Polignac-Whittaker.

Thanks to a student who wanted references on Legendre functions, I fell on a hidden Polignac statement in Whittaker & Watson's famous book!

Problem 29 in p. 333 of the edition of 1920:

29. Shew that $Q_n(z) = \dfrac{1}{2^n n!}\dfrac{d^n}{dz^n}\left\{(z^2-1)^n \log\dfrac{z+1}{z-1}\right\} - \dfrac{1}{2}P_n(z)\log\dfrac{z+1}{z-1}$.

Prove also that $Q_n(z) = \dfrac{1}{2}P_n(z)\log\dfrac{z+1}{z-1} - W_{n-1}(z)$,

where $W_{n-1}(z) = \dfrac{2n-1}{1.n}P_{n-1}(z) + \dfrac{2n-5}{3(n-1)}P_{n-3}(z) + \dfrac{2n-9}{5(n-2)}P_{n-5}(z) + \cdots$

$$= k_n + (k_n-1)\frac{n(n+1)}{1^2}\left(\frac{z-1}{2}\right) + \left(k_n - 1 - \frac{1}{2}\right)\frac{n(n-1)(n+1)(n+2)}{1^2.2^2}\left(\frac{z-1}{2}\right)^2 +$$
$$\left(k_n - 1 - \frac{1}{2} - \frac{1}{3}\right)\frac{n(n-1)(n-2)(n+1)(n+2)(n+3)}{1^2.2^2.3^2}\left(\frac{z-1}{2}\right)^3 + \cdots$$

where $k_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$.

(Math. Trip. 1898).

The first of these expressions for $W_{n-1}(z)$ *was given by Christoffel, Journal für Math. LV (1858).*

end of quotation

(W&W write f_{n-1} for W_{n-1}).

Maybe the last formula may give a hint towards a fast proof!

Also, did the writer of the tripoes of 1898 know of Polignac, of Catalan?

The **Answer** follows from another exercise (solved in the book, thank God) of W&W (p.223 in the ed. of 1902; p.318 in ed. of 1920): $Q_n(z) = \dfrac{(-2)^n n!}{(2n)!}\dfrac{d^n}{dz^n}\left\{(z^2-1)^n\displaystyle\int_z^\infty (v^2-1)^{-n-1}dv\right\}$.

Proof: consider the differential equation $(z^2-1)X'' - 2(n-1)zX' - 2nX = 0$ and derivate n times $(z^2-1)\dfrac{d^{n+2}X}{dz^{n+2}} + 2z\dfrac{d^{n+1}X}{dz^{n+1}} - n(n+1)\dfrac{d^nX}{dz^n} = 0$, which is the Legendre diff. equation. Now, as the first equation $[(z^2-1)^{-n+1}X']' - 2n(z^2-1)^{-n}X = [2nzY + (z^2-1)Y']' - 2nY = 2(n+1)zY' + (z^2-1)Y'' = 0$ if $X = (z^2-1)^nY$, has the solutions $X = (z^2-1)^n$ and $X = (z^2-1)^n\displaystyle\int^z (v^2-1)^{-n-1}dv$, the formula follows, considering that $Q_n(z) = \dfrac{2^n(n!)^2}{(2n+1)!z^{n+1}} + O(z^{-n-2})$ for large z. End of proof. □

We must show $Q_n(z) = \dfrac{1}{2^n n!}\dfrac{d^n}{dz^n}\left\{(z^2-1)^n\log\dfrac{z+1}{z-1}\right\} \underbrace{-\dfrac{1}{2}P_n(z)\log\dfrac{z+1}{z-1}}_{-Q_n(z)-W_{n-1}(z)}$, or $Q_n(z) =$

$$\frac{1}{2^{n+1}n!}\frac{d^n}{dz^n}\left\{(z^2-1)^n\log\frac{z+1}{z-1}\right\} - \frac{W_{n-1}(z)}{2}.$$

We proceed with the integral in the formula above $\displaystyle\int_z^\infty \frac{dv}{(v^2-1)^{n+1}} = \int_z^\infty \sum_{j=1}^{n+1}\frac{\text{const.}\ dv}{(v\pm 1)^j} =$ $\dfrac{R_n(z)}{(z^2-1)^n} + A_n\log\dfrac{z+1}{z-1}$ from simple fractions expansions, and where the residues at ∓ 1 are $\pm A_n$. So, $-R_n'(z)(z^2-1) + 2nzR_n(z) + 2A_n(z^2-1)^n \equiv 1$. Now I am on my own and become very clumsy again. As I want to recover the numerator polynomial W_{n-1} known by its expansion at $z=1$, let $R_n(z) = \sum_{k=0}^{2n-1} \alpha_k (z-1)^k$, so $-(z-1)[2+z-1] \sum_1^{2n-1} k\alpha_k (z-1)^{k-1} + 2n[1+z-1] \sum_0^{2n-1} \alpha_k (z-1)^k + 2A_n (z-1)^n [2+z-1]^n \equiv 1, 2(n-k)\alpha_k + (2n-k+1)\alpha_{k-1} = -A_n 2^{2n-k+1} \binom{n}{k-n} + \delta_{k,0}$: $\alpha_k = \dfrac{(-1)^k 2n(2n-1)\cdots(2n-k+1)}{2^{k+1} n(n-1)\cdots(n-k)}$ when $k<n, \Rightarrow \alpha_{n-1} = \dfrac{(-1)^{n-1} 2n(2n-1)\cdots(n+2)}{2^n n!} = -A_n 2^{n+1}/(n+1)$: $A_n = \dfrac{(-1)^n}{2^{2n+1}} \binom{2n}{n}$; $\alpha_{2n} = 0$,

$$\alpha_{k-1} = -A_n \frac{2^{2n-k+1} n!}{(k-n)!(2n-k+1)!} - \frac{2(n-k)}{2n-k+1} \alpha_k, k = 2n, 2n-1, \ldots : \alpha_{2n-1} = -2A_n,$$

$$\alpha_{2n-2} = -A_n \frac{4n!}{2!(n-1)!} - \frac{-2(n-1)}{2} \alpha_{2n-1} = -2(2n-1)A_n,$$

$$\alpha_{2n-3} = -A_n \frac{8n!}{3!(n-2)!} - \frac{-2(n-2)}{3} \alpha_{2n-2} = -\frac{4}{3}(n(n-1) + (n-2)(2n-1))A_n$$

$$= -\frac{4}{3}(n(n-1)+(n-2)n+(n-2)(n-1))A_n = -\frac{4}{3} n(n-1)(n-2) \left(\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} \right) A_n,$$

Aha! Check $\alpha_k = -\dfrac{2^{2n-k}}{(2n-k)!} n(n-1)\cdots(k-n+1) \left(\dfrac{1}{n} + \dfrac{1}{n-1} + \cdots + \dfrac{1}{k-n+1} \right) A_n, k \geqslant n$,

$$\alpha_{k-1} = -A_n \frac{2^{2n-k+1} n!}{(k-n)!(2n-k+1)!} - \frac{2(k-n)}{2n-k+1} \frac{2^{2n-k} n!}{(2n-k)!(k-n)!} \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{k-n+1} \right) A_n,$$

YES! So, $Q_n(z) = \dfrac{(-2)^n n!}{(2n)!} \dfrac{d^n}{dz^n} \left\{ (z^2-1)^n \int_z^\infty (v^2-1)^{-n-1} dv \right\}$

$$= \frac{(-2)^n n!}{(2n)!} \frac{d^n}{dz^n} \left\{ R_n(z) + A_n (z^2-1)^n \log \frac{z+1}{z-1} \right\}$$

$$= \frac{(-2)^n n!}{(2n)!} \frac{d^n}{dz^n} \sum_{k=0}^{2n-1} \alpha_k (z-1)^k + \frac{(-2)^n n!}{(2n)!} A_n \frac{d^n}{dz^n} \left\{ (z^2-1)^n \log \frac{z+1}{z-1} \right\}$$

$= \dfrac{(-2)^n n!}{(2n)!} \displaystyle\sum_{k=n}^{2n-1} \alpha_k k(k-1)\cdots(k-n+1)(z-1)^{k-n} + \dfrac{1}{2^{n+1} n!} \dfrac{d^n}{dz^n} \left\{ (z^2-1)^n \log \dfrac{z+1}{z-1} \right\}$. The first polynomial is $-\dfrac{(-2)^n n!}{(2n)!} \dfrac{(-1)^n}{2^{2n+1}} \dbinom{2n}{n} \displaystyle\sum_{k=n}^{2n-1} \dfrac{2^{2n-k} n! k!}{(2n-k)!((k-n)!)^2} \left(\dfrac{1}{n} + \dfrac{1}{n-1} + \cdots + \dfrac{1}{k-n+1} \right) (z-1)^{k-n}$

$$= -\frac{1}{2^{n+1} n!} \sum_{k=0}^{n-1} \frac{2^{n-k} n!(n+k)!}{(n-k)!(k!)^2} \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{k+1} \right) (z-1)^k = -\frac{W_{n-1}(z)}{2}. \quad \square$$

A research subject in 1875 (Polignac) and in 1891 (Catalan) became an exercise for (advanced) students in 1898. Catalan was very likely unaware of valuable British literature in 1891. And Polignac, who spent some time in London, may have picked the statement there, trying later on to make a proof of his own.