An instance of the Maroni's touch.

Let P_n and $\mathbb{P}_n$ be monic orthogonal polynomials related to the measures $d\mu(t)$ and $d\tilde{\mu}(t) = \dfrac{d\mu(t)}{t-c} + \kappa\delta(t-c)dt$. As $\int \mathbb{P}_n(t)P_m(t)d\mu(t) = \int \mathbb{P}_n(t)(t-c)P_m(t)\dfrac{d\mu(t)}{t-c} = \int \mathbb{P}_n(t)(t-c)P_m(t)d\tilde{\mu}(t) = 0$ when $m < n-1$ (the multiplication by $t-c$ kills the mass point at c!), we have the quasi orthogonality representation

$\mathbb{P}_n(x) = P_n(x) + c_n P_{n-1}(x)$,

what is c_n? The P_ns are kernel polynomials, as $d\mu(t) = (t-c)d\tilde{\mu}(t)$ (same killing), so, by Christoffel-Darboux (Pascal would have written "Darboux-Christoffel")

$(x-c)P_n(x) = \mathbb{P}_{n+1}(x) - \dfrac{\mathbb{P}_{n+1}(c)}{\mathbb{P}_n(c)}\mathbb{P}_n(x)$.

So, $(x-c)P_n(x) = P_{n+1}(x) + c_{n+1}P_n(x) - \dfrac{\mathbb{P}_{n+1}(c)}{\mathbb{P}_n(c)}(P_n(x) + c_nP_{n-1}(x))$, to compare to the P_n recurrence relation, for instance $a_n = -\dfrac{\mathbb{P}_{n+1}(c)}{\mathbb{P}_n(c)}c_n$, or the eq. for the c_ns $a_n(P_n(c) + c_nP_{n-1}(c)) = -(P_{n+1}(c) + c_{n+1}P_n(c))c_n$, etc.

Wimp & Kiesel [WK] 1.9, 1.15 ask when a combination of P_n and P_{n-1} satisfies a recurrence relation of the required form. We want $\mathbb{P}_{n+1}(x) = (x-\tilde{b}_n)\mathbb{P}_n(x) - \tilde{a}_n\mathbb{P}_{n-1}(x)$ and $P_{n+1}(x) = (x-b_n)P_n(x) - a_nP_{n-1}(x)$,

$$0 = \underbrace{P_{n+1}(x)}_{(x-b_n)P_n(x)-a_nP_{n-1}(x)} + c_{n+1}P_n(x) - (x-\tilde{b}_n)\{P_n(x) + c_nP_{n-1}(x)\} - \tilde{a}_n\left\{P_{n-1}(x) + c_{n-1}\left[P_{n-2}(x) = \frac{-P_n(x) + (x-b_{n-1})P_{n-1}(x)}{a_{n-1}}\right]\right\},$$ whence [WK] §2,

$$x - b_n + c_{n+1} - (x-\tilde{b}_n) + \frac{\tilde{a}_nc_{n-1}}{a_{n-1}} = -a_n - (x-\tilde{b}_n)c_n - \tilde{a}_n - \frac{\tilde{a}_nc_{n-1}(x-b_{n-1})}{a_{n-1}} \equiv 0.$$

They solve these equations, and more difficult ones too, by computer algebra.

Even the great Wolfgang Hahn has to struggle a short while with this problem [H52] p. 95-96, [HChr] eq. (7) and (8), finding that c_n is a ratio of solutions of the 3-term recurrence relation for P_n.

Pascal solves the problem in a matter of seconds [M] p. 225, in showing that "*il faut et il suffit*" that $\mathbb{P}_n$ be orthogonal to constants, so, that $c_n = -\dfrac{\int P_n(t)d\tilde{\mu}(t)}{\int P_{n-1}d\tilde{\mu}(t)}, n = 1, 2, \dots$, and that $\int P_n(t)d\mu(t)/(t-c)$ is the numerator polynomial, or first associated orthogonal polynomial $N_n(c)+$ a constant times $P_n(c)$.

About forms. Pascal writes Lf for the more familiar (to whom?) measure writing $\int f(t)d\mu(t)$. Let Mf be our $\int f(t)(t-c)^{-1}d\mu(t)$, $(x-c)M$ applied to a test function f gives Lf of course, but Maroni's $(x-c)^{-1}Lf$ is NOT Mf, it gives L applied to $(f(t)-f(c))/(t-c)$, corresponding to $d\mu(t)/(t-c) - m_0\delta(t-c)dt$, where m_0 is the zeroest moment of $d\mu/(t-c)$, reminding the formula of the numerator polynomials: in terms of Kiesel & Wimps's favorite representations by Stieltjes functions (my favorite too) $F(x) = \int d\mu(t)/(x-t)$, the numerator, or associated, polynomial N_n is the polynomial part of

$$F(x)P_n(x) = \int P_n(x)d\mu(t)/(x-t) = \underbrace{\int \frac{P_n(t)-P_n(x)}{t-x}d\mu(t)}_{N_n(x) = P^{(1)}_{n-1}(x)} + \underbrace{\int \frac{P_n(t)}{x-t}d\mu(t)}_{Q_n(x) = O(x^{-n-1})},$$

so $N_n(c) = (x-c)^{-1}LP_n$.

Where are forms coming from? They became often used since the 1980s [Brez80] [Draux] and of course [M], shall we speak of a *French touch*? Pascal thought highly of Geronimus [Geronimus].

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