Tentative asymptotics with Freud's equations.

Problem: get the $a_n$'s of $\exp(-f(x))$ (if supposed even for simplicity).

Suggested method: try a sequence ($... a_{n-1}, a_n, a_{n+1}, ...$) and look if $F_n(a)$ is close to $n$ ...

Here, $F_n(a) = a \int q_n q_n - \bar{f} \, d\sigma$, where the $q_n$'s and $d\sigma$ come from the solution of the moment problem for the proposed $a_n$'s. It is assumed that $F_n(a)$ depends mostly on the $a_n$'s with $m$ near $n$.

Simplest example: all the $a_m$'s for $m$ near $n$ are equal to $a_n$. Then,

$$d\sigma(x) = 1/2 \pi (4a_n^2 - x^2)^{-1/2}, \quad q_n(x) = 2 \pi (n+1) \cos(n\theta + \varphi) \quad \text{(Bernstein-Szegö)}$$

$$F_n(a) \sim \frac{1}{n} \int_{-2a_n}^{2a_n} (4a_n^2 - x^2)^{-1/2} \left[ \cos\varphi + \cos((2n-1)\theta + 2\varphi) \right] f'(x) \, dx$$

The first part is nothing else than the Mhaskar and Saff function! So we see how this function appears as a first approximation.

I hoped that the second part should give the oscillatory behaviour of $a_n$ for $f(x) = |x|^\alpha$ when $\alpha$ is not an even integer. Well, working the integrals (replacing $\varphi$ by $\varphi(n/2) = 0$ : the Van Hove singularity at $x=0$ creates the damped oscillation).

$$F_n(a) \sim \frac{\alpha}{n} \int_0^{\pi/2} \cos^\alpha \theta \, d\theta + \frac{\alpha}{n} \int_0^{\pi/2} \cos((2n-1)\theta) \cos^{\alpha-1} \theta \, d\theta \sim n,$$

$$\sim C(\alpha)(a_n)^\alpha \left[ 1 + (-1)^{n-1} \frac{\Gamma(n-\alpha/2)}{\Gamma(1+\alpha/2)} \frac{\Gamma(1+\alpha/2)}{\Gamma(n+\alpha/2)} \frac{\Gamma(n+\alpha/2)}{\Gamma(1-\alpha/2)} \right]$$

i.e. $C(\alpha)(a_n)^\alpha \left[ 1 + (-1)^{n-1} A(\alpha)/n^\alpha \right] \sim n$.

$$C(\alpha)(a_n)^\alpha - n \sim (-1)^n A(\alpha)n^{1-\alpha}, \quad \text{with} \quad A(\alpha) = \Gamma(\alpha/2)\Gamma(1+\alpha/2)\sin(\pi\alpha/2)/\pi$$

For $\alpha < 1$, the oscillatory part of $C(\alpha)(a_n)^\alpha - n$ behaves indeed like const.$n^{1-\alpha}$, but the constant is not the same, it is about two times $A(\alpha)$ for large $\alpha$, and the behaviour when $\alpha$ approaches 1 is quite interesting: I find the numerical formula

$$C(\alpha)(a_n)^\alpha - n \approx (\alpha-2)/(24n) - 13(\alpha-2)/(1530n^3) \ldots$$

$\alpha - 1/\alpha$
\[
\frac{(-1)^n B(a)}{a^{n+1}} \\
|a| \quad B(a) \quad (a^{-1}/a)B(a) \quad A(a) \\
7 \quad -3.5 \quad -24 \quad -12.3 \\
5 \quad 0.39 \quad 1.9 \quad 1.40 \\
3 \quad -0.085 \quad -0.23 \quad -0.375 \\
1.5 \quad 0.030 \\
1 \quad 0.035 \text{ for } a = 1, \text{ the perturbation is } (-1)^n B(1)/\log n \\
0.75 \quad 0.029 \\
0.6666 \quad 0.026 \\
0.5 \quad 0.016 \\
0.3333 \quad 0.0033 \\
\]

the non oscillatory part comes from Nevai et al. \((a=4)\) and Sheen \((a=6)\).

\[
\frac{(C)^{\frac{1}{2}}d_{a}}{m_{n}^{\frac{1}{2}}} = A + \frac{a^{-2}}{12a_{n}^{2}} - \frac{(a^{-2})(2a^{-1})(3a^{-2})}{720} m_{n}^{-1} \left( \frac{d_{a}^{2}}{m_{n}} \right) + \frac{(q^{-2})(2q^{-1})(5q^{-2})}{181440} m_{n}^{-6} \\
\]

the non oscillatory part comes from Nevai et al. \((a=4)\) and Sheen \((a=6)\).
1. Freud's weights and coefficients.

Let \( \{p_n(x) = p_n(x; d\mu(x))\}_{n=0}^{\infty} \) be the orthonormal polynomials with respect to an even measure (i.e., \( \int_{-\infty}^{\infty} x^{2m-1} d\mu(x) = 0, m = 1, 2, \ldots \)) on \( \mathbb{R} \). These polynomials then satisfy the recurrence relations

\[
a_{n+1}p_{n+1}(x) = xp_n(x) - a_n p_{n-1}(x), \quad n = 0, 1, \ldots \quad (a_0 = 0)
\]

(1)

One wishes to relate the behaviour of \( d\mu(x) \) for large \( x \) to the behaviour of the recurrence coefficients \( a_n \) for large \( n \) (Freud's programme [2], see also § 4.18 of [11]). Quite a number of dramatic achievements have been made recently, using advanced orthogonal polynomials theory (Christoffel functions), functional spaces theory and potential theory, see [3,4,5(Appendix),6,7,11,13,14(Chap.4)] as landmarks and surveys.

Freud remarked in [2] how one can use the identity

\[
\frac{n}{a_n} = \int_{-\infty}^{\infty} p_n'(x)p_{n-1}(x)e^{-Q(x)} \, dx = \int_{-\infty}^{\infty} p_n(x)p_{n-1}(x)Q'(x)e^{-Q(x)} \, dx, \quad n = 1, 2, \ldots
\]

(2)

when the polynomials \( p_n \) are orthonormal with respect to \( \exp(-Q(x)) \) (Freud's weight). Indeed, if \( Q \) is a polynomial, repeated applications of (1) in the right-hand side of (2) yields a polynomial in \( a_n, a_{n\pm1}, \ldots \), so that (2) turns as a set of equations for the recurrence coefficients \( \{a_n\} \). For instance, \( Q(x) = x^4 \) gives

\[
4a_n^2(a_{n-1}^2 + a_n^2 + a_{n+1}^2) = n, \quad n = 1, 2, \ldots
\]

(3)

a well worked example ([1,2,8,9,10,11,12(pp.470-471)]). Moreover, such equations allowed Freud and followers to establish the asymptotic behaviour of \( a_n \) for various exponential weights (exponentials of polynomials [1,2,8,9,10]), and to arrive naturally to a conjecture for non polynomial exponentials:

\[
\text{if } d\mu(x) = \exp(-|x|^\alpha) \, dx, \text{ then } a_n \sim \left( \frac{n}{C(\alpha)} \right)^{1/\alpha}, \quad (4)
\]

when \( n \to \infty \), with \( C(\alpha) = \frac{2^\alpha \Gamma((\alpha + 1)/2)}{\Gamma(1/2)\Gamma(\alpha/2)} \), for \( \alpha > 0 \). The proof of (4) appears as a special case of very powerful investigations which led to

\[
\text{if } d\mu(x) = \exp(-Q(x)) \, dx, \text{ then } a_n \sim \tilde{a}_n, \text{ with } \tilde{a}_n \int_0^{\pi} Q'(2\tilde{a}_n \cos \theta) \cos \theta \, d\theta = n\pi
\]

\[(4')\]

where \( Q \) is even, continuous and convex on \( \mathbb{R} \), \( Q'(x) > 0 \) for \( x > 0 \), among other conditions. The (positive) root \( \tilde{a}_n \) of the equation of (4') is called the Lubinsky-Mhaskar-Rahmanov-Saff's number. The method of proof of (4') can even give asymptotic estimates of \( a_1a_2 \ldots a_n([3,4,5,6,7,13]) \). Now, let us try to explore the subject further with Freud's equations.
2. Freud's functionals and equations.
Let us consider only positive sequences \( a = \{a_n\}_{n=1}^\infty \) with \( \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty \) so as to be sure that the related moment problem, amounting to finding \( d\mu \) such that

\[
\frac{1}{z - \frac{a_1^2}{z}} = \int_{-\infty}^{\infty} (z - x)^{-1} d\mu(x), \quad \forall z \notin \mathbb{R} \quad (5)
\]

has a unique solution. Then the Freud's functional related to \( Q \) and \( a \) is defined as

\[
F_n(Q; a) = a_n \int_{-\infty}^{\infty} p_n(x)p_{n-1}(x)Q'(x) \, d\mu(x), \quad n = 1, 2, \ldots \quad (6)
\]

where the \( p_n(x) = p_n(x; d\mu(x)) \)'s are the orthonormal polynomials related to the measure \( d\mu \) solving the moment problem (5). Thus, Freud's remark becomes: \( a \) is truly the sequence of recurrence coefficients related to the measure \( \exp(-Q(x)) \, dx \) on \( \mathbb{R} \) if

\[
F_n(Q; a) = n, \quad n = 1, 2, \ldots \quad (7)
\]

making the Freud's equations for \( a \) (when \( Q \) is an even function, or else one also must consider the functionals \( G_n = \int_{-\infty}^{\infty} p_n(x)Q'(x) \, d\mu(x) \) [9]). The functionals \( F_n \) are linear in \( Q \), but nonlinear in \( a \), so with \( Q(x) = x^4 \) one recovers the example (3).

3. Asymptotic expansions.
When \( Q \) is a polynomial, (6) is explicit in a finite number of neighbours of \( a_n \), and asymptotic expansions can be studied: so (4) has been completed as

\[
a_n \sim \left( \frac{n}{C(\alpha)} \right)^{1/\alpha} \sum_{k=0}^{\infty} \frac{A_k}{n^{2k}} \quad (8)
\]

with \( A_0 = 1 \), when \( \alpha \) is an even integer ([10], see also [1]).

The problem now is to extend (8) when \( \alpha \) is not an even integer, still using (6). Making the assumption that \( F_n(Q; a) \) still depends essentially on the close neighbours of \( a_n \) (technically, that \( \partial F_n(Q; a)/\partial a_k \to 0 \) when \( |n - k| \to \infty \), see further for more on \( \partial F_n/\partial a_k \), we approximate \( F_m(Q; a) \) for \( m \) near \( n \) by \( F_m(Q; a^{(n)}) \), where \( a^{(n)} \) is the constant sequence \( \ldots = a_{n-2}^{(n)} = a_{n-1}^{(n)} = a_{n}^{(n)} = a_{n+1}^{(n)} = \ldots = a_n \) (as we already know that \( a_{n+k}/a_n \to 1 \) when \( n \to \infty \)). The measure \( \tilde{\mu}^{(n)} \) is then

\[
d\tilde{\mu}^{(n)}(x) = \left( \frac{1}{\pi a_n} \right) \sqrt{1 - (x/(2a_n))^2} \, dx \quad (-2a_n, 2a_n].
\]

This does not mean that \( d\tilde{\mu}^{(n)} \) is close to \( d\mu \), but that \( F_m(Q; a^{(n)}) \) (probably) is close to \( F_m(Q; a) \) for \( m \) close to \( n \), and a lot of other trial measures would be as good. Proceeding with the computations (\( p_m(x) \) is the Chebyshev polynomial \( U_m(x/(2a_n)) \)), one finds

\[
F_m(Q; a^{(n)}) = a_n \int_{-2a_n}^{2a_n} U_m \left( \frac{x}{2a_n} \right) U_{m-1} \left( \frac{x}{2a_n} \right) Q'(x) \frac{1}{\pi a_n} \sqrt{1 - \left( \frac{x}{2a_n} \right)^2} \, dx
\]

\[
= \frac{\tilde{a}_n}{\pi} \int_{0}^{\pi} [\cos \theta - \cos(2m+1)\theta] Q'(2a_n \cos \theta) \, d\theta
\]
If \( Q \) is reasonably smooth, the \((2m+1)^{th}\) Fourier coefficient of \( Q'(2\tilde{a}_n \cos \theta) \) may be neglected for large \( m \) and we recover the \( LMR \) approximation \( \tilde{a}_n \) for \( a_n \) from (9). For instance, with \( Q(x) = |x|^\alpha \):

\[
F_m(|x|^\alpha; \tilde{a}^{(n)}) = \frac{2\tilde{a}_n}{\pi} \int_0^{\pi/2} \left[ \cos \theta - \cos(2m+1)\theta \right] (2\tilde{a}_n \cos \theta)^{\alpha-1} d\theta \\
= \frac{2\tilde{a}_n}{\pi} \left[ \Gamma((\alpha + 1)/2)\Gamma(1/2) - \Gamma(\alpha/2)\Gamma((\alpha + 1)/2)\Gamma(1/2) \right] \\
= \tilde{a}_n^{\alpha} C(\alpha) \left[ 1 - (-1)^m \sin(\pi\alpha/2) \Gamma(1 + m - \alpha/2) \Gamma(\alpha/2) \Gamma(1 + \alpha/2) \right] \\
\sim \tilde{a}_n^{\alpha} C(\alpha) \left[ 1 - (-1)^m \sin(\pi\alpha/2) \Gamma(\alpha/2) \Gamma(1 + \alpha/2) \right]
\]

This suggests an \((-1)^n/n^\alpha\) term in the expansion:

**Conjecture.** The recurrence coefficients of \( \exp(-|x|^\alpha) \) satisfy the asymptotic expansion

\[
a_n \sim \left( \frac{n}{C(\alpha)} \right)^{1/\alpha} \left( \sum_{k=0}^\infty \frac{A_k}{n^{i_k}} + (-1)^n \sum_{k=0}^\infty \frac{B_k}{n^{j_k}} \right) 
\]

with \( 0 = i_0 < i_1 < \ldots, A_0 = 1, \alpha = j_0 < j_1 < \ldots \ (\alpha > 1) \).

For more accurate predictions, one relates errors on \( a \) to errors on \( F(a) \) (\( F(a) \) is the sequence \( \{F_m(Q; a)\} \)) by \( F(a) - \hat{F}(a^{(n)}) \sim J(a - \hat{a}^{(n)}) \), where \( J \) is the Jacobian matrix of the partial derivatives \( \partial F_m(Q; a)/\partial \sigma_k \). The elements of this matrix (computed at \( \hat{a}^{(n)} \)) are

\[
J_{m,k} = \partial F_m(Q; \hat{a}^{(n)})/\partial \sigma_k^{(n)}
\]

\[
= 2\tilde{a}_n \int_R \int_R p_m(x)p_{m-1}(y)p_k(x)p_{k-1}(y) \frac{Q'(x) - Q'(y)}{x - y} \mu^{(n)}(x) \mu^{(n)}(y) dx dy 
\]

\[
= \frac{8\tilde{a}_n}{\pi^2} \int_0^{\pi} \int_0^{\pi} \sin(m+1)\theta \sin m\psi \sin(k+1)\theta \sin k\psi \frac{Q'(2\tilde{a}_n \cos \theta) - Q'(2\tilde{a}_n \cos \psi)}{2\tilde{a}_n (\cos \theta - \cos \psi)} d\theta d\psi
\]

Here again, keeping only the lowest order Fourier coefficient:

\[
J_{m,k} \sim \frac{2\tilde{a}_n}{\pi^2} \int_0^{\pi} \int_0^{\pi} \cos(m-k)\theta \cos(m-k)\psi \frac{Q'(2\tilde{a}_n \cos \theta) - Q'(2\tilde{a}_n \cos \psi)}{2\tilde{a}_n (\cos \theta - \cos \psi)} d\theta d\psi
\]

leaving a Toeplitz matrix of symbol \( \Phi(\varphi) = \sum_{p=-\infty}^{\infty} J_{p,0} \exp(ip\varphi) \) such that

\[
\frac{1}{\pi} \int_0^{\pi} \cos p\varphi \Phi(\varphi) d\varphi = \frac{2\tilde{a}_n}{\pi^2} \int_0^{\pi} \int_0^{\pi} \cos p\theta \cos p\psi \frac{Q'(2\tilde{a}_n \cos \theta) - Q'(2\tilde{a}_n \cos \psi)}{2\tilde{a}_n (\cos \theta - \cos \psi)} d\theta d\psi
\]

i.e.,

\[
\Phi(\varphi) = \frac{2\tilde{a}_n}{\pi} \int_0^{\pi} \frac{Q'(2\tilde{a}_n \cos \theta) - Q'(2\tilde{a}_n \cos (\theta + \varphi))}{2\tilde{a}_n (\cos \theta - \cos (\theta + \varphi))} d\theta
\]
\( J^{-1} \) is approximately the Toeplitz matrix of symbol \( 1/\hat{\Phi} \), so that for \( |x|^\alpha \):

\[
\alpha_n - \tilde{\alpha}_n^{(n)} \sim \sum_{p=-\infty}^{\infty} (J^{-1})_{n,n+p}(F_{n+p}(Q;\alpha) - F_{n+p}(Q;\tilde{\alpha}^{(n)})) \\
\sim \sum_{p=0}^{\infty} (J^{-1})_{p,0} \left( n + p - n + n(-1)^n + p \sin(\pi \alpha/2) \frac{\Gamma(\alpha/2)\Gamma(1+\alpha/2)}{\pi(n+p)^\alpha} \right) \\
\sim (-1)^n \sin(\pi \alpha/2) \frac{\Gamma(\alpha/2)\Gamma(1+\alpha/2)}{\pi \hat{\Phi}(\pi)} n^{1-\alpha}
\]

suggesting \( B_0 = (\alpha - 1) \sin(\pi \alpha/2) / (\Gamma(\alpha/2))^2 / (2\pi) \) in (10), using \( \hat{\Phi}(\pi) = \pi^{-1} (2\tilde{\alpha}_n)^{\alpha-1} \int_0^{\pi/2} (\cos \theta)^{\alpha-2} d\theta = \alpha (2\tilde{\alpha}_n)^{\alpha-1} \Gamma((\alpha-1)/2)/(\Gamma(1/2)\Gamma(\alpha/2)) \).

Now, some horribly wrong mistake must have occurred somewhere, because very high accuracy (up to 200 digits, on the IBM 3090 of the University) calculations of instances of \( \alpha_n \) for various values of \( \alpha \), followed by severe extrapolation devices designed to exhibit \( B_0 \), lead to the

**Problem.** Show that one has \( j_0 = \alpha \) and \( B_0 = (\alpha - 1) \sin(\pi \alpha/2) (1-1/\alpha)^\alpha (\Gamma(\alpha/2))^2 / (2\pi) \)
in (10) when \( \alpha > 1 \).

Where does this \( (1 - 1/\alpha)^\alpha \) come from? ? ? ? ?

**References**

8. Magnus A.P., *A proof of Fred's conjecture about orthogonal polynomials related to \( |x|^\alpha \exp(-x^2m) \) for integer \( m \), in Lecture Notes Math. 1171, Springer (1985), 362-372.

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