $c_1^{(i)}, c_2^{(i)} > 0$ which do not depend on α. The estimate on the size of α' follows from this sequence of inequalities.

Let $0 < \varepsilon < 1$ and let

$$\alpha = \begin{pmatrix} * & * & * \\ * & * & * \\ a & b & c \end{pmatrix} \in \mathrm{SL}_3(\mathbb{F}_q[t]).$$

Swapping columns if necessary, which is possible thanks to S_1, we can assume that α has an ε-large entry $\alpha_{i,3}$ in the last column.

Now we claim that without loss of generality, we can also assume that the lower right entry c is ε-large. Indeed, we can first assume, at the price of exchanging the first two columns, that $b \neq 0$ (otherwise if $a = b = 0$ then we can directly move to Claim 4 of this proof). Now multiplying α on the right by

$$\theta = \begin{pmatrix} 1 & 0 & 0 \\ t^{\deg \alpha_{i,3}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

has the effect of changing a into $a + t^{\deg \alpha_{i,3}} b$, which has degree at least $\deg \alpha_{i,3}$ because $b \neq 0$. But according to Lemma 2.16, α and $\alpha\theta$ are uniformly externally connected. Therefore we can suppose that the lower left entry is ε-large, and finally we just have to exchange the first and third columns to obtain an ε-large lower right entry.

Claim 1. α is uniformly externally connected to

$$\alpha_2 = \begin{pmatrix} * & * & * \\ * & * & * \\ a' & b' & c \end{pmatrix},$$

where a', b' verify $\gcd(a', b') = 1$.

Proof. Multiplying α by $e_{1,3}(1)$ if necessary, which has the effect of changing a into $a' = a + c$, we can assume that $a' \neq 0$. Let $p_1, \dots, p_d$ be the distinct prime divisors of a'. If we let m be the product of all the p_i's dividing neither b nor c, then it is easy to check that a' and $b' = b + mc$ are relatively prime. Now using Lemma 2.16 and the identity

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain that α is uniformly externally connected to

$$\begin{pmatrix} * & * & * \\ * & * & * \\ a' & b' & c \end{pmatrix} = \alpha_2.$$

□

We now want to perform an external trajectory between α_2 and

$$\alpha_3 = \begin{pmatrix} * & * & * \\ * & * & * \\ a' & b'' & c \end{pmatrix},$$

where the entry b'' is ε-large and satisfies $\gcd(a', b'') = 1$. There is nothing to do if $\deg b' \geq \deg c$. Otherwise we get this external trajectory by setting $b'' = b' + t^{\deg c} a'$ and using Lemma 2.16 and the identity

$$\begin{pmatrix} 1 & t^{\deg c} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -t^{\deg c} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Claim 2. α_3 is uniformly externally connected to

$$\alpha_4 = \begin{pmatrix} * & * & * \\ * & * & * \\ a' & 1 & b'' \end{pmatrix}.$$

Proof. First note that we can go in one step from α_3 to

$$\alpha_3' = \alpha_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ a' & -c & b'' \end{pmatrix}.$$

Let u, v be Bézout coefficients of small degree for the pair (a', b''), that is, $a'u + b''v = 1$. Using Lemma 2.16 one more time and the two identities already used above, we get that α_3' is uniformly externally connected to

$$\alpha_3' \begin{pmatrix} 1 & (c+1)u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & (c+1)v & 1 \end{pmatrix} = \alpha_3' \begin{pmatrix} 1 & (c+1)u & 0 \\ 0 & 1 & 0 \\ 0 & (c+1)v & 1 \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ a' & 1 & b'' \end{pmatrix}.$$

□

Claim 3. α_4 is uniformly externally connected to

$$\alpha_5 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 1 & 1 \end{pmatrix},$$

where the third column of α_5 remains ε-large.

Proof. It directly follows from Lemma 2.16 that α_4 is uniformly externally connected to

$$\alpha_4' = \alpha_4 \begin{pmatrix} 1 & 0 & 0 \\ -a' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 1 & b'' \end{pmatrix}.$$

Now we can find a polynomial P of small degree such that if we first externally connect α_4' to

$$\alpha_4'' = \alpha_4' \begin{pmatrix} 1 & P & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then the third column of

$$\alpha_5 = \alpha_4'' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1-b'' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 1 & 1 \end{pmatrix}$$

remains ε-large. $\square$

Now the vertex corresponding to α_5 in the Cayley graph is adjacent to

$$\alpha_6 = \alpha_5 e_{2,3}(-1) = \begin{pmatrix} \begin{matrix} & B & \end{matrix} & \begin{matrix} * \\ * \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix},$$

where $B \in \mathrm{SL}_2(\mathbb{F}_q[t])$.

Claim 4. α_6 is uniformly externally connected to

$$\alpha_7 = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

which concludes the proof of Lemma 2.17.

Proof. Note that since the matrix B^{-1} has determinant 1, the entries of its first row are relatively prime. It follows from the Euclidean algorithm applied to these entries that B^{-1} is a product of matrices

$$B^{-1} = \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\pm 1} \begin{pmatrix} 1 & 0 \\ q_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\pm 1} \cdots \begin{pmatrix} 1 & 0 \\ q_k & 1 \end{pmatrix},$$

where the polynomials q_i are the quotients appearing when performing the Euclidean algorithm. In particular the quantity $\sum \deg q_i$ is bounded by the maximum of the degrees of the entries of B^{-1}.

Now let us multiply α_6 on the right by

$$\begin{pmatrix} 1 & 0 & 0 \\ q_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\pm 1} \begin{pmatrix} 1 & 0 & 0 \\ q_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\pm 1} \cdots \begin{pmatrix} 1 & 0 & 0 \\ q_k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Each product by a monomial matrix consists in moving to an adjacent vertex in the Cayley graph. Now the fact that $\sum \deg q_i$ is well controlled, together with Lemma 2.16 applied for each product by a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ q_i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

yield that α_6 is uniformly externally connected to

$$\alpha_6 \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{B^{-1}}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \alpha_7.$$

□

□

2.3.2 Adding valuations

In this subsection we prove that the growth rate of the divergence function of the group $\mathrm{SL}_n(\mathcal{O}_{\mathcal{S}})$ cannot increase while adding valuations to the set $\mathcal{S}$.

Number theory. We now recall some basic definitions about valuations on a global function field $\Bbbk$.

Definition 2.18. A discrete valuation on $\Bbbk$ is a non-trivial homomorphism $\nu : \Bbbk^* \to \mathbb{R}$ satisfying $\nu(x+y) \geq \min(\nu(x), \nu(y))$ for all $x, y \in \Bbbk^*$ with $x + y \neq 0$. It is convenient to extend ν to a function defined on $\Bbbk$ by setting $\nu(0) = \infty$.

Example 2.19. Let $P \in \mathbb{F}_q[t]$ be an irreducible polynomial. Every nonzero element x of $\mathbb{F}_q(t)$ can be written in a unique way $x = P^n(a/b)$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{F}_q[t]$ are not divisible by P. We define a valuation on $\Bbbk = \mathbb{F}_q(t)$ putting $\nu_P(x) = n$. If $P = t - a$ then $\nu_P(x)$ is the order of vanishing of the rational function x at the point $a \in \mathbb{F}_q$.

Example 2.20. Besides the valuations ν_P defined above, we get another discrete valuation ν_∞ on $\mathbb{F}_q(t)$ by setting $\nu_\infty(a/b) = \deg b - \deg a$ for any two non-zero polynomials $a, b \in \mathbb{F}_q[t]$.

Remark 2.21. Every discrete valuation on $\Bbbk = \mathbb{F}_q(t)$ is equivalent to either ν_∞ or ν_P for some irreducible $P \in \mathbb{F}_q[t]$ [Wei95, Th.2 Chap.III.1], where two valuations ν_1, ν_2 are said to be equivalent if there exists $c > 0$ such that $\nu_1(x) = c\,\nu_2(x)$ for all $x \in \Bbbk$.

Given a valuation ν on $\Bbbk$, we define the associated norm by the formula $|x|_\nu = q^{-\nu(x)}$ for all $x \in \Bbbk$. It is easily checked that we get a metric on $\Bbbk$ by setting $d_\nu(x, y) = |x - y|_\nu$. By the completion of $\Bbbk$ with respect to the valuation ν we mean the completion of the metric space $(\Bbbk, d_\nu)$. Note that the field operations and the valuation ν on $\Bbbk$ extend to its completion.

Example 2.22. The completion of $\Bbbk = \mathbb{F}_q(t)$ with respect to ν_P is the field $\mathbb{F}_q(\!(P)\!)$, and its completion with respect to ν_∞ is $\mathbb{F}_q(\!(t^{-1})\!)$.

If $\mathcal{S}$ denotes a finite set of valuations on $\Bbbk$, we denote by $\mathcal{O}_\mathcal{S}$ the ring of $\mathcal{S}$-integer points of $\Bbbk$. Recall that $\mathcal{O}_\mathcal{S}$ is defined as the set of $x \in \Bbbk$ such that x is ν-integral for all valuations $\nu \notin \mathcal{S}$,

$$\mathcal{O}_\mathcal{S} = \{x \in \Bbbk \,:\, |x|_\nu \leq 1 \text{ for all valuations } \nu \notin S\}\,.$$

We have a natural diagonal embedding of $\Bbbk$ into

$$\Bbbk_\mathcal{S} = \prod_{\nu \in \mathcal{S}} \Bbbk_\nu,$$

where $\Bbbk_\nu$ denotes the completion of $\Bbbk$ with respect to the valuation ν. Note that $\mathcal{O}_\mathcal{S}$ has a discrete and cocompact image into $\Bbbk_\mathcal{S}$.

Example 2.23. Let $\Bbbk = \mathbb{F}_q(t)$ and let $\mathcal{S} = \{\nu_\infty\}$. Then $\mathcal{O}_\mathcal{S}$ is the polynomial ring $\mathbb{F}_q[t]$. It is a discrete cocompact subring of $\mathbb{F}_q(\!(t^{-1})\!)$.

Example 2.24. Let $\Bbbk = \mathbb{F}_q(t)$ and let ν_a denote the valuation associated to the polynomial $t - a$. If $\mathcal{S} = \{\nu_\infty, \nu_0, \nu_{-1}\}$ then $\mathcal{O}_\mathcal{S} = \mathbb{F}_q[t, t^{-1}, (t+1)^{-1}]$. It is a discrete cocompact subring of the locally compact ring $\mathbb{F}_q(\!(t^{-1})\!) \times \mathbb{F}_q(\!(t)\!) \times \mathbb{F}_q(\!(t+1)\!)$.

Proof of the result. From now and until the end of this subsection, $\mathcal{S}$ will denote a finite set of $s \geq 2$ pairwise non-equivalent valuations on $\Bbbk$ containing ν_∞. We will prove that the divergence function of the finitely generated group $\mathrm{SL}_n(\mathcal{O}_\mathcal{S})$ is linear. As in the previous subsection, we write down the arguments only for $n = 3$, the proof of the general case being a straightforward extension of the proof for this case.

We now choose a finite generating set of $\mathrm{SL}_3(\mathcal{O}_\mathcal{S})$. We choose similarly the sets S_0, S_1, S_2 defined at the beginning of Section 2.3.1. Recall that according to Dirichlet unit theorem, the group $\mathcal{O}_\mathcal{S}^\times$ of units of $\mathcal{O}_\mathcal{S}$ is the direct product of a free Abelian group of rank $s-1$, freely generated by $\lambda_1, \ldots, \lambda_{s-1}$, with a finite Abelian group generated by $\lambda_s, \ldots, \lambda_d$. We define S_3 as the following set of matrices

$$S_3 = \bigcup_{i=1}^{d} \left\{ \begin{pmatrix} \lambda_i & & \\ & \lambda_i^{-1} & \\ & & 1 \end{pmatrix}, \begin{pmatrix} \lambda_i & & \\ & 1 & \\ & & \lambda_i^{-1} \end{pmatrix} \right\}.$$

We now take $S = S_0 \cup S_1 \cup S_2 \cup S_3$ as finite generating set of $\mathrm{SL}_3(\mathcal{O}_\mathcal{S})$.

If $x \in \mathcal{O}_\mathcal{S}$ and γ is a matrix with entries in $\mathcal{O}_\mathcal{S}$, we denote by

$$|x| = \max_{\nu \in \mathcal{S}} |x|_\nu$$

and

$$\|\gamma\| = \max_{i,j} \{|\gamma_{i,j}|\}.$$

The control of the size of a matrix with respect to the word metric associated to S by the size of its entries, provided by [LMR00], still holds in this setting:

Proposition 2.25. *There exists $C > 0$ such that for any $\gamma \in \mathrm{SL}_3(\mathcal{O}_\mathcal{S})$,*

$$C^{-1}(1 + \log \|\gamma\|) \leq d_S(e, \gamma) \leq C(1 + \log \|\gamma\|).$$

We define similarly the notion of ε-large entry of an element of $\mathrm{SL}_3(\mathcal{O}_\mathcal{S})$, and the notion of external trajectories between two elements in the Cayley graph of $\mathrm{SL}_3(\mathcal{O}_\mathcal{S})$ associated to S. As in Section 2.3.1 for $\mathrm{SL}_3(\mathbb{F}_q[t])$, we prove that any two elements of $\mathrm{SL}_3(\mathcal{O}_\mathcal{S})$ are uniformly externally connected. Some points of the proof will be similar to what we did in the case of $\mathbb{F}_q[t]$, but the idea is that here we are in a more pleasant situation because diagonal matrices coming from units in $\mathcal{O}_\mathcal{S}$ provide some more place to move.

Lemma 2.26. *There exist $c_1', c_2' > 0$ such that any $\alpha \in \mathrm{SL}_3(\mathcal{O}_\mathcal{S})$ can be uniformly externally connected to an element*

$$\alpha' \in \begin{pmatrix} \mathrm{SL}_2(\mathcal{O}_\mathcal{S}) & & * \\ & & * \\ 0 & 0 & 1 \end{pmatrix}$$

satisfying

$$c_1' \leq \frac{1 + d_S(e, \alpha)}{1 + d_S(e, \alpha')} \leq c_2'.$$

Proof. As before, the control on the size of α' will come from the fact that we proceed in a finite number of steps to connect α to α', and that each intermediate point satisfies such a control. Let $\varepsilon > 0$ be fixed. Write

$$\alpha = \begin{pmatrix} * & * & * \\ * & * & * \\ a & b & c \end{pmatrix}.$$

First note that multiplying if necessary by diagonal elements of S_3, which has the effect of multiplying the columns of α, we can assume that the first two columns belong to $\mathbb{F}_q[t]$ and that the third column is ε-large. Now proceeding as in Claim 1 of the proof of Lemma 2.17 and using an analogue of Lemma 2.16, we obtain that α is uniformly externally connected to

$$\alpha_2 = \begin{pmatrix} * & * & * \\ * & * & * \\ a' & b' & c \end{pmatrix},$$

where a' and b' are coprime. Then swapping columns 2 and 3, multiplying if necessary by elements of S_3 to get an ε-large third column, and doing as in Claim 2 in 2.3.1, we get that α_2 is uniformly externally connected to

$$\alpha_3 = \begin{pmatrix} * & * & * \\ * & * & * \\ a' & 1 & b'' \end{pmatrix}.$$

Now using Lemma 2.16 we uniformly externally connect α_3 to

$$\alpha_4 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 1 & b'' \end{pmatrix}.$$

To obtain the desired result it is now sufficient to make the first column ε-large thanks to S_3, and use one more time Lemma 2.16 to connect α_4 to

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 1 & 0 \end{pmatrix},$$

and finally exchange columns 2 and 3. $\square$

To finish the proof we now have to show that any two

$$\gamma, \gamma' \in \begin{pmatrix} \mathrm{SL}_2(\mathcal{O}_S) & \begin{matrix} * \\ * \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

are uniformly externally connected. But this is straightforward because since $s \geq 2$, $\mathrm{SL}_2(\mathcal{O}_\mathcal{S})$ is finitely generated and quasi-isometrically embedded in $\mathrm{SL}_3(\mathcal{O}_\mathcal{S})$, so we easily get (by first making their third column large thanks to S_3) that each of them is uniformly externally connected to an element of the form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Now in order to see that any two elements of this form are uniformly externally connected, we can for example argue that they lie in the quasi-isometrically embedded subgroup

$$\begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \simeq (\mathcal{O}_\mathcal{S} \rtimes \mathcal{O}_\mathcal{S}^\times)^2,$$

which, as a direct product of two infinite finitely generated groups, trivially has a linear divergence.

2.4 On the asymptotic cone of SOL

Recall that the group SOL is the three-dimensional Lie group $\mathbb{R}^2 \rtimes \mathbb{R}$, where the action of $t \in \mathbb{R}$ on $\mathbb{R}^2$ is given by the diagonal matrix $\mathrm{diag}(e^t, e^{-t})$. It was proved in [Cor08, Section 9] that the asymptotic cones of SOL do not depend on the choice of ω and $\mathbf{s}$. More precisely, embedding the group SOL as a horosphere into the direct product $(\mathbb{R} \rtimes \mathbb{R}) \times (\mathbb{R} \rtimes \mathbb{R})$, it is proved that all these asymptotic cones are bi-Lipschitz equivalent to

$$\{(x, y) \in \mathbb{T} \times \mathbb{T} \,:\, b(x) + b(y) = 0\},$$

where $\mathbb{T}$ is a homogeneous real tree of continuum branching cardinality, and b is a Busemann function on $\mathbb{T}$. This metric space, defined up to bi-Lipschitz homeomorphism, will be denoted by Cone(SOL).

Proposition 2.27. *The metric space* Cone(SOL) *does not have cut-points.*

Proof. In order to show that a metric space does not have cut-points, it is enough to prove that given any two points, there exist two different paths joining them which intersect only at their extremities. For let $(x_1, y_1), (x_2, y_2) \in \mathrm{Cone(SOL)}$. We can always assume, composing by an isometry if necessary, that $b(x_1) = b(y_1) = 0$ and $b(x_2) > 0$. In this situation Figure 2.1 shows how to construct two convenient paths. $\square$

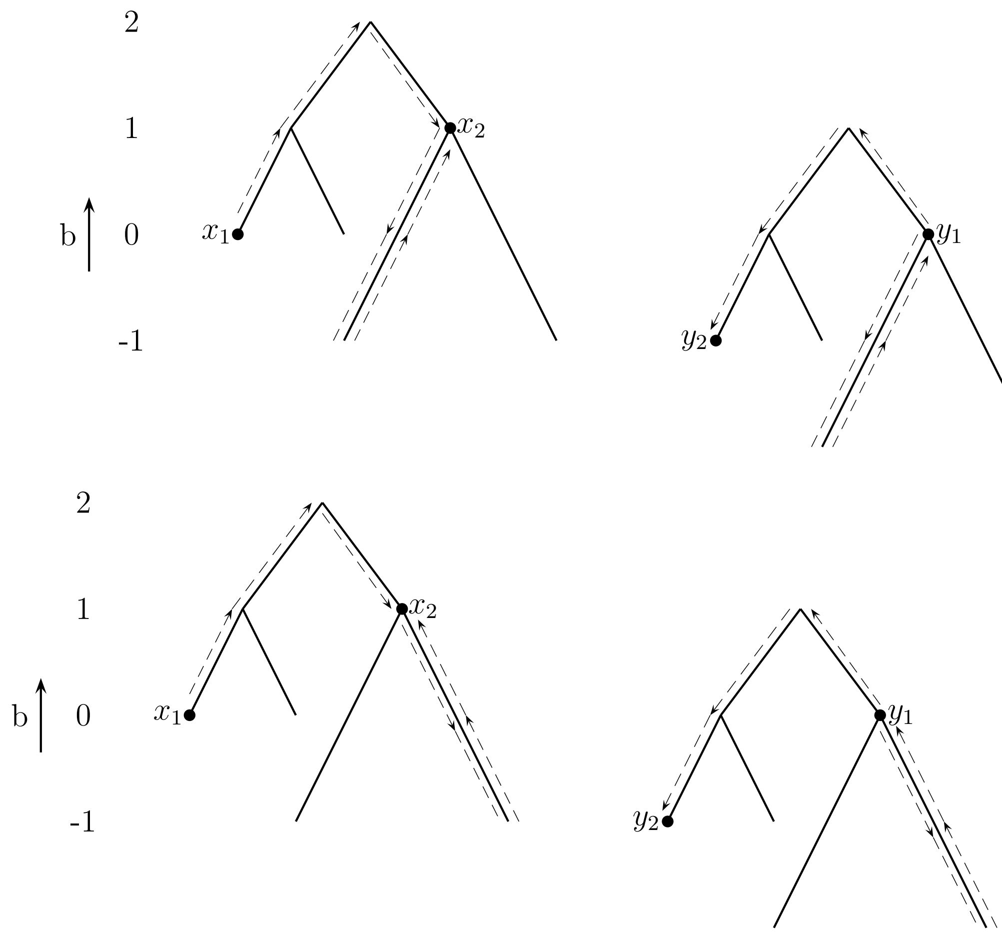


Figure 2.1 – Construction of a circle containing (x_1, y_1) and (x_2, y_2)

Now consider the three-dimensional real Heisenberg group $\mathrm{H}_3(\mathbb{R})$, i.e. the group of upper triangular unipotent matrices of size 3 with entries in $\mathbb{R}$, and its extension $\mathrm{H}_3(\mathbb{R}) \rtimes \mathbb{R}$, where

$$t \cdot \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^t x & z \\ & 1 & e^{-t} y \\ & & 1 \end{pmatrix}.$$

Note that this group is also a central extension of the group SOL by the group $\mathbb{R}$, and in particular has discrete infinite central subgroups.

It is not known whether the asymptotic cones of $\mathrm{H}_3(\mathbb{R}) \rtimes \mathbb{R}$ are bi-Lipschitz homeomorphic to Cone(SOL), and the aim of the end of this paragraph is to explain how a negative answer to an apparently more tractable question explained below, would yield a negative answer to this problem.

Proposition 2.28 (Drutu-Sapir)**.** *Let G be a compactly generated group with a discrete central infinite cyclic subgroup. Then every asymptotic cone of G satisfies the property that for every $\varepsilon > 0$, there exists an isometry sending every point to another point at distance exactly ε.*

Proof. See [DS05, Lemma 6.3]. The argument is given there for discrete groups, but the proof works verbatim for locally compact groups. □

The following lemma, whose proof is easy and left to the reader, explains how an isometry like in the conclusion of Proposition 2.28 is transported by a bi-Lipschitz homeomorphism.

Lemma 2.29. *Let $\psi : (X_1, d_1) \to (X_2, d_2)$ be a C-bi-Lipschitz homeomorphism between metric spaces, and let $\varepsilon > 0$. If $\varphi_1 \in \mathrm{Isom}(X_1)$ is such that $d_1(\varphi_1(x_1), x_1) = \varepsilon$ for every $x_1 \in X_1$, then $\varphi_2 = \psi \circ \varphi_1 \circ \psi^{-1}$ is a C^2-bi-Lipschitz homeomorphism of X_2 such that $C^{-1}\varepsilon \leq d_2(\varphi_2(x_2), x_2) \leq C\varepsilon$ for every $x_2 \in X_2$.*

So if the metric space Cone(SOL) does not admit a bi-Lipschitz self-homeomorphism φ so that $c_1 \leq d(\varphi(x), x) \leq c_2$ for some $c_1, c_2 > 0$ and every $x \in$ Cone(SOL), then according to Proposition 2.28 and Lemma 2.29, the asymptotic cones of $\mathrm{H}_3(\mathbb{R}) \rtimes \mathbb{R}$ are not bi-Lipschitz homeomorphic to Cone(SOL).

The question of the existence of a bi-Lipschitz self-homeomorphism of Cone(SOL) with this property is addressed at the end of this thesis.

2.5 Asymptotic cut-points of Lie groups and p-adic groups

In this section we give a rigidity result for connected Lie groups and linear algebraic groups over an ultrametric local field of characteristic zero, namely that if such a group has cut-points in one asymptotic cone, then it is Gromov-hyperbolic.

Actually our characterization will follow from a general statement about groups satisfying a law. Recall that a law is a non-trivial reduced word $w(x_1, \ldots, x_n)$ in some letters $x_1, \ldots, x_n$. A group G is said to satisfy the law $w(x_1, \ldots, x_n)$ if $w(g_1, \ldots, g_n) = 1$ in G for every $g_1, \ldots, g_n \in G$. Examples of groups satisfying a law are solvable groups, or groups of finite exponent. In [DS05], Drutu and Sapir proved that if a finitely generated non-virtually cyclic group G satisfies a law, then G does not have cut-points in any of its asymptotic cones. This result does not hold for locally compact compactly generated groups. For example for every local field $\mathbb{K}$, the affine group $\mathbb{K} \rtimes \mathbb{K}^*$ is solvable of class two, and is also non-elementary hyperbolic, and therefore all its asymptotic cones are real trees.

Nevertheless, applying Theorem 1.1 will allow us to obtain some generalization of the result of Drutu and Sapir to the locally compact setting, by proving that a locally compact group satisfying a law does not have cut-points in any of its asymptotic cones as soon as it is neither an elementary nor a focal hyperbolic group (see Theorem 2.33). Before doing this, let us derive the following consequence of Theorem 1.21.

Proposition 2.30. *Let G be a locally compact lacunary hyperbolic group. If G satisfies a law then G is hyperbolic.*

Proof. Let $\mathrm{Cone}^\omega(G, \mathbf{s})$ be an asymptotic cone of G that is a real tree. Note that since the group G satisfies a law, the same holds for the group $\mathrm{Precone}(G, \mathbf{s})$. Clearly we can assume that $\mathrm{Cone}^\omega(G, \mathbf{s})$ is not a point. If $\mathrm{Cone}^\omega(G, \mathbf{s})$ is a line, then by Lemma 1.29 the group G is elementary hyperbolic. So we may assume that $\mathrm{Cone}^\omega(G, \mathbf{s})$ is not a line, and it follows that the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathrm{Cone}^\omega(G, \mathbf{s})$ is either focal or of general type. But it cannot be of general type, because otherwise this would imply that $\mathrm{Precone}(G, \mathbf{s})$ contains a non-abelian free subgroup [CM87, Theorem 2.7], which is a contradiction with the fact that $\mathrm{Precone}(G, \mathbf{s})$ satisfies a law. Therefore the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathrm{Cone}^\omega(G, \mathbf{s})$ is focal, and it follows from Theorem 1.21 that G is a focal hyperbolic group. $\square$

Remark 2.31. Since the properties of being lacunary hyperbolic and of being hyperbolic are invariant under quasi-isometries, Proposition 2.30 still holds for groups quasi-isometric to a group satisfying a law.

Although it is not stated explicitly in these terms, the following result can be derived from the work of Drutu and Sapir. For an introduction to the concept of tree-graded spaces, we refer the reader to [DS05].

Proposition 2.32 (Drutu-Sapir)**.** *Let G be a locally compact compactly generated group satisfying a law. If $\mathcal{C} = \mathrm{Cone}^\omega(G, \mathbf{s})$ is an asymptotic cone of G with cut-points, then $\mathcal{C}$ must be a real tree.*

Proof. Since by assumption $\mathcal{C}$ has cut-points, it follows from Lemma 2.31 of [DS05] that $\mathcal{C}$ is tree-graded with respect to a collection of proper subsets. Assume by contradiction that $\mathcal{C}$ is not a real tree. Then we can apply Proposition 6.9 of [DS05] to the action of $\mathrm{Precone}(G, \mathbf{s})$ on $\mathcal{C}$, and we obtain that $\mathrm{Precone}(G, \mathbf{s})$ contains a non-abelian free subgroup. On the other hand since the group G satisfies a law, $\mathrm{Precone}(G, \mathbf{s})$ cannot contain a non-abelian free group. Contradiction. $\square$

The following theorem generalizes to the realm of locally compact compactly generated groups the aforementioned result of Drutu and Sapir about finitely generated groups satisfying a law.

Theorem 2.33. *Let G be a locally compact compactly generated group satisfying a law. If G has cut-points in one of its asymptotic cones, then G is either an elementary or a focal hyperbolic group.*

Proof. We let $\mathcal{C}$ be an asymptotic cone of G with cut-points. Since the group G satisfies a law, it follows from Proposition 2.32 that $\mathcal{C}$ is a real tree. Therefore G is lacunary hyperbolic, and the conclusion then follows from Proposition 2.30. $\square$

Since the property of having cut-points in one asymptotic cone is a quasi-isometry invariant, the following result follows immediately from the contrapositive of Theorem 2.33.

Corollary 2.34. *Let G be a compactly generated group that is quasi-isometric to a group satisfying a law. If G is not a hyperbolic group then G does not have cut-points in any of its asymptotic cones.*

In particular since connected-by-compact locally compact groups, or compactly generated linear algebraic groups over an ultrametric local field of characteristic zero, are quasi-isometric to a solvable group, we deduce the following result.

Corollary 2.35. *Let G be a locally compact compactly generated group. Assume that G is either connected-by-compact, or a linear algebraic group over an ultrametric local field of characteristic zero. If G is not a hyperbolic group then G does not have cut-points in any of its asymptotic cones.*

Note that by Corollary 3 of [CT11], we have a complete description of connected Lie groups or linear algebraic groups over a non-Archimedean local field of characteristic zero that are non-elementary hyperbolic. For example in the case of a connected Lie group, G is either isomorphic to a semidirect product $N \rtimes (K \times \mathbb{R})$, where N is a simply connected nilpotent Lie group, K is a compact connected Lie group and the action of $\mathbb{R}$ on N is contracting; or the quotient of G by its maximal compact normal subgroup is isomorphic to a rank one simple Lie group with trivial center. So it follows from Corollary 2.35 that if a connected Lie group is not of this form, then it does not have cut-points in any of its asymptotic cones.

Remark 2.36. Here is another proof of Corollary 2.35 when G is a connected-by-compact locally compact group. Argue by contradiction and assume that G admits one asymptotic cone $\mathcal{C}$ with cut-points. Since G is quasi-isometric to a solvable group, according to Proposition 2.32 the asymptotic cone $\mathcal{C}$ must be a real tree. Now since connected-by-compact groups are compactly presented (see for example [CH15, Proposition 8.A.19]), the group G must be hyperbolic by Corollary 1.37. Contradiction.

Chapter 3

Neretin's group and its generalizations

The group $\mathcal{N}_d$ of almost automorphisms of a non-rooted regular tree of degree $d+1 \geq 3$ was introduced by Neretin in connection with his work in representation theory [Ner92]. He proved that $\mathcal{N}_d$ can be seen as a p-adic analogue of the diffeomorphism group of the circle, in the sense that some of the features of the representation theory of the latter are inherited by $\mathcal{N}_d$. Inspired by a simplicity result of the diffeomorphism group of the circle $\mathrm{Diff}^+(\mathbb{S}^1)$ [Her71], Kapoudjian later proved that the group $\mathcal{N}_d$ is abstractly simple [Kap99].

Recently, Bader, Caprace, Gelander and Mozes proved that $\mathcal{N}_d$ does not have any lattice [BCGM12]. This result is remarkable for the reason that all the familiar examples of simple locally compact groups (which are unimodular), e.g. real or p-adic Lie-groups, or the group of type preserving automorphisms of a locally finite regular tree, are known to have lattices. Actually $\mathcal{N}_d$ turned out to be the first example of a locally compact simple group without lattices.

In this chapter we investigate a large family of groups which appear as generalizations of Neretin's group. In Section 3.1 we set some terminology, recall some background on Higman-Thompson groups, and define the groups $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ and their topology. Section 3.2 deals with the study of compact presentability and the Dehn function of these groups, as well as other examples of groups of almost automorphisms. Finally in Section 3.3 we study metric properties and embeddings of these groups.

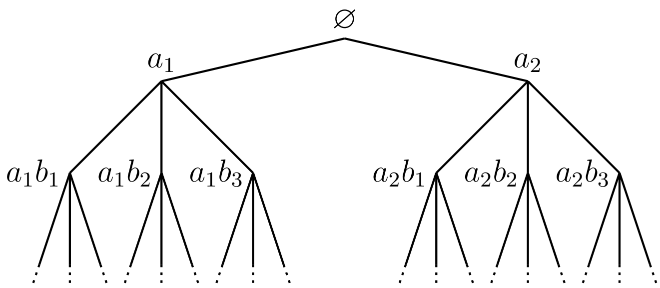


Figure 3.1 – A picture of the tree $\mathcal{T}_{3,2}$.

3.1 Introduction

3.1.1 Definitions

The quasi-regular rooted tree $\mathcal{T}_{d,k}$ and its boundary. Let A and B be finite sets of cardinality respectively $k \geq 2$ and $d \geq 2$. Consider the set of finite words $\{\varnothing\} \cup \{ab_1 \cdots b_n : a \in A, b_i \in B\}$ over the alphabet $X = A \cup B$ being either empty or beginning by an element of A. This set is naturally the vertex set of a rooted tree, where the root is the empty word $\varnothing$ and two vertices are adjacent if they are of the form v and vx, $x \in X$. We will denote this tree by $\mathcal{T}_{d,k}$. In the case when $k = d$ it will be denoted by $\mathcal{T}_d$ for simplicity. See Figure 3.1.1 for the case $k = 2$, $d = 3$.

For any vertex v, we will also denote by $\mathcal{T}_{d,k}^v$ the subtree of $\mathcal{T}_{d,k}$ spanned by vertices having v as a prefix. The distance between a vertex and the root will be called its level, and the number of its neighbours will be called its degree. If v is a vertex of level $n \geq 0$, then its neighbours of level $n+1$ are called the descendants of v. By construction, the root of $\mathcal{T}_{d,k}$ has degree k, and a vertex of level $n \geq 1$ has degree $d+1$: it has one distinguished neighbour pointing toward the root, and d descendants. Any subtree of $\mathcal{T}_{d,k}$ spanned by a vertex of level $n \geq 1$ and its d neighbours of level $n+1$ will be called a caret.

The boundary $\partial_\infty \mathcal{T}_{d,k}$ of the tree $\mathcal{T}_{d,k}$ is defined as the set of infinite words $ab_1 \cdots b_n \cdots$, i.e. infinite geodesic rays in $\mathcal{T}_{d,k}$ started at the root. We define the distance between two such words ξ, ξ' by $d(\xi, \xi') = d^{-|\xi \wedge \xi'|}$, where $|\xi \wedge \xi'|$ is the length of the longest common prefix of ξ and ξ'. Equipped with this distance, the boundary at infinity $\partial_\infty \mathcal{T}_{d,k}$ turns out to be homeomorphic to the Cantor set.

From now and for the rest of this chapter, we fix an embedding of $\mathcal{T}_{d,k}$ in the oriented plane. This embedding induces a canonical way of ordering, say from left to right, the descendants of any vertex. In particular we obtain a total ordering on the boundary at infinity $\partial_\infty \mathcal{T}_{d,k}$, defined by declaring that

$\xi \leq \xi'$ if the first letter of ξ following the longest common prefix of ξ and ξ', is smaller than the one of ξ'.

The group $\mathrm{AAut}(\mathcal{T}_{d,k})$ **of almost automorphisms of** $\mathcal{T}_{d,k}$**.** Recall that the group $\mathrm{Aut}(\mathcal{T}_{d,k})$ of automorphisms of the rooted tree $\mathcal{T}_{d,k}$ is defined as the group of bijections of the set of vertices fixing the root and preserving the edges. In particular every automorphism of $\mathcal{T}_{d,k}$ induces a homeomorphism of $\partial_\infty \mathcal{T}_{d,k}$. We now introduce a larger subgroup of the homeomorphism group of $\partial_\infty \mathcal{T}_{d,k}$, namely the group of homeomorphisms of $\partial_\infty \mathcal{T}_{d,k}$ which are piecewise tree automorphisms.

Definition 3.1. A finite subtree T of $\mathcal{T}_{d,k}$ is a rooted complete subtree if it contains the root as a vertex of degree k and if any other vertex which is not a leaf has degree $d+1$.

If T is a finite rooted complete subtree of $\mathcal{T}_{d,k}$ then its complement is a forest composed of finitely many copies of the tree $\mathcal{T}_d$. If T, T' are subtrees of $\mathcal{T}_{d,k}$, a map $\psi : \mathcal{T}_{d,k} \setminus T \to \mathcal{T}_{d,k} \setminus T'$ will be called a forest isomorphism if it maps each connected component of $\mathcal{T}_{d,k} \setminus T$ to a connected component of $\mathcal{T}_{d,k} \setminus T'$, and induces a tree isomorphism on each of these connected components. Note that such a forest isomorphism naturally induces a homeomorphism of $\partial_\infty \mathcal{T}_{d,k}$.

Definition 3.2. The group $\mathrm{AAut}(\mathcal{T}_{d,k})$ is defined as the set of equivalence classes of triples (ψ, T, T'), where T, T' are finite rooted complete subtrees such that $|\partial T| = |\partial T'|$ and $\psi : \mathcal{T}_{d,k} \setminus T \to \mathcal{T}_{d,k} \setminus T'$ is a forest isomorphism, where two triples are said to be equivalent if they give rise to the same homeomorphism of $\partial_\infty \mathcal{T}_{d,k}$. The multiplication in $\mathrm{AAut}(\mathcal{T}_{d,k})$ is inherited from the composition in $\mathrm{Homeo}(\partial_\infty \mathcal{T}_{d,k})$.

We mention the following result, whose proof is easy and left to the reader, which gives an alternative definition of the group of almost automorphisms $\mathrm{AAut}(\mathcal{T}_{d,k})$.

Lemma 3.3. *Let T_1, T_1', T_2, T_2' be finite complete rooted subtrees of $\mathcal{T}_{d,k}$. Then two triples (ψ_1, T_1, T_1'), (ψ_2, T_2, T_2') are equivalent if and only if there exist finite rooted complete subtrees T, T' so that T (resp. T') contains both T_1 and T_2 (resp. T_1' and T_2') and $\psi_1, \psi_2 : \mathcal{T}_{d,k} \setminus T \to \mathcal{T}_{d,k} \setminus T'$ are equal.*

Remark 3.4. By the previous lemma, when considering a triple (ψ, T, T') representing an element of $\mathrm{AAut}(\mathcal{T}_{d,k})$, we can always assume that T and T' both contain a given finite subtree of $\mathcal{T}_{d,k}$.

Note that since the only automorphism of $\mathcal{T}_{d,k}$ inducing the trivial homeomorphism on $\partial_\infty \mathcal{T}_{d,k}$ is the identity, the group $\mathrm{AAut}(\mathcal{T}_{d,k})$ contains a copy of the group $\mathrm{Aut}(\mathcal{T}_{d,k})$ of automorphisms of the tree $\mathcal{T}_{d,k}$.

3.1.2 Higman-Thompson groups

History. R. Thompson introduced in 1965 three groups $F \leq T \leq V$, an introduction to which can be found in [CFP96], while constructing a finitely generated group with unsolvable word problem. The groups T and V turned out to be the first examples of finitely presented infinite simple groups. Higman then generalized Thompson group V to an infinite family of groups (which were originally denoted by $G_{d,k}$, but we will use the notation $V_{d,k}$ to keep in mind the analogy with Thompson group V, which is nothing else than $V_{2,1}$). The proof of the following result, which will be used in 3.2.2, is due to Higman [Hig74] (see also [Bro87]).

Theorem 3.5. *Higman-Thompson groups $V_{d,k}$ are finitely presented.*

K. Brown later generalized Higman's construction to an infinite family of groups $F_{d,k} \leq T_{d,k} \leq V_{d,k}$ such that $F_{2,1} \simeq F$ and $T_{2,1} \simeq T$. These groups were originally defined as automorphism groups of certain free algebras. We refer the reader to [Bro87] for an introduction from this point of view.

The definition of the groups $V_{d,k}$ we give below is in terms of homeomorphism groups of the boundary of the quasi-regular rooted tree $\mathcal{T}_{d,k}$. From this point of view, elements of these groups can be represented either as homeomorphisms of $\partial_\infty \mathcal{T}_{d,k}$ or by combinatorial diagrams, and we will use the interplay between these two representations.

Higman-Thompson groups $V_{d,k}$ as subgroups of $\mathrm{AAut}(\mathcal{T}_{d,k})$. The point of view which is adopted here to define Higman-Thompson groups $V_{d,k}$ is mostly borrowed from [CDM11]. Note that when $k = d$ the group $V_{d,d}$ will be denoted by V_d for simplicity.

Definition 3.6. An element of $\mathrm{AAut}(\mathcal{T}_{d,k})$ is called locally order-preserving if it can be represented by a triple (ψ, T, T') such that T, T' are complete rooted subtrees of $\mathcal{T}_{d,k}$ and $\psi : \mathcal{T}_{d,k} \setminus T \to \mathcal{T}_{d,k} \setminus T'$ preserves the order of the boundary at infinity on each connected component.

It follows from the order-preserving condition that such a forest isomorphism ψ is uniquely determined by the induced bijection between the leaves of T and the leaves of T'. Locally order-preserving almost automorphisms are easily checked to form a subgroup of $\mathrm{AAut}(\mathcal{T}_{d,k})$.

Definition 3.7. The Higman-Thompson group $V_{d,k}$ is defined as the subgroup of $\mathrm{AAut}(\mathcal{T}_{d,k})$ of locally order-preserving elements.

Every locally order-preserving $v \in V_{d,k}$ has a unique representative (ψ, T, T') so that T, T' are complete rooted subtrees of $\mathcal{T}_{d,k}$, the map ψ : $\mathcal{T}_{d,k} \setminus T \to \mathcal{T}_{d,k} \setminus T'$ preserves the order of the boundary on each connected component, and T is minimal for the inclusion. Then T' is also minimal and (ψ, T, T') will be called the canonical representative of v. This notion coincides with the classical notion of reduced tree pair diagrams commonly used to study Thompson groups. The tree T will be called the domain tree of v and T' the range tree. When considering a triple representing a locally order-preserving element, we will without further mention assume that this is the canonical representative.

Saturated subsets. We now introduce a notion of saturated subsets inside the group $V_{d,k}$, which will be used in the proof of Theorem 3.17 to perform the cost estimates carefully.

Definition 3.8. A subset $\Sigma \leq V_{d,k}$ is said to be saturated if for every $\sigma = (\psi, T, T') \in \Sigma$ and every $u \in \mathrm{Aut}(\mathcal{T}_{d,k})$, all the elements of $V_{d,k}$ of the form $(\psi', u(T), T')$ belong to Σ.

Lemma 3.9. *Every finite subset $\Sigma \leq V_{d,k}$ is contained in a finite saturated subset.*

Proof. Let Σ' be the subset of $V_{d,k}$ consisting of elements of the form $(\psi, u(T), T')$, where $u \in \mathrm{Aut}(\mathcal{T}_{d,k})$ and T is the domain tree of the canonical representative of some element of Σ. Clearly Σ' contains Σ and is saturated. Since Σ is finite, the number of such trees T is finite, and so is the set of $u(T)$, $u \in \mathrm{Aut}(\mathcal{T}_{d,k})$. The result then follows from the following observation: if T is a fixed finite complete rooted subtree of $\mathcal{T}_{d,k}$, then there are only finitely many elements of $V_{d,k}$ having a canonical representative of the form (ψ, T, T'). $\square$

A lower bound for the word metric in $V_{d,k}$. Here we give a lower bound for the word metric in the group $V_{d,k}$ in terms of a combinatorial data contained in the diagrams (ψ, T, T') representing elements of $V_{d,k}$.

Recall that a caret in $\mathcal{T}_{d,k}$ is a subtree spanned by a vertex of level $n \geq 1$ and its d neighbours of level $n+1$. We insist on the fact that we do not consider the subtree of $\mathcal{T}_{d,k}$ spanned by the root and its k neighbours as a caret. If T is a finite complete rooted subtree of $\mathcal{T}_{d,k}$ with κ carets, then the number of leaves of T is $(d-1)\kappa + k$. In particular if $v \in V_{d,k}$ has canonical representative (ψ, T, T'), then T, T' have the same number of leaves, and consequently they also have the same number of carets. By abuse we will call it the number of carets of v and denote it by $\kappa(v)$.

Metric properties of Higman-Thompson groups of type F and T can be essentially understood in terms of the number of carets of tree diagrams, the latter being quasi-isometric to the word-length associated to some finite generating set. The use of this point of view shed light on some interesting large scale geometric properties of these groups (see [Bur99], [BCS01], [BCST09]). However metric properties of Higman-Thompson groups of type V are far less well understood, as it follows from the work of Birget [Bir04] that the number of carets is no longer quasi-isometric the the word-length in Thompson group V.

Nevertheless, the following lemma gives a lower bound for the word metric in $V_{d,k}$ in terms of the number of carets. Note that the same result appears in [Bir04] for the case of Thompson group V.

Proposition 3.10. *For any finite generating set Σ of $V_{d,k}$, there exists a constant $C_\Sigma > 0$ such that for any $v \in V_{d,k}$, we have $\kappa(v) \leq C_\Sigma |v|_\Sigma$.*

Proof. Define $C_\Sigma = \max_{\sigma\in\Sigma} \kappa(\sigma)$. Now remark that when multiplying, say on the right, an element $v \in V_{d,k}$ by an element $\sigma \in \Sigma$, we obtain an element $v\sigma$ having a canonical representative with trees having at most $\kappa(v) + C_\Sigma$ carets. This is because when expanding the domain tree of v to get a common expansion with the range tree of σ, we have to add at most C_Σ carets. So it follows from a straightforward induction that every element of length at most n with respect to the word metric associated to Σ has a canonical representative with at most $C_\Sigma n$ carets, and the proof is complete. □

3.1.3 Generalizations of Neretin's group

Almost automorphisms of $\mathcal{T}_{d,k}$ are homeomorphisms of the boundary $\partial_\infty \mathcal{T}_{d,k}$ which are piecewise tree automorphisms. In this subsection we introduce a family of subgroups of $\mathrm{AAut}(\mathcal{T}_{d,k})$ consisting of almost automorphisms whose local action satisfies a rigidity condition.

Almost automorphisms of type $W(D)$. Let $D \leq \mathrm{Sym}(d)$ be a subgroup of the symmetric group on d elements. Recall that D is known to give rise to a closed subgroup of the automorphism group $\mathrm{Aut}(\mathcal{T}_d)$, by considering the infinitely iterated permutational wreath product $W(D) = (\ldots \wr D) \wr D$. More precisely, define recursively $D_1 = D$, seen as the subgroup of $\mathrm{Aut}(\mathcal{T}_d)$ acting on level one; and $D_{n+1} = D \wr D_n$ for every $n \geq 1$, where the permutational wreath product is associated with the natural action of D_n on the set of vertices of level n of $\mathcal{T}_d$. We now let $W(D)$ be the closed subgroup generated by the family (D_n). Elements of $W(D)$ are rooted automorphisms whose local action is prescribed by D. For any $k \geq 2$ we let $W_k(D)$ be the subgroup of $\mathrm{Aut}(\mathcal{T}_{d,k})$ fixing pointwise the first level and acting by an element of $W(D)$ in each subtree rooted at level one. The group $W_k(D)$ is naturally isomorphic to the product of k copies of the group $W(D)$.

Definition 3.11. An almost automorphism of $\mathcal{T}_{d,k}$ is said to be piecewise of type $W(D)$ if it can be represented by a triple (ψ, T, T') such that T, T' are finite rooted complete subtrees of $\mathcal{T}_{d,k}$ and $\psi : \mathcal{T}_{d,k} \setminus T \to \mathcal{T}_{d,k} \setminus T'$ belongs to $W(D)$ on each connected component, after the natural identification of each connected component of $\mathcal{T}_{d,k} \setminus T$ and $\mathcal{T}_{d,k} \setminus T'$ with $\mathcal{T}_d$.

We observe that by construction of $W(D)$, if a triple (ψ_1, T_1, T_1') is such that $\psi_1 : \mathcal{T}_{d,k} \setminus T_1 \to \mathcal{T}_{d,k} \setminus T_1'$ belongs to $W(D)$ on each connected component, then for any equivalent triple (ψ_2, T_2, T_2') such that T_2 (resp. T_2') contains T_1 (resp. T_1'), then $\psi_2 : \mathcal{T}_{d,k} \setminus T_2 \to \mathcal{T}_{d,k} \setminus T_2'$ belongs to $W(D)$ on each connected component.

Proposition 3.12. *The set of almost automorphisms* $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ *which are piecewise of type* $W(D)$ *is a subgroup of* $\mathrm{AAut}(\mathcal{T}_{d,k})$*.*

Proof. The only non-trivial fact that one needs to check is that $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is closed under multiplication, but this follows from the previous observation and from the fact that $W(D)$ is a subgroup of $\mathrm{Aut}(\mathcal{T}_d)$. □

Roughly, the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ consists in homeomorphisms of $\partial_\infty \mathcal{T}_{d,k}$ that are piecewise tree automorphisms whose local action is prescribed by D. This family of groups generalizes Neretin's groups because when D is the full permutation group $\mathrm{Sym}(d)$, the group $W(D)$ is the full automorphism group of $\mathcal{T}_d$, and we can check that $\mathrm{AAut}_{\mathrm{Sym}(d)}(\mathcal{T}_{d,2}) \simeq \mathcal{N}_d$. On the opposite, if D is the trivial group, then being piecewise trivial means being locally order-preserving and $\mathrm{AAut}_D(\mathcal{T}_{d,k}) = V_{d,k}$. It is straightforward from the definition that if D' contains D then $\mathrm{AAut}_{D'}(\mathcal{T}_{d,k})$ contains $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. In particular we note that for every subgroup $D \leq \mathrm{Sym}(d)$, the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ always contains $V_{d,k}$.

The groups $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ appear in [CDM11], where a careful study of the abstract commensurator group of self-replicating profinite wreath branch groups is carried out (we refer to [BEW11] for an introduction to abstract commensurators of profinite groups). Under the additional assumption that $D \leq \mathrm{Sym}(d)$ is transitive and is equal to its normaliser in $\mathrm{Sym}(d)$, the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ turns out to be isomorphic to the abstract commensurator group of $W_k(D)$. In particular Neretin's group $\mathcal{N}_d$ is the abstract commensurator group of $W_2(\mathrm{Sym}(d))$, or equivalently the group of germs of automorphisms of $\mathrm{Aut}(T_{d+1})$ in the language of [CDM11].

Topology on $\mathrm{AAut}_D(\mathcal{T}_{d,k})$**.** By definition the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ also contains a copy of the tree automorphism group $W_k(D)$. The latter comes equipped with a natural group topology, which is totally disconnected and compact, defined by saying that the pointwise stabilizers of vertices of level n form a basis of neighbourhoods of the identity. We would like to extend this topology to the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, i.e. define a group topology on $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ for which the subgroup $W_k(D)$ is an open subgroup. For, let us first recall the following well known lemma, a proof of which can be consulted in [Bou71, Chapter 3].

Lemma 3.13. *Let G be a group and let $\mathcal{F}$ be a family of subgroups of G which is filtering, i.e. so that the intersection of any two elements of $\mathcal{F}$ contains an element of $\mathcal{F}$. Assume moreover that for every $g \in G$ and every $U \in \mathcal{F}$, there exists $V \in \mathcal{F}$ so that $V \subset gUg^{-1}$. Then there exists a (unique) group topology on G for which $\mathcal{F}$ is a base of neighbourhoods of the identity.*

Recall that a subgroup H of a group G is said to be commensurated by a subset K of G if for every $k \in K$, the subgroup $kHk^{-1} \cap H$ has finite index in both H and kHk^{-1}. The following easy lemma, whose proof is left to the reader, provides an easy way to check commensurability.

Lemma 3.14. *Let G be a group and S a generating set of G. Then a subgroup H of G is commensurated by G if and only if it is commensurated by S.*

Now let $\mathcal{F}$ be the family of open subgroups of $W_k(D)$, which is a base of neighbourhoods of the identity in $W_k(D)$. It follows from Lemma 3.13 that if G is a group containing $W_k(D)$ as a subgroup, then there exists a group topology on G such that the inclusion of $W_k(D)$ in G is continuous and open as soon as $W_k(D)$ is commensurated in G.

Now remark that Lemma 3.14 together with Proposition 3.20 (a corollary of which is that $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is generated by $W_k(D)$ and $V_{d,k}$) imply that $W_k(D)$ is commensurated by $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, because it is trivially commensurated by itself and commensurated by $V_{d,k}$ by Lemma 3.19. We therefore obtain:

Proposition 3.15. *There exists a (unique) group topology on $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ turning $W_k(D)$ into a compact open subgroup. In particular $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is a t.d.l.c. group (which is discrete if and only if $W_k(D)$ is trivial, if and only if D is trivial).*

Remark 3.16. 1) It is interesting to point out that whereas the topology on $\mathrm{Aut}(\mathcal{T}_{d,k})$ coincides with the compact-open topology induced from $\mathrm{Homeo}(\partial_\infty \mathcal{T}_{d,k})$, this is no longer true for the group $\mathrm{AAut}(\mathcal{T}_{d,k})$. Indeed, the inclusion $\mathrm{AAut}(\mathcal{T}_{d,k}) \hookrightarrow \mathrm{Homeo}(\partial_\infty \mathcal{T}_{d,k})$ is continuous but has a non-closed image. In other words, the topology on $\mathrm{AAut}(\mathcal{T}_{d,k})$ is strictly finer than the compact-open topology. Actually the image of $\mathrm{AAut}(\mathcal{T}_{d,k}) \hookrightarrow \mathrm{Homeo}(\partial_\infty \mathcal{T}_{d,k})$ is even dense, because one can check that the group $V_{d,k}$ is a dense subgroup of the homeomorphism group of $\partial_\infty \mathcal{T}_{d,k}$ with respect to the compact-open topology.

2) We also insist on the fact that for any permutation group D, the inclusion $\mathrm{AAut}_D(\mathcal{T}_{d,k}) \hookrightarrow \mathrm{AAut}(\mathcal{T}_{d,k})$ is always continuous, but its image is never closed unless D is the full permutation group $\mathrm{Sym}(d)$. Indeed, $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ contains the subgroup $V_{d,k}$ which is dense in $\mathrm{AAut}(\mathcal{T}_{d,k})$ by Remark 3.21, and therefore $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is never closed inside $\mathrm{AAut}(\mathcal{T}_{d,k})$ unless it is the whole group.

3.2 Compact presentability and Dehn function

3.2.1 Background

Recall that a locally compact group G is said to be compactly presented if it admits a compact generating subset S such that G has a presentation, as an abstract group, with S as set of generators and relators of bounded length (but possibly infinitely many relators). When the group G is discrete, this amounts to saying that G is finitely presented, and like in the discrete case, for a locally compact group, being compactly presented does not depend on the choice of the compact generating set S.

Compact presentability can be interpreted in terms of coarse simple connectedness of the Cayley graph of the group with respect to some compact generating subset. In particular, among compactly generated locally compact groups, being compactly presented is preserved by quasi-isometries. For a proof of this result see for instance [CH15, Proposition 8.A.3].

Having obtained compact presentability of a locally compact group G naturally leads to the study of an invariant of G, having both geometric and combinatorial flavors, called the Dehn function of G.

From the geometric point of view, the Dehn function $\delta_G(n)$ is the supremum of areas of loops in G of length at most n. In other words, it is the best isoperimetric function, where isoperimetric function can be understood like for simply connected Riemannian manifolds.

From the combinatorial perspective, the Dehn function is a quantified version of compact presentability: $\delta_G(n)$ is the supremum over all relations w of length at most n in the group, of the minimal number of relators needed to convert w to the trivial word.

If G is compactly presented and if S is a compact generating set, then for some $k \geq 1$ the group G has the presentation $\langle S \mid R_k \rangle$, where R_k is set of relations in G of length at most k. The area $a(w)$ of a relation w, i.e. a word in the letters of S which represents the identity in G, is the smallest integer m so that w can be written in the free group F_S as a product of m conjugates of relators of R_k. Now define the Dehn function of G by

$$\delta_G(n) = \sup \{a(w) : w \text{ relation of length at most } n\}.$$

This function depends on the choice of S and k, but its asymptotic behavior does not, and is actually a quasi-isometry invariant of G.

3.2.2 Presentation of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$

Statement of the theorem. One of the reasons why combinatorial group theorists became interested in Thompson groups is because of the combination of simplicity and finiteness properties. While simplicity results for $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ have recently been obtained in [CDM11], here we settle in the positive the question if whether or not these groups satisfy the locally compact version of being finitely presented, i.e. being compactly presented, and give the following upper bound on their Dehn function.

Theorem 3.17. *For any $k, d \geq 2$, and any subgroup $D \leq \mathrm{Sym}(d)$, the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is compactly presented, and the Dehn function of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is asymptotically bounded by that of $V_{d,k}$.*

As mentioned earlier, the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ contains a dense copy of the Higman-Thompson finitely presented group $V_{d,k}$. Here we insist on the fact that for a locally compact group, although having a dense finitely generated subgroup is a sufficient condition for being compactly generated, this does not hold for compact presentation, i.e. having a dense finitely presented subgroup does not imply compact presentation of the ambient group. For example, for any non-Archimedean local field $\mathbb{K}$, the group $\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K})$ has a central extension with non-compactly generated kernel, and is therefore not compactly presented (see for instance [CH15, Proposition 8.A.26]). However the reader can check that this group admits dense finitely generated free subgroups.

We also emphasize the fact that for the case of Neretin's group, Theorem 3.17 cannot be obtained by proving finite presentation of a discrete cocompact subgroup because these do not exist [BCGM12]. However we note that it seems to be unknown whether Neretin's group $\mathcal{N}_d$ is quasi-isometric to a finitely generated group.

As a by-product of Theorem 3.17 and the main result of [BCGM12], we also obtain that locally compact simple groups without lattices also exist in the realm of compactly presented groups.

On the other hand, the Dehn function of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is not linear because having a linear Dehn function characterizes Gromov-hyperbolic groups among compactly presented groups, and the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is easily seen not to be Gromov-hyperbolic. So by a general argument (see for example [Bow91]), the Dehn function of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ has a quadratic lower bound.

In the case $d = 2$, all the groups $V_{2,k}$ turn out to be isomorphic to Thompson group V. While the Dehn function of Thompson group F has been proved to be quadratic [Gub06], it is not known whether the Dehn function of V is quadratic or not. However, using a result of Guba [Gub00] who showed the upper bound $\delta_V \preccurlyeq n^{11}$, we obtain:

Corollary 3.18. *Neretin's group $\mathcal{N}_2$ has a polynomially bounded Dehn function ($\preccurlyeq n^{11}$).*

We believe that the result of Guba could be extended to the family of groups $V_{d,k}$, i.e. that every group $V_{d,k}$ satisfies a polynomial isoperimetric inequality. By Theorem 3.17 this would imply that the Dehn function of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is polynomially bounded for arbitrary $k, d \geq 2$ and $D \leq \mathrm{Sym}(d)$.

Preliminary results. In this subsection we establish preliminary results about the groups $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. Recall that $D \leq \mathrm{Sym}(d)$ is a finite permutation group, and we define recursively a family of finite subgroups of $\mathrm{Aut}(\mathcal{T}_d)$ by $D_1 = D$ and $D_{n+1} = D \wr D_n$ for every $n \geq 1$, where the permutational wreath product is associated with the natural action of D_n on the d^n vertices of level n of $\mathcal{T}_d$. We denote by D_∞ the subgroup generated by the family (D_n) (which also coincides with the increasing union of the family (D_n)) and by $W(D)$ the closure of D_∞ in $\mathrm{Aut}(\mathcal{T}_d)$. For $n \in \{1, \ldots, \infty\}$ we will also denote by D_n^k the subgroup of $W_k(D)$ fixing pointwise the first level of $\mathcal{T}_{d,k}$ and acting by an element of D_n on each subtree rooted at the first level. Note that these groups have a natural decomposition $D_n^k = D_n^{(1)} \times \ldots \times D_n^{(k)}$, where $D_n^{(i)}$ is the subgroup of elements acting only on the ith subtree rooted at level one.

If $\sigma = (\psi, T, T') \in V_{d,k}$, we let $W_k(D)_\sigma$ be the subgroup of $W_k(D)$ consisting of automorphisms which are the identity on the subtree T. Note $W_k(D)_\sigma$ always contains some neighbourhood of the identity and is consequently an open subgroup of $W_k(D)$. The latter being compact, we obtain that $W_k(D)_\sigma$ is a finite index subgroup of $W_k(D)$.

Lemma 3.19. *For every $\sigma \in V_{d,k}$, we have the inclusion*

$$\sigma\, W_k(D)_\sigma\, \sigma^{-1} \subset W_k(D),$$

and $\sigma\, W_k(D)_\sigma\, \sigma^{-1}$ is an open subgroup of $W_k(D)$. In particular $W_k(D)$ is commensurated by $V_{d,k}$.

Proof. If $\sigma = (\psi, T, T')$ and $u \in W_k(D)_\sigma$, the reader will easily check that the element $\sigma u \sigma^{-1} \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$ is represented by a triple (ψ', T', T'), where ψ' permutes trivially the connected components of $\mathcal{T}_{d,k} \setminus T'$. Now if we consider the tree automorphism $u' \in W_k(D)$ being the identity on T' and acting on $\mathcal{T}_{d,k} \setminus T'$ like ψ', it is clear that u' is represented by the triple (ψ', T', T'), and therefore $\sigma u \sigma^{-1} = u' \in W_k(D)$. □

The next result yields a decomposition of the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ in terms of the two subgroups $W_k(D)$ and $V_{d,k}$. It will be essential for proving Theorem 3.17.

Proposition 3.20. *For any $g \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$ there exists $(u, v) \in W_k(D) \times V_{d,k}$ such that $g = uv$.*

Proof. Let (ψ, T, T') be a triple representing $g \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$. Let us consider the element $v \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$ represented by the triple (ξ, T, T') where ξ is defined by declaring that each tree of the forest $\mathcal{T}_{d,k} \setminus T$ is globally sent on its image by ψ, but so that ξ is order-preserving on each connected component of $\mathcal{T}_{d,k} \setminus T$. Clearly we have $v \in V_{d,k}$. Now the default between g and v can be filled by performing the rooted tree automorphism induced by g on each subtree rooted at a leaf of T'. But all of these can be achieved at the same time by an element of $W_k(D)$, namely the automorphism being the identity on T' and acting as the desired rooted tree automorphism on each connected component of $\mathcal{T}_{d,k} \setminus T'$. □

Remark 3.21. Actually in Proposition 3.20, $W_k(D)$ can be replaced by the pointwise stabilizer of the nth level of $\mathcal{T}_{d,k}$ in $W_k(D)$, for every $n \geq 1$, the proof being the same. It yields in particular that $V_{d,k}$ is a dense subgroup of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$.

Now given $g \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$, there is not a unique $(u, v) \in W_k(D) \times V_{d,k}$ such that $g = uv$ because the two subgroups $W_k(D)$ and $V_{d,k}$ have a non-trivial intersection (as soon as D is non-trivial). The measure of how this decomposition fails to be unique naturally leads to the study of the intersection of these two subgroups.

Lemma 3.22. *The intersection between $V_{d,k}$ and $W_k(D)$ in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is D_∞^k.*

Proof. D_n^k lies inside $V_{d,k}$ and $W_k(D)$ for any $n \geq 1$, so the inclusion $D_\infty^k \subset V_{d,k} \cap W_k(D)$ is clear. To prove the reverse inclusion, let g be an element of $V_{d,k} \cap W_k(D)$. Such an element g is an automorphism of $\mathcal{T}_{d,k}$ and therefore does act on the tree fixing setwise each level, so it is enough to prove that there exists an element of D_∞^k acting like g on $\mathcal{T}_{d,k}$. Since $g \in W_k(D)$, for every $n \geq 1$ there exists $g_n \in D_n^k$ acting like g on the first n levels of $\mathcal{T}_{d,k}$. But now since $g \in V_{d,k}$, it is eventually order-preserving and therefore $g = g_n$ for n large enough, which completes the proof. □

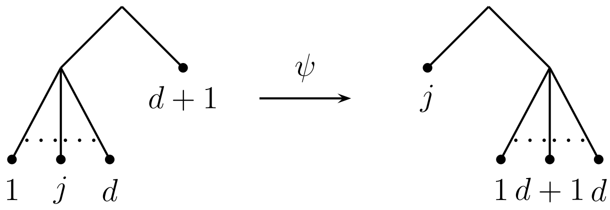


Figure 3.2 – The diagram of $\delta_{1,j}$ when $k = 2$.

The end of this paragraph is devoted to establishing Lemma 3.23, which will be applied in the proof of Lemma 3.26. Roughly, the idea is to find a finite set of elements $\Delta \leq V_{d,k}$, so that given an element $u \in D_\infty^{(i)}$, we can find $\delta \in \Delta$ so that conjugating by δ increases by one the level of the action of u.

If $i = 1 \dots k$, recall that $\mathcal{T}_{d,k}^{a_i}$ denotes the full subtree of $\mathcal{T}_{d,k}$ rooted at a_i, and that $(a_1, \dots a_k)$ (resp. $(a_i b_1, \dots, a_i b_d)$) denotes the ordered vertices of level one of $\mathcal{T}_{d,k}$ (resp. $\mathcal{T}_{d,k}^{a_i}$). In what follows, by convention indexes will be taken modulo k (for example a_{k+1} will denote the vertex a_1).

For every $i = 1 \dots k$ and $j = 1 \dots d$, we define an element $\delta_{i,j} = (\psi, T, T') \in V_{d,k}$ by the following manner:

- T is the smallest finite complete rooted subtree containing the d descendants of a_i;
- T' is the smallest finite complete rooted subtree containing the d descendants of a_{i+1};
- ψ is defined by the formulas
  - $\psi(a_\ell) = a_\ell$ for every $\ell \notin \{i, i+1\}$;
  - $\psi(a_{i+1}) = a_{i+1} b_j$;
  - $\psi(a_i b_\ell) = a_{i+1} b_\ell$ for every $\ell \neq j$;
  - $\psi(a_i bj) = a_i$.

For example the diagram of $\delta_{1,j}$ is represented in Figure 3.2.2 in the case $k = 2$. We denote by Δ the set of $\delta_{i,j}$, for $i = 1 \dots k$, $j = 1 \dots d$.

Lemma 3.23. *For any $i = 1 \dots k$, $j = 1 \dots d$, any $n \geq 1$ and any element $u \in D_\infty^{(i)}$ being an automorphism of $\mathcal{T}_{d,k}^{a_i b_j}$ with at most $n+1$ carets, the element $\delta_{i,j} u \delta_{i,j}^{-1}$ is an automorphism of $\mathcal{T}_{d,k}^{a_i}$ and has at most n carets.*

Proof. This is a direct consequence of the fact that $\delta_{i,j}$ maps the subtree $\mathcal{T}_{d,k}^{a_i b_j}$ to the the subtree $\mathcal{T}_{d,k}^{a_i}$. $\square$

Proofs of the theorem. The remaining of this subsection consists in writing down an explicit presentation of the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ for any $k \geq 1, d \geq 2$ and $D \leq \mathrm{Sym}(d)$, and proving Theorem 3.17.

Let Σ denote a finite generating set of the group $V_{d,k}$, which is supposed to contain Δ. Enlarging Σ if necessary, we can also assume that Σ is saturated by Lemma 3.9. This implies the following:

Lemma 3.24. *We have the inclusion $\Sigma\, W_k(D) \subset W_k(D)\, \Sigma$.*

Proof. It follows from the proof of Proposition 3.20 that any $\sigma_1 u_1 \in \Sigma\, W_k(D)$ can be written $u_2\sigma_2$ with $u_2 \in W_k(D)$ and $\sigma_2 \in V_{d,k}$ being of the form $(\psi, u^{-1}(T), T')$, where T is the domain tree of σ. Since Σ is saturated, σ_2 belongs to Σ and therefore $\sigma_1 u_1 \in W_k(D)\, \Sigma$. $\square$

According to Proposition 3.20, the set $S = \Sigma \cup W_k(D)$ is a generating set of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. The strategy to prove Theorem 3.17 will be to list some particular relations between the elements of S satisfied in the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, and then to prove that they generate all the relations in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$.

(R_Σ) According to Theorem 3.5 there exists a finite set of words $R_\Sigma \leq \Sigma^*$ so that $\langle \Sigma \mid R_\Sigma \rangle$ is a presentation of $V_{d,k}$.

(R_D) We let R_D be the set of words of the form $u_1 u_2 u_3^{-1}$, $u_i \in W_k(D)$, whenever the relation $u_1 u_2 = u_3$ is satisfied in the group $W_k(D)$.

(R_1) The set of relations R_1 will correspond to commensurating relations in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. Recall that if $\sigma \in V_{d,k}$ and $u \in W_k(D)_\sigma$ then $\sigma u \sigma^{-1} \in W_k(D)$ by Lemma 3.19. We let R_1 be the set of words of the form $\sigma u_1 \sigma^{-1} u_2^{-1}$, where $\sigma \in \Sigma, u_1 \in W_k(D)_\sigma, u_2 \in W_k(D)$, whenever the relation $\sigma u_1 \sigma^{-1} = u_2$ holds in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$.

(R_2) We add relations corresponding to the fact that the subgroup D_1^k of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ lies in the intersection of $V_{d,k}$ and $W_k(D)$. More precisely, for every $i \in \{1, \ldots, k\}$ and every $u \in D_1^{(i)}$, we choose a word $w_u \in \Sigma^*$ so that $u = w_u$ in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. We denote by R_2 the set of words uw_u^{-1}, and by r_i the maximum word length of the words w_u when u ranges over $D_1^{(i)}$.

(R_3) By Lemma 3.24, for every $\sigma_1 \in \Sigma$ and $u_1 \in W_k(D)$ we can pick some $u_2 \in W_k(D)$ and $\sigma_2 \in \Sigma$ so that $\sigma_1 u_1 = u_2 \sigma_2$ in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. We denote by R_3 the set of words $\sigma_1 u_1 \sigma_2^{-1} u_2^{-1}$.

Denote by $R = R_\Sigma \cup R_D \cup_i R_i$ the union of all these relations. Note that elements of R have bounded length with respect to the compact generating set $S = \Sigma \cup W_k(D)$ of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$.

We let G be the group defined by the presentation $\langle S \mid R\rangle$, that is we have a short exact sequence

$$1 \to \mathcal{R} = \langle\langle R\rangle\rangle \to F_S \to G \to 1,$$

where F_S is the free group over the set S and $\langle\langle R\rangle\rangle$ is the normal subgroup generated by R. Denote by $a : F_S \to [0, +\infty]$ the corresponding area function, which by definition associates to $w \in \mathcal{R}$ the least integer n so that w is a product of at most n conjugates of elements of R, and $a(w) = +\infty$ if $w \notin \mathcal{R}$. We also define the associated cost function $c : F_S \times F_S \to [0, +\infty]$ by $c(w_1, w_2) = a(w_1^{-1}w_2)$. This function estimates the cost of converting w_1 to w_2, or the cost of going from w_1 to w_2, in the sense that $c(w_1, w_2)$ is the distance in F_S between w_1 and w_2 with respect to the word metric associated to the union of conjugates of R. In particular the cost function is symmetric and satisfies the triangular inequality $c(w_1, w_3) \leq c(w_1, w_2) + c(w_2, w_3)$ for every $w_1, w_2, w_3 \in F_S$. This, combined with the bi-invariance of the cost function, yields the following inequality, which will be used repeatedly: for every $\ell \geq 1$ and every $w_1, \ldots, w_\ell, w'_1, \ldots, w'_\ell \in F_S$, we have:

$$c(w_1 \ldots w_\ell, w'_1 \ldots w'_\ell) \leq \sum_{i=1}^{\ell} c(w_i, w'_i).$$

Two words $w_1, w_2 \in F_S$ are said to be homotopic if they represent the same element of G, i.e. if $c(w_1, w_2) < +\infty$. A word w is said to be null-homotopic if it represents the identity, i.e. if $w \in \mathcal{R}$.

We are now able to state the main theorem of this paragraph, which proves Theorem 3.17.

Theorem 3.25. *The natural map $G \to \mathrm{AAut}_D(\mathcal{T}_{d,k})$ is an isomorphism. Furthermore, the Dehn function of the presentation $\langle S \mid R\rangle$ is asymptotically bounded by that of $V_{d,k}$.*

It is clear that the map from G to $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is a well defined morphism because relations $R_\Sigma, R_D, (R_i)$ are satisfied in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, and it is onto because S generates the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. So proving the first claim comes down to proving that this morphism is injective, i.e. any word in F_S representing the identity in the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ already represents the trivial element in the group G. This will be achieved, as well as the proof of the upper bound on the Dehn function, in Proposition 3.29, using both geometric and combinatorial arguments.

The goal of Lemma 3.26 and Corollary 3.27 is to prove that relations in the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ coming from the fact that the subgroups $W_k(D)$ and $V_{d,k}$ intersect non-trivially, are already satisfied in the group G, and to obtain a precise estimate of their cost.

Lemma 3.26. *Fix $i \in \{1, \dots, k\}$ and let $C_i = 2d + \max(2, r_i)$ (recall that r_i has been defined with the set of relators R_2). Then for every $n \geq 0$ and every $u \in D_\infty^{(i)}$ having at most n carets, there exists a word $w \in \Sigma^*$ of length at most $C_i n$ so that the relation $u = w$ holds in G and has cost at most $C_i n$.*

Proof. We use induction on n. The result is trivially true for $n = 0$ because the only element of $D_\infty^{(i)}$ with zero caret is the identity, and is true for $n = 1$ thanks to the set of relators R_2.

The idea of the proof of the induction step is the following. Given $u \in D_\infty^{(i)}$ with at most $n+1$ carets, we begin by multiplying it by an element of $D_1^{(i)}$ in order to ensure that it acts trivially on the first level of $\mathcal{T}_{d,k}^{a_i}$. The resulting automorphism has a natural decomposition into a product of d elements of $D_\infty^{(i)}$, coming from its action on the subtrees $\mathcal{T}_{d,k}^{a_i b_1}, \dots, \mathcal{T}_{d,k}^{a_i b_d}$, with a nice control on the number of carets of each element of this product. We then apply the induction hypothesis to each of these elements, after having reduced their number of carets by conjugating by an element of Δ, which has the effect of increasing by 1 the level of the subtree on which they act.

Henceforth we assume that $u \in D_\infty^{(i)}$ is an element having at most $n + 1$ carets, with $n \geq 1$. If we let $\bar{u}$ denote the element of $D_1^{(i)}$ acting like u on the first level of $\mathcal{T}_{d,k}^{a_i}$, it is clear that $u' = u\bar{u}^{-1}$ stabilizes pointwise the first level of $\mathcal{T}_{d,k}^{a_i}$. Using relators from R_2, we pick a word $w_{\bar{u}} \in \Sigma^*$ so that the relation $\bar{u} = w_{\bar{u}}$ holds in G and has cost at most one.

Now in the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, since u' acts trivially on the first level of $\mathcal{T}_{d,k}^{a_i}$, it has a natural decomposition $u' = u_1 \dots u_d$, where each $u_\ell \in D_\infty^{(i)}$ acts on the subtree of $\mathcal{T}_{d,k}^{a_i b_\ell}$. Note that each u_ℓ has at most $n+1$ carets and that $\sum_\ell \kappa(u_\ell) \leq \kappa(u) + d - 1 \leq n + d$, because the caret corresponding to the root of $\mathcal{T}_{d,k}^{a_i}$ can appear d times in this sum, whereas it is counted only once in $\kappa(u)$. Note also that thanks to the set of relators R_D, the relation $u' = u_1 \dots u_d$ also holds in the group G and has cost at most d.

Remark that by construction of the set Δ, every element of $W_k(D)$ acting trivially on the second level of $\mathcal{T}_{d,k}$ lies inside $W_k(D)_\delta$ for every $\delta \in \Delta$. In particular if $\ell \in \{1, \dots, d\}$ and if $\delta_\ell = \delta_{i,\ell}$, we have $u_\ell \in W_k(D)_{\delta_\ell}$ and thanks to R_1, the word $\delta_\ell u_\ell \delta_\ell^{-1}$ represents in the group G an element $\tilde{u}_\ell \in D_\infty^{(i)}$ with at most $\kappa(u_\ell) - 1 \leq n$ carets according to Lemma 3.23. Note in particular that

$$\sum_\ell \kappa(\tilde{u}_\ell) \leq \sum_\ell (\kappa(u_\ell) - 1) \leq \sum_\ell \kappa(u_\ell) - d \leq n. \tag{3.1}$$

For every $\ell \in \{1, \dots, d\}$, we now apply the induction hypotheses to $\tilde{u}_\ell$ and obtain a word $\tilde{w}_\ell$ of length at most $C_i \kappa(\tilde{u}_\ell)$ so that $\tilde{u}_\ell = \tilde{w}_\ell$ in G and $c(\tilde{u}_\ell, \tilde{w}_\ell) \leq C_i \kappa(\tilde{u}_\ell)$. If we denote by $w_\ell = \delta_\ell^{-1} \tilde{w}_\ell \delta_\ell$, then the relation $u_\ell = w_\ell$

holds in the group G and has cost

$$c(u_\ell, w_\ell) \le c(u_\ell, \delta_\ell^{-1}\tilde{u}_\ell\delta_\ell) + c(\delta_\ell^{-1}\tilde{u}_\ell\delta_\ell, w_\ell) \le 1 + C_i\kappa(\tilde{u}_\ell).$$

We now want to put all these pieces together and conclude the proof of the induction step. For, let $w = w_1 \dots w_d w_{\bar{u}} \in \Sigma^*$. Its length easily satisfies

$$|w|_\Sigma \le \sum_{\ell=1}^{d} |w_\ell|_\Sigma + |w_{\bar{u}}|_\Sigma \le \sum_{\ell=1}^{d} (2+|\tilde{w}_\ell|_\Sigma) + r_i \le C_i \sum_{\ell=1}^{d} \kappa(\tilde{u}_\ell) + 2d + r_i \le C_i(n+1),$$

because $\sum_\ell \kappa(\tilde{u}_\ell) \le n$ according to (3.1), and $C_i \ge 2d + r_i$. Furthermore, we claim that the relation $u = w$ is satisfied in G and has cost at most $C_i(n+1)$, which follows from the following summation of cost estimates:

$$\begin{aligned}
c(u, w) &\le c(u, u'\bar{u}) + c(u'\bar{u}, w) \\
&\le 1 + c(u', w_1 \dots w_d) + c(\bar{u}, w_{\bar{u}}) \\
&\le 1 + c(u', u_1 \dots u_d) + c(u_1 \dots u_d, w_1 \dots w_d) + 1 \\
&\le 2 + d + \sum c(u_\ell, w_\ell) \\
&\le 2 + d + \sum (1 + C_i\kappa(\tilde{u}_\ell)) \\
&\le 2 + 2d + C_i \sum \kappa(\tilde{u}_\ell) \\
&\le 2 + 2d + C_i n \\
&\le C_i(n+1),
\end{aligned}$$

so the proof of the induction step is complete. □

Corollary 3.27. *There exists a constant $C > 0$ such that for every $u \in D_\infty^k$, there exists a word $w \in \Sigma^*$ of length at most $C\kappa(u)$ so that the relation $u = w$ holds in G and has cost at most $k + \kappa(u)$.*

Proof. Let $C = \max_i C_i$, where the constant C_i is defined in Lemma 3.26. Any $u \in D_\infty^k$ can be written $u = u_1 \dots u_k$ in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, with $u_i \in D_\infty^{(i)}$ and $\kappa(u) = \kappa(u_1) + \cdots + \kappa(u_k)$. Applying Lemma 3.26 to u_i, we get a word w_i of length at most $C_i\kappa(u_i)$ so that the relation $u_i = w_i$ holds in G and has cost at most $\kappa(u_i)$. Let $w = w_1 \dots w_k$. Then

$$|w|_\Sigma \le \sum_{i=1}^{k} |w_i|_\Sigma \le \sum_{i=1}^{k} C_i\kappa(u_i) \le C \sum_{i=1}^{k} \kappa(u_i) = C\kappa(u).$$

Moreover the relation $u = u_1 \dots u_k$ holds in G thanks to the set of relators R_D. Consequently in G we have $u = w$ at a total cost of at most

$$\begin{aligned}
c(u, u_1 \dots u_k) + c(u_1 \dots u_k, w) &\le k + \sum_{i=1}^{k} c(u_i, w_i) \\
&\le k + \sum_{i=1}^{k} \kappa(u_i) = k + \kappa(u).
\end{aligned}$$

□

The next lemma will reduce the estimate of the area function to its estimate for words of the special form $W_k(D)\Sigma^*$.

Lemma 3.28. *There exists a constant $c_1 > 0$ such that for any n and any word $w \in S^*$ of length at most n, there exists a word $w' = u\sigma_1 \dots \sigma_j$ of length at most n, where $u \in W_k(D)$, $\sigma_1, \dots, \sigma_j \in \Sigma$, so that w' is homotopic to w and $c(w, w') \le c_1 n \log(n)$.*

Proof. For any word $w \in S^*$, define

$$\tau(w) = \inf \{c(w, w') : w' \in W_k(D)\Sigma^* \text{ and } w' \text{ is homotopic to } w\},$$

and

$$f(n) = \sup \{\tau(w) : w \in S^* \text{ has length at most } n\}.$$

Note that both τ and f take finite values thanks to relators from R_3 and R_D. We want to prove that $f(n) \le c_1 n \log(n)$ for some constant c_1.

We use an algorithmic strategy. Given a word w, we first divide it into two subwords, then apply the algorithm to each of them and finally merge the results. More precisely, let us consider a word w of length 2^{n+1}, and divide it into two subwords w_1, w_2 of length 2^n. By definition of the function f, there exists words $w_1', w_2' \in W_k(D)\Sigma^*$ such that $c(w_1, w_1'), c(w_2, w_2') \le f(2^n)$. Now in the word $\bar{w} = w_1' w_2' \in W_k(D)\Sigma^* W_k(D)\Sigma^*$ we can move the $W_k(D)$ part of w_2' to the left by applying at most $2^n - 1$ relators of R_3, and merge it with the $W_k(D)$ part of w_1' with cost 1 thanks to the set of relators R_D. We therefore get a word $w' \in W_k(D)\Sigma^*$ homotopic to w and so that $c(w, w') \le 2f(2^n) + (2^n - 1) + 1$, which implies that $\tau(w) \le 2f(2^n) + 2^n$. By definition of f, we obtain $f(2^{n+1}) \le 2f(2^n) + 2^n$, from which we easily get the inequality $f(2^n) \le n2^{n-1}$. The result then follows from this inequality together with the fact that f is non-decreasing. □

Proposition 3.29. *There exists a constant $c > 0$ such that if $w \in F_S$ is a null-homotopic word of length at most n in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, then w already represents the identity in G and has area*

$$a(w) \le cn \log(n) + \delta(cn),$$

where δ is the Dehn function of the presentation $\langle \Sigma, R_\Sigma \rangle$ of $V_{d,k}$.

Proof. We first apply Lemma 3.28 to w and get a word $w' = u\sigma_1 \dots \sigma_j$ so that $c(w, w') \le c_1 n \log(n)$. Since w is null-homotopic, so is w' and therefore the element u^{-1} belongs to $W_k(D) \cap V_{d,k} = D_\infty^k$, and has length at most n in the group $V_{d,k}$ because w' has length at most n. According to Proposition 3.10, we have $\kappa(u^{-1}) \le C_\Sigma n$. Applying Corollary 3.27 to u^{-1} yields a word

$w'' \in \Sigma^*$ of length at most $C\kappa(u^{-1}) \leq CC_\Sigma n$ so that the relation $u^{-1} = w''$ holds in G and has cost at most $k + C_\Sigma n$. Therefore we obtain that

$$\begin{aligned} a(w) \leq & c(w, w') + c(u^{-1}, w'') + c(w'', \sigma_1 \dots \sigma_j) \\ \leq & c_1 n \log(n) + (k + C_\Sigma n) + \delta\left(CC_\Sigma n + n\right), \end{aligned}$$

because w'' and $\sigma_1 \dots \sigma_j$ represents the same element in $V_{d,k}$ and $w''(\sigma_1 \dots \sigma_j)^{-1}$ has length at most $CC_\Sigma n + n$, so $c(w'', \sigma_1 \dots \sigma_j)$ is at most $\delta(CC_\Sigma n + n)$ by definition of the Dehn function. Therefore $a(w) \leq cn\log(n) + \delta(cn)$ for some constant c depending only on Σ, and the proof is complete. □

In particular we deduce from Proposition 3.29 that the Dehn function of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is $\preccurlyeq n \log n + \delta_{V_{d,k}}$. But now the group $V_{d,k}$ is not Gromov-hyperbolic since it has a $\mathbb{Z}^2$ subgroup, so its Dehn function is not linear and consequently at least quadratic [Bow91]. Therefore $n \log n \preccurlyeq \delta_{V_{d,k}}$, and the Dehn function of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is thus asymptotically bounded by $\delta_{V_{d,k}}$.

3.2.3 Compact presentability of Schlichting completions

In this paragraph, which is completely independent, we establish a general result about compact presentability of Schlichting completions. Our result, which we believe is of independent interest, will be applied in the next paragraph to almost automorphism groups associated with closed regular branch groups.

If Γ is a group with a commensurated subgroup Λ, the Schlichting completion process builds a t.d.l.c. group $\Gamma /\!\!/ \Lambda$ and a morphism $\Gamma \to \Gamma /\!\!/ \Lambda$, so that the image of Γ is dense and the closure of the image of Λ is compact open. It was formally introduced in [Tza03], following an idea appearing in [Sch80].

We would like to point out that Schlichting completions are sometimes called *relative profinite completions* [SW13, EW13], but we choose not to use this terminology in order to avoid confusion with the notion of *localised profinite completion* appearing in [Rei12]. Although we will not use this terminology, we also note that a group together with a commensurated subgroup is sometimes called a Hecke pair.

The main theorem of this paragraph is a general result about compact presentability of Schlichting completions:

Theorem 3.30. *Let Γ be a finitely presented group and let Λ be a finitely generated commensurated subgroup. Then the t.d.l.c. group $\Gamma /\!\!/ \Lambda$ is compactly presented.*

Before going into the proof, let us mention the following result which can derived from Theorem 3.30. As mentioned above, the notion of Schlichting completion is different but closely related to the notion of profinite completion of a group localised at a subgroup [Rei12]. More precisely, it is proved in [Rei12, Corollary 3, (vii)] that the Schlichting completion $\Gamma /\!\!/ \Lambda$ is the quotient of the profinite completion of Γ localised at Λ by a compact normal subgroup. Now since a locally compact group is compactly presented if and only if one of its quotient by a compact normal subgroup is, we obtain:

Corollary 3.31. *If Γ is a finitely presented group with a finitely generated commensurated subgroup Λ, then the profinite completion of Γ localised at Λ is compactly presented.*

From commensurated subgroups to t.d.l.c. groups. We start by recalling the definition of the process of Schlichting completion.

Let Γ be a group and let Λ be a subgroup of Γ. The left action of Γ on the coset space Γ/Λ yields a homomorphism $\Gamma \to \mathrm{Sym}(\Gamma/\Lambda)$, whose kernel is the normal core of Λ, i.e. the largest normal subgroup of Γ contained in Λ (or equivalently, the intersection of all conjugates of Λ). The Schlichting completion of Γ with respect to Λ, denoted $\Gamma /\!\!/ \Lambda$, is by definition the closure of the image of Γ in $\mathrm{Sym}(\Gamma/\Lambda)$, the latter group being equipped with the topology of pointwise convergence.

Recall that Λ is said to be commensurated by a subset K of Γ if for every $k \in K$, the subgroup $k\Lambda k^{-1} \cap \Lambda$ has finite index in both Λ and $k\Lambda k^{-1}$. We say that Λ is a commensurated subgroup if it is commensurated by the entire group Γ. It can be checked that if this holds, then the closure of the image of Λ in $\Gamma /\!\!/ \Lambda$ is a compact open subgroup. In particular $\Gamma /\!\!/ \Lambda$ is a t.d.l.c. group. Note that by construction the image of Γ in $\Gamma /\!\!/ \Lambda$ is a dense subgroup.

From now Λ will be a commensurated subgroup of a group Γ. We point out that although the map $\Gamma \to \Gamma /\!\!/ \Lambda$ is generally not injective, for the sake of simplicity we still use the notation Λ and Γ for their images in the group $\Gamma /\!\!/ \Lambda$.

The next two lemmas are straightforward, we provide proofs for completeness.

Lemma 3.32. *We have $\Gamma /\!\!/ \Lambda = \overline{\Lambda} \cdot \Gamma$.*

Proof. Since $\overline{\Lambda}$ is an open subgroup of $\Gamma /\!\!/ \Lambda$, $\overline{\Lambda} g$ is an open neighbourhood of g for any $g \in \Gamma /\!\!/ \Lambda$. Therefore the dense subgroup Γ intersects $\overline{\Lambda} g$, meaning that there exist $\gamma \in \Gamma$ and $\lambda \in \overline{\Lambda}$ so that $\lambda g = \gamma$, i.e. $g = \lambda^{-1}\gamma$. □

Lemma 3.33. *The subgroups $\overline{\Lambda}$ and Γ intersect along Λ.*

Proof. The subgroup Λ stabilizes the coset Λ in $\mathrm{Sym}(\Gamma/\Lambda)$, so every element of $\overline{\Lambda}$ must stabilizes this coset as well by definition of the topology. Therefore $\overline{\Lambda} \cap \Gamma \subset \Lambda$. The reverse inclusion is clear. $\square$

The following result is a useful tool to identify some t.d.l.c. group G with the Schlichting completion of one of its dense subgroup. It is due to Shalom and Willis [SW13, Lemmas 3.5-3.6].

Proposition 3.34. *Let G be a topological group with a compact open subgroup U. If Γ is a dense subgroup of G, then $\Gamma \cap U$ is commensurated in Γ and the embedding of Γ in G induces an isomorphism of topological groups $\varphi : \Gamma /\!\!/ (\Gamma \cap U) \to G/K_U$, where K_U is the normal core of U. In particular if U contains no non-trivial normal subgroup of G, then φ is an isomorphism between $\Gamma /\!\!/ (\Gamma \cap U)$ and G.*

Example 3.35. Elder and Willis [EW13] considered the Schlichting completion $G_{m,n}$ of the Baumslag-Solitar group $\mathrm{BS}(m,n) = \langle t, x \mid tx^m t^{-1} = x^n \rangle$ with respect to the commensurated subgroup $\langle x \rangle$. Theorem 3.30 can be applied and yields that $G_{m,n}$ is compactly presented. However in this case this can be seen more directly because $G_{m,n}$ coincides with the closure of $\mathrm{BS}(m,n)$ in the automorphism group of its Bass-Serre tree, and therefore $G_{m,n}$ acts on a locally finite tree with compact vertex stabilizers. It follows that $G_{m,n}$ is Gromov-hyperbolic, and consequently automatically compactly presented.

The next example shows that the almost automorphism group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is a Schlichting completion of the Higman-Thompson group $V_{d,k}$. In particular Neretin's group is the Schlichting completion of $V_{d,2}$ with respect to an infinite locally finite subgroup, a point of view which does not seem to appear in the literature. This will be generalized in Theorem 3.51.

Example 3.36. Let us consider the t.d.l.c. group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ and its compact open subgroup $W_k(D)$, which is easily seen not to contain any non-trivial normal subgroup of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. The Higman-Thompson group $V_{d,k}$ is a dense subgroup intersecting $W_k(D)$ along D_∞^k by Lemma 3.22. So it follows from Proposition 3.34 that the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ is isomorphic to the Schlichting completion $V_{d,k} /\!\!/ D_\infty^k$.

However, note that compact presentability of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ cannot be obtained by applying Theorem 3.30 because D_∞^k is not finitely generated.

Presentation of $\Gamma /\!\!/ \Lambda$. We will now prove the main result of this subsection, namely Theorem 3.30, which will follow from Proposition 3.38.

From now Γ is a finitely presented group and Λ a finitely generated commensurated subgroup. Recall that by abuse of notation, we still denote by Γ and Λ their images in the group $\Gamma/\!\!/\Lambda$. We let $S = \{s_1, \dots, s_n, \dots, s_m\}$ be a finite generating set of Γ such that the elements $s_1, \dots, s_n$ generate Λ. It follows from Lemma 3.32 that $S \cup \overline{\Lambda}$ is a compact generating set of $\Gamma/\!\!/\Lambda$. To prove that the group $\Gamma/\!\!/\Lambda$ is compactly presented, we will consider a set $R = R_1 \cup R_2 \cup R_3 \cup R_4$ of relations of bounded length in $\Gamma/\!\!/\Lambda$, and prove that it is a set of defining relations, i.e. that relations of R generate all the relations in $\Gamma/\!\!/\Lambda$.

We let R_1 be a set of words so that $\langle S \mid R_1 \rangle$ is a finite presentation of Γ.

Let us also consider relations corresponding to the inclusion $\Lambda \leq \overline{\Lambda}$ in the group $\Gamma/\!\!/\Lambda$. That is, for every $i \in \{1, \dots, n\}$, we let $\bar{s_i} \in \overline{\Lambda}$ be such that $s_i = \bar{s_i}$ in $\Gamma/\!\!/\Lambda$, and denote by R_2 the set of words $s_i \bar{s_i}^{-1}$.

We denote by R_3 the set of relations of the form $u_1 u_2 = u_3$, $u_i \in \overline{\Lambda}$.

Now let us define the abstract group $G_1 = \left\langle S \cup \overline{\Lambda} \mid R_1, R_2, R_3 \right\rangle$. Note that by construction there is a homomorphism $G_1 \to \Gamma/\!\!/\Lambda$.

Proposition 3.37. *Let w be a word in the elements of S and $u \in \overline{\Lambda}$. If the word $u^{-1}w$ represents the identity in $\Gamma/\!\!/\Lambda$, then it already represents the identity in G_1.*

Proof. The fact that $u^{-1}w$ represents the identity in $\Gamma/\!\!/\Lambda$ means that the element represented by w lies in $\Gamma \cap \overline{\Lambda}$, which is reduced to Λ according to Lemma 3.33. Therefore there exists a word w_Λ in the letters $s_1, \dots, s_n$, so that $w = w_\Lambda$ in $\Gamma/\!\!/\Lambda$. But thanks to R_1, the relation $w = w_\Lambda$ is also satisfied in G_1. Now for each letter of w_Λ we can apply a relation from R_2 to obtain a word $w_{\overline{\Lambda}}$ in the letters $\bar{s_1}, \dots, \bar{s_n}$, so that $w = w_{\overline{\Lambda}}$ in G_1. Consequently the relation $w_{\overline{\Lambda}} = u$ holds in $\Gamma/\!\!/\Lambda$, and thanks to R_3 this relation also holds in G_1, meaning that $w = u$ in G_1. □

We finally consider a last family of relations in $\Gamma/\!\!/\Lambda$. According to Lemma 3.32, for every $i \in \{1, \dots, m\}$ and $u \in \overline{\Lambda}$, we can pick some $u' \in \overline{\Lambda}$ and some word $w \in S^*$ so that $s_i u = u' w$ in $\Gamma/\!\!/\Lambda$. We denote by R_4 the set of corresponding relations.

Now let us define the abstract group $G_2 = \left\langle S \cup \overline{\Lambda} \mid R_1, R_2, R_3, R_4 \right\rangle$. Note that the group G_2 is a quotient of G_1.

Proposition 3.38. *The natural homomorphism $G_2 \to \Gamma/\!\!/\Lambda$ is an isomorphism.*

Proof. It is clear that this morphism is onto because $S \cup \overline{\Lambda}$ is a generating set of $\Gamma/\!\!/\Lambda$. So we only have to prove that it is injective. For, let us consider a word w in the elements of $S \cup \overline{\Lambda}$ representing the identity in $\Gamma/\!\!/\Lambda$. We want to prove that w represents the identity in G_2. Applying successively relators from R_4, we can move each occurrence of an element of $\overline{\Lambda}$ in w to the left, and obtain a word w' of the form $w' = u_1 \cdots u_k s_{i_1} \cdots s_{i_\ell}$, with

$u_i \in \overline{\Lambda}$, $s_j \in S$, so that $w = w'$ in G_2. Now thanks to R_3, the word w' can be transformed into a word w'' of the form $w'' = us_{i_1} \cdots s_{i_\ell}$, $u \in \overline{\Lambda}$. But now since w represents the identity in $\Gamma /\!\!/ \Lambda$, the same holds for w''. Therefore by Proposition 3.37, the word w'' represents the identity in G_1, and a fortiori it also represents the identity in G_2, the latter being a quotient of G_1. It follows that the word w represents the identity in G_2, and the proof is complete. □

Corollary 3.39. *Let G be a topological group with a compact open subgroup U. Assume that G admits a dense finitely presented subgroup intersecting U along a finitely generated group. Then G is compactly presented.*

Remark 3.40. Here we do not try to get an estimate on the Dehn function of $\Gamma /\!\!/ \Lambda$, because a careful reading of the proof reveals that the best we could hope in this level of generality is to obtain that the Dehn function of $\Gamma /\!\!/ \Lambda$ is bounded by the Dehn function of Γ. However this would be far from being sharp, as for example the Baumslag-Solitar group $\mathrm{BS}(1, n)$ has an exponential Dehn function for $n \geq 2$ (see for instance [GH01]), whereas its Schlichting completion $\mathbb{Q}_n \rtimes_n \mathbb{Z}$ is Gromov-hyperbolic, and therefore has a linear Dehn function.

3.2.4 Almost automorphism groups and branch groups

Statement of the result. In this subsection we restrict ourselves to the case $k = d$ for the sake of simplicity, but the results could naturally be extended to almost automorphism subgroups of $\mathrm{AAut}(\mathcal{T}_{d,k})$.

Almost automorphisms of $\mathcal{T}_d$ are homeomorphisms of the boundary $\partial_\infty \mathcal{T}_d$ which locally coincide with a tree automorphism. It seems natural to extend this definition by considering a subgroup $G \leq \mathrm{Aut}(\mathcal{T}_d)$, and homeomorphisms of $\partial_\infty \mathcal{T}_d$ which locally coincide with an element of G. In other words, we want to define homeomorphisms of $\partial_\infty \mathcal{T}_d$ which are piecewise in G. It turns out that the notion naturally appearing for G is self-similarity. The notions of self-similarity and branching appear naturally in the theory of groups acting on rooted trees. Basic definitions are recalled below, and we refer the reader to the surveys [Nek05], [BGŠ03] for more on self-similar and branch groups.

To any self-similar group $G \leq \mathrm{Aut}(\mathcal{T}_d)$ we naturally associate a subgroup $\mathrm{AAut}_G(\mathcal{T}_d) \leq \mathrm{AAut}(\mathcal{T}_d)$, consisting of almost automorphisms acting locally like an element of G. A more precise definition of this group is given in the sequel. The group $\mathrm{AAut}_G(\mathcal{T}_d)$ always contains the Higman-Thompson group V_d and is generated by V_d together with an embedded copy of G. It is worth noting that the definition of the group $\mathrm{AAut}_G(\mathcal{T}_d)$ makes sense when $G \leq \mathrm{Aut}(\mathcal{T}_d)$ is an abstract subgroup. In particular we make a priori neither topological (e.g. closed) nor finiteness (e.g. finitely generated) assumption on G.

The first example of such a group was studied by Röver in the case when G is the first Grigorchuk group. He proved that $\mathrm{AAut}_G(\mathcal{T}_d)$ is finitely presented and simple [Röv99]. The case of a general self-similar group was then studied by Nekrashevych [Nek04], who proved that these groups enjoy properties rather similar to the properties of the Higman-Thompson groups (see [Nek04], [Nek13]).

Later Barnea, Ershov and Weigel [BEW11] made use of Röver's simplicity result to prove that the profinite completion of the Grigorchuk group, which coincides with its topological closure in $\mathrm{Aut}(\mathcal{T}_2)$, embeds as an open subgroup in a topologically simple group, namely the group of almost automorphisms acting locally like an element of the closure of the Grigorchuk group.

Here we are interested in almost automorphism groups associated with closed regular branch groups. These can also be seen as generalizations of Neretin's group. It turns out that in this setting, the group $\mathrm{AAut}_G(\mathcal{T}_d)$ is naturally a totally disconnected locally compact group, admitting G as a compact open subgroup. Under some more assumptions on G, we prove:

Theorem 3.41. *Let $G \leq \mathrm{Aut}(\mathcal{T}_d)$ be the closure of some finitely generated, contracting regular branch group, branching over a congruence subgroup. Then $\mathrm{AAut}_G(\mathcal{T}_d)$ is a t.d.l.c. compactly presented group.*

Examples of groups covered by Theorem 3.41 include the aforementioned topologically simple group constructed in [BEW11], as well as other groups described below. As an application, we obtain that the profinite completion of the Grigorchuk group embeds as an open subgroup in a topologically simple compactly presented group.

Note that any group G appearing in Theorem 3.41 can be explicitly described in terms of the notions of patterns and finitely constrained groups, an introduction of which can be found in [Šun07]: G is the finitely constrained group defined by allowing all patterns of a fixed size appearing in the group of which it is the closure. See also the comment at the end of this subsection.

The proof of Theorem 3.41 consists in two separate steps. The first one is the main purpose of this paragraph, and consists in identifying our group with the Schlichting completion of one of its dense subgroup (see Theorem 3.51). The second one will consist in making use of a recent result of Nekrashevych, which will allow us to apply Theorem 3.30.

Preliminary results on branch groups. This paragraph is devoted to reviewing basic definitions and facts about self-similar and branch groups, and establishing some preliminary results. We refer the reader to [Nek05], [BGŠ03], for more on self-similar and branch groups.

Recall that vertices of $\mathcal{T}_d$ are labeled by words over a finite alphabet X of cardinality d, and we freely identify a vertex with the word associated to it.

If G is a subgroup of the automorphism group $\mathrm{Aut}(\mathcal{T}_d)$ and if $n \geq 0$, we will denote by G_n the nth level stabilizer of G, that is the subgroup of G fixing pointwise the nth level of $\mathcal{T}_d$. Note that G_n is always a finite index subgroup of G, but the converse is far from true because there may exist some finite index subgroup of G not containing any level stabilizer. This motivates the following definition.

Definition 3.42. A finite index subgroup of G is a congruence subgroup if it contains some level stabilizer.

If $g \in \mathrm{Aut}(\mathcal{T}_d)$ is an automorphism and $v \in X^*$ is a vertex of $\mathcal{T}_d$, the section of g at v is the unique automorphism g_v of $\mathcal{T}_d$ defined by the formula

$$g(vw) = g(v)g_v(w)$$

for every $w \in X^*$.

Definition 3.43. A subgroup $G \leq \mathrm{Aut}(\mathcal{T}_d)$ is self-similar if every section of every element of G is an element of G.

Self-similar groups appear naturally when studying holomorphic dynamics and fractal geometry. The study of self-similar groups is also motivated by the fact that this class contains examples of groups exhibiting some exotic behavior. Among self-similar groups is a class of groups which is better understood, namely contracting self-similar groups.

Definition 3.44. A self-similar group G is said to be contracting if there exists a finite subset $\mathcal{N} \leq G$ such that for every $g \in G$, there exists $k \geq 1$ so that all the sections of g of level at least k belong to $\mathcal{N}$.

Here we are interested in a particular class of self-similar groups, namely regular branch groups, whose definition is recalled below.

Definition 3.45. Let $G \leq \mathrm{Aut}(\mathcal{T}_d)$ be a self-similar group. By definition, G comes equipped with an injective homomorphism $\psi : G \to G \wr \mathrm{Sym}(d)$ (sometimes called the wreath recursion). We say that G is regular branch over its finite index subgroup K if $\psi(K)$ contains $K \times \ldots \times K$ as a subgroup of finite index.

Remark 3.46. We note that being regular branch is stable by taking the topological closure in $\mathrm{Aut}(\mathcal{T}_d)$. More precisely, if G is regular branch over K then the closure of G is regular branch over the closure of K. Note also that if K contains some level stabilizer of G then its closure contains the stabilizer of the same level in the closure of G, so being regular branch over a congruence subgroup is also stable by taking the topological closure.

The most popular example of a self-similar group is the Grigorchuk group of intermediate growth introduced in [Gri80]. It is a regular branch group, branching over a subgroup containing its stabilizer of level 3. Other examples are the Gupta-Sidki group as well as the Fabrykowski-Gupta group, which are regular branch over their commutator subgroup, the latter containing their level 2 stabilizer. For the definitions and properties of these groups we refer the reader to Sections 6 and 8 of [BG02]. In view of Theorem 3.41, we note that all these examples are contracting.

In the following standard lemma, a proof of which can be consulted in [Šun07, Lemma 10], the isomorphism is obtained via the wreath recursion, which is usually omitted.

Lemma 3.47. *Let $H \leq \mathrm{Aut}(\mathcal{T}_d)$ be a regular branch group, branching over a subgroup containing the level stabilizer H_s. Then for every $n \geq s$, the level stabilizer H_{n+1} is isomorphic to $H_n \times \ldots \times H_n$.*

If H is a subgroup of the automorphism group $\mathrm{Aut}(\mathcal{T}_d)$, it is in general very hard to describe its topological closure in $\mathrm{Aut}(\mathcal{T}_d)$. In the case of the Grigorchuk group, the closure has been described by Grigorchuk in [Gri05]. We will use a generalization of this result due to Sunic, which is the following:

Proposition 3.48. *Let $H \leq \mathrm{Aut}(\mathcal{T}_d)$ be a regular branch group, branching over a subgroup containing the level stabilizer H_s, and let G be the topological closure of H in $\mathrm{Aut}(\mathcal{T}_d)$. Then an element $\gamma \in \mathrm{Aut}(\mathcal{T}_d)$ belongs to G if and only if for every section γ_v of γ, there exists an element of H acting like γ_v up to and including level $s+1$.*

Proof. The statement is a reformulation of the implication $(ii) \Rightarrow (i)$ of Theorem 3 of [Šun07]. Note that the author requires level transitivity in the definition of a regular branch group, but the proof given there does not use this assumption. □

This description of the closure of a regular branch group allows us to deduce the following result, which does not seem to appear in the literature, and which may be of independent interest.

Proposition 3.49. *Let $H \leq \mathrm{Aut}(\mathcal{T}_d)$ be a regular branch group, branching over a congruence subgroup, and let G be the topological closure of H in $\mathrm{Aut}(\mathcal{T}_d)$. Then the intersection in $\mathrm{Aut}(\mathcal{T}_d)$ between G and $H \wr \mathrm{Sym}(d)$ is equal to H.*

Proof. By self-similarity the subgroup $H \wr \mathrm{Sym}(d)$ of $\mathrm{Aut}(\mathcal{T}_d)$ contains the group H, so the inclusion $H \subset G \cap (H \wr \mathrm{Sym}(d))$ is clear. To prove that equality holds, we prove that H and $G \cap (H \wr \mathrm{Sym}(d))$ have the same index in the group $H \wr \mathrm{Sym}(d)$.

Assume that H is branching over a subgroup containing H_s. By multiplicativity of the index, we have

$$[H \wr \mathrm{Sym}(d) : H_{s+1}] = [H \wr \mathrm{Sym}(d) : H] \times [H : H_{s+1}]\,,$$

that is

$$[H \wr \mathrm{Sym}(d) : H] = \frac{[H \wr \mathrm{Sym}(d) : H_{s+1}]}{[H : H_{s+1}]}.$$

Now the number of possibilities for the action of an element of $H \wr \mathrm{Sym}(d)$ on the first level is $|\mathrm{Sym}(d)| = d!$. Moreover the first level stabilizer of $H \wr \mathrm{Sym}(d)$ is $H \times \ldots \times H$, so

$$[H \wr \mathrm{Sym}(d) : H_{s+1}] = d!\,[H \times \ldots \times H : H_{s+1}]\,. \tag{3.2}$$

Furthermore we can apply Lemma 3.47 to obtain that H_{s+1} is equal to $H_s \times \ldots \times H_s$, which yields

$$[H \times \ldots \times H : H_{s+1}] = [H \times \ldots \times H : H_s \times \ldots \times H_s] = [H : H_s]^d\,.$$

Going back to (3.2), we obtain

$$[H \wr \mathrm{Sym}(d) : H] = \frac{d!\,[H : H_s]^d}{[H : H_{s+1}]}.$$

Let us now compute the index of $G \cap (H \wr \mathrm{Sym}(d))$ in $H \wr \mathrm{Sym}(d)$. According to Proposition 3.48, an element $\gamma \in \mathrm{Aut}(\mathcal{T}_d)$ belongs to G if and only if for every section γ_v of γ, there exists an element of H acting like γ_v up to level $s+1$. Since elements of $H \wr \mathrm{Sym}(d)$ have all their sections of level at least 1 in H, it follows that an element $\gamma \in H \wr \mathrm{Sym}(d)$ belongs to G if and only if there exists an element of H acting like γ up to level $s+1$. It follows that the index of $G \cap (H \wr \mathrm{Sym}(d))$ in $H \wr \mathrm{Sym}(d)$ is the number of possibilities for the action on level $s+1$ for $H \wr \mathrm{Sym}(d)$, divided by the number of possibilities for the action on level $s+1$ for H. The latter being $[H : H_{s+1}]$ and the former being $d!\,[H : H_s]^d$, we have

$$[H \wr \mathrm{Sym}(d) : G \cap (H \wr \mathrm{Sym}(d))] = \frac{d!\,[H : H_s]^d}{[H : H_{s+1}]}.$$

□

Definition of the groups. Let $G \leq \mathrm{Aut}(\mathcal{T}_d)$ be a self-similar group. We will say that an almost automorphism of $\mathcal{T}_d$ is piecewise of type G if it can be represented by a triple (ψ, T, T') such that T, T' are finite rooted complete subtrees of $\mathcal{T}_d$ and $\psi : \mathcal{T}_d \setminus T \to \mathcal{T}_d \setminus T'$ belongs to G on each connected component, after the natural identification of each connected component of $\mathcal{T}_d \setminus T$ and $\mathcal{T}_d \setminus T'$ with $\mathcal{T}_d$. We observe that by self-similarity, if a triple (ψ_1, T_1, T_1') is such that $\psi_1 : \mathcal{T}_d \setminus T_1 \to \mathcal{T}_d \setminus T_1'$ belongs to G on each connected component, then for any equivalent triple (ψ_2, T_2, T_2') such that T_2 (resp. T_2') contains T_1 (resp. T_1'), then $\psi_2 : \mathcal{T}_d \setminus T_2 \to \mathcal{T}_d \setminus T_2'$ belongs to G on each connected component. It follows from this observation that the set of almost automorphisms which are piecewise of type G is a subgroup of $\mathrm{AAut}(\mathcal{T}_d)$, which will be denoted by $\mathrm{AAut}_G(\mathcal{T}_d)$. Note that $\mathrm{AAut}_G(\mathcal{T}_d)$ obviously contains the group G.

It is worth pointing out that the definition of the group does not require any topological (e.g. closed) or finiteness (e.g. finitely generated) assumption on G.

Following [Nek13], we let $L(G) \leq \mathrm{Aut}(\mathcal{T}_d)$ be the embedded copy of G acting on the subtree hanging below the first vertex of level 1. Since the Higman-Thompson group V_d acts transitively on the set of proper balls of $\partial_\infty \mathcal{T}_d$, it is not hard to see that the group $\mathrm{AAut}_G(\mathcal{T}_d)$ is generated by V_d together with $L(G)$. See Lemma 5.12 in [Nek13] for details. In particular if G is a finitely generated self-similar group, then $\mathrm{AAut}_G(\mathcal{T}_d)$ is finitely generated as well.

The first example of such a group was considered by Röver when G is the first Grigorchuk group. He proved that $\mathrm{AAut}_G(\mathcal{T}_d)$ is finitely presented and simple [Röv99]. Then Nekrashevych [Nek04] introduced the group $\mathrm{AAut}_G(\mathcal{T}_d)$ for an arbitrary self-similar group G and generalized both simplicity and finiteness results (see Theorem 4.7 in [Nek13] and Theorem 3.56 cited below).

Remark 3.50. It is worth noting that $\mathrm{AAut}_G(\mathcal{T}_d)$ is always a dense subgroup of $\mathrm{AAut}(\mathcal{T}_d)$, since it contains the subgroup V_d which is already dense. In particular if N is a non-trivial normal subgroup of $\mathrm{AAut}_G(\mathcal{T}_d)$, then the closure of N in $\mathrm{AAut}(\mathcal{T}_d)$ is normalized by the closure of $\mathrm{AAut}_G(\mathcal{T}_d)$, which is $\mathrm{AAut}(\mathcal{T}_d)$. By simplicity of the latter, the closure of N has to be equal to $\mathrm{AAut}(\mathcal{T}_d)$. This proves that any non-trivial normal subgroup of $\mathrm{AAut}_G(\mathcal{T}_d)$ is dense in $\mathrm{AAut}(\mathcal{T}_d)$. In particular G can not contain any non-trivial normal subgroup of $\mathrm{AAut}_G(\mathcal{T}_d)$.

Almost automorphism groups arising as Schlichting completions. The main result of this paragraph is the following.

Theorem 3.51. *Let $H \leq \mathrm{Aut}(\mathcal{T}_d)$ be a regular branch group, branching over a congruence subgroup, and let G be the topological closure of H in $\mathrm{Aut}(\mathcal{T}_d)$. Then the inclusion of $\mathrm{AAut}_H(\mathcal{T}_d)$ in $\mathrm{AAut}_G(\mathcal{T}_d)$ induces an isomorphism of topological groups between $\mathrm{AAut}_H(\mathcal{T}_d)/\!\!/H$ and $\mathrm{AAut}_G(\mathcal{T}_d)$.*

For example this brings a new perspective to the topologically simple group constructed in [BEW11]: this is the Schlichting completion of Röver's group [Röv99] with respect to the Grigorchuk group.

Theorem 3.51 will be proved at the end of this paragraph. We begin by showing how to endow the group $\mathrm{AAut}_G(\mathcal{T}_d)$ with a natural topology when G is a closed regular branch group. We will need the following:

Proposition 3.52. *Any regular branch group $G \leq \mathrm{Aut}(\mathcal{T}_d)$ is commensurated in $\mathrm{AAut}_G(\mathcal{T}_d)$.*

Proof. Since $\mathrm{AAut}_G(\mathcal{T}_d)$ is generated by V_d and $L(G)$, it is enough to prove that these two subgroups commensurate G.

Let us first prove that V_d commensurates G. Henceforth we assume that K is a branching subgroup of G. For every finite rooted complete subtree T of $\mathcal{T}_d$, we denote by K_T the subgroup of $\mathrm{Aut}(\mathcal{T}_d)$ fixing pointwise T and acting by an element of K on each subtree hanging below a leaf of T. Since G is regular branch over K, K_T is a finite index subgroup of G for every finite rooted complete subtree T. Now if $\sigma \in V_d$ and if T, T' are respectively the domain and range tree of the canonical representative triple of σ, we easily check that $\sigma K_T \sigma^{-1} = K_{T'}$. So conjugation by σ sends a finite index subgroup of G to another finite index subgroup of G, which exactly means that σ commensurates G.

Now let us prove that $L(G)$ commensurates G. It is classic that since K is a finite index subgroup of G, there exists a finite index subgroup N of K that is normal in G. Therefore $\psi(G)$ contains $N \times \ldots \times N$ as a finite index subgroup, and the latter is normalized by $L(G)$ because N is normal in G. This proves an even stronger result that commensuration, namely the existence of a finite index subgroup of G which is normalized by $L(G)$. $\square$

Now assume that $G \leq \mathrm{Aut}(\mathcal{T}_d)$ is a closed regular branch group. Examples of such groups include the topological closure of any of the finitely generated regular branch groups mentioned earlier. In this context, the group G comes equipped with a profinite topology, inherited from the profinite topology of $\mathrm{Aut}(\mathcal{T}_d)$. The fact that G is commensurated in $\mathrm{AAut}_G(\mathcal{T}_d)$ together with Lemma 3.13 allows us to extend the topology of G to the larger group $\mathrm{AAut}_G(\mathcal{T}_d)$:

Proposition 3.53. *Assume that $G \leq \mathrm{Aut}(\mathcal{T}_d)$ is a closed regular branch group. Then there exists a (unique) group topology on $\mathrm{AAut}_G(\mathcal{T}_d)$ turning G into a compact open subgroup. In particular $\mathrm{AAut}_G(\mathcal{T}_d)$ is a t.d.l.c. compactly generated group.*

We now prove some preliminary results which will be used in the proof of Theorem 3.51.

Until the end of this subsection, $H \le \mathrm{Aut}(\mathcal{T}_d)$ *is a regular branch group, branching over a congruence subgroup, and* G *is the topological closure of* H *in* $\mathrm{Aut}(\mathcal{T}_d)$*.*

Proposition 3.54. $\mathrm{AAut}_H(\mathcal{T}_d)$ *is a dense subgroup of* $\mathrm{AAut}_G(\mathcal{T}_d)$*.*

Proof. We let L be a branching congruence subgroup of H, and we denote by K the closure of L in $\mathrm{Aut}(\mathcal{T}_d)$. For every finite rooted complete subtree T of $\mathcal{T}_d$, we still denote by K_T the subgroup of $\mathrm{Aut}(\mathcal{T}_d)$ fixing pointwise T and acting by an element of K on each subtree hanging below a leaf of T. Note that since L contains some level stabilizer of H, the subgroup K contains some level stabilizer of G and is therefore an open subgroup of G. It follows that (K_T) forms a basis of neighbourhoods of the identity in G, when T ranges over all finite rooted complete subtrees. By definition of the topology, it is also a basis of neighbourhoods of the identity in $\mathrm{AAut}_G(\mathcal{T}_d)$.

Let g be an element of G. By definition there exists a sequence (h_n) of elements of H converging to g. Since L has finite index in H, we may assume that all the elements h_n lie in the same left coset of L, that is there exists $h \in H$ such that $h_n \in hL$ for every n. From this we deduce that $g \in hK$.

Now let γ be an element of $\mathrm{AAut}_G(\mathcal{T}_d)$. We will prove that $\mathrm{AAut}_H(\mathcal{T}_d)$ intersects every neighbourhood of γ. Let (ψ, T, T') be a triple representing γ such that $\psi : \mathcal{T}_d \setminus T \to \mathcal{T}_d \setminus T'$ belongs to G on each connected component of $\mathcal{T}_d \setminus T$. This means that for every leaf v of T, there exists an element $g_v \in G$ so that ψ sends the subtree hanging below the leaf v to a subtree hanging below a leaf of T' via the element g_v. According to the above remark, there exists some element $h_v \in H$ such that $h_v^{-1} g_v \in K$. Now let us consider the almost automorphism $\hat{\gamma}$ represented by the triple $(\hat{\psi}, T, T')$, where $\hat{\psi}$ induces the same bijection between the leaves of T and the leaves of T', but does not act on the subtree hanging below the leaf v by g_v but by the element h_v. By construction, we have $\hat{\gamma} \in \mathrm{AAut}_H(\mathcal{T}_d)$ and $\hat{\gamma}^{-1}\gamma \in K_T$. Since on the one hand we can choose T to be as large as we want, and on the other hand (K_T) is a basis of neighbourhoods of the identity, we obtain that $\mathrm{AAut}_H(\mathcal{T}_d)$ intersects every neighbourhood of γ. □

Proposition 3.55. *The intersection in* $\mathrm{AAut}_G(\mathcal{T}_d)$ *between* G *and* $\mathrm{AAut}_H(\mathcal{T}_d)$ *is equal to* H*.*

Proof. The inclusion $H \subset G \cap \mathrm{AAut}_H(\mathcal{T}_d)$ being clear, we only have to prove the reverse inclusion. First note that the intersection between $\mathrm{Aut}(\mathcal{T}_d)$ and $\mathrm{AAut}_H(\mathcal{T}_d)$ is the increasing union for $n \ge 0$ of the subgroups $H \wr \mathrm{Aut}_n$, where

Aut_n is the subgroup of $\mathrm{Aut}(\mathcal{T}_d)$ consisting of elements whose sections of level n are trivial; and the permutational wreath product is associated to the action of Aut_n on the vertices of level n. In particular

$$G \cap \mathrm{AAut}_H(\mathcal{T}_d) = \bigcup_{n \geq 0} G \cap (H \wr \mathrm{Aut}_n).$$

Let us prove by induction on $n \geq 0$ that $G \cap (H \wr \mathrm{Aut}_n)$ is reduced to H. This is true for $n = 0$ by definition, and true for $n = 1$ according to Proposition 3.49. Assume that this is true for some $n \geq 1$, and let $\gamma \in G \cap (H \wr \mathrm{Aut}_{n+1})$. Then every section of level 1 of γ lies in $H \cap (H \wr \mathrm{Aut}_n)$, which is reduced to H by induction hypotheses. Therefore $\gamma \in G \cap (H \wr \mathrm{Sym}(d))$, which is also equal to H by Proposition 3.49. So we have proved the induction step, namely $G \cap (H \wr \mathrm{Aut}_{n+1}) = G$, and consequently $G \cap \mathrm{AAut}_H(\mathcal{T}_d) = H$. $\square$

We are now ready to prove the main result of this paragraph.

Proof of Theorem 3.51. The group $\mathrm{AAut}_G(\mathcal{T}_d)$ admits $\mathrm{AAut}_H(\mathcal{T}_d)$ as a dense subgroup by Proposition 3.54, and the latter intersects the compact open subgroup G along H according to Proposition 3.55. Moreover Remark 3.50 prevents G from containing any non-trivial normal subgroup of $\mathrm{AAut}_G(\mathcal{T}_d)$, so the conclusion follows from Proposition 3.34. $\square$

Proof of Theorem 3.41. We conclude by proving Theorem 3.41. The only missing argument is a recent result of Nekrashevych, generalizing the previous example of Röver [Röv99].

Theorem 3.56 ([Nek13], Theorem 5.9)**.**
If $H \leq \mathrm{Aut}(\mathcal{T}_d)$ is a finitely generated, contracting self-similar group, then $\mathrm{AAut}_H(\mathcal{T}_d)$ is finitely presented.

Proof of Theorem 3.41. Let H be a finitely generated, contracting regular branch group, branching over a congruence subgroup, having G for topological closure in $\mathrm{Aut}(\mathcal{T}_d)$. Then by Theorem 3.51 $\mathrm{AAut}_G(\mathcal{T}_d)$ is isomorphic to the Schlichting completion $\mathrm{AAut}_H(\mathcal{T}_d)/\!\!/H$. Now according to Theorem 3.56 the group $\mathrm{AAut}_H(\mathcal{T}_d)$ is finitely presented, and H is finitely generated by assumption, so the conclusion follows from Theorem 3.30. $\square$

We make a brief comment on the fact that any group G appearing in Theorem 3.41 can be explicitly described in terms of the group of which it is the topological closure. Indeed, if H is a finitely generated, contracting regular branch group, branching over a subgroup containing H_s, having G for topological closure in $\mathrm{Aut}(\mathcal{T}_d)$; then Proposition 3.48 yields that elements of G are exactly the automorphisms having all their sections acting like an element of H up to level $s + 1$. One can rephrase this in terms of patterns and finitely constrained groups (see [Šun07]), by saying that G is the finitely constrained group defined by allowing all the patterns of size $s+1$ appearing in H.

3.3 Metric properties, commensurating action and embeddings

3.3.1 D-diagrams

In this subsection we introduce a notion of diagrams to represent elements of the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. From now and until the end of this subsection, we fix some integers $k, d \geq 2$, a finite permutation group $D \leq \mathrm{Sym}(d)$ and an embedding of the quasi-regular tree $\mathcal{T}_{d,k}$ in the oriented plane. We will freely use the terminology introduced in 3.1.1.

Let T be a finite complete rooted subtree of $\mathcal{T}_{d,k}$ with n leaves. The planarity of $\mathcal{T}_{d,k}$ induces a canonical way, say from left to right, of labeling the leaves of T. In particular the n connected components of the complement of T in $\mathcal{T}_{d,k}$ naturally inherit a labeling $T^1, \ldots, T^n$ coming from the labeling of the leaves of T.

Definition 3.57. A *D-diagram* $[\varphi, T_-, T_+]$, is the data of a pair of finite complete rooted subtrees (T_-, T_+) of $\mathcal{T}_{d,k}$ with the same number of leaves, together with a forest isomorphism $\varphi : \mathcal{T}_{d,k} \backslash T_- \to \mathcal{T}_{d,k} \backslash T_+$ acting like $W(D)$ on each connected component. In other words, if $\mathcal{T}_{d,k} \backslash T_- = T_-^1 \cup \ldots \cup T_-^n$ and $\mathcal{T}_{d,k} \setminus T_+ = T_+^1 \cup \ldots \cup T_+^n$, the map φ is given by a permutation $\sigma \in \mathrm{Sym}(n)$ together with n isomorphisms $\varphi_i : T_-^i \to T_+^{\sigma(i)}$ belonging to $W(D)$, after the natural identifications of T_-^i and $T_+^{\sigma(i)}$ with $\mathcal{T}_d$. The permutation σ will be called the permutation of the D-diagram, and the tree isomorphisms $\varphi_1, \ldots, \varphi_n$ will be called the coordinates. The tree T_- will be called the domain tree and T_+ the range tree.

Remark 3.58. By definition the range and the domain tree of a D-diagram have the same number of leaves, and a fortiori the same number of carets. By abuse we refer to the number of carets of any of these two tree as the number of carets of the D-diagram.

It is straightforward from the definition that any D-diagram gives rise to an element of the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. However, given a D-diagram, we can easily build a new one representing the same element of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, by the process of unfolding a leaf and decomposing the associated tree automorphism. Consequently in order to obtain a one-to-one correspondence between D-diagrams and elements of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, we have to isolate a subclass of D-diagrams.

Definition 3.59. If $[\varphi, T_-, T_+]$ and $\left[\varphi', T'_-, T'_+\right]$ are D-diagrams, we say that $\left[\varphi', T'_-, T'_+\right]$ is a simple expansion of $[\varphi, T_-, T_+]$ if:

— T'_- is obtained from T_- by folding out the ith leaf of T_-, and T'_+ is obtained from T_+ by folding out the leaf $\sigma(i)$ of T_+;

— φ' coincides with φ on the unaffected rooted subtrees $T^k_-, k \neq i$, sends the new leaves of T_2 on their images by φ_i and acts on the associated rooted subtrees according to φ_i.

An expansion of $[\varphi, T_-, T_+]$ is a D-diagram $\left[\varphi', T'_-, T'_+\right]$ obtained by making a finite number of simple expansions. When this is so, $[\varphi, T_-, T_+]$ is said to be a reduction of $\left[\varphi', T'_-, T'_+\right]$. A D-diagram admitting no other reduction than itself is said to be *D-reduced*. The reason why we use the terminology *D-reduced* rather than *reduced* is that this notion strongly depend on the finite permutation group D.

Saying that two D-diagrams are equivalent if they have a common expansion defines an equivalence relation on the set of D-diagrams. We leave to the reader the verification of the fact that two D-diagrams are equivalent if and only if they represent the same homeomorphism of the boundary $\partial_\infty \mathcal{T}_{d,k}$, and that each equivalence class contains a unique D-reduced representative.

Definition 3.60. If $g \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$, we denote by $\mathcal{C}_D(g)$ the number of carets of the unique D-diagram representing g that is D-reduced . When $D = \mathrm{Sym}(d)$, for simplicity we use the notation $\mathcal{C}(g)$ instead of $\mathcal{C}_{\mathrm{Sym}(d)}(g)$.

Remark that the subgroup of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ consisting of elements that can be represented by a D-diagram with trivial coordinates is nothing but Higman-Thompson's group $V_{d,k}$. Elements of $V_{d,k}$ are usually represented by a combinatorial data called a *tree pair diagram*, an introduction of which can be found in [CFP96] (see also [Bur99, BCS01]). However we point out that in general the notion of tree pair diagram *does not* correspond to our notion of D-diagram. To illustrate this difference, let us consider the subgroup $\Gamma = V_{d,k} \cap W_k(D)$ of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, which has been described in Lemma 3.22: Γ is a locally finite subgroup, which is infinite as soon as D is non trivial. It readily follows from the definition that any D-reduced D-diagram associated to an element of Γ must have zero caret, whereas we easily check that tree pair diagrams associated to elements of Γ have unbounded size. Actually one can check that the notion of D-diagram corresponds to the notion of tree pair diagram if and only if the permutation group D is trivial, in which case we have $\mathrm{AAut}_D(\mathcal{T}_{d,k}) = V_{d,k}$.

A subgroup of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ of considerable interest is the group of elements that can be represented by a D-diagram with trivial coordinates and trivial permutation. This group is usually denoted $F_{d,k}$, and was introduced by Brown [Bro87] as a generalization of Thompson's group F. Obviously $F_{d,k}$ is a subgroup of $V_{d,k}$. As mentioned above, although D-diagrams are in general quite different from tree pair diagrams, in the case of $F_{d,k}$ we have the following result.

Lemma 3.61. *Let $k, d \geq 2$, $D \leq \mathrm{Sym}(d)$ and $f \in F_{d,k}$. If $[\varphi, T_-, T_+]$ is the unique D-reduced D-diagram representing f, then (T_-, T_+) is the unique reduced tree pair diagram associated to f.*

Proof. Saying that f belongs to $F_{d,k}$ is equivalent to saying that all the coordinates and the permutation of $[\varphi, T_-, T_+]$ are trivial. It follows that the pair (T_-, T_+) is a tree pair diagram representing f. Assume by contradiction that (T_-, T_+) is not reduced in the sense of tree pair diagrams. This means that a caret can be removed in both trees T_-, T_+, and at the level of the D-diagram we obtain a new D-diagram representing f, still having all its coordinates equal to the identity. Since D surely contains the identity, this implies that the D-diagram $[\varphi, T_-, T_+]$ we started from was not D-reduced, which is a contradiction. $\square$

3.3.2 Metric properties in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$

In this subsection we explain how our construction of D-diagrams gives rise to a pseudo-metric on the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, and compare it to the word-metric associated to a compact generating subset. This allows us to deduce that some remarkable subgroups of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ are quasi-isometrically embedded in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$.

A length function on $\mathrm{AAut}_D(\mathcal{T}_{d,k})$**.** Recall that a length function on a group Γ is a map $\mathcal{L} : \Gamma \to \mathbb{R}_+$ vanishing at the identity, and satisfying $\mathcal{L}(g^{-1}) = \mathcal{L}(g)$ and $\mathcal{L}(gh) \leq \mathcal{L}(g) + \mathcal{L}(h)$ for every $g, h \in \Gamma$. Note that any length function on a group Γ gives rise to a left invariant pseudo-metric on Γ defined by $\mathrm{dist}(g, h) = \mathcal{L}(g^{-1}h)$.

If $f, g : X \to \mathbb{R}_+$ are two functions on a set X, we say that f is dominated by g if there exists some constant $c > 0$ such that $f(x) \leq cg(x)$ for every $x \in X$.

If G is a locally compact group, a length function on G is said to be locally bounded if for every compact set K, $\sup\{L(k) : k \in K\} < \infty$. The proof of the following result is easy, and we leave it to the reader.

Lemma 3.62. *Let G be a locally compact compactly generated group. Then any locally bounded length function on G is dominated by any word-length function.*

It readily follows from the definition that the map $\mathcal{C}_D : \mathrm{AAut}_D(\mathcal{T}_{d,k}) \to \mathbb{R}_+$ vanishes on the compact open subgroup $W_k(D)$ and satisfies $\mathcal{C}_D(g^{-1}) = \mathcal{C}_D(g)$ for every $g \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$. Now if $g, h \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$, one can obtain a common expansion of the domain tree of g and the range tree of h by adding at most $\mathcal{C}_D(h)$ carets to the domain tree of g. This means that we can construct a D-diagram for the element gh having at most $\mathcal{C}_D(g)+\mathcal{C}_D(h)$ carets. A fortiori the D-reduced D-diagram associated to gh has at most $\mathcal{C}_D(g)+\mathcal{C}_D(h)$ carets, which proves that the function $\mathcal{C}_D$ is sub-additive. We therefore obtain the following:

Proposition 3.63. *For every $k, d \geq 2$ and finite permutation group $D \leq \mathrm{Sym}(d)$, the map $\mathcal{C}_D : \mathrm{AAut}_D(\mathcal{T}_{d,k}) \to \mathbb{R}_+$ is a locally bounded length function on $\mathrm{AAut}_D(\mathcal{T}_{d,k})$.*

Word metric in the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. The aim of this paragraph is to prove the following estimate for the word-metric in the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, and to show that the length function $\mathcal{C}_D(g)$ is not comparable to the word metric in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$.

Proposition 3.64. *Let $k, d \geq 2$, $D \leq \mathrm{Sym}(d)$ and let S be a compact generating subset of $\mathrm{AAut}(\mathcal{T}_{d,k})$. Then we have*

$$\mathcal{C}_D \preccurlyeq |\cdot|_S \preccurlyeq \mathcal{C}_D \log(1+\mathcal{C}_D).$$

Note that the lower bound immediately follows from Proposition 3.63 together with Lemma 3.62. The upper bound can be obtained by using a suitable decomposition of an element of $\mathrm{AAut}(\mathcal{T}_{d,k})$ as a product of an element of $W_k(D)$ times an element of $V_{d,k}$, and invoking the analoguous result for elements of $V_{d,k}$ that has been proved by Birget [Bir04]. However Birget's proof is rather laborious, and here we choose not to go into this direction. Nevertheless we give below a very elementary proof of the upper bound in the case when D is a transitive subgroup of $\mathrm{Sym}(d)$.

For every $n \geq 1$, we let O_n be the subgroup of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ consisting of elements that can be represented by a D-diagram of the form $[\varphi, B_n, B_n]$, where B_n is the subtree of $\mathcal{T}_{d,k}$ spanned by vertices of level at most n. Note that the subtree B_n has kd^{n-1} leaves, and the subgroup O_n admits a natural decomposition $O_n = U_n \rtimes \mathrm{Sym}(kd^{n-1})$, where U_n is the subgroup of $W_k(D)$ fixing pointwise the nth level of $\mathcal{T}_{d,k}$. By abuse we freely identify $\mathrm{Sym}(kd^{n-1})$ with the subgroup of O_n associated to this decomposition, and we think of $\mathrm{Sym}(kd^{n-1})$ as acting on the leaves of B_n, which are labeled from 1 to kd^{n-1}. Note that $(O_n)_n$ is an increasing sequence of compact open subgroups of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, ascending to the subgroup O of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ preserving the visual measure on the boundary $\partial_\infty \mathcal{T}_{d,k}$.

The content of the following lemma is essentially contained in the proof of Proposition 3.9 of [Bir04].

Lemma 3.65. *Let $k, d \geq 2$, and $D \leq \mathrm{Sym}(d)$. There exists a constant $c > 0$ such that for every $g \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$, there exist an integer $n \leq c \log(1 + \mathcal{C}_D(g))$ and $(f_1, o_n, f_2) \in F_{d,k} \times O_n \times F_{d,k}$ so that*

$$g = f_1 o_n f_2$$

and $\mathcal{C}_D(f_1), \mathcal{C}_D(f_1) \leq \mathcal{C}_D(g)$.

For every $n \geq 2$, we let $t_n \in O_n$ be the transposition $(d^{n-1}, d^{n-1} + 1)$.

Lemma 3.66. *Let $k, d \geq 2$ and let D be a transitive subgroup of $\mathrm{Sym}(d)$. Then for every $n \geq 2$, the set $S_n = \{t_n, W_k(D)\}$ is a compact generating subset of O_n, and*

$$\sup_{\gamma \in O_n} |\gamma|_{S_n} \leq 9kd^{n-1}.$$

Proof. Recall that for every $\gamma \in O_n$, there exists $u \in W_k(D)$ such that $u\gamma \in \mathrm{Sym}(kd^{n-1})$. We claim that it is enough to prove that the subgroup generated by S_n contains every transposition of $\mathrm{Sym}(kd^{n-1})$ and that transpositions have length at most 9. Indeed, since every permutation of $\mathrm{Sym}(kd^{n-1})$ is a product of at most $kd^{n-1} - 1$ transpositions, and by the above remark, this will prove that every $\gamma \in O_n$ has length at most $9(kd^{n-1} - 1) + 1 \leq 9kd^{n-1}$.

So let $1 \leq x < y \leq kd^{n-1}$. We prove that the transposition (x, y) belongs to $\langle S_n \rangle$ and has word length at most 9. We distinguish several cases according to the position of x and y.

First assume that $x \leq d^{n-1} < y$. Since the group D is transitive, there exists $u_{x,y} \in W_k(D)$ sending d^{n-1} to x and $d^{n-1} + 1$ to y, and it follows that $(x, y) = u_{x,y} t_n u_{x,y}^{-1}$ has word length at most 3 with respect to S_n.

Now if $x, y \leq d^{n-1}$, then we can find $u_x \in W_k(D)$ (resp. u_y) fixing $d^{n-1}+1$ and sending d^{n-1} to x (resp. y). Then we obtain that $(x, d^{n-1}+1) = u_x t_n u_x^{-1}$ and $(y, d^{n-1} + 1) = u_y t_n u_y^{-1}$ have length at most 3, and using the identity

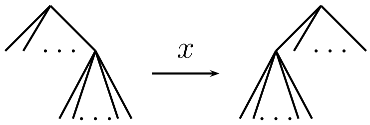
$$(x, y) = (x, d^{n-1} + 1)(y, d^{n-1} + 1)(x, d^{n-1} + 1)$$

we see that (x, y) has length at most 9.

The case $x, y \geq d^{n-1} + 1$ is similar. □

The following lemma explains how the transposition t_n can be constructed.

Lemma 3.67. *The exists $f \in F_{d,k}$ such that for every $n \geq 2$, $f^{n-2} t_2 f^{-(n-2)} = t_n$.*



Proof. Let us denote by x the element of F_d defined by the tree diagram:

We consider the direct product $F_d \times \cdots \times F_d$ of k copies of F_d embedded in $F_{d,k}$ in the natural way, and denote by f the element whose coordinates are $(x^{-1}, x, id, \ldots, id)$. The verification of the fact that f satisfies $f^{n-2} t_2 f^{-(n-2)} = t_n$ for every $n \geq 2$ is an easy computation, and we leave it to the reader. $\square$

In particular this last result implies that for a fixed generating subset of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, the length of t_n grows at most linearly with n. This, together with Lemma 3.66, immediately implies the following:

Corollary 3.68. *Let $k, d \geq 2$, let $D \leq \mathrm{Sym}(d)$ be a transitive permutation group and let S be a compact generating subset S of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. Then*

$$\sup_{\gamma \in O_n} |\gamma|_S \preccurlyeq n d^n.$$

We now give the proof of the upper bound of Proposition 3.64 in the case when $D \leq \mathrm{Sym}(d)$ is transitive.

Proof of Proposition 3.64. Let S be a compact generating subset of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, and let $g \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$. According to Lemma 3.65, there exists an integer $n \leq c_1 \log(1 + \mathcal{C}_D(g))$ for some constant $c_1 > 0$, and $(f_1, o_n, f_2) \in F_{d,k} \times O_n \times F_{d,k}$ so that $g = f_1 o_n f_2$ and $\mathcal{C}_D(f_1), \mathcal{C}_D(f_2) \leq \mathcal{C}_D(g)$. Now we can apply Theorem 3.71 to each f_i, which together with Lemma 3.61 imply that $|f_i|_S \leq c_2 \mathcal{C}_D(f_i)$ for some constant $c_2 > 0$. Therefore we obtain

$$|g|_S \leq |f_1|_S + |f_2|_S + |o_n|_S \preccurlyeq \mathcal{C}_D(f_1) + \mathcal{C}_D(f_2) + |o_n|_S \preccurlyeq \mathcal{C}_D(g) + + |o_n|_S \preccurlyeq \mathcal{C}_D(g) + n d^n,$$

where the last inequality follows from Corollary 3.68. Since $n \leq c_1 \log(1 + \mathcal{C}_D(g))$, we obtain

$$|g|_S \preccurlyeq \mathcal{C}_D(g) + \mathcal{C}_D(g) \log(1 + \mathcal{C}_D(g)) \simeq \mathcal{C}_D(g) \log(1 + \mathcal{C}_D(g)),$$

and the proof is complete. $\square$

The end of this paragraph is devoted to the proof of the fact that the length function $\mathcal{C}_D$ is not comparable to the word metric in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. Towards this result, we need the following volume computation. Note that similar estimates appear in [BCGM12].

Lemma 3.69. *Let $k, d \geq 2$, $D \leq \mathrm{Sym}(d)$ and let μ be a left-invariant Haar measure on $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ normalized so that $\mu(W_k(D)) = 1$. Then for every $n \geq 1$, we have*

$$\mu(O_n) = \frac{(kd^{n-1})!}{|D|^{e_n}},$$

where

$$e_n = k\frac{d^{n-1}-1}{d-1}.$$

Proof. Recall that we denote by U_n the subgroup of $W_k(D)$ fixing pointwise the ball B_n of radius n around the root of $\mathcal{T}_{d,k}$, which has kd^{n-1} leaves. A direct computation shows that the sequence $x_n = (W_k(D) : U_n)$ satisfies $x_1 = 1$ and $x_{n+1} = |D|^{kd^{n-1}} x_n$ for every $n \geq 1$. We easily infer from this that $x_n = |D|^{e_n}$ for every $n \geq 1$, and in particular the Haar measure of U_n is

$$\mu(U_n) = \mu(W_k(D)) \times x_n^{-1} = |D|^{-e_n}.$$

Now since U_n is a normal subgroup of O_n such that the quotient group O_n/U_n is isomorphic to the symmetric group $\mathrm{Sym}(kd^{n-1})$, we have

$$\mu(O_n) = |\mathrm{Sym}(kd^{n-1})| \times \mu(U_n) = (kd^{n-1})! \times |D|^{-e_n}.$$

□

Proposition 3.70. *Let $k, d \geq 2$, $D \leq \mathrm{Sym}(d)$ and let S be a compact generating subset of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$. For every $n \geq 1$, set*

$$\ell_S(n) := \sup |\gamma|_S,$$

where γ ranges over the compact open subgroup O_n. Then

$$\ell_S(n) \succcurlyeq nd^n.$$

Proof. Let μ be a left-invariant Haar measure on $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, normalized so that $\mu(W_k(D)) = 1$. By definition of $\ell_S(n)$, the subgroup O_n lies in the ball of radius $\ell_S(n)$ for the word metric associated to S. It follows that the Haar measure of O_n is at most $\mu(S^{\ell_S(n)})$, so there exists $\alpha \geq 1$ such that $\mu(O_n) \leq \alpha^{\ell_S(n)}$. Combined with Lemma 3.69, this inequality yields

$$(kd^{n-1})! \leq \alpha^{\ell_S(n)} \times |D|^{e_n},$$

where

$$e_n = k\frac{d^{n-1}-1}{d-1}.$$

Now taking the log and using that $x \log x \leq 2 \log x!$, we obtain that

$$(n-1)d^{n-1} \preccurlyeq \ell_S(n) + e_n \simeq \ell_S(n) + d^n,$$

and the conclusion easily follows. □

Non-distortion of certain subgroups. In this paragraph we show how the interpretation of elements of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ in terms of D-diagrams yields insights about the study of distortion of certain subgroups in $\mathrm{AAut}_D(\mathcal{T}_{d,k})$.

Recall that $F_{d,k}$ is the subgroup of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ of elements that can be represented by a D-diagram with trivial coordinates and trivial permutation. Burillo showed [Bur99] that Thompson's group F satisfies the pleasant property that the size of a reduced tree pair diagram representing an element is comparable to the length of the element with respect to some finite generating subset. This result has been generalized to the groups $F_{d,k}$ in [BCS01, Theorem 5].

Theorem 3.71. *For every $k, d \geq 2$, the number of carets of the unique tree pair diagram representing an element $f \in F_{d,k}$ is comparable to the word length of f with respect to any finite generating subset of $F_{d,k}$.*

Combined with the lower bound in Proposition 3.64 and the fact that the notion of reduced tree pair diagrams correspond to the notion of D-reduced D-diagrams for elements of $F_{d,k}$ (see Lemma 3.61), we immediately deduce the following result.

Proposition 3.72. *For every $d, k \geq 2$ and $D \leq \mathrm{Sym}(d)$, the group $F_{d,k}$ is quasi-isometrically embedded inside $\mathrm{AAut}_D(\mathcal{T}_{d,k})$.*

Since the group $F_{d,k}$ is known to have quasi-isometrically embedded $\mathbb{Z}^n$-subgroups for all $n \geq 1$, the same is true in the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ for every $k, d \geq 2$ and $D \leq \mathrm{Sym}(d)$. In particular Neretin's group has this property.

For the remaining of this subsection we take $D = \mathrm{Sym}(d)$. Recall that if $g \in \mathrm{AAut}(\mathcal{T}_{d,k})$, we denote by $\mathcal{C}(g)$ the number of carets of the unique $\mathrm{Sym}(d)$-diagram representing g that is $\mathrm{Sym}(d)$-reduced. Recall also that the set of ordered vertices of level one in $\mathcal{T}_{d,k}$ is denoted $\{a_1, \ldots, a_k\}$. Note that the subtree of $\mathcal{T}_{d,k}$ obtained by forgetting vertices that do not hang below either a_1 or a_2, is isomorphic to the tree $\mathcal{T}_{d,2}$. In particular we have a natural embedding of $\mathrm{AAut}(\mathcal{T}_{d,2})$ inside $\mathrm{AAut}(\mathcal{T}_{d,k})$. Now recall that $\mathrm{AAut}(\mathcal{T}_{d,2})$ is nothing but the group $\mathrm{AAut}(T_{d+1})$ of almost automorphisms of the *non-rooted* regular tree T_{d+1} of degree $d+1 \geq 3$, so that we have an embedding of $\mathrm{AAut}(T_{d+1})$ inside $\mathrm{AAut}(\mathcal{T}_{d,k})$. This embedding can be thought of as the following. We fix an edge e_0 of T_{d+1}, whose vertices are denoted v_1 and v_2. We glue the tree T_{d+1} on $\mathcal{T}_{d,k}$ by putting the middle point of the edge e_0 on the root of $\mathcal{T}_{d,k}$, and by gluing the half tree of T_{d+1} emanating from e_0 and containing v_1 (resp. v_2) onto the subtree of $\mathcal{T}_{d,k}$ hanging below a_1 (resp. a_2). From now we freely identify the tree T_{d+1} and its copy inside $\mathcal{T}_{d,k}$ that we just described. Note in particular that the original edge e_0 is divided into two edges in $\mathcal{T}_{d,k}$.

In particular we have an embedding of the automorphism group $\mathrm{Aut}(T_{d+1})$ inside $\mathrm{AAut}(\mathcal{T}_{d,k})$, and the aim of this paragraph is to prove the following result.

Proposition 3.73. *For every $k, d \geq 2$, the group $\mathrm{Aut}(T_{d+1})$ is quasi-isometrically embedded inside $\mathrm{AAut}(\mathcal{T}_{d,k})$.*

The first step towards Proposition 3.73 is the following description of the unique $\mathrm{Sym}(d)$-reduced $\mathrm{Sym}(d)$-diagram associated to an element of $\mathrm{Aut}(T_{d+1})$. For every $g \in \mathrm{Aut}(T_{d+1})$, we let $D(g)$ be the unique minimal complete subtree of T_{d+1} containing e_0 and $g^{-1}(e_0)$, and let $R(g)$ be the image of $D(g)$ by g. By construction $D(g)$ and $R(g)$ both contain the edge e_0, and g maps the complement of $D(g)$ in T_{d+1} onto the complement of $R(g)$.

The distance between a vertex v and an edge e is by definition the minimal length of a path from v to a vertex of e, and the distance between two edges e and e' is the maximum of $d(v_1, e)$ and $d(v_2, e)$, where v_1, v_2 are the vertices of e'.

Lemma 3.74. *Let $k, d \geq 2$. For every $g \in \mathrm{Aut}(T_{d+1})$, the domain and range trees of the unique $\mathrm{Sym}(d)$-diagram representing g that is $\mathrm{Sym}(d)$-reduced, are respectively $D(g)$ and $R(g)$. In particular we have $\mathcal{C}(g) = d(e_0, g(e_0))$.*

Proof. Write $T_{d+1} \setminus D(g) = T_-^1 \cup \ldots \cup T_-^n$ and $T_{d+1} \setminus R(g) = T_+^1 \cup \ldots \cup T_+^n$, where each $T_\pm^i$ is isomorphic to a rooted d-regular tree. By construction g maps the complement of $D(g)$ in T_{d+1} onto the complement of $R(g)$, so there exists a permutation $\sigma \in \mathrm{Sym}(n)$ and automorphisms of rooted d-regular trees $g_1, \ldots, g_n$ such that g maps T_-^i onto $T_+^{\sigma(i)}$ via g_i for each i. Clearly the diagram having for domain and range trees $D(g)$ and $R(g)$, for permutation σ and for coordinates the g_i's represents the element g. We leave the reader to convince himself that this diagram is reduced. $\square$

The following lemma is easy, so we omit the proof.

Lemma 3.75. *Let e_0 be a fixed edge of T_d. Then for any compact generating subset S of $\mathrm{Aut}(T_{d+1})$, there exists $c > 0$ (depending on e_0 and S) so that for any $g \in \mathrm{Aut}(T_{d+1})$,*

$$c^{-1}(1 + d(e_0, g \cdot e_0)) \leq |g|_S \leq c(1 + d(e_0, g \cdot e_0)).$$

We now give the proof of the non-distortion of $\mathrm{Aut}(T_{d+1})$ inside $\mathrm{AAut}(\mathcal{T}_{d,k})$.

Proof of Proposition 3.73. Let e_0 still denote a fixed edge of T_{d+1}, let S_1 be a compact generating subset of $\mathrm{Aut}(T_{d+1})$ and S_2 a compact generating subset of $\mathrm{AAut}(\mathcal{T}_{d,k})$. According to the lower bound in Proposition 3.64, there exists $c_1 > 0$ such that for every $g \in \mathrm{Aut}(T_{d+1})$, we have $\mathcal{C}(g) \leq c_1 |g|_{S_2}$. Now by Lemma 3.74, the number $\mathcal{C}(g)$ is equal to $d(e_0, g(e_0))$. But by Lemma 3.75, we have $|g|_{S_1} \leq c_2(d(e_0, g(e_0)) + 1)$ for some constant $c_2 > 0$, so we finally obtain $|g|_{S_1} \leq c_2(1 + c_1|g|_{S_2})$. This proves the statement. $\square$

A remark on quasi-automorphism groups. Consider the set of permutations of vertices of $\mathcal{T}_{d,k}$ fixing the root and preserving the tree structure almost everywhere, i.e. sending all but finitely many edges onto edges. Permutations of these types are called quasi-automorphisms. The set $\mathrm{QAut}(\mathcal{T}_{d,k})$ of quasi-automorphisms of $\mathcal{T}_{d,k}$ is a group admitting a natural topology, defined by saying that a basis of neighbourhoods of the identity is given by a basis of neighbourhoods of the identity in the automorphism group $\mathrm{Aut}(\mathcal{T}_{d,k})$. Therefore it is a totally disconnected locally compact group, and it is not hard to see that it is compactly generated.

The almost automorphism group $\mathrm{AAut}(\mathcal{T}_{d,k})$ is isomorphic to the quotient of $\mathrm{QAut}(\mathcal{T}_{d,k})$ by its discrete normal subgroup of finitary permutations. In other words, we have a short exact sequence of locally compact groups

$$1 \longrightarrow \mathrm{Sym}_0(\mathcal{T}_{d,k}) \longrightarrow \mathrm{QAut}(\mathcal{T}_{d,k}) \longrightarrow \mathrm{AAut}(\mathcal{T}_{d,k}) \longrightarrow 1.$$

In particular we deduce from Proposition 3.73 that $\mathrm{Aut}(T_{d+1})$ is quasi-isometrically embedded inside $\mathrm{QAut}(T_d)$. Similarly one can check that the group $F_{d,k}$ admits a section in $\mathrm{QAut}(\mathcal{T}_{d,k})$, which must be quasi-isometrically embedded by Proposition 3.72.

3.3.3 A commensurating action

For all this subsection we fix two integers $k, d \geq 2$ and a finite permutation group $D \leq \mathrm{Sym}(d)$.

Construction. Recall that if G is a group and X a G-set, a subset A of X is commensurated if for every $g \in G$, the symmetric difference $gA \triangle A$ has finite cardinality.

Although it has never been explicitly written, the idea of the following construction already appeared in the literature [Far03, Nav02].

Recall that we denote by a_1 the first vertex of $\mathcal{T}_{d,k}$ of level one. For every vertex v, we denote by $T^{(v)}$ the subtree of $\mathcal{T}_{d,k}$ hanging below the vertex v. We will say that an almost automorphism sends $\partial T^{(v)}$ onto $\partial T^{(v')}$ D-isometrically if g sends $\partial T^{(v)}$ onto $\partial T^{(v')}$ via an element of $W(D)$. Write $G = \mathrm{AAut}_D(\mathcal{T}_{d,k})$, and let H be the open subgroup of G stabilizing $\partial T^{(a_1)}$ and acting on it D-isometrically. More generally we let A be the subset of G consisting of almost automorphisms sending $\partial T^{(a_1)}$ D-isometrically onto $\partial T^{(v)}$ for some vertex v of level at least one. The subset $A \subset G$ is a union of left cosets of the subgroup H, so that we will freely identify the subset A with its image in G/H.

Proposition 3.76. *For every $k, d \geq 2$ and $D \leq \mathrm{Sym}(d)$, the action of $G = \mathrm{AAut}_D(\mathcal{T}_{d,k})$ on G/H commensurates the subset A. More precisely, we have*

$$\#(gA \triangle A) = 2\mathcal{C}_D(g)$$

for every $g \in G$.

Proof. Write $A = \cup_v g_v H$, where v ranges over the of vertices of $\mathcal{T}_{d,k}$ of level at least one, and g_v is an element of G sending $\partial T^{(x_1)}$ onto $\partial T^{(v)}$ D-isometrically.

Let $g \in G$, and $[\varphi, T_-, T_+]$ the unique D-reduced D-diagram representing g, and let v be a vertex of level at least one. If the vertex v is not a node of the domain tree T_-, then by definition the almost automorphism g sends $\partial T^{(v)}$ onto $\partial T^{(v')}$ for some vertex v' via a an element of $W(D)$. It follows the element gg_v sends $\partial T^{(x_1)}$ D-isometrically onto $\partial T^{(v')}$, and therefore $gg_v H \in A$. Conversely if v is a node of T_-, then since the diagram $[\varphi, T_-, T_+]$ is supposed to be reduced, this exactly means that the element g does not send $\partial T^{(v)}$ D-isometrically onto $\partial T^{(v')}$ for some vertex v', and therefore $gg_v H \notin A$. It follows that the number of elements of A which are not sent into A by g is equal to the number of nodes in the tree T_-, which is nothing but $\mathcal{C}_D(g)$. By applying the same argument to the inverse of g, we obtain that the number of elements of gA which does not belong to A is equal to $\mathcal{C}_D(g)$ as well, and the proof is complete. □

Remark 3.77. By a general principle (see for instance [Cor13]), we deduce from the previous construction that there exist a continuous action of $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ on a CAT(0) cube complex, and a vertex x_0 such that in the ℓ^1-metric $d(x_0, gx_0) = 2\mathcal{C}_D(g)$ for every $g \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$. From this point of view, the fact that the function $\mathcal{C}_D$ is not comparable to the word metric, which has been proved in the previous subsection, means that the orbital map from $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ into its CAT(0) cube complex is not a quasi-isometric embedding.

Application to L^p-space compression. In this paragraph we explain how one can combine the results established previously with the techniques developed in [AGS06] to study the L^p-space compression of the groups $\mathrm{AAut}_D(\mathcal{T}_{d,k})$.

Recall that if G is a locally compact compactly generated group, a map $f : G \to L^p(X, \mu)$ is a *coarse embedding* if for every $r > 0$,

$$\sup \{\|f(g) - f(h)\| : d_S(g, h) \leq r\} < \infty,$$

and such that the associated *compression function*

$$\rho_f(r) = \inf\{\|f(g) - f(h)\| \,:\, d_S(g,h) \geq r\}$$

is proper.

The L^p-space compression exponent $\alpha_p(G)$ of G is the supremum of all $\alpha \in [0,1]$ for which there exists a coarse embedding $f : G \to L^p(X,\mu)$ with $\rho_f(r) \geq Cr^\alpha$ for some constant $C > 0$. Similarly, the equivariant L^p-space compression exponent $\alpha_p^\sharp(G)$ is obtained when restricting to G-equivariant coarse embeddings into a space $L^p(X,\mu)$ equipped with a G-affine isometric action. Clearly, $\alpha_p^\sharp(G) \leq \alpha_p(G)$.

Arzhantseva, Guba and Sapir proved in [AGS06] that Thompson's group F has a Hilbert space compression exponent equal to $1/2$, by exhibiting a family of *cubes* in F and using a geometric inequality from [Enf69]. Actually they proved something stronger, namely that any coarse embedding of F into a Hilbert space has compression function $\preccurlyeq x^{1/2}\log(x)$. Their techniques immediately adapt to the groups $F_{d,k}$ and the case of L^p-spaces for $p \in [1,2]$, using the fact that such a space has roundness p (see [Enf69]), which yields that any coarse embedding of $F_{d,k}$ into a L^p-space has compression function $\preccurlyeq x^{1/p}\log(x)$ for $p \in [1,2]$. Now since the group $F_{d,k}$ is quasi-isometrically embedded inside $\mathrm{AAut}_D(\mathcal{T}_{d,k})$ according to Proposition 3.72, a fortiori the same result holds for the group $\mathrm{AAut}_D(\mathcal{T}_{d,k})$, and in particular one has $\alpha_p(\mathrm{AAut}_D(\mathcal{T}_{d,k})) \leq 1/p$.

On the other hand, using a general argument (see for example [Cor13]), one can deduce from the construction of the previous paragraph an equivariant embedding $f : \mathrm{AAut}_D(\mathcal{T}_{d,k}) \to \ell^p(X)$ for every $p \geq 1$, such that $\|f(g)\|_p = (2\mathcal{C}_D(g))^{1/p}$ for every $g \in \mathrm{AAut}_D(\mathcal{T}_{d,k})$. One can check using the upper bound in Proposition 3.64 that such an embedding has compression function $\succcurlyeq x^\alpha$ for every $\alpha < 1/p$, and in particular $\alpha_p^\sharp(\mathrm{AAut}_D(\mathcal{T}_{d,k})) \geq 1/p$ for every $p \geq 1$.

Therefore we have proved the following result.

Proposition 3.78. *For every $d,k \geq 2$, $D \leq \mathrm{Sym}(d)$ and $p \in [1,2]$, we have*

$$\alpha_p^\sharp(\mathrm{AAut}_D(\mathcal{T}_{d,k})) = \alpha_p(\mathrm{AAut}_D(\mathcal{T}_{d,k})) = 1/p.$$

Chapter 4

Groups acting on trees with almost prescribed local action

In this chapter we define and study a family of groups acting on a regular tree, and whose local action depends on some finite permutation groups. In Section 4.1 we establish some preliminary results, and exhibit some simple groups in this family. In Section 4.2 we make the observation that these groups have asymptotic dimension one, and prove that most of their subgroups are not compactly presented. Finally in Section 4.3 we prove that, under the assumption that F is transitive, the group $G(F)$ has property (PW).

I am very grateful to the authors of [BCGM12] who pointed out to my attention the definition of the groups under consideration here, and especially to Pierre-Emmanuel Caprace whose comments improved the content of this chapter.

4.1 Introduction

4.1.1 Notation and terminology

Let $d \geq 3$ be an integer, and let $\Omega = \{1, \ldots, d\}$ be the set of positive integers which are at most d. Every partition of Ω gives rise to a subgroup of $\mathrm{Sym}(\Omega)$ consisting of permutations of Ω stabilizing each block of the partition. Such a subgroup is called a Young subgroup of $\mathrm{Sym}(\Omega)$, and is naturally isomorphic to the direct product of the symmetric groups on each block of the partition. In particular when $F \leq \mathrm{Sym}(\Omega)$ is a finite permutation group, we can consider the Young subgroup $\hat{F} \leq \mathrm{Sym}(\Omega)$ associated to the partition of Ω into F-orbits, which is nothing but the subgroup of $\mathrm{Sym}(\Omega)$ stabilizing the orbits of F. Note that we always have $F \leq \hat{F}$, and $\hat{F} = \mathrm{Sym}(\Omega)$ if and only if the permutation group F is transitive.

Let T_d be the regular tree of degree d, whose vertex set will be denoted $\mathsf{V}(T_d)$ and set of non-oriented edges $\mathsf{E}(T_d)$. Let us choose and fix a coloring $c : \mathsf{E}(T_d) \to \Omega$ such that neighbouring edges haves different colors. In other words, for each vertex $v \in \mathsf{V}(T_d)$, the map c restricts to a bijection c_v from the set $\mathsf{E}(v)$ of edges containing v to Ω. For each $g \in \mathrm{Aut}(T_d)$ and for each vertex $v \in \mathsf{V}(T_d)$, the automorphism g induces a bijection $g_v : \mathsf{E}(v) \to \mathsf{E}(gv)$, which gives rise to a permutation $\sigma(g, v)$ of Ω defined by $\sigma(g, v) = c_{gv} \circ g_v \circ c_v^{-1}$. These permutations satisfy the rules

$$\sigma(gh, v) = \sigma(g, h(v))\sigma(h, v) \text{ and } \sigma(g^{-1}, v) = \sigma(g, g^{-1}(v))^{-1} \tag{4.1}$$

for every $g, h \in \mathrm{Aut}(T_d)$ and every $v \in \mathsf{V}(T_d)$.

Remark 4.1. We easily see that an automorphism $g \in \mathrm{Aut}(T_d)$ is uniquely determined by the value of some vertex together with the collection of permutations $\sigma(g, v)$, where $v \in \mathsf{V}(T_d)$. This observation will be used repeatedly in this chapter.

For every vertex $v \in \mathsf{V}(T_d)$ and every $i \in \Omega$, we denote by $e_i(v)$ the unique edge emanating from v having color i, and by v^i the vertex of T_d connected to v by $e_i(v)$. For every $n \geq 0$, we also let $\mathcal{B}(v, n)$ be the subtree of T_d spanned by vertices at distance at most n from the vertex v.

We fix an edge e_0 of T_d such that $c(e_0) = 1$, whose vertices will be denoted v_0 and v_1. To every vertex $v \in \mathsf{V}(T_d)$, we associate the subtree of T_d consisting of vertices whose projection to the geodesic between v and e_0 is the vertex v. This subtree is naturally isomorphic to an infinite regular rooted tree of degree d, and will be denoted $\mathsf{L}(v)$.

A vertex v of a subtree T of T_d is called a leaf of T if v has only one neighbour in T. A subtree T of T_d is said to be complete if all of its vertices which are not leaves have degree d in T. If T is a complete subtree of T_d, we denote by $\mathsf{IV}(T)$ the set of internal vertices of T, i.e. vertices of T which are not leaves.

For every subtree T of T_d and every group G acting on T_d, we denote by G_T the subgroup of G fixing T pointwise. For example if $T = e$ is a single edge, then G_e is the subgroup of G fixing both vertices of e. The subgroup of G generated by the subgroups G_e, where e ranges over the set of edges of T_d, will be denoted G^+. Note that G^+ is a normal subgroup of G, and if G is endowed with the topology induced from $\mathrm{Aut}(T_d)$, then G^+ is open in G.

4.1.2 A local rigidity condition almost everywhere

Definitions. In the remainder of this section, we fix an integer $d \geq 3$ and a subgroup $F \leq \mathrm{Sym}(\Omega)$ of the symmetric group on the set $\Omega = \{1, \ldots, d\}$.

Recall that we denote by $\hat{F}$ the subgroup of $\mathrm{Sym}(\Omega)$ stabilizing the orbits of F.

The universal Burger-Mozes' group [BM00] is defined as the subgroup of automorphisms of T_d whose local action is prescribed by F,

$$U(F) = \{g \in \mathrm{Aut}(T_d) \, : \, \sigma(g, v) \in F \text{ for all } v \in \mathsf{V}(T_d)\} \,.$$

It is a closed subgroup of $\mathrm{Aut}(T_d)$, which is discrete if and only if the permutation group F acts freely on Ω, and whose conjugacy class in $\mathrm{Aut}(T_d)$ does not depend on the choice of the coloring $c : \mathsf{E}(T_d) \to \Omega$. Clearly $U(F')$ is a subgroup of $U(F)$ when $F' \leq F \leq \mathrm{Sym}(\Omega)$. Combined with the fact that the group $U(\{1\})$ acts transitively on the set $\mathsf{V}(T_d)$, this observation implies that $U(F)$ is always vertex-transitive.

One can relate the combinatorics of the finite permutation group F with the properties of the group $U(F)$. For example one shows that the group $U(F)^+$ has finite index in $U(F)$ if and only if the group F acts transitively on Ω and is generated by its point stabilizers. When this is so the subgroup $U(F)^+$ has index two in $U(F)$. Moreover the group $U(F)$ acts transitively on the boundary $\partial_\infty T_d$ if and only if $U(F)^+$ acts transitively on $\partial_\infty T_d$, which is equivalent to asking that the group F is 2-transitive on Ω (see for example [BM00, Lemma 3.1.1]).

The definition of the groups under consideration here can be seen as a relaxation of the definition of the groups $U(F)$, in the sense that the local action is prescribed almost everywhere only. More precisely, we let

$$G(F) = \{g \in \mathrm{Aut}(T_d) \, : \, \sigma(g, v) \in F \text{ for all but finitely many } v \in \mathsf{V}(T_d)\} \,.$$

It readily follows from the multiplication rules (4.1) that $G(F)$ is a subgroup of $\mathrm{Aut}(T_d)$, and of course one has $U(F) \leq G(F)$.

For every $g \in G(F)$, we let $T(g)$ be the unique complete subtree of T_d such that $\sigma(g, v) \in F$ for every vertex $v \in \mathsf{V}(T_d)$ that is not an internal vertex of $T(g)$, and being minimal for this property. Equivalently, $T(g)$ can be defined as the 1-neighbourhood of the complete subtree of T_d spanned by vertices v such that $\sigma(g, v) \notin F$.

Lemma 4.2. *Let $g \in G(F)$, and denote by $T = T(g)$, $U_T = U(F)_T$ and $U_{g(T)} = U(F)_{g(T)}$. Then one has $gU_Tg^{-1} = U_{g(T)}$.*

Proof. Noting that $g(T) = T(g^{-1})$, we see that by symmetry it is enough to prove that $gU_Tg^{-1} \subset U_{g(T)}$. The fact that gU_Tg^{-1} fixes pointwise $g(T)$ is easy, so the only thing that needs to be checked is that gU_Tg^{-1} lies in $U(F)$. So let $u \in U_T$ and $v \in \mathsf{V}(T_d)$. According to (4.1), one has

$$\sigma(gug^{-1}, v) = \sigma(g, ug^{-1}(v)) \, \sigma(u, g^{-1}(v)) \, \sigma(g, g^{-1}(v))^{-1}. \qquad (4.2)$$

As observed previously, the element gug^{-1} fixes pointwise $g(T)$, so we only have to deal with the case when v is not an internal vertex of $g(T)$, i.e. when $g^{-1}(v)$ is not an internal vertex of T. This implies that $ug^{-1}(v)$ is not an internal vertex of T either, and by construction of T we deduce that $\sigma(g, g^{-1}(v)), \sigma(g, ug^{-1}(v)) \in F$. Since $\sigma(u, g^{-1}(v)) \in F$ by definition of u, it follows from 4.2 that $\sigma(gug^{-1}, v) \in F$. $\square$

This implies (see Lemma 3.13) that there exists a group topology on $G(F)$ such that the inclusion of $U(F)$ in $G(F)$ is continuous and open. In particular the group $G(F)$ is discrete if and only if F acts freely on Ω. We point out that in general this topology on $G(F)$ is not the topology induced from $\mathrm{Aut}(T_d)$ (see Corollary 4.6).

Let $v \in \mathsf{V}(T_d)$ being fixed. For every $n \geq 0$, we denote by $K_n(v)$ the set of automorphisms $g \in \mathrm{Aut}(T_d)$ fixing the vertex v and such that $\sigma(g, w) \in F$ for every vertex w that is not in $\mathcal{B}(v, n)$. Again, it follows from (4.1) that $K_n(v)$ is a subgroup of $G(F)$. For example $K_0(v)$ is the subgroup of $\mathrm{Aut}(T_d)$ consisting of elements g fixing v and such that $\sigma(g, w) \in F$ for every vertex $w \neq v$. Each $K_n(v)$ contains the stabilizer of the vertex v in $U(F)$ as a finite index subgroup. The latter being compact open, $K_n(v)$ is a compact open subgroup of $G(F)$. The stabilizer of the vertex v in $G(F)$ is nothing but the increasing union

$$G(F)_v = \bigcup_{n \geq 0}^{\nearrow} K_n(v),$$

so in particular $G(F)_v$ is a locally elliptic open subgroup of $G(F)$.

Remark 4.3. One can check that the index two subgroup $G(F) \cap \mathrm{Aut}(T_d)^+$ of $G(F)$ is generated by the vertex stabilizers $G(F)_v$, $v \in \mathsf{V}(T_d)$. Those being locally elliptic open subgroups, it follows that $G(F)$ is generated by its compact subgroups. Since any continuous homomorphism from a compact group to the group $\mathbb{R}$ must vanish, we deduce that any continuous homomorphism $G(F) \to \mathbb{R}$ must be trivial. In particular the group $G(F)$ is unimodular.

Preliminary results. The following result shows that, although elements of $G(F)$ are not required to act locally like F everywhere, their local action exhibits some rigidity.

Lemma 4.4. *For every $g \in G(F)$ and every vertex $v \in \mathsf{V}(T_d)$, the permutation $\sigma(g, v)$ stabilizes the orbits of F in Ω. In other words, the group $G(F)$ is contained in $U(\hat{F})$.*

Proof. For a given $g \in G(F)$, let us consider the set V_g of vertices for which the conclusion does not hold. We want to prove that V_g is empty. It is not hard to see that every vertex in V_g must have at least two neighbours that also belong to V_g. It follows that if V_g is not empty, then it must contain an infinite subtree, which is a impossible by definition of $G(F)$. $\square$

For every permutation group $F' \leq \mathrm{Sym}(\Omega)$ such that $F \leq F' \leq \hat{F}$, we denote by $G(F, F')$ the subgroup of $G(F)$ consisting of elements $g \in G(F)$ such that $\sigma(g, v) \in F'$ for all $v \in \mathsf{V}(T_d)$, i.e. $G(F, F') = G(F) \cap U(F')$. Clearly we have $G(F, F') \leq G(F, F'')$ as soon as $F' \leq F''$, and $G(F, F) = U(F)$ and $G(F, \mathrm{Sym}(\Omega)) = G(F)$. Therefore the family of subgroups $G(F, F') \leq G(F)$ interpolates between $U(F)$ and $G(F)$ when F' ranges over subgroups of $\hat{F}$ containing F. Note that $G(F, F')$ is always an open subgroup of $G(F)$.

In some sense, the following result can be seen as a converse of Lemma 4.4.

Lemma 4.5. *Let $v \in \mathsf{V}(T_d)$ and $n \geq 0$. If $h \in \mathrm{Aut}(T_d)$ is such that $\sigma(h, w) \in F'$ for every vertex w in $\mathcal{B}(v, n)$, then there exists $g \in G(F, F')$ such that g and h coincide on $\mathcal{B}(v, n+1)$ and $\sigma(g, w) \in F$ for every vertex w that is not in $\mathcal{B}(v, n)$.*

Proof. We denote by $x_1, \ldots, x_k$ the vertices of T_d which are at distance exactly n from the vertex v. For every $j \in \{1, \ldots, k\}$, we denote by $V_j(T)$ the set of vertices w such that the unique path between v and w contains the vertex x_j.

Since the group $G(F, F')$ acts transitively on the set of vertices of T_d (as it is already the case for $U(F)$), we may assume that h fixes the vertex v. So we impose that g fixes v as well, and therefore giving the value of $\sigma(g, w)$ for every vertex w is enough to define the element g. Naturally we put $\sigma(g, w) = \sigma(h, w)$ for every vertex w in $\mathcal{B}(v, n)$. This implies that g and h coincide on $\mathcal{B}(v, n+1)$, and we must explain how extend the definition of g to an element of $G(F, F')$.

For every $j \in \{1, \ldots, k\}$ and every $i \in \Omega$, there exists $\sigma_{i,j} \in F$ such that $\sigma(h, x_j)(i) = \sigma_{i,j}(i)$ because $\sigma(h, x_j)$ stabilizes the orbits of F. Now for every vertex $w \in V_j(T)$ different from x_j, we let $\sigma(g, w) = \sigma_{i(w),j}$, where $i(w)$ is the color of the unique edge emanating from x_j and separating x_j and w. The verification that the definition of the element g is consistent and that $g \in G(F, F')$ is easy, and we leave it to the reader. $\square$

Recalling that a basis of neighbourhoods for the topology on the group $\mathrm{Aut}(T_d)$ is given by pointwise stabilizers of finite sets, we immediately deduce the following result.

Corollary 4.6. *The closure of $G(F, F')$ in the topological group* $\mathrm{Aut}(T_d)$ *is the group $U(F')$. In particular $G(F)$ is dense in* $\mathrm{Aut}(T_d)$ *if and only if the permutation group F is transitive.*

For every $n \geq 0$ and every vertex $v \in \mathsf{V}(T_d)$, we denote by $K_{n,F'}(v)$ the intersection between $K_n(v)$ and $G(F, F')$. This is the compact open subgroup of $G(F, F')$ consisting of elements $g \in G(F, F')$ fixing v and such that $\sigma(g, w) \in F$ for all vertices $w \in \mathsf{V}(T_d)$ at distance at least $n + 1$ from v.

Proposition 4.7. *Let $k \geq 0$, and $g \in G(F, F')$ such that there are at most k vertices $v \in \mathsf{V}(T_d)$ such that $\sigma(g, v) \notin F$. Then there exist vertices $v_1, \ldots, v_k \in \mathsf{V}(T_d)$ and elements $\gamma \in U(F)$, $g_i \in K_{0,F'}(v_i)$, such that $g = \gamma g_1 \cdots g_k$.*

Proof. We argue by induction on the number k. The result is clear when $k = 0$ by definition. Now let $g \in G(F, F')$ having at most $k + 1$ vertices w for which $\sigma(g, w) \notin F$, and let v be one of these vertices. Since the group $U(F)$ is transitive on the set of vertices, there exists $\gamma_1 \in U(F)$ such that $g' = \gamma_1 g$ fixes v. Note that since $\gamma_1 \in U(F)$, for every vertex w we have $\sigma(g, w) \notin F$ if and only if $\sigma(g', w) \notin F$. According to Lemma 4.5 applied with $n = 0$, there exists $g_v \in K_{0,F'}(v)$ acting like g' on the star around the vertex v. Therefore if we let $g'' = g' g_v^{-1} = \gamma_1 g g_v^{-1}$, then there are at most k vertices w for which $\sigma(g'', w) \notin F$. So by induction hypothesis there exist $v_1, \ldots, v_k \in \mathsf{V}(T_d)$ and $\gamma_2 \in U(F)$, $g_{v_i} \in K_{0,F'}(v_i)$, such that $g'' = \gamma_2 g_{v_1} \cdots g_{v_k}$, which can be rewritten $g = (\gamma_1^{-1}\gamma_2) g_{v_1} \cdots g_{v_k} g_v$. $\square$

Corollary 4.8. *The group $G(F, F')$ is compactly generated.*

Proof. Let S_F denote a compact generating subset of $U(F)$, and $S = K_{0,F'}(v_0) \cup S_F \subset G(F, F')$. Since the group $U(F)$ is vertex-transitive, the subgroup of $G(F, F')$ generated by S contains all the subgroups $K_{0,F'}(v)$, for $v \in \mathsf{V}(T_d)$. Since all these subgroups together with $U(F)$ generate the group $G(F, F')$ according to Proposition 4.7, one has $\langle S \rangle = G(F, F')$. $\square$

The following proposition gives an explicit characterization of the finite permutation groups $F' \leq \hat{F}$ for which the inclusion $U(F) \leq G(F, F')$ is proper.

Proposition 4.9. *Given $F \leq \mathrm{Sym}(\Omega)$, the following statements are equivalent:*

(i) $U(F) = G(F, F')$;

(ii) $F = F'$;

(iii) $G(F, F')$ *is a closed subgroup* $\mathrm{Aut}(T_d)$;

(iv) vertex stabilizers $G(F, F')_v$ are compact.

In particular when F is transitive, the inclusion $U(F) \leq G(F)$ is proper as soon as F is a proper subgroup of $\mathrm{Sym}(\Omega)$.

Proof. The implications $(ii) \Rightarrow (i) \Rightarrow (iv)$ being clear, it is enough to prove $(iv) \Rightarrow (iii) \Rightarrow (ii)$.

$(iv) \Rightarrow (iii)$. This is a general argument: since $G(F, F')$ is locally compact and its action on T_d is continuous and proper, the subgroup $G(F)$ must be closed in $\mathrm{Aut}(T_d)$.

$(iii) \Rightarrow (ii)$ Since $G(F, F')$ is closed, according to corollary 4.6 we have $G(F, F') = U(F')$, and it is not hard to see that this implies that $F = F'$. □

We end this subsection with the following result, which characterizes the permutation groups F for which the group $G(F)$ is transitive on $\partial_\infty T_d$. It turns out that the condition arising on F is the same as for the group $U(F)$ [BM00] (compare with Proposition 4.17).

Proposition 4.10. *The group $G(F)$ acts transitively on $\partial_\infty T_d$ if and only if F is 2-transitive.*

Proof. Assume that $G(F)$ is transitive on $\partial_\infty T_d$. For every $i \neq j \in \Omega$, we want to prove that there is an element $\sigma \in F$ such that $\sigma(1) = i$ and $\sigma(2) = j$. Let us consider an infinite ray ξ_1 whose edges have color $(1, 2, 3, 1, 2, 3 \ldots)$, and another ray ξ_2 whose sequence of edges is colored by $(i, j, i, j, i, j \ldots)$. By assumption there is an element $g \in G(F)$ sending the boundary point defined by ξ_1 to the one defined by ξ_2. This means that ξ_1 admits an infinite subray ξ_1', whose first edge can be assumed to have color 1, which is sent by g onto an infinite subray ξ_2' of ξ_2 whose first edge is either colored i or j. For every $n \geq 0$, we denote by $\sigma_n = \sigma(g, \xi_1'(n))$, where $\xi_1'(n)$ is the vertex of ξ_1' at distance n from its starting point. Since $g \in G(F)$, upon replacing ξ_1' by an infinite subray, we may assume that $\sigma_n \in F$ for every $n \geq 0$. Now two cases may occur: if the first edge of ξ_2' has color i then the permutation $\sigma_1 \in F$ sends 1 to i and 2 to j, and if the first edge of ξ_2' has color j then the same argument applies for $\sigma_4 \in F$.

The converse implication comes from the fact that the subgroup $U(F)$ is already transitive on $\partial_\infty T_d$ when the group F is 2-transitive [BM00]. □

4.1.3 Simplicity

A simplicity criterion. Recall that if T is a simplicial tree, we say that the action of a group G on T is minimal if G does not stabilize any proper subtree of T. If T' is a subtree of T, we denote by $G_{T'}$ the pointwise stabilizer of T' in G. We also let G^+ be the subgroup of G generated by the set of subgroups G_e, where e ranges over the set of edges of T.

Tits introduced in [Tit70] a simplicity criterion for groups acting on trees, usually referred to as Tits' independence property (P), whose definition is now recalled. Let T be a simplicial tree, and let $G \leq \mathrm{Aut}(T)$. We denote by X a non-empty (finite or infinite) path in T. For every $x \in X$, the group G_X induces a permutation group of the set of vertices v whose projection on the path X is the vertex x, which is denoted $G_X(x)$. We have a natural injective homomorphism $\varphi_X : G_X \to \prod_x G_X(x)$, and we say that the group G satisfies Tits' independence property (P) if φ_X is an isomorphism for every choice of X.

The main result of [Tit70] says that if G satisfies Tits' independence property (P) and acts minimally on T without fixing any end of T, then the group G^+ is simple as soon as it is not trivial. This remarkable result has been extensively used to establish simplicity of various groups. For example the group $U(F)^+$ is simple as soon as the permutation group F does not act freely on Ω.

The goal of this paragraph is to prove a simplicity criterion, namely Theorem 4.15, by weakening the assumption that the group satisfies Tits' independence property (P). Our motivation comes from the fact that the groups $G(F, F')$ do not satisfy Tits' independence property (P) in general. More precisely, one can check that the map $\varphi_X : G(F, F')_X \to \prod_x G(F, F')_X(x)$ is never onto when F is a proper subgroup of F' and X is an infinite path.

Definition 4.11. We say that a group $G \leq \mathrm{Aut}(T)$ satisfies the *weak Tits' independence property* if, with the same notation as before, the map $\varphi_X : G_X \to \prod_x G_X(x)$ is an isomorphism as soon as X is a path of length one.

The weak Tits' independence property is strictly weaker than Tits' independence property (P). However, one can check that these are equivalent for closed subgroups of $\mathrm{Aut}(T)$ (see for instance [Ama03, Lemma 10]).

The following two lemmas are standard.

Lemma 4.12. *Let G be a subgroup of* $\mathrm{Aut}(T)$. *Assume that the action of G on T is minimal, that G does not fix any end of T and that G^+ is non-trivial. Then the action of G^+ on T is minimal and of general type. Moreover the same holds for any non-trivial subgroup $N \leq \mathrm{Aut}(T)$ normalized by G^+.*

Proof. The fact that the action of G^+ is minimal immediately follows from the fact that G^+ is normal in G, which implies that any G^+-invariant subtree is G-invariant. Now observe that since G^+ is non-trivial by assumption, the tree T is neither a point nor a line. In particular G^+ cannot fix a point or a line in T. Since G^+ is normal in G, if G^+ has a fixed point in the boundary of T then the same holds for G, contradiction. So the action of G^+ on T must be of general type.

The proof of the statement for a subgroup N normalized by G^+ follows exactly the same lines, and we leave it to the reader. □

If e is an edge of T and v a vertex of e, we denote by $T_e(v)$ the subtree of T spanned by vertices whose projection on the edge e is the vertex v. A subtree T' of T is called a half-tree if $T' = T_e(v)$ for some edge e and vertex v.

Lemma 4.13. *Let G be a subgroup of $\mathrm{Aut}(T)$ whose action on T is minimal and of general type. Given any half-tree $T' \subset T$, there exists a hyperbolic element in G whose axis is contained in T'.*

Proof. Let X be the set of hyperbolic elements of G. First remark that there exists $x \in X$ having an endpoint in $\partial T'$. Indeed, otherwise we would have a G-invariant subtree (namely the union of the axes of the elements of X) contained in the complement of T', which is a contradiction with the fact that G acts minimally on T. Now the conclusion follows from the fact that if $y \in X$ does not have any endpoint in common with x, then there exists some integer $k \in \mathbb{Z}$ such that the axis of $x^k y x^{-k}$ is contained in T'. □

The following result plays an essential role in the proof of Theorem 4.15. The idea in the proof of using double commutators already appears for example in [Gri00, Theorem 4].

Lemma 4.14. *Let G be a subgroup of $\mathrm{Aut}(T)$, and N a subgroup of $\mathrm{Aut}(T)$ normalized by G. Let T' be a half-tree in T. Assume that N contains a hyperbolic element whose axis is contained in T'. Then N contains the derived subgroup of $G_{T'}$.*

Proof. Let $\gamma \in N$ be a hyperbolic element whose axis is contained in T'. We let e be the edge of T and v the vertex of T such that T' is the subtree emanating from e containing v. We denote by w the projection of the vertex v on the axis of γ. We denote by L the maximal subtree of T containing w but not its neighbours on the axis of γ. By construction the subtrees L and $\gamma(L)$ are disjoint, and $\gamma^{\pm 1}(L) \subset T'$. This implies that for every $g \in G_{T'}$, the element $[g, \gamma] = g\gamma g^{-1}\gamma^{-1} \in N$ acts like g on L, like $\gamma g^{-1}\gamma^{-1}$ on $\gamma(L)$, and is the identity elsewhere. It follows that for every $h \in G_{T'}$, the element $[[g, \gamma], h]$ (which remains in N) acts like $[g, h]$ on L and is the identity elsewhere, and therefore this element is equal to $[g, h]$. □

The difference between the following result and Tits' theorem [Tit70] is that the independence assumption is strictly weaker here. This is counterbalanced by the fact that we need to impose a condition on stabilizers of edges.

Theorem 4.15. *Let G be a subgroup of* $\mathrm{Aut}(T)$ *such that:*

(a) G acts minimally on T and does not fix any end of T;

(b) G satisfies the weak Tits' independence property;

(c) G_e is a perfect group for every edge e of T.

Then any non-trivial subgroup of G normalized by G^+ must contain G^+. In particular G^+ is simple (or trivial).

Proof. We may assume that G^+ is non-trivial, which implies in particular that T is neither a point nor a line. Let N be a non-trivial subgroup of G normalized by G^+. We want to prove that $G^+ \leq N$. Given an edge e of T, we will prove that the subgroup G_e lies in N. We denote by T' and T'' the two half-trees emanating from the edge e. It follows from the assumption that G satisfies the weak Tits' independence property that G_e is equal to the direct product of the subgroups $G_{T'}$ and $G_{T''}$, and we will prove that both lie in N. By symmetry it is enough to prove that $G_{T'} \leq N$. Moreover since G_e is perfect by assumption, the group $G_{T'}$ is perfect as well, so we only need to prove that the derived subgroup of $G_{T'}$ is contained in N.

By assumption G acts minimally on T and does not fix any end, so it follows from Lemma 4.12 that the action of N on T is minimal and of general type. Therefore we are in position to apply Lemma 4.13, which ensures the existence of a hyperbolic element of N whose axis is contained in T', and the conclusion then follows from Lemma 4.14. □

Application to the groups $G(F, F')$. In this paragraph we investigate the subgroup $G(F, F')^+ \leq G(F, F')$, and characterize permutation groups F, F' such that $G(F, F')^+$ has finite index in $G(F, F')$.

Lemma 4.16. *Let $e \in \mathsf{E}(T_d)$ and $g \in G(F, F')_e$. Then for all $v \in \mathsf{V}(T_d)$, the element $\sigma(g, v)$ belongs to the subgroup of F' generated by its point stabilizers.*

Proof. The proof goes by induction on the distance n between the vertex v and the edge e. If $n = 0$ then the result is clear because $\sigma(g, v)$ fixes a point of Ω. Now let v be a vertex at distance $n+1$ from the edge e. We let w be the unique neighbour of v at distance n from e, and denote by $i \in \Omega$ the color of the edge between w and v. The elements $\sigma(g, w)$ and $\sigma(g, v)$ satisfy $\sigma(g, w)(i) = \sigma(g, v)(i)$, which proves the induction step. □

Of particular interest is the case when the subgroup $G(F, F')^+$ has finite index in $G(F, F')$. Note that in the particular case $F' = \mathrm{Sym}(\Omega)$, the condition on F is strictly weaker for the group $G(F)$ than for the group $U(F)$ [BM00, Proposition 3.2.1].

Proposition 4.17. *The following conditions are equivalent:*

(i) $G(F, F')^+$ *has index two in* $G(F, F')$*;*

(ii) $G(F, F')^+$ *has finite index in* $G(F, F')$*;*

(iii) F *is transitive and* F' *is generated by its points stabilizers.*

Proof. $(ii) \Rightarrow (iii)$. We let $\Omega_1, \ldots, \Omega_r$ be the orbits of F in Ω. For every $i \in \Omega$, we let $w(i)$ be the unique integer such that $i \in \Omega_{w(i)}$. We identify the tree T_d with the Cayley graph of the free product of d copies of the group of order two $\Gamma = \langle x_1, \ldots, x_d \,|\, x_i^2 = 1\rangle$. Let us consider the quotient Γ_F of Γ defined by adding the relation $x_i = x_j$ when i and j are in the same F-orbit, i.e. when $w(i) = w(j)$. The Cayley graph T_r of Γ_F is a regular tree of degree r, and we have a natural projection $p_F : T_d \to T_r$. Let $g \in G(F, F')$ fixing some vertex and let $v, v' \in \mathsf{V}(T_d)$ such that $v' = g(v)$. Note that since g fixes a vertex, the distance between v and v' must be even. We consider the unique path from v to v', whose sequence of colors of edges is denoted by $(i_1, \ldots, i_{2n})$. Since the element g fixes a vertex and stabilizes the orbits of F on the set of edges, the word $w(i_1) \cdots w(i_{2n})$ is a palindromic word, which implies that the vertices v and v' have the same image by the projection $p_F : T_d \to T_r$. In other words, we have proved that any $g \in G(F, F')$ fixing some vertex must stabilize the fibers of vertices of the map p_F, and a fortiori the same holds for the group $G(F, F')^+$. Argue by contradiction and assume that F is not transitive, i.e. $r \geq 2$. Then the tree T_r is infinite, and therefore $G(F, F')^+$ must have infinitely many orbits of vertices in T_d, which prevents $G(F, F')^+$ from being of finite index in $G(F, F')$. Contradiction.

Now we want to prove that F' is generated by its point stabilizers, or equivalently that the subgroup H of F' generated by point stabilizers is transitive on Ω. We carry out the same construction as in the previous paragraph by replacing F-orbits by H-orbits. According to Lemma 4.16, all the elements of $G(F, F')^+$ must stabilize the fibers of the projection, and the conclusion follows by the same argument.

$(iii) \Rightarrow (i)$. The fact that F' is transitive and generated by its point stabilizers implies that for every vertex $v \in \mathsf{V}(T_d)$, the group $G(F, F')^+_v$ is transitive on the set of edges around v. We easily deduce that $G(F, F')^+$ is transitive on the set of non-oriented edges of T_d, and therefore has index two in $G(F, F')$. □

Proposition 4.18. *Assume that* F *is transitive, and that point stabilizers in* F *and* F' *are perfect. Then stabilizers of edges in the group* $G(F, F')$ *are perfect.*

Proof. We denote respectively by F_a and F'_a a point stabilizer in F and F'. The stabilizer of an edge in $G(F, F')$ is isomorphic to the direct product of two copies of the increasing union of infinitely iterated permutational wreath products

$$C_n = \ldots \wr F_a \wr F'_a \wr \ldots \wr F'_a,$$

where the group F'_a appears n times. Clearly, it is enough to show that C_n is a perfect group for every $n \geq 0$. Since F_a and F'_a are perfect groups by assumption, we are in position to apply [Nik04, Corollary 1.4] to deduce that the group C_n has bounded commutator width. So the derived subgroup of C_n is closed. But since F_a and F'_a are perfect, this derived subgroup contains all the finite groups

$$C_n^{(m)} = F_a \wr \ldots \wr F_a \wr F'_a \wr \ldots \wr F'_a,$$

where the group F_a appears $m \geq 1$ times. Therefore the derived subgroup of C_n is at the same time closed, and contains the increasing union for $m \geq 1$ of the subgroups $C_n^{(m)}$, which is a dense subgroup of C_n, so C_n must be perfect. □

The following result shows that the family of groups under consideration in this chapter yields some examples of totally disconnected locally compact compactly generated simple groups.

Theorem 4.19. *Let $d \geq 3$, and let $F \leq \mathrm{Sym}(\Omega)$ be a transitive permutation group whose point stabilizers are perfect. Assume also that point stabilizers in F' are perfect and generate F'. Then the group $G(F, F')^+$ is compactly generated and abstractly simple.*

Proof. The fact that $G(F, F')$ is compactly generated has been proved in Corollary 4.8. Now since F is transitive and F' is generated by its point stabilizers, the group $G(F, F')^+$ has index two in $G(F, F')$ by Proposition 4.17, and is therefore compactly generated as well. Now since point stabilizers in F and F' are perfect, according to Proposition 4.18 the stabilizers of edges in $G(F, F')$ are perfect. Moreover the group $G(F, F')$ satisfies the weak Tits's independence property, so we are in position to apply Theorem 4.15 to deduce that $G(F, F')^+$ is simple. □

When specifying to discrete groups, i.e. when the permutation group F is moreover assumed to act freely on Ω, we obtain the following result. Note that in this case Proposition 4.18 is obvious.

Corollary 4.20. *Let $d \geq 3$, and let $F \leq \mathrm{Sym}(\Omega)$ be a permutation group whose action on Ω is simply transitive. Assume that point stabilizers in F' are perfect and generate F'. Then the group $G(F, F')^+$ is a finitely generated simple group.*

Example 4.21. Examples of permutation groups satisfying the assumptions of Corollary 4.20 are

$$F = \langle (1, \ldots, d) \rangle \text{ and } F' = \mathrm{Alt}(d)$$

for any $d \geq 7$ odd, or

$$F = \left\langle (1, \ldots, 2n)(2n+1, \ldots, 4n), \prod_{i=1}^{2n} (i, 2n+i) \right\rangle \text{ and } F' = \mathrm{Alt}(4n)$$

for any $n \geq 2$. Note that on the opposite $\mathrm{Alt}(4n+2)$ does not contain any element of order two without any fixed point, and therefore cannot contain a simply transitive subgroup.

4.2 Further properties of the groups $G(F)$

4.2.1 Asymptotic dimension

Recall that a metric space X has asymptotic dimension at most $n \geq 0$ if for every (large) $r > 0$, one can find $n+1$ uniformly bounded families $X_0, \ldots, X_n$ of r-disjoint sets, whose union is a cover of the space X. Two sets are said to be r-disjoint if any point in the first is at distance at least r from any point in the second. The asymptotic dimension of X is the smallest integer n such that X has asymptotic dimension at most n. Asymptotic dimension is an invariant of metric coarse equivalence, so that if G is a locally compact compactly generated group, the asymptotic dimension of G is well defined.

Proposition 4.22. *Let G be a locally compact compactly generated group acting on a locally finite tree X such that all vertex stabilizers in G are locally elliptic open subgroups. Then G has asymptotic dimension at most one.*

Proof. Let x_0 be a vertex of X, and let $H = G_{x_0}$. Since the tree X is locally finite, for every $r > 0$, the coarse stabilizer $W_r(x_0) = \{g \in G : d(gx_0, x_0) \leq r\}$ of x_0 is a finite union of left cosets of H. Since the subgroup H is locally elliptic, it has asymptotic dimension zero [CH15, Proposition 4.D.4], and therefore by the previous observation $W_r(x_0)$ (endowed with the induced topology) has asymptotic dimension zero as well. So we are in position to apply Theorem 2 from [BD01], which implies that G has asymptotic dimension at most one. Note that the result is stated there for discrete groups, but the same proof works in the locally compact setting. □

This result applies notably to the group $G(F)$, which clearly does not have asymptotic dimension zero.

Corollary 4.23. *The group $G(F)$ has asymptotic dimension one.*

Since asymptotic dimension of a closed subgroup is always bounded above by the asymptotic dimension of the ambient group, it follows that all the groups $G(F, F')$ have asymptotic dimension one. When specifying to the permutation groups satisfying the assumptions of Corollary 4.20, we obtain:

Corollary 4.24. *Under the assumptions of Corollary 4.20, the group $G(F, F')^+$ is a finitely generated simple group of asymptotic dimension one.*

4.2.2 Compact presentability

Acting on T_d transitively and with compact stabilizers on the set of vertices, the group $U(F)$ is quasi-isometric to the tree T_d, and in particular $U(F)$ is compactly presented. We show that closed unimodular subgroups of $G(F)$ which are compactly presented must actually satisfy the much stronger property to act properly on T_d.

We will make use of the following well-known lemma.

Lemma 4.25. *Let G be locally compact compactly generated unimodular group, admitting a proper and continuous action on a tree X. Then compact open subgroups of G have uniformly bounded Haar measure. In particular G does not have non-compact locally elliptic open subgroups.*

Proof. Upon replacing X by a minimal G-invariant subtree, one may assume that X is a locally finite tree on which G acts with finitely many orbits of vertices. Since the action is proper, vertex stabilizers are compact open, and by the previous remark there are only finitely many conjugacy classes of vertex stabilizers. Moreover G is unimodular, so vertex stabilizers have a finite number of possible Haar measures. Since every compact open subgroup has a subgroup of index at most two that is contained in a vertex stabilizer, the first statement is proved. The second statement follows because any non-compact locally elliptic open subgroup would be a strictly increasing union of compact open subgroups, which cannot happen. □

Proposition 4.26. *Let G be a closed unimodular subgroup of $G(F)$. If G is compactly presented, then the action of G on T_d is proper.*

In particular, the group $G(F, F')$ is never compactly presented as soon as F is a proper subgroup of F'.

Proof. Since $G(F)$ has asymptotic dimension one by Corollary 4.23, it follows that G must have asymptotic dimension zero or one. If G has asymptotic dimension zero then G is compact, so we may assume that

G has asymptotic dimension one. We now make use of a result of Fujiwara and Whyte [FW07, Theorem 1.1], which characterizes locally compact compactly presented groups with asymptotic dimension one as those which are quasi-isometric to an unbounded tree. So the group G must be quasi-isometric to a tree. Using Bass-Serre theory and accessiblity results (see [Cor12, Theorem 4.A.1] and references therein), one obtains that the group G must actually act geometrically on a locally finite tree. Since G is unimodular, it follows from Lemma 4.25 that every locally elliptic open subgroup of G must be compact. In particular vertex stabilizers in G for its action on T_d are compact, so the first statement is proved.

The second statement follows Proposition 4.9. □

Corollary 4.27. *If F is a proper subgroup of F', and if Γ is a lattice in $G(F, F')$, then Γ is not finitely presented.*

Proof. Let $H = G(F, F')_v$ be a vertex stabilizer in $G(F, F')$. According to Proposition 4.9, the subgroup H is non-compact. Assume by contradiction that Γ is a finitely presented lattice in $G(F)$. According to Proposition 4.26, this implies that Γ is virtually free. Therefore the intersection between Γ and the locally elliptic open subgroup H must be finite. But since Γ is a lattice in $G(F, F')$, the subgroup $\Gamma \cap H$ is a lattice in H, so the group H must be compact. Contradiction. □

Here is another proof of the non-compact presentability of $G(F)$ not making use of asymptotic dimension, in the particular case when F is 2-transitive. The main ingredient is the following topological version of a theorem of Bieri and Strebel.

Theorem 4.28 (Bieri-Strebel)**.** *Let $G = N \rtimes_\varphi \mathbb{Z}$ be a locally compact compactly generated group, where φ is a topological automorphism of the locally elliptic open subgroup N. If G is compactly presented, then (upon replacing φ by its inverse) there exists a compact open subgroup K of N such that $\varphi(K) \subset K$ and $N = \cup_{n \geq 0} \varphi^{-n}(K)$.*

Proof. See for instance [CH15, Corollary 8.C.19]. □

Proposition 4.29. *Let $d \geq 3$, and let $F \leq \mathrm{Sym}(\Omega)$ be a 2-transitive proper subgroup. Then the group $G(F)$ is not compactly presented.*

Proof. Let $\xi \in \partial_\infty T_d$ be any boundary point, and let $H = G(F)_\xi$ be the stabilizer of ξ in $G(F)$. Since F is 2-transitive, the group $G(F)$ acts transitively on $\partial_\infty T_d$ by Proposition 4.10, and it follows that H is a cocompact subgroup of $G(F)$. Compact presentability being preserved by passing to a cocompact subgroup, it is enough to prove that H is not compactly presented.

Let (ξ_n) be the sequence of vertices starting at v_0 and defining the point ξ. If $h \in H$ is a translation of minimal length, then the group H admits a semi-direct product decomposition $H = N \rtimes \langle h \rangle$, where the subgroup N consists of elliptic elements of H. It is not hard to check that such an element h can be chosen inside $U(F)$. For every $n \geq 0$, we let N_n be the subgroup consisting of elements $g \in N$ fixing the infinite subray of ξ starting at ξ_n, and such that $\sigma(g, v) \in F$ for every vertex v at distance at least n^2 from ξ_n. One can check that (N_n) is an increasing union of compact open subgroups ascending to N, so the subgroup N is locally elliptic. Consequently we are in position to apply Theorem 4.28, which implies that if H is compactly presented then there is a compact open subgroup K of N such that the $\langle h \rangle$-conjugates of K cover the entire N. Since K is compact there exists some integer n_0 such that $K \subset N_{n_0}$. In particular N is the union of the $\langle h \rangle$-conjugates of N_{n_0}, and since $h \in H$, this implies that for every element of $g \in H$, there are at most n_0 vertices v such that $\sigma(g, v) \notin F$. In particular vertex stabiliers in H are compact, which is a contradiction with the fact that $F \neq \mathrm{Sym}(\Omega)$ according to the implication $(iii) \Rightarrow (ii)$ of Proposition 4.9. □

4.3 Diagrams and commensurating actions

In this section we explain how the group $G(F)$, which is defined in terms of its action on the tree T_d, can be profitably studied by using a notion of diagrams introduced below. In the case when the group F is transitive, one shows that this combinatorial data yields an estimate of the word-metric in the group $G(F)$ (see Proposition 4.37). We moreover show that the group $G(F)$ admits a commensurating action so that the corresponding cardinal definite function is given by the size of the diagrams, and deduce, by a general argument, that $G(F)$ admits a proper action on a CAT(0) cube complex.

4.3.1 Diagrams

Recall that if T is a complete subtree of T_d, we denote by $\mathsf{IV}(T)$ the set of internal vertices of T, i.e. vertices of T all of whose neighbours in T_d belong to T.

Let $g \in G(F)$. It readily follows from the definition of the group $G(F)$ that there exists a unique finite complete subtree $T_d^-(g)$ of T_d such that :

(i) $T_d^-(g)$ contains the edges e_0 and $g^{-1}(e_0)$;

(ii) for every vertex v that is not an internal vertex of $T_d^-(g)$, we have $\sigma(g, v) \in F$;

and being minimal for this property. We let $T_d^+(g)$ be the image of $T_d^-(g)$ by g, and denote by $\mathcal{N}(g)$ the number of internal vertices or $T_d^-(g)$. Note that by construction, $\mathcal{N}(g)$ is also the number of internal vertices of $T_d^+(g)$. One easily check that $\mathcal{N}(g) = 0$ if and only if $g \in U(F)$ and g stabilizes the edge e_0.

Lemma 4.30. *Let $g \in G(F)$ and $v \in \mathsf{V}(T_d)$. Then the following statements are equivalent:*

(i) $g(\mathsf{L}(v)) = \mathsf{L}(g(v))$ and $\sigma(g, w) \in F$ for every vertex w in $\mathsf{L}(v)$;

(ii) v is not an internal vertex of $T_d^-(g)$.

Proof. By construction g sends the complement of $T_d^-(g)$ onto the complement of $T_d^+(g)$ locally like F, so it clear that if v is not an internal vertex of $T_d^-(g)$ then $g(\mathsf{L}(v)) = \mathsf{L}(g(v))$ and $\sigma(g, w) \in F$ for every vertex w in $\mathsf{L}(v)$. For the converse implication, remark that if v is an internal vertex of $T_d^-(g)$, then either there is a vertex w in $\mathsf{L}(v)$ such that $\sigma(g, w) \notin F$, or the edge $g^{-1}(e_0)$ belongs to $\mathsf{L}(v)$. This last property implies that $g(\mathsf{L}(v))$ contains the edge e_0, and therefore cannot be equal to $\mathsf{L}(g(v))$. $\square$

Recall that a length function on a group Γ is a map $\mathcal{L} : \Gamma \to \mathbb{R}_+$ satisfying $\mathcal{L}(1) = 0$, $\mathcal{L}(g^{-1}) = \mathcal{L}(g)$ and $\mathcal{L}(gh) \leq \mathcal{L}(g) + \mathcal{L}(h)$ for every $g, h \in \Gamma$. We say that a length function is locally bounded if it is bounded on compacts subsets.

Proposition 4.31. *The map $\mathcal{N} : G(F) \to \mathbb{R}_+$ is a locally bounded length function on $G(F)$.*

Proof. By definition it is clear that the function $\mathcal{N}$ satisfies $\mathcal{N}(1) = 0$ and $\mathcal{N}(g) = \mathcal{N}(g^{-1})$ for every $g \in G(F)$. Now let $g, h \in G(F)$, and let us prove the inequality $\mathcal{N}(gh) \leq \mathcal{N}(g) + \mathcal{N}(h)$. Let T be the subtree of T_d spanned by $T_d^+(h)$ and $T_d^-(g)$, and let T_d^- be the preimage by h of the subtree T. Note that T is a finite rooted complete subtree with at most $\mathcal{N}(g) + \mathcal{N}(h)$ internal vertices, and a fortiori the same holds for T_d^-. By construction, the subtree T_d^- contains the edges e_0 and $(gh)^{-1}(e_0)$, and the product gh acts locally like F outside T_d^-. By minimality it follows that $T_d^-(gh)$ must be a subtree of T_d^-, and in particular the number of internal vertices of $T_d^-(gh)$ is at most $\mathcal{N}(g) + \mathcal{N}(h)$.

Finally $\mathcal{N}$ is bounded on compact sets because $\mathcal{N}$ is equal to zero on the compact open subgroup $U(F)_{e_0}$. $\square$

So in particular the map $\mathcal{N} : G(F) \to \mathbb{R}_+$ gives rise to a left invariant pseudo-metric on $G(F)$ defined by $\mathrm{dist}(g, h) = \mathcal{N}(g^{-1}h)$, and the aim of the rest of this subsection is to prove that when F is transitive, this pseudo-metric is quasi-isometric to the word metric in $G(F)$.

Lemma 4.32. *For every $g \in G(F)$, there exist $\gamma \in U(F)$ and $g' \in G(F)_{v_1}$ such that $g = \gamma g'$ and $\mathcal{N}(g') \leq \mathcal{N}(g) + 1$.*

Proof. Let $g \in G(F)$. Since the group $U(F)$ is transitive on the set of vertices of T_d, we can choose some $\gamma \in U(F)$ such that $\gamma(v_1) = g(v_1)$, and set $g' = \gamma^{-1} g$. Clearly, $g' \in G(F)_{v_1}$. Let us consider the finite complete subtree T_d^- of T_d obtained by adjoining if necessary the star around the vertex v_1 to the subtree $T_d^-(g)$. We check that T_d^- contains the edges e_0 and $g'^{-1}(e_0)$, and that g' acts locally like F at every vertex of T_d which is not an internal vertex of T_d^-. It follows that $\mathcal{N}(g')$ is at most equal to the number of internal vertices of T_d^-, which by construction is at most $\mathcal{N}(g) + 1$. □

From now and until the end of 4.3.1, we assume that F is transitive.

For every $i \in \Omega \setminus \{1\}$, we choose some $\sigma_i \in F$ such that $\sigma_i(1) = i$. Let us consider the bi-infinite line ℓ_i in T_d defined by saying that ℓ_i contains the edge e_0, and the edge of ℓ_i in $\mathsf{L}(v_0)$ (resp. $\mathsf{L}(v_1)$) at distance n from e_0 has color $\sigma_i^{-n}(1)$ (resp. $\sigma_i^n(1)$). We let h_i be the hyperbolic isometry of T_d of translation length 1 whose axis is ℓ_i, and such that for every $v \in \mathsf{V}(T_d)$, we have $\sigma(h_i, v) = \sigma_i$. Note that h_i belongs to $U(F)$ and sends the subtree $\mathsf{L}(v_1)$ onto the subtree $\mathsf{L}(v_1^i)$, and h_i^{-1} sends $\mathsf{L}(v_0)$ onto $\mathsf{L}(v_0^i)$. Let $S_H = \{h_2, \ldots, h_d\}$ and $S = S_H \cup K_0(v_0) \cup K(v_1)$.

Lemma 4.33. *For every $g \in G(F)_{\mathsf{L}(v_0)}$, we have $|g|_S \leq 3(d-1)\mathcal{N}(g) + 1$.*

Proof. First note that any element fixing $\mathsf{L}(v_0)$ must fixe the vertex v_1 as well. Let us argue by induction on $\mathcal{N}(g)$. The case when $\mathcal{N}(g) = 0$ is easily settled, because $\mathcal{N}(g) = 0$ easily implies that $g \in U(F)_{v_1} \subset K(v_1)$, so we have $|g|_S \leq 1$. Now assume that the result holds for every $g \in G(F)_{\mathsf{L}(v_0)}$ with $\mathcal{N}(g) \leq n$ for some integer $n \geq 0$, and let $g \in G(F)_{\mathsf{L}(v_0)}$ such that $\mathcal{N}(g) = n + 1$. We want to prove that the word length of g is at most $3(d-1)(n+1) + 1$. There exists an element $u \in K(v_1)$ such that $g' = ug$ fixes the star around the vertex v_1. Consequently the element g' can be written as a product $g' = g_2 \cdots g_d$, where $g_i \in G(F)$ acts trivially outside $\mathsf{L}(v_1^i)$. Moreover we have $\mathcal{N}(g_i) \leq \mathcal{N}(g)$ and $\sum_i \mathcal{N}(g_i) \leq \mathcal{N}(g) + d - 2$, because the vertex v_1 can be counted $d-1$ times in the sum, whereas it is counted only once in $\mathcal{N}(g)$. This last inequality can be rewritten as $\sum_i (\mathcal{N}(g_i) - 1) \leq n$. Now for each g_i different from the identity, let us consider the element $g_i' = h_i g_i h_i^{-1}$. By construction of the element h_i, the element g_i' belongs to $G(F)_{\mathsf{L}(v_0)}$ and satisfies $\mathcal{N}(g_i') = \mathcal{N}(g_i) - 1 \leq n$. By induction hypothesis, the word length of g_i' is at most $3(d-1)\mathcal{N}(g_i') + 1$. It follows that the word length of g_i is at most $2 + 3(d-1)\mathcal{N}(g_i') + 1$, and we obtain

$$
\begin{aligned}
|g|_S &\leq 1 + \sum_{g_i \neq id} |g_i|_S \leq 1 + \sum_{g_i \neq id} (2 + 3(d-1)\mathcal{N}(g_i') + 1) \\
&\leq 1 + 3(d-1) + 3(d-1) \sum_{g_i \neq id} (\mathcal{N}(g_i) - 1) \\
&\leq 1 + 3(d-1) + 3(d-1)n = 3(d-1)(n+1) + 1.
\end{aligned}
$$

□

Remark 4.34. Note that the conclusion of Lemma 4.33 also holds for elements of $G(F)_{\mathsf{L}(v_1)}$, just by replacing $K(v_1)$ in the proof by $K(v_0)$, and each h_i by its inverse.

Corollary 4.35. *For every $g \in G(F)_{v_1}$, we have $|g|_S \leq 3(d-1)\mathcal{N}(g) + 3$.*

Proof. Let $g \in G(F)$ fixing the vertex v_1. By definition of $K(v_1)$, there exists $u \in K(v_1)$ such that $g' = ug \in G(F)$ fixes the edge e_0. Note that since the element u acts locally like F at every vertex different from v_1, we have $\mathcal{N}(g') \leq \mathcal{N}(g)$. Now the element g' can be written $g' = g_0' g_1'$, where $g_0' \in G(F)_{\mathsf{L}(v_0)}$, $g_1' \in G(F)_{\mathsf{L}(v_1)}$ satisfy $\mathcal{N}(g_0') + \mathcal{N}(g_1') = \mathcal{N}(g')$. Therefore Lemma 4.33 can be applied to these elements, and we obtain

$$
|g|_S \leq 1+|g_0'|_S+|g_1'|_S \leq 1+3(d-1)\mathcal{N}(g_0')+1+3(d-1)\mathcal{N}(g_1')+1 \leq 3(d-1)\mathcal{N}(g)+3.
$$

□

Lemma 4.36. *For every $\gamma \in U(F)$, we have $|\gamma|_S \leq d(\gamma(v_1), v_1) + 1$.*

Proof. We argue by induction on $d(\gamma(v_1), v_1)$. If γ fixes v_1 then γ belongs to $K(v_1)$ and therefore $|\gamma|_S \leq 1$. Assume that $|\gamma|_S \leq d(\gamma(v_1), v_1) + 1$ for every $\gamma \in U(F)$ such that $d(\gamma(v_1), v_1) \leq n$, and let $\gamma \in U(F)$ such that $d(\gamma(v_1), v_1) = n + 1$. If the vertex $\gamma(v_1)$ belongs to the branch $\mathsf{L}(v_1)$, then there exists some integer i such that $\gamma' = h_i^{-1}\gamma \in U(F)$ satisfies $d(\gamma'(v_1), v_1) \leq n$. By induction hypothesis, the word length of γ' is at most $n + 1$, and we deduce that $|\gamma|_S \leq n + 2$. Now if $\gamma(v_1)$ belongs to $\mathsf{L}(v_0)$ then the same argument can be applied to $h_i\gamma$ for some integer i. □

We are finally able to give the following precise estimate of the word metric in $G(F)$.

Proposition 4.37. *Assume that $F \leq \mathrm{Sym}(\Omega)$ is transitive. Then S is a compact generating subset of $G(F)$, and for every $g \in G(F)$, we have*

$$
\mathcal{N}(g) \leq |g|_S \leq (3d-2)\mathcal{N}(g) + 3d + 2.
$$

Proof. The lower bound easily follows from the fact that the function $\mathcal{N}$ is subadditive and takes value 0 or 1 on elements of S. Let us prove the upper bound. According to Lemma 4.32, one can write $g = \gamma g'$ with $\gamma \in U(F)$ and $g' \in G(F)_{v_1}$ such that $\mathcal{N}(g') \leq \mathcal{N}(g) + 1$. It follows from Lemma 4.36 that the word length of γ satisfies

$$|\gamma|_S \leq d(\gamma(v_1), v_1) + 1 = d(g(v_1), v_1) + 1 \leq \mathcal{N}(g) + 2.$$

On the other hand, we can apply Corollary 4.35 to the element g', which yields

$$|g'|_S \leq 3(d-1)\mathcal{N}(g') + 3 \leq 3(d-1)(\mathcal{N}(g) + 1) + 3.$$

We finally obtain

$$|g|_S \leq |\gamma|_S + |g'|_S \leq \mathcal{N}(g) + 2 + 3(d-1)(\mathcal{N}(g)+1) + 3 = (3d-2)\mathcal{N}(g) + 3d + 2.$$

$\square$

4.3.2 A commensurating action of $G(F)$

Recall that e_0 is a fixed edge of T_d having color $c(e_0) = 1$. Let H denote the open subgroup of $G(F)$ consisting of elements g stabilizing $\mathsf{L}(v_0)$ and such that $\sigma(g, w) \in F$ for every vertex w in $\mathsf{L}(v_0)$. For every vertex $v \in \mathsf{V}(T_d)$, we let M_v be the set of elements $g \in G(F)$ such that $g(\mathsf{L}(v_0)) = \mathsf{L}(v)$ and $\sigma(g, w) \in F$ for every vertex w in $\mathsf{L}(v_0)$.

Lemma 4.38. *For every $v \in \mathsf{V}(T_d)$, M_v is either empty or equal to a single H-coset.*

Actually Lemma 4.38 follows immediately from the following two observations.

Lemma 4.39. *For every $v \in \mathsf{V}(T_d)$, we have $M_v H = M_v$.*

Proof. If $g \in M_v$ and $h \in H$, then $(gh)(\mathsf{L}(v_0)) = g(\mathsf{L}(v_0)) = \mathsf{L}(v)$, and for every vertex w in $\mathsf{L}(v_0)$, we have $\sigma(gh, w) = \sigma(g, h(w))\sigma(h, w)$. Now $\sigma(h, w) \in F$ because $h \in H$, and $\sigma(g, h(w)) \in F$ because $h(w)$ remains in $\mathsf{L}(v_0)$ and $g \in M_v$. So $\sigma(gh, w) \in F$, and we have proved that $gh \in M_v$. $\square$

Lemma 4.40. *For every $v \in \mathsf{V}(T_d)$, all the elements of M_v are in the same left coset of H.*

Proof. If $g_1, g_2 \in M_v$, then $(g_1^{-1}g_2)(\mathsf{L}(v_0)) = g_1^{-1}(\mathsf{L}(v)) = \mathsf{L}(v_0)$ and

$$\sigma(g_1^{-1}g_2, w) = \sigma(g_1, g_1^{-1}g_2(w))^{-1}\sigma(g_2, w) \in F$$

for every vertex w in $\mathsf{L}(v_0)$. So $g_1^{-1}g_2 \in H$. $\square$

Lemma 4.41. *For every $v \in \mathsf{V}(T_d)$, the set M_v is empty if and only if the color of the unique edge of $\mathsf{E}(v)$ that is not in $\mathsf{L}(v)$ is not in the F-orbit of 1.*

Proof. Let $v \in \mathsf{V}(T_d)$, and let e_v be the unique edge of $\mathsf{E}(v)$ that is not in $\mathsf{L}(v)$, whose color is denoted by i_v.

Assume that M_v is non-empty, and let $g \in M_v$. Since g sends the subtree $\mathsf{L}(v_0)$ onto $\mathsf{L}(v)$, we have $g(e_0) = e_v$. But according to Lemma 4.4, the permutation $\sigma(g, v_0)$ preserves the orbits of F, so it follows that i_v is in the F-orbit of $c(e_0) = 1$.

For the converse implication, let $\sigma \in F$ such that $\sigma(1) = i_v$. We let γ be the automorphism of T_d define by declaring that $\gamma(v_0) = v$, and $\sigma(\gamma, w) = \sigma$ for every vertex w. Clearly $\gamma \in U(F)$ and by construction γ must send $\mathsf{L}(v_0)$ onto $\mathsf{L}(v)$ because $\sigma(\gamma, v_0)(1) = i_v$. So $\gamma \in M_v$, which is therefore non-empty. □

From now we assume that F is transitive.

According to Lemma 4.41, this assumption ensures that the set M_v is non-empty for every $v \in \mathsf{V}(T_d)$.

Lemma 4.42. *Let $v, v' \in \mathsf{V}(T_d)$ and $g \in G(F)$. Then gM_v and $M_{v'}$ are either disjoint or equal, and $gM_v = M_{v'}$ if and only if $v' = g(v)$ and v is not an internal vertex of $T_d^-(g)$.*

Proof. The fact that gM_v and $M_{v'}$ are either disjoint or equal follows immediately from Lemma 4.38. Assume that $gM_v = M_{v'}$. Since the subset M_v is non-empty, there exists $g_v \in G(F)$ such that $g_v(\mathsf{L}(v_0)) = \mathsf{L}(v)$ and $\sigma(g_v, w) \in F$ for every vertex w in $\mathsf{L}(v_0)$. Since $gg_v \in M_{v'}$ by assumption, we have $gg_v(\mathsf{L}(v_0)) = \mathsf{L}(v')$ and $\sigma(gg_v, w) \in F$ for every vertex w in $\mathsf{L}(v_0)$. In particular we have $g(\mathsf{L}(v)) = \mathsf{L}(v')$, so $v' = g(v)$. Since $\sigma(gg_v, w) = \sigma(g, g_v(w))\sigma(g_v, w)$ and $\sigma(gg_v, w), \sigma(g_v, w) \in F$, we obtain that $\sigma(g, w') \in F$ for every vertex w' in $\mathsf{L}(v)$. According to $(i) \Rightarrow (ii)$ of Lemma 4.30, this implies that the vertex v is not an internal vertex of $T_d^-(g)$.

Conversely assume that v is not an internal vertex of $T_d^-(g)$. According to the implication $(ii) \Rightarrow (i)$ of Lemma 4.30, we have $g(\mathsf{L}(v)) = \mathsf{L}(g(v))$ and $\sigma(g, w) \in F$ for every vertex w in $\mathsf{L}(v)$. By the same argument as above, it follows that $gM_v \subset M_{g(v)}$, and therefore $gM_v = M_{g(v)}$. □

We denote by $M \subset G(F)$ the union of the subsets M_v, when v ranges over the set of vertices $\mathsf{V}(T_d)$. Since M is a union of left cosets of H, we freely identity the subset M of $G(F)$ with its image in $G(F)/H$.

Recall that if G is a group acting on a set X, a subset $A \subset X$ is commensurated if $\#(gA \triangle A)$ is finite for every $g \in G$.

Proposition 4.43. *The action of $G(F)$ on $G(F)/H$ commensurates the subset M. More precisely, we have $\#(gM\triangle M) = 2\mathcal{N}(g)$ for every $g \in G(F)$.*

Proof. Let $g \in G(F)$. According to Lemma 4.42, the subset $gM\backslash M$ is the union of gM_v, where v ranges over the set of internal vertices of $T_d^-(g)$. Since none of these gM_v is empty, this union consists exactly in $\mathcal{N}(g)$ left cosets of H, and therefore $\#(gM\backslash M) = \mathcal{N}(g)$. By applying the same argument to g^{-1}, we obtain

$$\#(gM\triangle M) = \#(gM\backslash M) + \#(M\backslash gM) = \mathcal{N}(g) + \mathcal{N}(g^{-1}) = 2\mathcal{N}(g).$$

□

By a general principle (see for instance [Cor13]) and using Proposition 4.37, we deduce the following result.

Corollary 4.44. *There exist a CAT(0) cube complex C on which $G(F)$ acts properly, and a vertex $x_0 \in C$ such that in the ℓ^1-metric, $d(gx_0, x_0) = 2\mathcal{N}(g)$ for every $g \in G(F)$. In particular the orbital map $G(F) \to C$, $g \mapsto gx_0$, is a quasi-isometric embedding.*

Remark 4.45. The action of $G(F)$ on this CAT(0) cube complex is not cocompact when F is a proper subgroup of $\mathrm{Sym}(\Omega)$, and more generally one cannot hope that $G(F)$ acts properly and cocompactly by isometries on a CAT(0) metric space. The reason is that the existence of such an action would imply that $G(F)$ is coarsely simply connected, a contradiction with Proposition 4.26.

Chapter 5

Measuring relations in finitely generated groups

In this chapter we introduce the notion of *relation range* of a finitely generated group, which in some sense measures the size of independent relations appearing in the group. In Section 5.1 we introduce the definitions and establish some stability properties of this notion. In Section 5.2 we show how the process of iterating a non-injective epimorphism yields a large class of examples of groups with relation range as large as possible. Finally in Section 5.3 we establish a link between the relation range of a finitely generated group and simple connectedness of its asymptotic cones.

5.1 Relation range

5.1.1 Definition

Let G be a finitely generated group and S a finite generating subset, so that we have a short exact sequence

$$1 \longrightarrow N \longrightarrow F_S \longrightarrow G \longrightarrow 1,$$

where F_S is the free group on S. For every $k \geq 0$, we let $N(k)$ be the normal subgroup of F_S generated as such by elements of N or word length at most k. By definition $(N(k))$ is an increasing sequence of normal subgroups of F_S ascending to the subgroup N. For every $k \geq 0$, we let $x_{G,S}(k)$ be the shortest length of a relation in G that is not generated by relations of length at most $k-1$, that is

$$x_{G,S}(k) = \min \{|n|_S : n \in N \setminus N(k-1)\} \in \overline{\mathbb{R}}_+.$$

Note that the function $x_{G,S} : \mathbb{N} \to \overline{\mathbb{R}}_+$ is non-decreasing, and $x_{G,S}(k) \geq k$ for every $k \geq 0$. Note also that one has $x_{G,S}(k) = \infty$ if and only if N is generated as a normal subgroup by its elements of length at most $k-1$, so in particular $x_{G,S}(k) = \infty$ for k large enough if and only if the group G is finitely presented.

We denote by $\mathcal{R}_S(G)$ the set of finite values taken by the function $x_{G,S}$. This is the set of integers $\ell \geq 0$ for which there exists a relation in G of length ℓ that is not generated by relations of length at most $\ell - 1$. We call $\mathcal{R}_S(G)$ the *relation range* of the group G with respect to the finite generating subset S. Note that since $x_{G,S}(0) = 0$ by construction, the set $\mathcal{R}_S(G)$ always contains 0, and is therefore non-empty. Note also that $\mathcal{R}_S(G)$ is finite if and only if the group G is finitely presented.

If X, Y are non-empty subsets of $\mathbb{N}$, we say that X and Y are equivalent, and write $X \sim Y$, if $\log(X)$ and $\log(Y)$ are at bounded Hausdorff distance from each other. It is easy to see that this defines an equivalence relation on the set of non-empty subsets of $\mathbb{N}$, and the class of a subset X will be denoted $[X]$. Note that the union operation on the set of equivalence classes $[X] \cup [Y] = [X \cup Y]$ is well defined, and the relation defined by $[X] \subset [Y]$ if there exists $X' \sim X$ and $Y' \sim Y$ such that $X' \subset Y'$ yields an order on the set of equivalence classes.

Remark 5.1. If G, G' are finitely generated groups and S, S' two finite generating subsets, then one can check that $\mathcal{R}_S(G) \sim \mathcal{R}_{S'}(G')$ if and only if there exists some constants $c_1, c_2 > 0$ so that has

$$c_1 x_{G,S}(c_1 k) \leq x_{G',S'}(k) \leq c_2 x_{G,S}(c_2 k)$$

for every $k \geq 0$. This observation will be used repeatedly in the sequel.

Definition 5.2. A finitely generated group G is said to be *fully presented* by a finite generating subset S if $\mathcal{R}_S(G) \sim \mathbb{N}$. Otherwise G is said to be *lacunary presented* by S.

The following result says that the relation range of a finitely generated group G is an asymptotic invariant of G, in the sense that the equivalence class of $\mathcal{R}_S(G)$ does not depend on the choice of the finite generating subset S.

Proposition 5.3. *Let G be a finitely generated group, and S, T two finite generating subsets of G. Then $\mathcal{R}_S(G) \sim \mathcal{R}_T(G)$.*

Proof. We denote by N (resp. N') the kernel of the natural morphism $F_S \to G$ (resp. $F_T \to G$). For every element $s \in S$, we fix a word $w_s \in F_T$ such that the equality $w_s = s$ holds in the group G. Similarly, for every $t \in t$, we choose a word $w_t \in F_S$ such that the equality $w_t = t$ is satisfied in G. Since the number of words under consideration is finite, there exists a constant $C \geq 1$ such that $|w_s|_T, |w_t|_S \leq C$ for every $s \in S, t \in T$.

Given any word $w \in F_S$, we call a substitution of type (i) the action of replacing every letter s in w by the corresponding word w_s in the elements of T. Similarly, a substitution of type (ii) is the replacement of every letter of a word in F_T by the corresponding word in the elements of S. Note that given $w \in F_S$, a substitution of type (i) followed by a substitution of type (ii) yields a new word $\bar{w}$ representing the same element of G. More precisely, since S and T are finite, there exists an integer $k_0 \geq 1$ such that for every $w \in F_S$, the two substitution process builds a word $\bar{w}$ such that $\bar{w}w^{-1} \in N(k_0)$.

Now let $w \in N$, and let w' denote the word obtained from w after having performed substitutions of type (i). Note that the length of w' is $|w'|_T \leq C|w|_S$, and since $w \in N$, the word w' belongs to N'. Assume that w' belongs to $N'(m)$ for some integer m such that $Cm \geq k_0$. This means that w' can be written as a product of conjugates of elements of N' of length at most m. Performing substitutions of type (ii), we build a word $\bar{w} \in F_S$ which is a product of conjugates of elements of N of length at most Cm, i.e. $\bar{w} \in N(Cm)$. On the other hand it follows from the above observation that $\bar{w}w^{-1} \in N(k_0) \subset N(Cm)$ because $Cm \geq k_0$, so the word w we started from belongs to $N(Cm)$.

Now let $k \geq k_0+1$. We let $w \in N \setminus N(k-1)$ be a word such that $|w|_S = x_S(k)$, and w' denote the word obtained from w after having performed substitutions of type (i). Since w does not belong to $N(k-1)$, it follows from the last paragraph that w' does not belong to $N'(m)$ for $m = \lceil (k-1)/C \rceil$, so in particular

$$x_{G,T}\left(\lceil (k-1)/C \rceil + 1\right) \leq |w'|_T \leq C|w|_S = Cx_{G,S}(k),$$

and the result follows by symmetry. $\square$

If G is a finitely generated group, we will denote by $\mathcal{R}(G) = [\mathcal{R}_S(G)]$, where S is a finite generating subset of G, and call it the *relation range* of the group G. We will say that the group G is *fully presented* if $\mathcal{R}(G) \sim \mathbb{N}$, and *lacunary presented* otherwise.

5.1.2 Stability properties

In this paragraph we establish stability properties of the relation range.

Recall that a retract H of a group G is a subgroup such that there exists an epimorphism $G \twoheadrightarrow H$ whose restriction to H is the identity of H.

Proposition 5.4. *Let G be a finitely generated group, and let H be a retract of G. Then $\mathcal{R}(H) \subset \mathcal{R}(G)$.*

Proof. Let S' be a finite generating subset of G containing a finite generating subset S of H, and let $\varphi : G \twoheadrightarrow H$ be an epimorphism whose restriction to H is the identity of H. Let w be a relation in the group H in the letters of S. A fortiori w is a relation in G. If w admits a decomposition in $F_{S'}$ as a product of conjugates of elements of smaller length, then replacing each letter of S' by its image by φ, we would obtain a decomposition of w in F_S as a product of conjugates of the same elements since φ is the identity on H. It follows that if w is not generated by relations in smaller length in F_S, then the same holds in $F_{S'}$, and the result is proved. $\square$

The next result shows that the relation range behaves well with respect to direct and free products.

Proposition 5.5. *Let G, H be finitely generated groups. Then*

$$\mathcal{R}(G * H) \sim \mathcal{R}(G \times H) \sim \mathcal{R}(G) \cup \mathcal{R}(H).$$

Proof. Since both G and H are retracts of $G * H$ and $G \times H$, it follows from the previous lemma that $\mathcal{R}(G) \cup \mathcal{R}(H) \subset \mathcal{R}(G * H), \mathcal{R}(G \times H)$. Now by definition every relation in $G * H$ comes either from a relation in G or in H, so $\mathcal{R}(G * H) \sim \mathcal{R}(G) \cup \mathcal{R}(H)$. For the direct product of G and H, a presentation of it can be obtained by taking presentations of G and H, and by adding relation forcing generators of G to commute with those of H. In such a presentation, every relation that is not generated by relations of length at most 4 comes either from a relation in G or in H, so the conclusion follows. $\square$

In the next result we establish that the relation range of a finitely generated group is invariant under the operations of modding out by finite normal subgroup and passing to a finite index subgroup.

Proposition 5.6. *Let G be a finitely generated group.*

(a) If $K \lhd G$ is a finite normal subgroup of G and $Q = G/K$, then $\mathcal{R}(Q) \sim \mathcal{R}(G)$;

(b) If H is a finite index subgroup of G then $\mathcal{R}(H) \sim \mathcal{R}(G)$.

Proof. (a). Let S be a finite generating subset of G, whose image in Q is still denoted by S for simplicity. We let N be the kernel of the morphism $\pi : F_S \to G$, and $N' = \pi^{-1}(K)$ the kernel of the natural morphism $F_S \to Q$. Since the subgroup K is finite, there exist $x_1, \ldots, x_j \in N'$ such that N' is the union of left cosets $x_i N$. We fix an integer $k_0 \geq 1$ such that $|x_i|_S \leq k_0$ for every i.

Let $k \geq k_0 + 1$ and $w' \in N' \setminus N'(k-1)$ such that $|w'|_S = x_{Q,S}(k)$. The element w' belongs to N', so there exists x_i and $w \in N$ such that

$w' = x_i w$. Since x_i has length at most $k_0 \leq k-1$, we have $x_i \in N'(k-1)$. So $w \notin N'(k-1)$, because otherwise $w' = x_i w$ would belong to $N'(k-1)$ as well, which is a contradiction. A fortiori $w \notin N(k-1)$ and we deduce that

$$x_{G,S}(k) \leq |w|_S \leq |w'|_S + k_0 = x_{Q,S}(k) + k_0 \leq 2x_{Q,S}(k).$$

We now prove a reverse inequality. Before going into the proof, we make the observation that for every $x_{i_1}, \ldots, x_{i_\ell}$ and every $\alpha_1, \ldots, \alpha_\ell \in F_S$, if the element

$$w = \prod_{i=1}^{\ell} \alpha_i x_{k_i} \alpha_i^{-1}$$

belongs to N then it belongs to $N(q)$ for some bounded integer $q \geq 1$. Indeed, the fact that w belongs to N means that

$$\pi(w) = \prod_{i=1}^{\ell} \pi(\alpha_i)\pi(x_{k_i})\pi(\alpha_i)^{-1}$$

is the identity in G. But now each element $\pi(\alpha_i)\pi(x_{k_i})\pi(\alpha_i)^{-1}$ is an element of K in G, and this can be seen by imposing a finite number of relations because K is a finite normal subgroup of G. So the fact that $\pi(w)$ is the identity in G is just a relation between elements of K, and since K is finite all these relations can be deduced from a finite number of them. This proves the observation.

Now let $k \geq k_0 + 1$ and $w \in N \setminus N(k-1)$ such that $|w|_S = x_{G,S}(k)$. Assume that w belongs to $N'(m)$ for some integer $m \geq 1$. Then w can be written

$$w = \prod_{i=1}^{\ell} \alpha_i w_i' \alpha_i^{-1}$$

with $w_i' \in N'$ of length at most m. Now every w_i' can be written $w_i' = x_{k_i} w_i$ with $w_i \in N$ and $|w_i|_S \leq m + k_0$. By substitution we obtain that w can be written

$$w = \prod_{i=1}^{\ell} \beta_i x_{k_i} \beta_i^{-1} \prod_{i=1}^{\ell} \alpha_i w_i \alpha_i^{-1}.$$

Now both the element w and $\prod \alpha_i w_i \alpha_i^{-1}$ belong to N, so it follows that the element $\prod \beta_i x_{k_i} \beta_i^{-1}$ belongs to N as well. According to the previous observation, this element belongs to $N(m)$ as soon as m is larger than q. From this we deduce that both terms in the decomposition of w belong to $N(m+k_0)$, and it follows that $w \in N(m+k_0)$. But by definition w does not belong to $N(k-1)$, so this implies that $m + k_0 < k - 1$. In particular we have proved that $w \notin N'(k-k_0-1)$, and this implies that $x_{Q,S}(k-k_0-1) \leq |w|_S = x_{G,S}(k)$ for every k large enough.

(b). The case of a finite index subgroup uses some rather similar techniques, and we leave the proof to the reader. □

5.2 Direct limits of non-Hopfian groups

5.2.1 Preliminaries

Let us consider the following procedure. Let G be a non-Hopfian finitely generated group, and let $\varphi : G \to G$ be a morphism from G onto G with non-trivial kernel. Such a data provides an increasing sequence of normal subgroups of G, namely $K_k = \ker(\varphi^k)$, whose union will be denoted by

$$K_\varphi = \bigcup_{k \geq 0} K_k.$$

We let $G_k = G/K_k$ be the quotient of G by the k-th kernel, and $G_\varphi = G/K_\varphi$ be the limit group. In other words, we have a sequence of groups and epimorphisms

$$G = G_0 \xrightarrow{\pi_0} G_1 \xrightarrow{\pi_1} \cdots \longrightarrow G_k \xrightarrow{\pi_k} \cdots$$

whose direct limit is G_φ.

5.2.2 Direct limits of non-Hopfian groups are fully presented

With the above notation, it is easy to see that since φ has non-trivial kernel, the inclusion $K_k \subset K_{k+1}$ is proper for every $k \geq 0$. This implies that K_φ is not finitely generated as a normal subgroup, and the group G_φ is therefore not finitely presented. The following result can be seen as a strong strengthening of this fact.

Proposition 5.7. *Let G be a finitely generated non-Hopfian group and let φ be a non-injective epimorphism. Then the group G_φ is fully presented.*

Proof. Let S be a finite generating subset of G, and $\pi : F_S \twoheadrightarrow G$ the canonical projection. For every $k \geq 0$, we denote by $N_k = \pi^{-1}(\ker \varphi^k)$. The increasing union N_∞ of the subgroups N_k is nothing but the kernel of the natural map from F_S to the group G_φ:

$$1 \longrightarrow N_\infty \longrightarrow F_S \longrightarrow G_\varphi \longrightarrow 1.$$

For every $k \geq 1$ we let $N_\infty(k)$ be the normal subgroup of F_S generated as such by elements of N_∞ of length at most k. We also set

$$\alpha(k) = \min \left\{ |w|_S \;\middle|\; w \in N_\infty \setminus N_{k-1} \right\}$$

and

$$x(k) = \min \left\{ |w|_S \;\middle|\; w \in N_\infty \setminus N_\infty(k-1) \right\}.$$

We want to prove that $x(k)$ grows at most linearly.

For any integer $k \geq 1$, we let m_k be the smallest integer so that the set of elements of N_∞ of word length at most k lies in N_{m_k}. Note that since N_{m_k} is a normal subgroup, we have $N_\infty(k) \subset N_{m_k}$.

Let w be a word in $N_\infty \setminus N_{m_k-1}$ of minimal length, i.e. $|w|_S = \alpha(m_k)$. Since the homomorphism φ is onto, the set $\varphi(S)$ generates the group G. We denote by $\ell \geq 1$ the length of the element $g = \pi(w) \in G$ with respect to the finite generating set $\varphi(S)$. This means that there exist $s_1, \ldots, s_\ell \in S$ such that the equality $g = \varphi(s_1) \ldots \varphi(s_\ell)$ holds in the group G. By construction the word $w' = s_1 \ldots s_\ell$ lies in $N_\infty \setminus N_{m_k}$, and a fortiori w' does not belong to $N_\infty(k)$. This implies that $x(k+1)$, which is by definition the smallest length of an element in $N_\infty \setminus N_\infty(k)$, is at most the length of w':

$$x(k+1) \leq |w'|_S \leq \ell = |g|_{\varphi(S)}\,.$$

Since two finite generating sets yield bi-Lipschitz equivalent word metrics, there exists some constant $C > 0$ such that $|g|_{\varphi(S)} \leq C|g|_S$. But $g = \pi(w)$, so the word length of g with respect to S is at most $|w|_S = \alpha(m_k)$. Therefore we have proved that $x(k+1) \leq C\alpha(m_k)$. But now by definition of the integer m_k, there exists an element of length at most k in $N_\infty \setminus N_{m_k-1}$. This implies that $\alpha(m_k) \leq k$, and combined with the above inequality we obtain $x(k+1) \leq Ck$. $\square$

Remark 5.8. In [GM97, p. 204], a slight variation of the procedure of iterating a non-injective epimorphism is considered. One can check that the proof of Proposition 5.7 readily adapts to this construction. This implies in particular that the Grigorchuk group introduced in [Gri80], which can be obtained by following the aforementioned procedure [GM97, Theorem 4], is fully presented. Anticipating Proposition 5.17, this implies in particular that none of its asymptotic cones are simply connected.

5.2.3 Residually finite and metabelian limits

In this subsection me make the (probably well known) obervations that when the group G_φ has certain property (e.g. being residually finite or metabelian), then actually G_φ is the largest quotient of G having this property. This will be illustrated by various examples in the next paragraph.

If G is group, we denote by N_{res} the intersection of all finite index normal subgroups of G, and by $G_{res} = G/N_{res}$ its *residualization.* The group G_{res} is the largest residually finite quotient of G, in the sense that any $G \twoheadrightarrow H$ factors through N_{res} provided that H is residually finite. We also denote by G_{met} the metabelianization of G, i.e. the quotient of G by its second derived subgroup G''.

We will make use of the following elementary lemma.

Lemma 5.9. *Let G be a finitely generated group, and φ a morphism from G onto G. Then the kernel of φ lies inside N_{res}.*

Proof. We prove that for every $k \geq 1$, the kernel of φ is contained in all subgroups of G of index k. For, we make use of a result due to M. Hall, which says that the finitely generated group G has a finite number of subgroups $M_1, \ldots, M_{i_k}$ of index k. For $i = 1, \ldots, i_k$, set $L_i = \varphi^{-1}(M_i)$. Since φ is onto, it can be checked that all the subgroups L_i are different and have index k in G. By the pigeonhole principle, we obtain that the set of subgroups L_i is exactly $\{M_1, \ldots, M_{i_k}\}$. Being contained in all the L_i, the kernel of φ is contained in all the M_i. □

Remark 5.10. The above proof also shows that the normal subgroup N_{res} is invariant by any morphism from G onto G.

Applying Lemma 5.9 to all positive powers of φ, we immediately obtain the following result.

Proposition 5.11. *If G is a finitely generated group and φ a homomorphism from G onto G, then we have the inclusion $N_\varphi \subset N_{res}$. In particular, the group G_φ is residually finite if and only if this inclusion is an equality, i.e. if and only if G_φ is the residualization of G.*

Remark 5.12. Let G be a finitely generated group. Since finitely generated metabelian groups are residually finite by a result of P.Hall [Hal59], one has the inclusion $N_{res} \leq G''$. It follows that if G is non-Hopfian and φ is a non-injective epimorphism such that the group G_φ is metabelian, then G_φ is exactly the metabelianization of the group G. This situation is illustrated for instance in Example 5.13.

5.2.4 Examples

In this paragraph we give various examples of groups that can be obtained as direct limits of finitely generated non-Hopfian groups by iterating a non-injective epimorphism. In particular we highlight the fact that this class of finitely generated groups seems to be very broad. It follows from Proposition 5.7 that all the groups appearing in the following examples are fully presented.

Example 5.13. Let $r \geq 1$, and let $m, n \geq 2$ be two coprime integers. Let us consider the group $\mathrm{C}(m, n, r)$ defined by the presentation

$$\mathrm{C}(m,n,r) = \left\langle x, t \mid t^r x^m t^{-r} = x^n, [txt^{-1}, x] = \ldots = [t^{r-1}xt^{-(r-1)}, x] = 1 \right\rangle .$$

This family of groups appeared in [BS76] as a generalization of Baumslag-Solitar groups, which correspond to the case $r = 1$. Since $m, n \geq 2$ are chosen relatively prime, the group $G = \mathrm{C}(m,n,r)$ is non-Hopfian. More precisely, the homomorphism given by $\varphi(x) = x^m$ and $\varphi(t) = t$, is well defined, surjective but non-injective. In this case the group G_φ is the metabelian group $\mathbb{Z}[1/mn]^r \rtimes_M \mathbb{Z}$, where $\mathbb{Z}[1/mn]$ is the additive group of rational numbers with denominator a power of mn, and the action is defined by the $r \times r$ companion matrix

$$M = \begin{pmatrix} 0 & \cdots & 0 & n/m \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix}.$$

Example 5.14. Let us recall the construction from [Mei82] of non-Hopfian HNN-extensions. Let A be a finitely generated group and let $\mu, \nu : A \to A$ be two injective homomorphisms with proper image. Assume that

(a) $\mu \circ \nu = \nu \circ \mu$;

(b) $\langle \mu(A), \nu(A) \rangle = A$;

(c) $\exists\, \alpha \notin \mu(A)$, $\beta \notin \nu(A)$; $[\nu(\alpha), \mu(\beta)] = 1$.

Then the HNN-extension of A associated to the subgroups $\mu(A)$ and $\nu(A)$

$$G = \left\langle A, t \mid t\mu(a)t^{-1} = \nu(a) \ \ \forall a \in A \right\rangle$$

is non-Hopfian. More precisely, the map defined by $\varphi(t) = t$ and $\varphi(a) = \mu(a)$ for every $a \in A$ is a non-injective homomorphism from G onto G. One can check that if moreover A is nilpotent of class s, then the group G_φ is (nilpotent of class s)-by-cyclic.

When $A = \mathbb{Z}^2$ and $\mu(x,y) = (x, ny)$, $\nu(x,y) = (nx, y)$ for some integer $n \geq 2$, then the limit group is the metabelian group $\mathbb{Z}[1/n]^2 \rtimes \mathbb{Z}$, where the action is the multiplication by (n, n^{-1}), which is a well known example of a finitely generated infinitely presented metabelian group.

When $A = H_3(\mathbb{Z})$ is the three dimensional Heisenberg group, and μ, ν are defined by

$$\mu : \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & pz \\ & 1 & py \\ & & 1 \end{pmatrix},$$

$$\nu : \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & px & pz \\ & 1 & y \\ & & 1 \end{pmatrix},$$

where p is a prime, then one can check that the group G_φ is the group

$$A_3(\mathbb{Z}[1/p]) = \begin{pmatrix} 1 & \mathbb{Z}[1/p] & \mathbb{Z}[1/p] \\ & p^{\mathbb{Z}} & \mathbb{Z}[1/p] \\ & & 1 \end{pmatrix}.$$

It is an example of a 3-solvable finitely generated group with non-finitely generated center.

Example 5.15. Let $n \geq 2$, and denote by $\mathbb{F}_n$ the ring $\mathbb{Z}/(n)$. Consider the group

$$\Gamma = \begin{pmatrix} 1 & \mathbb{F}_n[t^{\pm 1}] & \mathbb{F}_n[t^{\pm 1}] \\ & t^{\mathbb{Z}} & \mathbb{F}_n[t^{\pm 1}] \\ & & 1 \end{pmatrix}$$

and its central subgroup

$$Z = \begin{pmatrix} 1 & 0 & \mathbb{F}_n[t] \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

The conjugation by the matrix $\mathrm{diag}(t, 1, 1)$ is an isomorphism of Γ, strictly mapping Z into itself, and thus induces a non-injective epimorphism φ of $G = \Gamma/Z$. The union of the iterated kernels of φ is easily seen to be the group $Z(\Gamma)/Z$, and thus G_φ is isomorphic to the quotient of Γ by its center. Now this group is $\mathbb{F}_n[t^{\pm 1}]^2 \rtimes_{(t,t^{-1})} \mathbb{Z}$, and is isomorphic to $\mathbb{F}_n^2 \wr \mathbb{Z}$. It is a subgroup of index two in the group $\mathbb{F}_n \wr \mathbb{Z}$.

The same procedure can be carried out when considering the subgroup of $\mathrm{GL}_3(\mathbb{R})$ generated by

$$\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ & \lambda & 0 \\ & & 1 \end{pmatrix},$$

where λ is any transcendental real number. In this situation we obtain $\mathbb{Z}^2 \wr \mathbb{Z}$ as limit group, which has index two in $\mathbb{Z} \wr \mathbb{Z}$.

5.3 Asymptotic cones of fully presented groups

The following terminology and notation are essentially borrowed from [Pap96]. We let $\mathbb{D} = [0,1] \times [0,1]$ be the unit Euclidean square of dimension two, and denote by $\mathbb{S}^1$ its boundary. A collection of squares $D_1, \ldots, D_k$ is partition of $\mathbb{D}$ if $D_i \cap D_j = \partial D_i \cap \partial D_j$ whenever $i \neq j$, and if $\mathbb{D}$ is the union of the squares D_i.

If X is a geodesic metric space, a loop in X is by definition a continuous map $c : \mathbb{S}^1 \to X$, and we freely identify a loop with its image in X. A partition π of c is a continuous map extending c to $\partial D_1 \cup \ldots \cup \partial D_k$, where $D_1, \ldots, D_k$ is a partition of $\mathbb{D}$. We define the *mesh* of π as the maximal length of the paths $\pi(\partial D_i)$.

Lemma 5.16. *Let X be a geodesic metric space, and let $\mathcal{C} = \mathrm{Cone}^\omega(X, (x_n), (s_n))$ be an asymptotic cone of X. Then*

(a) any loop in $\mathcal{C}$ is the ω-limit of a sequence of loops in X;

(b) if c is a loop in $\mathcal{C}$ being the ω-limit of a sequence of loops (c_n) in X, then any partition of c is the ω-limit of a sequence of partitions of c_n.

Proof. Statement (a) is proved in [Ken12, Lemma 2.2] for paths rather than loops, but the proof can be easily adapted to realize any loop in $\mathcal{C}$ as the ω-limit of a sequence of loops in X.

Statement (b) is obtained similarly, working in each square of the partition. □

Recall that by a theorem of Gromov, if a finitely generated group G has all its asymptotic cones simply connected, then G is finitely presented (and has a polynomially bounded Dehn function) [Gro93]. Of course one cannot hope to obtain the same conclusion if we weaken the hypothesis by requiring that *one* asymptotic cone of G is simply connected, as for example any non-hyperbolic lacunary hyperbolic group is infinitely presented [OOS09, Appendix]. Nevertheless, the following result says that if G has one asymptotic cone that is simply connected, then the group G is lacunary presented.

Proposition 5.17. *Let G be a fully presented finitely generated group. Then none of the asymptotic cones of G are simply connected.*

Proof. Let S be a finite generating subset of G, and let N be the kernel of the canonical projection $F_S \twoheadrightarrow G$. We argue by contradiction, and assume that there exist a non-principal ultrafilter ω and a scaling sequence (s_n) such that $\mathrm{Cone}^\omega(G, (s_n))$ is simply connected. By assumption there exist a constant $C > 0$ and a sequence of relations $r_n \in N$ so that

$$s_n + 1 \leq |r_n|_S \leq C(s_n + 1),$$

and r_n does not belong to $N(s_n)$. By construction, the sequence of loops $c_n : \mathbb{S}^1 \to \mathrm{Cay}(G, S)$ parametrized proportionally to the length associated to r_n yields a loop $c : \mathbb{S}^1 \to \mathrm{Cone}^\omega(G, (s_n))$. Since $\mathrm{Cone}^\omega(G, (s_n))$ is supposed to be simply connected, the map $c : \mathbb{S}^1 \to \mathrm{Cone}^\omega(G, (s_n))$ can be extended to a continuous function $\sigma : \mathbb{D} \to \mathrm{Cone}^\omega(G, (d_n))$. Now since $\mathbb{D}$ is compact, the map σ is uniformly continuous, and there exists $\eta > 0$ so that $d_\omega(\sigma(t), \sigma(u))$ is at most $1/5$ as soon as the distance between t and u is at most η. Let us consider the partition of $\mathbb{D}$ given by the net $\{(a\eta, b\eta) : a, b \in \mathbb{Z},\ 0 \leq a, b \leq 1/\eta\}$. Since the mesh of this partition is equal to 4η, the restriction of σ to this partition yields a partition of the loop c in $\mathrm{Cone}^\omega(G, (s_n))$ of mesh at most $4/5$.

From a geometric point of view, the fact that r_n does not belong to the normal subgroup of N generated by elements of length at most s_n implies that the mesh of any partition of the loop c_n is larger than d_n. Now we claim that this implies that the mesh of any partition of the loop c is at least 1. Indeed according to Lemma 5.16, any partition π of the loop c is the ω-limit of a sequence of partitions π_n of the loop c_n in $\mathrm{Cay}(G, S)$. Being the limit over the ultrafilter ω of the mesh of π_n rescaled by d_n, the mesh of π is at least 1 by the previous observation. This readily gives a contradiction with the previous paragraph. $\square$

The following is an immediate consequence of Proposition 5.17.

Corollary 5.18. *Any finitely generated lacunary hyperbolic group is lacunary presented.*

In view of Proposition 5.7, we also deduce the following result.

Corollary 5.19. *Let G be a finitely generated non-Hopfian group and let φ be a non-injective epimorphism. Then none of the asymptotic cones of the group G_φ are simply connected.*

So Proposition 5.17 establishes a link between the relation range of a finitely generated group and asymptotic properties of the group, and thus indicates that investigating further the relation range could provide an important tool to study the asymptotic geometry of finitely generated groups.

Open questions

1. Does the metric space Cone(SOL) admit a bi-Lipschitz self-homeomorphism φ so that there exist $c_1, c_2 > 0$ such that $c_1 \leq d(\varphi(x), x) \leq c_2$ for every $x \in \text{Cone(SOL)}$? A negative answer to this question would imply that the asymptotic cones of the Lie group $\mathrm{H}_3(\mathbb{R}) \rtimes \mathbb{R}$ are not bi-Lipschitz homeomorphic to Cone(SOL) (see Section 2.4).
2. Theorem 3.17 asserts that the Dehn function of the group $\mathrm{AAut}(\mathcal{T}_{d,k})$ is asymptotically bounded by the Dehn function of $V_{d,k}$. Is the Dehn function of $\mathrm{AAut}(\mathcal{T}_{d,k})$ strictly smaller than the one of $V_{d,k}$?

   On the one hand, we observed in Section 3.2 that the group $\mathrm{AAut}(\mathcal{T}_{d,k})$ is the Schlichting completion of $V_{d,k}$ with respect to a well understood commensurated subgroup, and it sometimes happen that the Dehn function becomes smaller after performing such a (non-trivial) completion. For example the Baumslag-Solitar group $\mathrm{BS}(1, n)$ has an exponential Dehn function for $n \geq 2$, while its Schlichting completion $\mathbb{Q}_n \rtimes \mathbb{Z}$ is Gromov-hyperbolic, and therefore has a linear Dehn function.

   On the other hand, it seems not clear how to deal with the loops in $V_{d,k}$ having a large area (see [Gub00]) more efficiently in the group $\mathrm{AAut}(\mathcal{T}_{d,k})$ than in $V_{d,k}$.
3. Are the asymptotic cones of $\mathrm{AAut}(\mathcal{T}_{d,k})$ (resp. $V_{d,k}$) simply connected ? It seems likely that the results from Section 3.2 (see in particular the proof of Proposition 3.29) imply that a positive answer for $V_{d,k}$ would yield a positive answer for $\mathrm{AAut}(\mathcal{T}_{d,k})$.
4. The following question is motivated by Remark 3.77. Does there exist a CAT(0) metric space X on which the group $\mathrm{AAut}(\mathcal{T}_{d,k})$ acts properly by isometries, and such that for some $x_0 \in X$, the orbital map $g \mapsto g \cdot x_0$ is a quasi-isometric embedding ? More generally, does there exist a quasi-isometric embedding of $\mathrm{AAut}(\mathcal{T}_{d,k})$ into a CAT(0) metric space ? The same questions hold for $V_{d,k}$ as well.
5. Can $\mathrm{AAut}(T_d)$ be quasi-isometric to $\mathrm{AAut}(T_{d'})$ for $d \neq d'$? Is $\mathrm{AAut}(T_d)$ quasi-isometric to a finitely generated group ?

6. It is a well-known open problem to decide whether there exist finitely presented groups of intermediate growth. The following question is strictly weaker: does there exist finitely generated groups of intermediate growth that are lacunary presented ?

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