Classical biorthogonal and hypergeometric

Preliminary notes, incomplete and unfinished

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Jes' Fine

Fremount (the boy bug) of Walt Kelly's *Pogo*.

Classical orthogonal polynomials are simple hypergeometric expansions, they have funny names, as numerous as the craters of the Moon, but what about biorthogonal rational functions?

1. Biorthogonal rational functions and rational interpolation.

Biorthogonality refers to two sequences of objects, possibly of different kind, but able to interact through a bilinear form $\langle\ ,\ \rangle$: $\{\mathcal{A}_n\}$ and $\{\mathcal{B}_m\}$ are biorthogonal if $\langle \mathcal{A}_n, \mathcal{B}_m \rangle = 0, m \neq n$. Here, $\mathcal{A}_n$ and $\mathcal{B}_m$ are rational functions

$$\frac{A_n(x)}{(x-a_0)\cdots(x-a_n)} \quad \text{and} \quad \frac{B_m(x)}{(x-b_0)\cdots(x-b_m)},$$

and the bilinear form is $\int_c^d f(t)g(t)d\mu(t)$, (possibly formal).

'Normally', A_n (resp. B_m) is orthogonal to all polynomials of smaller degrees with respect to $d\mu(t)/((t-a_0\cdots(t-a_n)(t-b_0)\cdots(t-b_{n-1}))$ (resp. $d\mu(t)/((t-a_0\cdots(t-a_{m-1})(t-b_0)\cdots(t-b_m)))$. This cries for C_n of degree n orthogonal to all polynomials of smaller degrees with respect to $d\mu(t)/((t-a_0\cdots(t-a_n)(t-b_0)\cdots(t-b_n))$.

Let $P_{2n} = C_n$ and $P_{2n+1} = A_n$, $n = 0,1,\ldots, P_{n+1}(x) = P_n(x) + r_n\left(\begin{cases}(x-a_{n/2}) \text{ if } n \text{ is even}\\(x-b_{(n-1)/2}) \text{ if } n \text{ is odd}\end{cases}\right) P_{n-1}(x), n=0,1,\ldots,P_{-1}=0$ Ismail & Masson, Spiridonov & Zhedanov

A_n etc. are ***denominators of rational interpolants*** of $f(x)=\int_c^d (x-t)^{-1}d\mu(t)$: $A_n(x)f(x) =$ pol. interpolant at $a_0,\cdots,a_n,b_0,\ldots,b_{n-1}+$ remainder, and the degree of the interpolant $=\int_c^d A_n(t)[1-(x-a_0)\ldots/(t-a_0)\ldots)](x-t)^{-1}d\mu(t)$ is only n instead of $2n$ thanks to orthogonality.

$$f(x)/f(a_0) = 1+\cfrac{r_0(x-a_0)}{1+\cfrac{r_1(x-b_0)}{1+\cfrac{r_2(x-a_1)}{1+\cfrac{r_3(x-b_1)}{1+\ddots}}}}$$ Multipoint Padé: Goncar, Lopez, Rahmanov, Stahl

CLASSICAL functions must satisfy simple explicit recurrence relations, difference equations and relations, hypergeometric expansions, perhaps Rodrigues formulas.

2. The exponential function.

$$e^x = 1 + \cfrac{x}{1 - \cfrac{x}{2 + \cfrac{x}{3 - \cfrac{x}{2 \ddots}}}} = 1 + \cfrac{x}{1 + \cfrac{-x/2}{1 + \cfrac{x/6}{1 + \cfrac{-x/6}{1 + \cfrac{x/10}{1 + \cfrac{-x/10}{1 + \ddots}}}}}}, \text{ Cuyt Perron}$$

Wall, from $f(x) = f_0(x), f_0'(x) = f_0(x)$ and $f_0(x) = 1 + \dfrac{x}{f_1(x)}$, the differential equation for f_1 is $\dfrac{f_1(x) - xf_1'(x)}{f_1^2(x)} = 1 + \dfrac{x}{f_1(x)}$, or $xf_1'(x) = -f_1^2(x) + (1-x)f_1(x)$, etc. Euler 24 years old! (Knuth **2** §4.5.3, answers of §4.5.3, 16, Cretney 2014

$$\frac{1}{1}, \frac{1+x}{1}, \frac{1+x/2}{1-x/2}, \frac{1+2x/3+x^2/6}{1-x/3}, \frac{1+x/2+x^2/12}{1-x/2+x^2/12}, \ldots, \frac{{}_1F_1(-m,-m-n,x)}{{}_1F_1(-n,-m-n,-x)}$$

Khovanskii quotes Lagrange 1776 for more examples.

Rational interpolation on two lattices with the same step, say, $a+nh'$ and $b+nsh'$, where $s=1$ or -1.

$\exp(x-a) = 1 + \cfrac{r_0(x-a)}{1+\cfrac{r_1(x-b)}{1+\cfrac{r_2(x-a-h')}{1+\ddots}}}$ so, $r_0 = \dfrac{\exp(b-a)-1}{b-a}, r_1 = \dfrac{\dfrac{r_0 h'}{\exp(h')-1}-1}{a+h'-b}, \cdots$ more and more complicated expressions, solved in some cases:

when $b=a+h'/2=a+h$, one interpolates on integer multiples of $h'/2=h$ Iserles 1981

we write everything in expansions of $(x-a)(x-a-h)(x-a-2h)\cdots(x-a-kh)$: let the denominator be $P_n(x)=\sum_0^n d_k(x-a)\cdots(x-a-(k-1)h)$, multiply by $\exp(x-a)$ replaced in the k^{th} term by the related Gregory interpolatory expansion $\exp(x-a)=\exp(kh)\exp(x-a-kh)=\exp(kh)[1+e^h-1]^{(x-a)/h-k}=\sum_p \dfrac{((x-a)/h-k)\cdots((x-a)/h-k-p+1)}{p!}\exp(kh)(e^h-1)^p$ whence the full expansion of the product

$$P_n(x)\exp(x-a) = \sum_r \left[\sum_{k=0}^{n} d_k \frac{\exp(kh)(e^h-1)^{r-k}}{h^{r-k}(r-k)!}\right](x-a)\cdots(x-a-(r-1)h)$$

must have vanishing coefficients for $r = m+1,\ldots,m+n$. As the expression in square brackets is $(e^h-1)^r h^{-r}/r!$ times a polynomial of degree n in r, it must factorize as const. $(r-m-1)\cdots(r-m-n) = \sum_k (-1)^{n-k}\frac{(m+n-k)!}{m!}\binom{n}{k} r(r-1)\cdots(r-k+1)$. This leads to $d_k = (-1)^{n-k}\frac{(m+n-k)!}{m!}\binom{n}{k}(1-e^{-h})^k h^{-k}$,

$P_n(x) = {}_2F_1(-n,(a-x)/h;-m-n;1-e^{-h})$, $N_m(x) = {}_2F_1(-m,(a-x)/h;-m-n;1-e^{-h})$,

$r_n = \dfrac{(-1)^n\exp((-1)^n h/2)}{h\cosh(h/2)[(n+(1+(-1)^n)/2)\coth(h/2)-1]}$, $n=1,2,\ldots$

When $s=-1$ and $b=a-h$, we again have interpolation at equidistant points $nh, n\in\mathbb{Z}$, and a simple formula: $r_n = \dfrac{(-1)^n}{(2n+1+(-1)^n)\dfrac{h\exp(h)}{\exp(h)-1}}$

Do we have similar tricks when $b \neq a+h'/2$?

We still have similar asymptotic behaviors when n is large: $s=1$, $r_n \sim \dfrac{2(-1)^n\exp((-1)^n h/4)\sinh(h/4))}{h\cosh^2(h/4)(n+s_n)}$, with $s_n \approx \dfrac{1+(-1)^n}{2} - \dfrac{2(b-a)}{h}\tanh\dfrac{h}{4}$ when $b-a$ and h are small

r_n is a rational function of n when $(b-a)/h - 1/2 \in \mathbb{Z}$:

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(b-a)/h       1/2  -1/2  3/2  -3/2
num degree     0    1     3    6
den degree     1    2     4    7
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$$r_n = \cfrac{(-1)^n X \exp((-1)^n h/4)}{n + (1+(-1)^n)/2 - Y - \cfrac{(b-a-h/2)(-1)^n X \exp(-(-1)^n h/4)}{n + (1-3(-1)^n)/2 + Y - \cfrac{(b-a+h/2)(-1)^n X'}{n + (1+5(-1)^n)/2 + ? - \cfrac{?}{n + ? - (b-a-3h/}}}}$$

$X = 2\tanh(h/4)/[h\cosh(h/4)], Y = 2(b-a)\tanh(h/4)/h$, $X' = 1 - (-1)^n h/8 - h^2/48 + (-1)^n h^3/192 + h^4/1920 + \cdots$

3. Jacobi and Hahn's weights.

Jacobi

Gauss's ratio of hypergeometric functions Perron 1913, 1929 §59, 1957 §24; Wall chap. XVIII §89. $_2F_1(\alpha+1,1;\alpha+\beta+2;x) = \cfrac{1}{1-\cfrac{\frac{\alpha+1}{\alpha+\beta+2}x}{1-\cfrac{\frac{\beta+1}{(\alpha+\beta+2)(\alpha+\beta+3)}x}{1-\cfrac{\frac{(\alpha+\beta+2)(\alpha+2)}{(\alpha+\beta+3)(\alpha+\beta+4)}x}{1-\cfrac{\frac{2(\beta+2)}{(\alpha+\beta+4)(\alpha+\beta+5)}x}{1-\cfrac{\frac{(\alpha+\beta+3)(\alpha+3)}{(\alpha+\beta+5)(\alpha+\beta+6)}x}{1-\ddots}}}}}}$

Denominators of approximants of order $2n$ and $2n-1$ are $x^nP_n^{(\alpha+1,\beta)}(1-2/x)$ and $x^nP_n^{(\alpha,\beta)}(1-2/x) = \ldots x^n{}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;1/x)$, NIST 18.5.7

Erdelyi *IT* **2** §16.4(4) p. 284 $\int_{-1}^{1}(z-x)^{-1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+n+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+2n+2)(z-1)^{n+1}}{}_2F_1(n+1,\alpha+n+1;\alpha+\beta+2n+2;2/(z-1))$

For the Jacobi measure $d\mu(t) = ((d-t)^{\alpha}(t-c)^{\beta}dt$ classical 2-point Padé only if exponents are half integer, Heun JAT 2021

Hahn

Let $\rho_k = \dfrac{(\beta+1)\cdots(\beta+k)(\alpha+1)\cdots(\alpha+N-k-1)}{k!(N-1-k)!}$, $k=0,\ldots,N-1$.

One has $\mu_0 = \sum_0^{N-1} \rho_k = (\alpha+\beta+2)\cdots(\alpha+\beta+N)/(N-1)!$,

$\mu_1 = \sum_0^{N-1} k\rho_k = (\beta+1)(\alpha+\beta+3)\cdots(\alpha+\beta+N)/(N-2)!$,

$\mu_2 = \sum_0^{N-1} k^2\rho_k = \mu_1 + (\beta+1)(\beta+2)(\alpha+\beta+4)\cdots(\alpha+\beta+N)/(N-3)!$,

$$f(x) = \sum_0^{N-1} \frac{\rho_k}{x-c-kh} = \frac{\mu_0}{x-c} + \frac{h\mu_1}{(x-c)^2} + \frac{h^2\mu_2}{(x-c)^3} + \cdots$$

$$= \cfrac{\mu_0}{x-c-\cfrac{\frac{h(N-1)(\beta+1)}{\alpha+\beta+2}}{x-c-\cfrac{\frac{h(N+\alpha+\beta+1)(\alpha+1)}{(\alpha+\beta+2)(\alpha+\beta+3)}}{x-c-\ddots}}}$$ (leads to Hahn polynomials by Padé),

See that $(k+1)(\alpha+N-k-1)\rho_{k+1} = (\beta+k+1)(N-1-k)\rho_k$. Niki §2.4.6 Pearson

$$(x-c+h)(x-d-\alpha h+h)f(x+h)$$
$$=\sum_{-1}^{N-2}\frac{[(x-c-kh)(x+kh-d-\alpha h+2h)+(k+1)(k-N-\alpha+1)h^2]\rho_{k+1}}{x-c-kh}=\sum_0^{N-1}(x+$$
$$(k-1)h+2h-d-\alpha h)\rho_k+\sum_0^{N-1}\frac{h^2(\beta+k+1)(k-N+1)\rho_k}{x-c-kh}=(x+h-d-\alpha h)\mu_0+h\mu_1+$$
$$(x-c+\beta h+h)(x-c-Nh+h)f(x)-(x-d+\beta h+2h)\mu_0-h\mu_1$$
$=(x-c+\beta h+h)(x-d+h)f(x)-(\alpha+\beta+1)h\mu_0,$ with $d=c+Nh$.

Interpolation at $x=c-\beta h, c-\beta h+h, c-\beta h+2h, \ldots$:

$$f(c-\beta h)=\frac{-(\alpha+\beta+1)h\mu_0}{-\beta h(-N-\alpha-\beta)h}, \text{ etc.}$$

Let $f_0(x)=f(x)/f(c-\beta h)$, $f_1(x)=\dfrac{r_0(x-c+\beta h)}{f_0(x)-1}$, $r_1=(f_1(c-\beta h+2h)-1)/h=\dfrac{\alpha+\beta-1}{h(2\beta-3+(2-\beta)N-(\beta-1)(\alpha+\beta))}$, etc.

$$r_0=\frac{\alpha+\beta}{h(\beta-1)(\alpha+\beta+N-1)}, r_{2n}=-n\frac{(\alpha+n)(N-n)}{h\ \Xi_n\Upsilon_{n+1}},$$
$$r_{2n-1}=-\frac{(\beta-n)(\alpha+\beta-n)(\alpha+\beta+N-n)}{h\ \Xi_n\Upsilon_n}, n=1,2,\ldots$$

with $\Xi_n = \beta(\alpha+\beta+N-3n) - n(\alpha+2N) + 3n^2$,
$\Upsilon_n = \beta(\alpha+\beta+N-3n+1) - n\alpha - (2n-1)N + 3n^2 - 3n + 1$
$= (\beta-n)(\alpha+\beta+N-2n+1) - (n-1)N + (n-1)^2$

Check $N \to \infty$ and $h = (d-c)/N \to 0$: $r_{2n} \to -\dfrac{n(\alpha+n)}{(d-c)(2n-\beta)(2n+1-\beta)}$, $r_{2n-1} \to -\dfrac{(n-\beta)(n-\alpha-\beta)}{(d-c)(2n-\beta)(2n-1-\beta)}$.

$P_0 = P_1 = 1$, $P_{2n}(x) = P_{2n-1}(x) + r_{2n-1}(x-c+\beta h-(2n-1)h)P_{2n-2}(x) = r_1r_3\cdots r_{2n-1}(x-c+\beta h)^n + \cdots$,

$$P_{2n+1}(x) = P_{2n}(x) + r_{2n}(x-c+\beta h-2nh)P_{2n-1}(x)$$
$$= r_2r_4\cdots r_{2n}(x-c+\beta h)^n \underbrace{\sum_0^n \frac{r_1r_3\cdots r_{2k-1}}{r_2r_4\cdots r_{2k}}}_{-\frac{r_1r_3\cdots r_{2n+1}\Xi_{n+1}}{hr_2r_4\cdots r_{2n}(\alpha+\beta-1)}\text{ no proof}} + \cdots$$

$P_{2n}(x) =$ const. ${}_3F_2(-n, \alpha+\beta-n, (x-c)/h+\beta-2n; \beta-2n, \alpha+\beta+N-2n; 1)$,

$P_{2n+1}(x) =$ const. ${}_3F_2(-n, \alpha+\beta-n-1, (x-c)/h+\beta-2n-1; \beta-2n-1, \alpha+\beta+N-2n-1; 1)$,

4.Classical biorthogonal rational functions.

Δ	$f(x+h)-f(x)$
∇	$f(x)-f(x-h)$
δ	$f(x+h/2)-f(x-h/2)$
G	$f(qx)-f(x)$
H	$f(qx+\omega)-f(x)$
W	$f((t+i/2)^2)-f((t-i/2)^2)$, $x=t^2$
AW	$f(\cos(\theta+\lambda))-f(\cos(\theta-\lambda))$, $x=\cos\theta$
NSU	$f(x(s+1))-f(x(s))$, $x(s)=c_1q^s+c_2q^{-s}+c_3$
	$x(s)=c_1s^2+c_2s+c_3$
E	$f(\mathcal{E}(s+1))-f(\mathcal{E}(s))$, where $\mathcal{E}$ is an elliptic function.

Table 1: Some difference operators of ANSUW+E type. First column gives the name, or the context: G is for geometric or Jackson, H for Hahn,W for Wilson, AW for Askey-Wilson with $q=\exp(2i\lambda)$, NSU for Nikiforov-Suslov-Uvarov, E for elliptic (Baxter, Spiridonov & Zhedanov).

Conjecture. Let $f(x) = \sum_0^{N-1} \frac{(\mathfrak{y}(c+k) - \mathfrak{y}(c+k-1))\rho_k}{x - \mathfrak{x}(c+k)}$, where $\mathfrak{x}(c+k), c \in \mathbb{C}, k \in \mathbb{Z}$ is a lattice, or grid, of ANSUW+E kind, and $U(\mathfrak{x}(c+k))\frac{\rho_{k+1} - \rho_k}{\mathfrak{x}(c+k+1) - \mathfrak{x}(c+k)} = V(\mathfrak{x}(c+k))\frac{\rho_{k+1} + \rho_k}{2}$, with U and V of degree $\leqslant 2$ (?) $\left(\mathfrak{y}(s) = \frac{\alpha\mathfrak{x}(s+1/2)+\beta}{\gamma\mathfrak{x}(s+1/2)+\delta}\right)$ JCAM 2009

If $\{a_0, \ldots, a_m\} \bigcup \{b_0, \ldots, b_n\}$ is a lattice with parameters in arithmetic progression $\{\mathfrak{x}(s_n - m), \mathfrak{x}(s_n - m + 1), \ldots, \mathfrak{x}(s_n + n + 1)\}$, i.e., where a_0 and b_0 are NOT independent separate starting points ($m = n$ or $m = n \pm 1$), and if one of the endpoints is a singular point, i.e. $\frac{U(\mathfrak{x}(s))}{\mathfrak{x}(s+1) - \mathfrak{x}(s)} \pm \frac{V(\mathfrak{x}(s))}{2} = 0$, then, numerators and denominators of rational interpolants have simple hypergeometric expansions.

Use $(\mathcal{D}f)(\mathfrak{y}((s)) = \frac{f(\mathfrak{x}(s+1)) - f(\mathfrak{x}(s))}{\mathfrak{x}(s+1) - \mathfrak{x}(s)}$
$= -\sum \frac{(\mathfrak{y}(c+k) - \mathfrak{y}(c+k-1))\rho_k}{F(\mathfrak{x}(c+k), \mathfrak{y}(s)) = (\mathfrak{y}(s) - \mathfrak{y}(c+k))(\mathfrak{y}(s) - \mathfrak{y}(c+k-1))}$
$= \sum \frac{\rho_{k+1} - \rho_k}{\mathfrak{y}(s) - \mathfrak{y}(c+k)}$, etc.

Wait! it's not finished. The conjecture is not an if-and-only-if

5. Known hypergeometric instances with separate $\{a_k\}$ and $\{b_k\}$ lattices. An interpolatory example, M. Rahman, Families of biorthogonal rational functions in a discrete variable, *SIAM Journal on Mathematical Analysis* Vol **12** issue 3 (May 1981) pp. 355–367.

We interpolate $f = f_0$ at $a, b, a+h, b+h, a+2h, b+2h, \ldots$, where f depends on two parameters ρ and σ, by

$f(x) = C\sum_{-\infty}^{\infty} \dfrac{\Gamma(k+1-a_0/h)\Gamma(k+1-b_0/h)}{\Gamma(k-\rho/h)\Gamma(k-\sigma/h)} \dfrac{1}{x/h-k}$. Special values are

$$f(a) = C\frac{-\pi^2\Gamma(\alpha)/h}{\sin(\pi a/h)\sin(\pi 0/h)\Gamma((a-\rho)/h)\Gamma((a-\sigma)/h)\Gamma((b-\rho)/h-1)\Gamma((b-\sigma)/h-1)},$$

$$f(b) = C\frac{-\pi^2\Gamma(\alpha)/h}{\sin(\pi a/h)\sin(\pi b/h)\Gamma((a-\rho)/h-1)\Gamma((a-\sigma)/h-1)\Gamma((b-\rho)/h)\Gamma((b-\sigma)/h)},$$

where $\alpha = (a+b-\rho-\sigma)/h - 2 > 0$. Dougall With C such that $f(a) = 1, f(b) = \dfrac{(a-\rho-h)(a-\sigma-h)}{b-\rho-h)(b-\sigma-h)}$. One also shows

$(x-\rho)(x-\sigma)f(x+h) = (x-a+h)(x-b+h)f(x) + (a-h-\rho)(a-h-\sigma)$

One finds $f(x) = 1 + \cfrac{r_0(x-a)}{1 + \cfrac{r_1(x-b)}{1 + \cfrac{r_2(x-a-h)}{\ddots}}}$, with $r_0 = -\dfrac{h\alpha}{(b-h-\rho)(b-h-\sigma)}$,

$$r_{2n} = \frac{nh}{(b-h-\rho)(b-h-\sigma)},\ r_{2n-1} = \frac{h(\alpha+n)}{(b-h-\rho)(b-h-\sigma)},\ n = 1, 2, \dots$$

Curiously, if one interpolates the same function at the single sequence $\{a, a+h, a+2h, \dots\}$, we still have closed forms, in agreement with the conjecture, but with much more complicated formulas for the r_ns.

$$\frac{P_{2n}(x)}{(x-a-h)\cdots(x-a-nh)}$$
$$= \frac{(\alpha+1)\cdots(\alpha+n)h^n}{(b-h-\rho)^n(b-h-\sigma)^n}\ {}_3F_2(-n, (a-\rho)/h, (a-\sigma)/h; \alpha+1, (a-x)/h+1; 1)$$

$$\frac{P_{2n+1}(x)}{(x-b-h)\cdots(x-b-nh)}$$
$$= \frac{(\alpha+2)\cdots(\alpha+n+1)h^n}{(b-h-\rho)^n(b-h-\sigma)^n}\ {}_3F_2(-n, (b-\rho)/h, (b-\sigma)/h; \alpha+2, (b-x)/h+1; 1)$$

Rahman has an example with opposite directions too: $a+kh$ and $b-kh$

biorthogonal 4F3 from Rahman 1981 $\rho = b_1,\ \sigma = b_2,\ b/h = -a_3 - 1, a/h = 2 + b_1 + b_2 - a_3$: $\rho + \sigma = (a-b)/h - 3$

Rahman ends his 1981 paper with complicated ${}_4F_3$ rational functions $R_n(x) = {}_4F_3\left[\begin{matrix} -n, n+\rho+\sigma+1, -M-x/h, a/h-N-\sigma-1 \\ \rho+1, -M-N, (a-x)/h \end{matrix}\right]$ and $S_n(x) = {}_4F_3\left[\begin{matrix} -n, n+\rho+\sigma+1, x/h-N, -b/h-M-1-\rho \\ \sigma+1, -M-N, (x-b)/h \end{matrix}\right]$ of poles $a+kh$ and $b-kh$, and a measure made of masses

$$w_k = \frac{\Gamma(M+\rho+k+1)\Gamma(N+1+\sigma-k)\Gamma(k+1-a/h)}{\Gamma(M+1+k)\Gamma(N+1-k)\Gamma(k-b/h)},$$ on $k = -M, -M+1, \ldots, N$.

$$w_{k+1} = \frac{(M+\rho+k+1)(N-k)(k-a/h+1)}{(M+1+k)(N+\sigma-k)(k-b/h)} w_k, k = -M, -M+1, \ldots N-1.$$

Remark that w_k automatically vanishes at integers $< -M$ or $> N$.

$$(n+\rho)(n+\rho+\sigma)(M+N-n+1)(2n+\rho+\sigma-2)(x-a-(n-1)h)R_n(x)$$
$$= \{(2n+\rho+\sigma-1)[(2n+\rho+\sigma)(2n+\rho+\sigma-2)a - h[(2M-2N-\rho-3\sigma)n^2+\cdots]]x + n^4 + \cdots\}R_{n-1}(x) - (n+\sigma-1)(n-1)(2n+\rho+\sigma)(n+M+N+\rho+\sigma)(x-a+(n+\rho+\sigma)h)R_{n-2}(x)$$

$$(n+\sigma)(n+\rho+\sigma)(M+N-n+1)(2n+\rho+\sigma-2)(x-b+(n-1)h)S_n(x)$$
$$= -(2n+\rho+\sigma-2)(2n+\rho+\sigma-1)(2n+\rho+\sigma)(x(b+\cdots)+\cdots)S_{n-1}(x)$$
$$-(n+\rho-1)(n-1)(2n+\rho+\sigma)(n+M+N+\rho+\sigma)(x-b-(n+\rho+\sigma)h)S_{n-2}(x)$$

?

The Stieltjes function $f(x) = \sum_{k=-M}^{N} \frac{w_k}{x-kh} = \frac{\sum_{k=-M}^{N} w_k}{x} + \frac{\sum_{k=-M}^{N} khw_k}{x^2} + \cdots$ satisfies $(x+(M+1)h)(x-Nh-\sigma h)(x-b)f(x+h)$ $-(x+(M+\rho+1)h)(x-Nh)(x-a+h)f(x)$ = a rational function turning to be the polynomial of first degree $(\sum_{-M}^{N} w_k)(x+(\rho+\sigma+1)b+(\rho+1)(M+\rho+\sigma+2)h-(\sigma+1)Nh)$. From Rahman (5.9) M*ple, $\sum w_k = \frac{\Gamma(\rho+1)\Gamma(\sigma+1)\Gamma((a-b)/h+1+M+N)\Gamma(-b/h-\rho+N)\Gamma(-a/h-M)}{\Gamma((a-b)/h+1)(M+N)!\Gamma(-b/h+1+N)\Gamma(-b/h-\rho-M)}$.

So, $f(a) = (\sum w_k)\frac{a-h+(\rho+\sigma+1)b+(\rho+1)(M+\rho+\sigma+2)h-(\sigma+1)Nh}{(a-h+(M+1)h)(a-h-Nh-\sigma h)(a-h-b)}$

$$= (\sum w_k)\frac{(\rho+\sigma+2)(b+h)+(\rho+1)(M+\rho+\sigma+2)h-(\sigma+1)Nh}{(b+(M+\rho+\sigma+3)h)(b+(\rho+2-N)h)(\rho+\sigma+2)h},$$
$$f(b) = -(\sum w_k)\frac{b+(\rho+\sigma+1)b+(\rho+1)(M+\rho+\sigma+2)h-(\sigma+1)Nh}{(b+(M+\rho+1)h)(b-Nh)(b-a+h=-(\rho+\sigma+2)h)}.$$

$f(x)/f(a) = 1+\cfrac{r_0(x-a)}{1+\cfrac{r_1(x-b)}{1+\cfrac{r_2(x-a-h)}{1+\dots}}}$, $r_0 = \dfrac{f(a)-f(b)}{(a-b)f(a)}$ does not seem to work

??

For $q-$analogues, Rahman & Suslov. See also

From D. R. Masson, The Last of the Hypergeometric Continued Fractions , in *Mathematical Analysis, Wavelets, and Signal Processing An International Conference on Mathematical Analysis and Signal Processing January 3-9, 1994 Cairo University, Cairo, Egypt*, M. E. H. Ismail & $al.$, editors, *Contemporary Mathematics* Volume **190**, 1995, p. 287–294 (what a title!):

let X be a monic polynomial of degree 8 (amazing!) with zeros $\mathfrak{a}q^2/s, \mathfrak{a}^2q^2/s, \mathfrak{a}^2q^3/s, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}$, with $s = \mathfrak{a}^3q^3/(\mathfrak{b}\mathfrak{c}\mathfrak{d}\mathfrak{e}\mathfrak{f})$ (an appropriate balance condition in relevant basic hypergeometric functions). Then, $X(0)$ = product of the 8 zeros $= \mathfrak{a}^5q^7\mathfrak{b}\mathfrak{c}\mathfrak{d}\mathfrak{e}\mathfrak{f}/s^3 = \mathfrak{a}^8q^{10}/s^4$. Let also $Y(x) = (\mathfrak{a}q^2x - s)(x-\mathfrak{b})\cdots(x-\mathfrak{f})$, $Z(x) = (x-\mathfrak{b})\cdots(x-\mathfrak{f})$. Let

$$\rho_{2n} = \frac{s(sq^{n-2}/\mathfrak{a})^6X(\mathfrak{a}q^{2-n}/s)}{(1-sq^{2n-1})(1-sq^{2n-2})Y(1)}, \rho_{2n+1} = \frac{X(\mathfrak{a}q^{n+1})s^3}{\mathfrak{a}^6q^{2n+7}Y(1)(1-sq^{2n})(1-sq^{2n-1})},$$

Check that $1+\rho_{2n}+\rho_{2n+1}$ has no residue at $1-sq^{2n-1}=0$, or $s = q^{1-2n}$, residue value is $\frac{s(sq^{n-2}/\mathfrak{a})^6X(\mathfrak{a}q^{2-n}/s)}{(1-sq^{2n-2}=1-1/q)Y(1)} + \frac{X(\mathfrak{a}q^{n+1})s^3}{\mathfrak{a}^6q^{2n+7}Y(1)(1-sq^{2n}=1-q)} = s\frac{X(\mathfrak{a}q^{n+1})}{(1-q)Y(1)}[-q^{-5-6n}\mathfrak{a}^{-6} + q^{-5-6n}\mathfrak{a}^{-6}] = 0.$

Denominators of $f = \cfrac{1}{1+\cfrac{\rho_1}{1+\cfrac{\rho_2}{1+\ddots}}}$ $D_0 = D_1 = 1, D_2 = 1+\rho_1 = 1 + \frac{(s-\mathfrak{a}q)(s-\mathfrak{a}q^2)Z(\mathfrak{a}q)}{\mathfrak{a}^3q^3Y(1)(s-1)}$, etc.

Start of basic hypergeometric expansions

$$D_{2n}(x) = \frac{q^{2n(n-1)}(1-\mathfrak{a}q)\cdots(1-\mathfrak{a}q^{n-1})Y(1/q)\cdots Y(1/q^{n-1})}{(1-\mathfrak{a}q^{n+1})\cdots(1-\mathfrak{a}q^{2n-1})(Y(1))^{n-1}}$$

$$+\frac{(1-q^n)(1-\mathfrak{a}q)\cdots(1-\mathfrak{a}q^{n-2})(1-\mathfrak{a}q^{2(n-1)})(s-\mathfrak{a}q)(s-\mathfrak{a}q^{3-n})Z(\mathfrak{a}q^n)Y(1/q)\cdots Y(1/q^{n-2})}{(1-q)(1-\mathfrak{a}q^{n+1})\cdots(1-\mathfrak{a}q^{2n-1})(sq^{2(n-1)}-1)\mathfrak{a}^3q^?(Y(1))^{n-1}}+$$

$\cdots$

$$D_{2n+1}(x) = \frac{q^{2n(n+1)}(1-\mathfrak{a}q)\cdots(1-\mathfrak{a}q^{n})Y(1/q)\cdots Y(1/q^{n})}{(1-\mathfrak{a}q^{n+2})\cdots(1-\mathfrak{a}q^{2n+1})(Y(1))^{n}}$$

$$-\frac{(1-q^n)\ldots Y(\mathfrak{a}q^{n+1})Y(1/q)\cdots Y(1/q^{n-1})}{(1-q)\ldots(Y(1))^n}+\cdots$$

Interpolation setting: replace two zeros of X, say, $\mathfrak{b}$ and $\mathfrak{c}$ by $\sqrt{\mathfrak{bc}}\; e^{\xi}$ and $\sqrt{\mathfrak{bc}}\; e^{-\xi}$ keeping the same product. Then, with $\begin{cases} z_{2n} & = \mathfrak{a}q^{2-n}/s \\ z_{2n+1} & = \mathfrak{a}q^{n+1}\end{cases}$, the two corresponding factors of $\dfrac{X(z_n)}{X(1)}$ make $\dfrac{z_n^2-2z_n\sqrt{\mathfrak{bc}}\cosh\xi+\mathfrak{bc}}{1-2\sqrt{\mathfrak{bc}}\cosh\xi+\mathfrak{bc}} = z_n + \dfrac{(z_n-1)(z_n-\mathfrak{bc})}{1-2\sqrt{\mathfrak{bc}}\cosh\xi+\mathfrak{bc}} =$ $(z_n-1)(z_n-\mathfrak{bc})\left(x+\dfrac{z_n}{(z_n-1)(z_n-\mathfrak{bc})}\right)$, with $x=\dfrac{1}{1-2\sqrt{\mathfrak{bc}}\cosh\xi+\mathfrak{bc}}$ ($x = \sinh\xi$ in Masson §7 p.293)

So, $\rho_{2n} = r_{2n}(x-a_n), a_n = -\dfrac{z_{2n}}{(z_{2n}-1)(z_{2n}-\mathfrak{bc})} = -\dfrac{\mathfrak{a}q^{2-n}/s}{(\mathfrak{a}q^{2-n}/s-1)(\mathfrak{a}q^{2-n}/s-\mathfrak{bc})}$,

$\rho_{2n+1} = r_{2n+1}(x-b_n), b_n = -\dfrac{z_{2n+1}}{(z_{2n+1}-1)(z_{2n+1}-\mathfrak{bc})} = -\dfrac{\mathfrak{a}q^{n+1}}{(\mathfrak{a}q^{n+1}-1)(\mathfrak{a}q^{n+1}-\mathfrak{bc})}$.

Elliptic.

From Spiridonov, V. P.; & Zhedanov, A. S., Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids, *in* Bustoz, Joaquin (ed.) et *al.*, (2001), the fully elliptic setting $R_n(z) = {}_{10}E_9(\dots)$, of poles $\alpha_1, \dots, \alpha_n$ Thm 3, eq. (4.12):

$$(z-\alpha_0)\cdots(z-\alpha_{n+1})R_{n+1} = \left(z-\alpha_{n+1}+\frac{\epsilon_{n-1}\mathfrak{b}_n(z-\beta_{n-1})}{\epsilon_n\mathfrak{a}_n}+\frac{\mathfrak{c}_n(z-\lambda_1)}{\epsilon_n\mathfrak{a}_n}\right)$$

$$(z-\alpha_0)\cdots(z-\alpha_n)R_n - \frac{\epsilon_{n-1}\mathfrak{b}_n(z-\beta_{n-1})(z-\alpha_n)}{\epsilon_n\mathfrak{a}_n}(z-\alpha_0)\cdots(z-\alpha_{n-1})R_{n-1}$$

$$c_n = \frac{\mathfrak{c}_n(z-\lambda_1)}{\epsilon_n\mathfrak{a}_n}+z-\alpha_{n+1}+\frac{\epsilon_{n-1}\mathfrak{b}_n(z-\beta_{n-1})}{\epsilon_n\mathfrak{a}_n},\ d_n = \frac{\epsilon_{n-1}\mathfrak{b}_n(z-\beta_{n-1})(z-\alpha_n)}{\epsilon_n\mathfrak{a}_n}$$

We have $c_n = (1+\rho_{2n}+\rho_{2n+1})\xi_{n+1}/\xi_n$, $d_n = \rho_{2n-1}\rho_{2n}\xi_{n+1}/\xi_{n-1}$, with $\xi_{n+1}/\xi_n = \mathfrak{c}_n(z-\lambda_1)/(\epsilon_n\mathfrak{a}_n)$.

With respect to n, $\epsilon_n\mathfrak{a}_n/\mathfrak{c}_n =$ a polynomial in the [ ] functions (θ functions) divided by $[1-x_2+2n][2-x_2+2n]$, and $\epsilon_{n-1}\mathfrak{b}_n/\mathfrak{c}_n =$ a polynomial divided by $[1-x_2+2n][-x_2+2n]$

So, ρ_n is a polynomial (depending on the evenness of n) divided by $[n-x_2][n+1-x_2]$.