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# The factorization method for the Askey–Wilson polynomials

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**Abstract**

A special Infeld–Hull factorization is given for the Askey–Wilson second order $q$-difference operator. It is then shown how to deduce a generalization of the corresponding Askey–Wilson polynomials. 

*Keywords:* Infeld–Hull factorization method; Askey–Wilson polynomial

## 1. Introduction

A given second order $q$-difference operator

$$H(z;n) = u_2(qz;n)\boldsymbol{E}_q^2 + u_1(qz;n)\boldsymbol{E}_q + u_0(qz;n) \tag{1}$$

where

$$\boldsymbol{E}_q^i(\Omega(z;n)) = \Omega(zq^i;n), \quad z \in C,\ i,n \in Z, \tag{2}$$

is said to admit an *Infeld–Hull factorization* (this is a $q$-version) if and only if the following product can be performed [9,18]:

$$H(z;n) - \mu(n) = (\eta(z;n)\boldsymbol{E}_q + g(z;n))\,(\zeta(z;n)\boldsymbol{E}_q + f(z;n)),$$

$$H(z;n+1) - \mu(n) = (\zeta(z;n)\boldsymbol{E}_q + f(z;n))\,(\eta(z;n)\boldsymbol{E}_q + g(z;n)). \tag{3}$$

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The operators $\eta(z;n)\boldsymbol{E}_q + g(z;n)$ and $\zeta(z;n)\boldsymbol{E}_q + f(z;n)$ are called '*lowering*' and '*raising*', respectively. For convenience, the integer $n$ will be said to play the role of the '*variable of factorization*'. The Infeld–Hull factorization (IHF) is thus a particular case of the well known Darboux transformation (where $n$ is no more a variable but all simply an index) [8,17,22,25].

Supposing that $u_2$ does not depend on $n$, let us reduce (3) in the following:

$$\hat{H}(z;n) - \hat{\mu}(n) = (u_2(z)\boldsymbol{E}_q + \hat{g}(z;n))\,(u_2(z)\boldsymbol{E}_q + \hat{f}(z;n)),$$

$$\hat{H}(z;n+1) - \hat{\mu}(n) = (u_2(z)\boldsymbol{E}_q + \hat{f}(z;n))\,(u_2(z)\boldsymbol{E}_q + \hat{g}(z;n)), \tag{4}$$

where $\hat{H}(z;n) = u_2(z)H(z;n)$. The corresponding system then reads:

$$\hat{f}(qz;n+1) + \hat{g}(z;n+1) = \hat{f}(z;n) + \hat{g}(qz;n),$$

$$\hat{f}(z;n+1)\hat{g}(z;n+1) = \hat{f}(z;n)\hat{g}(z;n) + \hat{\mu}(n) - \hat{\mu}(n+1). \tag{5}$$

An analogous system obtained by an adequate factorization of the operator

$$L = v_2(x)E + v_1(x;t) + v_0(x;t)E^{-1}, \tag{6}$$

where $E^i(h(x)) = h(x+i)$; $i \in Z$ and $t \in Z$, the 'variable of factorization', has got interesting studies in [24,27]. In our situation, even if we are more interested in applications rather than in the intrinsic structure of the considered system, let us make the following observations.

For convenience, rewrite first the system (5) replacing $z$ by $q^s$, $s \in Z$:

$$\begin{aligned} &\bar{f}(s+1;n+1) + \bar{g}(s;n+1) = \bar{f}(s;n) + \bar{g}(s+1;n),\\ &\bar{f}(s;n+1)\bar{g}(s;n+1) = \bar{f}(s;n)\bar{g}(s;n) + \hat{\mu}(n) - \hat{\mu}(n+1), \end{aligned} \tag{7}$$

where

$$\bar{f}(s;n) := \hat{f}(q^s;n), \qquad \bar{g}(s;n) := \hat{g}(q^s;n). \tag{8}$$

Next, in the system (7), 'integrate' the second equation relatively to $n$ and then 'differentiate' it relatively to $s$. We obtain

$$\begin{aligned} &\tilde{f}(s+1;n+1) + \tilde{g}(s+1;n) = \tilde{f}(s;n) + \tilde{g}(s;n+1),\\ &\tilde{f}(s+1;n)\tilde{g}(s+1;n) = \tilde{f}(s;n)\tilde{g}(s;n) + \tilde{\mu}(s) - \tilde{\mu}(s+1), \end{aligned} \tag{9}$$

where

$$\tilde{f}(s;n) := \bar{f}(s;n), \qquad \tilde{g}(s;n) := -\bar{g}(s;n),$$

$$\tilde{\mu}(s) := \tilde{\mu}(-1) - \bar{f}(-1;-1)\bar{g}(-1;-1) + \bar{f}(s;-1)\bar{g}(s;-1). \tag{10}$$

So, while the system (7) is an IHF system with $n$ as the 'variable of factorization', the one in Eq. (9) is an IHF system with now $s$ as the 'variable of factorization'. Next, from Eqs. (7), (9) and (10) we see that the correspondence

$$\breve{f}(s;n) := \bar{f}(n;s), \qquad \breve{g}(s;n) := -\bar{g}(n;s),$$

$$\breve{\mu}(n) := \breve{\mu}(-1) - \bar{f}(-1;-1)\bar{g}(-1;-1) + \bar{f}(n;-1)\bar{g}(n;-1), \tag{11}$$

(remark that $\hat{\mu}(n) := \hat{\mu}(-1) + \bar{f}(-1;-1)\bar{g}(-1;-1) - \bar{f}(-1;n)\bar{g}(-1;n)$) is a *symmetry* of the IHF system (7) i.e. the functions in left-hand side of Eq. (11) are solutions of (7) as well. This symmetry is probably the main characteristic of the system under consideration. We will give our interpretation of all that in Section 4.

In [3,23] were given special IHF of the following difference operator:

$$\tilde{\sigma}(s)\Delta\nabla + \tilde{\tau}(s;l)\Delta - \tilde{\lambda}(n;l) \tag{12}$$

solutions of which are the $l$th difference of the discrete hypergeometric polynomials on a linear lattice $P_n^{(l)}(s) := \Delta^l(P_n(s))$ [20]. Here $\Delta = \boldsymbol{T} - 1$, $\nabla = 1 - \boldsymbol{T}^{-1}$ ($\boldsymbol{T}^i p(s) := p(s+i)$), $\tilde{\sigma}$ and $\tilde{\tau}$ being polynomials (in $s$) of degree $\leqslant 2$ and 1, respectively, $\tilde{\lambda}$ a constant (in $s$). In [23] the role of the 'variable of factorization' (as $n$ in Eq. (3)) is played by the order $l$ of the corresponding difference ($n$ remaining fixed) while in [3] that role is played by the degree of the corresponding polynomials ($l$ remaining fixed (to zero, for simplicity)). Evidently, the IHF from [23] can be extended to the case of ($q$)-nonlinear lattices.

The main objective of the present work is to give an extension of [3] to the case of $q$-nonlinear lattices:

We consider the second-order divided difference operator [20]:

$$\sigma(s)\frac{\Delta}{\Delta y(s-\frac{1}{2})}\cdot\frac{\nabla}{\nabla y(s)} + \tau(s)\frac{\Delta}{\Delta y(s)}, \tag{13}$$

where

$$\sigma(s) = \hat{\sigma}(y(s)) - \tfrac{1}{2}\hat{\tau}(y(s))\Delta y(s-\tfrac{1}{2}),$$

$$\tau(s) = \hat{\tau}(y(s)); \tag{14}$$

$\hat{\sigma}$ and $\hat{\tau}$ being polynomials of degree $\leqslant 2$ and 1, respectively, and

$$y(s) = c_1 q^s + c_2 q^{-s} + c_3, \tag{15}$$

the lattice.

The form of the operator in Eq. (13) being invariant under the transformations

$$y \to Ay + B; \qquad s \to s - s_0, \tag{16}$$

we can use them to transform the lattice in Eq. (15) in its canonical form

$$y(s) = \tfrac{1}{2}(q^s + q^{-s}). \tag{17}$$

In that case, up to a multiplication by a constant, the operator in Eq. (13) can, after simple computations, be reduced to

$$\mathscr{L} = \frac{1}{z - z^{-1}}(\mathscr{A}(z)\boldsymbol{E}_q - [\mathscr{A}(z) + \mathscr{B}(z)] + \mathscr{B}(z)\boldsymbol{E}_q^{-1}), \tag{18}$$

where

$$\mathscr{A}(z) = \frac{A_{-2}z^{-2} + A_{-1}z^{-1} + A_0 + A_1 z + A_2 z^2}{qz - z^{-1}},$$

$$\mathscr{B}(z) = \frac{A_2 z^{-2} + A_1 z^{-1} + A_0 + A_{-1}z + A_{-2}z^2}{z - qz^{-1}},$$

$$A_{-2} = 1; \quad A_{-1} = -(a+b+c+d); \quad A_0 = ab + ac + ad + bc + bd + cd,$$

$$A_1 = -(abc + abd + bcd + acd); \qquad A_2 = abcd,$$

$$z = q^s$$

$$a = q^{s_1}; \quad b = q^{s_2}; \quad c = q^{s_3}; \quad d = q^{s_4},$$

$$\boldsymbol{E}_q^i(k(z)) = k(q^i z), \quad i \in Z, \tag{19}$$

$s_1, s_2, s_3$ and $s_4$ being the (mutually different) roots of $\sigma(s)$.

The operator $\mathscr{L}$ can also be written as [12]:

$$\mathscr{L}(z) = v(z)\boldsymbol{E}_q - (v(z) + v(z^{-1})) + v(z^{-1})\boldsymbol{E}_q^{-1},$$

$$v(z) = \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}. \tag{20}$$

Letting $\mathscr{D}_q$,

$$\mathscr{D}_q h(\chi(z)) = \frac{h(\chi(q^{\frac{1}{2}}z)) - h(\chi(q^{-\frac{1}{2}}z))}{\chi(q^{\frac{1}{2}}z) - \chi(q^{-\frac{1}{2}}z)},$$

$$\chi(z) = \frac{z + z^{-1}}{2}, \tag{21}$$

be the Askey–Wilson first-order divided difference operator [1], one can also write $\mathscr{L}$ as

$$\mathscr{L} = \left[ \frac{(q-1)^2}{4q} \frac{z^2-1}{z\omega(z)} \mathscr{D}_q \frac{z^2-1}{z} v(q^{-\frac{1}{2}}z)\omega(q^{-\frac{1}{2}}z) \right] \mathscr{D}_q, \tag{22}$$

where

$$\frac{\omega(qz)}{\omega(z)} = \frac{v(z)}{v((qz)^{-1})}. \tag{23}$$

The Askey–Wilson polynomials $\mathscr{P}_n(\chi(z))$ (see Eq. (27) below), satisfy the second order $q$-difference equation:

$$\mathscr{L}\mathscr{P}_n(\chi(z)) = \lambda(n)\mathscr{P}_n(\chi(z)), \tag{24}$$

where

$$\lambda(n) = -(1 - q^{-n})(1 - abcdq^{n-1}). \tag{25}$$

The operator $\mathscr{L}$ was shown in [11] to be self-adjoint (more precisely, the operator in square brackets, in the r.h.s. of Eq. (22), is adjoint to the first-order Askey–Wilson divided difference operator) in the space $S_{a,b,c,d}$ of real polynomials in $\chi(z)$ with the inner product

$$(h_1, h_2) := \frac{1}{2\pi i} \oint_C h_1(\chi(z))h_2(\chi(z))\omega(z)\frac{\mathrm{d}z}{z}, \tag{26}$$

where $\omega(z)$ is given by Eq. (23) and $C$ is a deformation of the unit circle.

Considering the representation in Eq. (22) and the equation in (24), we will refer to $\mathscr{L}$, everywhere in this work, as the *Askey–Wilson second order q-difference operator*.

In this work, we apply the IHF method to the operator $\mathscr{L}-\lambda$, where only the term $\lambda$ (independent of $z$) depends of the 'variable of factorization'.

An alternative to the IHF is the *Inui* factorization [10] where the 'up-stair' (raising) and 'down-stair' (lowering) operators are not obtained from a direct factorization of the studied second order difference operator itself but from some difference or contiguous relations satisfied by its eigenfunctions given in a general a priori known form. An Inui type factorization for the Askey–Wilson second order $q$-difference operator can be found in [11,19] (see also [7], with a non-canonical form of the operator) where interesting applications were also obtained (in [11,19], the basic form of the factorization chain (i.e. when the 'variable of factorization' is zero), reads (in essence) as the product in Eq. (22)). In [26,27] IHF techniques were applied to the three-term recurrence relations for orthogonal polynomials.

The IHF that we are going to carry out for the Askey–Wilson second order $q$-difference operator will furnish various formulas such as *strict* (i.e. without any perturbation of parameters) difference relations, recurrence relations and Rodrigues type formulas for the general sequences of eigenfunctions of the Askey–Wilson second order $q$-difference operator.

Next, a somewhat converse reasoning will suffice to attain the most ambitious aim of this work: to implement the IHF system (5) as an independent source of the Askey–Wilson Polynomials (a discrete version of the system (5) was applied to generate the Charlier and Meixner-Kravchuk polynomials already in [18]).

All those discussions will be held in the following section. In the third one, we will extract a Rodrigues type formula allowing to produce (starting at the difference hypergeometric functions obtained in [2]) a sequence of functions generalizing the Askey–Wilson polynomials. The last section is essentially a concluding one. It is devoted to some explanations concerning the interconnection between the symmetry (11) and the usual duality relation for orthogonal polynomials and that between the IHF method and the Laguerre–Hahn approach [15,16]. Some outlooks are also given for the method under consideration.

## 2. The factorization method for the Askey–Wilson second order $q$-difference operator

The Askey–Wilson polynomials (AWP) [1], $\mathscr{P}_n(\chi(z))$, $\chi(z)=\frac{1}{2}(z+z^{-1})$ are defined by:

$$\mathscr{P}_n(\chi(z))=\frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},abcdq^{n-1},az,az^{-1}\\ ab,ac,ad\end{matrix}\middle|\,q;q\right), \tag{27}$$

where the basic hypergeometric (or $q$-hypergeometric) series ${}_r\phi_s$ read:

$${}_r\phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix}\middle|\,q;z\right):=\sum_{k=0}^{\infty}\frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k}(-1)^{(1+s-r)k}q^{(1+s-r)\binom{k}{2}}\frac{z^k}{(q;q)_k}$$

with

$$(a_1,\dots,a_r;q)_k:=(a_1;q)_k\dots(a_r;q)_k,$$

while

$$(\sigma;q)_0:=1,(\sigma;q)_k:=\prod_{i=0}^{k-1}(1-\sigma q^i),\quad k=1,2,3,\dots \tag{28}$$

As already noted, they satisfy the following second order $q$-difference equation [12]:

$$\mathscr{L}\mathscr{P}_n(\chi(z)) = \lambda(n)\mathscr{P}_n(\chi(z)). \tag{29}$$

In anticipation on the coming $q$-integration and basing ourself on the experience from [3], we are led to reformulate, before the factorization, Eq. (29) as follows:

$$\{\mathscr{A}(qz)\boldsymbol{E}_q^2 - [\mathscr{A}(qz) + \mathscr{B}(qz) + \mathscr{K}(qz)\lambda(n)]\boldsymbol{E}_q + \mathscr{B}(qz)\}\mathscr{P}_n(\chi(z)) = 0, \tag{30}$$

where $\mathscr{A}(z)$ and $\mathscr{B}(z)$ are given in Eq. (19) and $\mathscr{K}(z) = z - z^{-1}$.

Consider next the following $q$-difference operator

$$\mathscr{H}(z;n) = \mathscr{A}(z)[\mathscr{A}(qz)\boldsymbol{E}_q^2 - [\mathscr{A}(qz) + \mathscr{B}(qz) + \mathscr{K}(qz)\lambda(n)]\boldsymbol{E}_q + \mathscr{B}(qz)]. \tag{31}$$

The central result of this work is the following

**Proposition 1.** *The operator $\mathscr{H}$ in Eq.* (31) *factorizes into*

$$\mathscr{H}(z;n) - \mu(n) = (\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{G}(z;n))(\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{F}(z;n)),$$

$$\mathscr{H}(z;n+1) - \mu(n) = (\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{F}(z;n))(\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{G}(z;n)) \tag{32}$$

*with $\mathscr{F}$ and $\mathscr{G}$ such that the class of the Askey–Wilson polynomials becomes invariant under the action of the 'raising' and 'lowering'* operators.

**Proof.** The operatorial relations (32) are equivalent to the system:

$$\mathscr{F}(qz;n) + \mathscr{G}(z;n) = -(\mathscr{A}(qz) + \mathscr{B}(qz) + \mathscr{K}(qz)\lambda(n)),$$

$$\mathscr{F}(z;n)\mathscr{G}(z;n) = \mathscr{A}(z)\mathscr{B}(qz) - \mu(n),$$

$$\Delta_q(\mathscr{F}(z;n) - \mathscr{G}(z;n)) = (\lambda(n+1) - \lambda(n))\mathscr{K}(qz), \tag{33}$$

where $\Delta_q h(z) = h(qz) - h(z)$.

Using the $q$-integration, we first transform the system (33) in:

$$\mathscr{F}(z;n) + \mathscr{F}(qz;n) = -(\mathscr{A}(qz) + \mathscr{B}(qz)) - \left(\beta_{-1} - \frac{\lambda(n)}{q}\right)z^{-1} - \beta_0 - (\beta_1 + \lambda(n)q)z,$$

$$\mathscr{F}(z;n)\mathscr{G}(z;n) = \mathscr{A}(z)\mathscr{B}(qz) - \mu(n),$$

$$\mathscr{G}(z;n) - \mathscr{F}(z;n) = \sum_{i=-1}^{i=1} \beta_i z^i, \tag{34}$$

where $\beta_{-1} = \{\lambda(n+1) - \lambda(n)\}/(1-q)$; $\beta_1 = q\beta_{-1}$; $\beta_0$ remaining arbitrary for the moment. Observing the first equation in (34) and then using the last one, it becomes sensible to search $\mathscr{F}(z;n)$ and then

$\mathscr{G}(z;n)$ under the forms:

$$\mathscr{F}(z;n) := \frac{F_{-2}z^{-2} + F_{-1}z^{-1} + F_0 + F_1 z + F_2 z^2}{qz - z^{-1}},$$

$$\mathscr{G}(z;n) := \frac{(F_{-2} - \beta_{-1})z^{-2} + (F_{-1} - \beta_0)z^{-1} + F_0 + (F_1 + \beta_0 q)z + (F_2 + \beta_1 q)z^2}{qz - z^{-1}}. \tag{35}$$

Taking $\mu$, $\beta_0$, $F_{-2}$, $F_{-1}$, $F_0$, $F_1$, $F_2$ as unknowns, the system (33) will then be transformed in an algebraic system of 16 equations for 7 unknowns. To solve it (by hand), one first determines those unknowns from 7 equations and then very delicately ensures himself that they satisfy all the remaining 9 equations.

The result is:

$$F_{-2}(n){:}\ \frac{\lambda(n) - q\lambda(n+1)}{q^2 - 1} - \frac{q + A_2}{q^2 + q} \tag{36}$$

$$F_2(n){:}\ \frac{\lambda(n)q - \lambda(n+1)}{1 - q^2}q^2 - \frac{q^2 + qA_2}{q+1}$$

$$\beta_0(n) :\ \frac{1-q}{(\lambda(n) - \lambda(n+1))q^3}\left\{\left(2\frac{\lambda(n)q - \lambda(n+1)}{1-q^2}q^2 + \frac{\lambda(n+1) - \lambda(n)}{1-q}q^2 \right.\right.$$
$$\left.\left. - 2\frac{q^2 + qA_2}{1+q}\right)(A_1 + qA_{-1}) + (2A_1 q^2 + 2A_2 A_{-1} q)\right\}$$

$$F_{-1}(n){:}\ \frac{\beta_0(n)}{2} - \frac{A_1 + qA_{-1}}{2q}$$

$$F_1(n){:}\ -\frac{q\beta_0(n)}{2} - \frac{A_1 + qA_{-1}}{2}$$

$$F_0(n){:}\ \frac{1}{q+q^2}\{q^2 - q^3 - A_0(q + q^2) + A_2(q-1) + q^2(\lambda(n) + \lambda(n+1))\}$$

$$\mu(n){:}\ A_0 + A_1 A_{-1} q^{-1} + A_0 A_2 q^{-2} + F_0(n)\beta_{-1}(n) + F_{-1}(n)\beta_0(n) - 2F_{-2}(n)F_0(n) - F_{-1}^2(n).$$

Thus, Eq. (35) is determined and a solution of the IHF problem (32) is found. It remains to prove the invariance of the class of the AWP under the action of the 'raising' and 'lowering' operators thus obtained. From Eq. (32), the intertwining relations follow:

$$\mathscr{H}(z;n+1)(\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{F}(z;n)) = (\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{F}(z;n))\mathscr{H}(z;n),$$

$$\mathscr{H}(z;n)(\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{G}(z;n)) = (\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{G}(z;n))\mathscr{H}(z;n+1). \tag{37}$$

Let us note that those relations, as well as the IHF (32) are in fact valid $\forall n \in Z$. So, if for a given number $n_0 \in Z$, a given function $\Psi_{n_0}(z)$ is an eigenvector of the operator $\mathscr{H}(z;n_0)$, corresponding to a given eigenvalue $v$ (take $v = 0$, in our situation), then the sequence of functions

$$^{\uparrow}\Psi_{n_0+k}(z) := \prod_{i=0}^{k-1}(\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{F}(z;n_0+i))\Psi_{n_0}(z), \quad k = 1,2,3,\ldots \tag{38}$$

are, respectively, eigenvectors of $\mathscr{H}(z;n_0+k)$ corresponding to the same eigenvalue $v=0$. The following formulas will be useful:

$$\mathscr{F}(z;n)-\mathscr{G}(z;n-1)=[2A_2q^{-1}q^n-2q^{-n}]\chi(z)-\frac{\beta_0(n)+\beta_0(n-1)}{2},\quad (\forall n\in Z), \tag{39}$$

$$(A(z)\boldsymbol{E}_q+\mathscr{F}(z;0))(1)=[2A_2q^{-1}-2]\chi(z)-\frac{\beta_0(0)}{2}+\frac{A_1-qA_{-1}}{2q}. \tag{40}$$

Considering Eqs. (38) and (32), we obtain the following difference relations (we let ${}^{\uparrow}\Psi_{n_0}(z):=\Psi_{n_0}(z)$):

$$\begin{aligned}&{}^{\uparrow}\Psi_{n+1}(z)=(A(z)\boldsymbol{E}_q+\mathscr{F}(z;n))^{\uparrow}\Psi_n(z),\\&-\mu(n)^{\uparrow}\Psi_n(z)=(A(z)\boldsymbol{E}_q+\mathscr{G}(z;n))^{\uparrow}\Psi_{n+1}(z),\quad n=n_0,n_0+1,n_0+2,\ldots.\end{aligned} \tag{41}$$

Now, using Eqs. (41) and (39), we obtain the following three-term recurrence relations:

$$\begin{aligned}&{}^{\uparrow}\Psi_{n+1}(z)+\mu(n-1)^{\uparrow}\Psi_{n-1}(z)\\&\quad=\left([2A_2q^{-1}q^n-2q^{-n}]\chi(z)-\frac{\beta_0(n)+\beta_0(n-1)}{2}\right)^{\uparrow}\Psi_n(z),\quad n=n_0+1,n_0+2,\ldots.\end{aligned} \tag{42}$$

Considering the particular case of Eq. (38) when $\Psi_{n_0}(z)$ is the AWP of order $n_0$, $\mathscr{P}_{n_0}(\chi(z))$, the invariance of the class of the AWP under the 'raising' operation means that $\Psi_{n_0+k}(z)$ will be a non vanishing constant multiple of $\mathscr{P}_{n_0+k}(\chi(z))$, the AWP of order $n_0+k$, $k\geqslant 1$.

It follows from Eqs. (40) and (42) (with $n_0=0$) that there exists a sequence of polynomial (in $\chi(z)$) eigenfunctions (of degrees $0,1,2,3,\ldots$) of the Askey–Wilson second order $q$-difference operator $\mathscr{L}$, invariant under the 'raising' operation. On the other side, according to Eq. (41), the same sequence is invariant, up to a multiplication by a constant, under the 'lowering' operation (as long as $n\geqslant 0$). But as the constant multiple of AWP are the unique non trivial polynomial eigenfunctions (of degrees $0,1,2,3,\ldots$) of the Askey–Wilson second order $q$-difference operator (see Theorem 3.4 in [7]), this suffices to show the cited invariance and the proposition is completely proved.

Thus, if in Eq. (38) we consider the case of AWP and set $n_0=0$, we obtain the Rodrigues type formula:

$$\tilde{c}(n)\mathscr{P}_n(\chi(z))=\prod_{i=0}^{n-1}[\mathscr{A}(z)\boldsymbol{E}_q+\mathscr{F}(z;i)](1),\quad n=1,2,3,\ldots \tag{43}$$

$\tilde{c}(n)$ being some no null constant (in $z$). It is not difficult to see that Eq. (43) is equivalent to

$$\tilde{c}(n)\mathscr{P}_n(\chi(z))=\frac{1}{\rho(z)}\prod_{i=0}^{n-1}[\boldsymbol{E}_q+\mathscr{F}(z;i)](\rho(z)),\quad n=1,2,3,\ldots, \tag{44}$$

where

$$\frac{\rho(qz)}{\rho(z)}=\mathscr{A}(z),$$

a formula similar to that from [3].

Now, remembering that the relations (37) as well as the factorization (32) are valid $\forall n \in Z$, let us give the following notes for completeness:

Remark first that as in Eq. (38), the functions

$$^{\downarrow}\Psi_{n_0-1-k}(z) := \prod_{i=0}^{k}(\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{G}(z; n_0 - 1 - i))\Psi_{n_0}(z), \quad k = 0, 1, 2, \ldots \tag{45}$$

are respectively eigenfunctions of the operators $\mathscr{H}(z; n_0 - 1 - k)$. They satisfy the difference relations (letting $^{\downarrow}\Psi_{n_0}(z) := \Psi_{n_0}(z)$):

$$-\mu(n)^{\downarrow}\Psi_{n+1}(z) = (A(z)\boldsymbol{E}_q + \mathscr{F}(z; n))^{\downarrow}\Psi_n(z)$$

$$^{\downarrow}\Psi_n(z) = (A(z)\boldsymbol{E}_q + \mathscr{G}(z; n))^{\downarrow}\Psi_{n+1}(z), \quad n = n_0 - 1, n_0 - 2, \ldots \tag{46}$$

and the recurrence relations

$$\mu(n)^{\downarrow}\Psi_{n+1}(z) +^{\downarrow} \Psi_{n-1}(z)$$
$$= \left([2q^{-n} - 2A_2q^{-1}q^n]\chi(z) + \frac{\beta_0(n) + \beta_0(n-1)}{2}\right)^{\downarrow} \Psi_n(z), \quad n = n_0 - 1, n_0 - 2, \ldots . \tag{47}$$

For $n_0 = 0$ and $\psi_0(z) = 1$ (polynomial case), Eq. (45) gives an 'extension' of Eq. (43):

$$^{\downarrow}\Psi_{-(n+1)}(z) := \prod_{i=0}^{n}(\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{G}(z; -(i + 1)))(1), \quad n = 0, 1, 2, \ldots . \tag{48}$$

But a direct verification of the relation

$$(\mathscr{A}(z)\boldsymbol{E}_q + \mathscr{G}(z; -1))(1) = 0 \tag{49}$$

shows, without surprise, that the functions produced in Eq. (48), 'extending' the AWP relatively to $n$ (as an index, from $Z^+$ to $Z^-$), are all vanishing ($^{\downarrow}\Psi_{-(n+1)}(z) \equiv 0,\ n = 0, 1, 2, \ldots$).

The possibility of unifying the 'raising' and 'lowering' operations in Eqs. (43) and (48) in types (41) or (46) pair of difference relations (and so in types (42) or (47) recurrence relations) with $n = \ldots - 2, -1, 0, 1, 2, \ldots$, is subordinated to the constraint $\mu(-1) = 0$ and this is actually the case.

We are now led to make a somewhat converse reasoning:

We remark first that

$$\mathscr{F}(z; -1) = -\mathscr{B}(qz), \qquad \mathscr{G}(z; -1) = -\mathscr{A}(z), \quad \mu(-1) = 0. \tag{50}$$

This leads together with the first equation in Eq. (33) to a new form of Eq. (30):

$$\{\mathscr{G}(qz; -1)\boldsymbol{E}_q^2 - [\mathscr{F}(qz; n) + \mathscr{G}(z; n)]\boldsymbol{E}_q + \mathscr{F}(z; -1)\}\mathscr{P}_n(\chi(z)) = 0. \tag{51}$$

We recall that this equation is verified by $\mathscr{F}$ and $\mathscr{G}$ given by (36) and (35) and the AWP given by (43) or (44) for $n \geqslant 1$ while $\mathscr{P}_0 = 1$ (the unit is a solution of (51) for $n = 0$ thanks to the first equation in the system (5)). Let us return now the situation taking the IHF system (5) as a starting point. Our point resides in the following

**Proposition 2.** *Consider the IHF system* (5) *with the unique initial condition* $\tilde{\mu}(-1) = 0$ *and let* $\tilde{\mathscr{F}}$ *and* $\tilde{\mathscr{G}}$ *be any of its solutions, then the equation*

$$\{\tilde{\mathscr{G}}(qz; -1)\boldsymbol{E}_q^2 - [\tilde{\mathscr{F}}(qz; n) + \tilde{\mathscr{G}}(z; n)]\boldsymbol{E}_q + \tilde{\mathscr{F}}(z; -1)\}\tilde{\mathscr{P}}_n(z) = 0 \tag{52}$$

*admits a sequence of solutions given by a 'Rodrigues type formula' (similar to (38)) applied to $\tilde{\mathscr{P}}_0 = 1$ and satisfying strict difference relations and three-term recurrence relations. Morever, if we add the condition*

$$\tilde{\mathscr{F}}(z;n) - \tilde{\mathscr{G}}(z;n-1) = c_0(n)\chi(z) + c_1(n) \tag{53}$$

*for some constants (in $z$) $c_0$ and $c_1$, the obtained sequence is of polynomial (in $\chi(z)$) type.*

**Proof.** To prove this proposition, one needs essentially to remark that if $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{G}}$ are solutions of the system (5) with initial condition $\tilde{\mu}(-1) = 0$, then the operator

$$\tilde{\mathscr{H}}(z;n) = \tilde{\mathscr{G}}(z;-1)[\tilde{\mathscr{G}}(qz;-1)\boldsymbol{E}_q^2 - [\tilde{\mathscr{F}}(qz;n) + \tilde{\mathscr{G}}(z;n)]\boldsymbol{E}_q + \tilde{\mathscr{F}}(z;-1)] \tag{54}$$

admits a type (32) IHF. The remaining part of the reasoning is similar to that of the preceding proposition, reason for which we omit it and we assume that the proposition is proved.

It is clear that if we add the analogous of the two first initial conditions in (50), we obtain the AWP. In other words, we remark in our satisfaction that the IHF system (5) can be considered as an independent source of the AWP.

## 3. The generalization of the Askey–Wilson polynomials

We envisage now to show how to produce a sequence of functions $\Phi_n(z,t)$ generalizing the AWP in that sense that they are eigenfunctions of the Askey–Wilson second order $q$-difference operator $\mathscr{L}$ and

$$\Phi_n(z,t) \to \mathscr{P}_n(\chi(z)), \quad t \to 1. \tag{55}$$

We showed above that if the eigenvalue of the Askey–Wilson second order $q$-difference operator is $\lambda(n) = -(1-q^{-n})(1-abcdq^{n-1})$, then an IHF as in Eq. (32) is realizable. Now, using Eq. (33), one finds that conversely, the possibility of a type (32) IHF implies necessarily that the eigenvalue in action is ${}^t\lambda(n) = -(1-tq^{-n})(1-abcdt^{-1}q^{n-1})$. On the other side, it is clear that the presence of the parameter $t$ in ${}^t\lambda(n)$ in no way annoys the IHF (32) (thanks to the specific dependence of ${}^t\lambda(n)$ at $t$). This means that if in the expressions (36), we substitute $\lambda(n)$ by ${}^t\lambda(n)$, the resulting expressions for $\mathscr{F}$ and $\mathscr{G}$, say ${}^t\mathscr{F}$ and ${}^t\mathscr{G}$, solve a type (32) IHF problem for an operator, say ${}^t\mathscr{H}(z;n)$, obtained from $\mathscr{H}(z;n)$ replacing $\lambda(n)$ by ${}^t\lambda(n)$.

Let us consider next the equation:

$$\mathscr{L}y(z,t) = {}^t\lambda(0)y(z,t), \tag{56}$$

or equivalently

$${}^t\mathscr{H}(z;0)y(z,t) = 0. \tag{57}$$

This equation is not new: A little wider equation (where the Askey–Wilson second order $q$-difference operator is replaced by the corresponding operator in Eq. (13)) has been explicitly solved in [2]. In our situation, according to [2], one first solves the non-homogeneous equations:

$$(\mathscr{L} - {}^t\lambda(0))u(z,t,\alpha) = G(z,t,\alpha) \tag{58}$$

where

$$G(z,t,\alpha)=\frac{(\alpha t, abcd\alpha t^{-1}q^{-1};q)_\infty}{\alpha(ab\alpha, ac\alpha, ad\alpha, \alpha q;q)_\infty}(az, az^{-1};q)_\infty \tag{59}$$

and $(\sigma;q)_\infty := \lim_{i\to\infty}(\sigma;q)_i$, for the following values of $\alpha$:

$$\alpha := 1 \quad \alpha := \frac{q}{ab} \quad \alpha := \frac{q}{ac} \quad \alpha := \frac{q}{ad}.$$

The corresponding solutions are the functions [2]:

$$u(z,t,\alpha)=\frac{(az, az^{-1};q)_\infty}{(a\alpha z, a\alpha z^{-1};q)_\infty}\sum_{i=0}^{\infty}\frac{(\alpha t, abcd\alpha t^{-1}q^{-1}, a\alpha z, a\alpha z^{-1};q)_i}{(ab\alpha, ac\alpha, ad\alpha, \alpha q;q)_i}q^i. \tag{60}$$

Let us remark that the AWP appear here as the functions $u(z, q^{-n}, 1)$, $n = 0, 1, 2, \ldots$ .

The solutions of the homogeneous equation (56) or (57) are then obtained by adequate (considering Eq. (59)) linear combinations of any two of the functions (60).

Among the solutions of the homogenous equation (56) or (57), we are interested in those having a constant limit when $t$ converges to the unit. Let us take the 'adequate' linear combination of $u(z,t,1)$ and $u(z,t,\alpha)$, $\alpha \neq 1$:

$$y(z,t) := u(z,t,1)-\alpha\frac{(t, abcdt^{-1}q^{-1}, ab\alpha, ac\alpha, ad\alpha, \alpha q;q)_\infty}{(ab, ac, ad, q, \alpha t, abcd\alpha t^{-1}q^{-1};q)_\infty}u(z,t,\alpha). \tag{61}$$

It is easily seen that $y(z,1)=1$. However, if $\tilde{y}(z,t)$ is another 'adequate' linear combination of $u(z,t,\alpha_1)$ and $u(z,t,\alpha_2)$ where $\alpha_1$ and $\alpha_2$ differ both from the unit and each other, then $\tilde{y}(z,1) \neq$ *constant*. In other words $\tilde{y}(z,1)$ is the nonconstant solution of Eq. (29) for $n=0$.

Our point here resides in the following

**Proposition 3.** *If $y(z,t)$ is the solution in* (61), *of* (56) *or* (57), *then the functions*

$$\Phi_0(z,t) := y(z,t),$$

$$\Phi_n(z,t) := \prod_{i=0}^{n-1}[\mathscr{A}(z)\boldsymbol{E}_q + {}^t\mathscr{F}(z;i)]y(z,t), \quad n=1,2,3,\ldots, \tag{62}$$

*generalize the AWP in that sense that they are eigenfunctions of the Askey–Wilson second order $q$-difference operator $\mathscr{L}$ (corresponding to ${}^t\lambda(n)$) and they converge to them when $t$ converges to the unit.*

**Proof.** As already noted, the operator ${}^t\mathscr{H}$ admits a type (32) IHF but now with $\mathscr{F}$ and $\mathscr{G}$ replaced by ${}^t\mathscr{F}$ and ${}^t\mathscr{G}$. Consequently, the fact that the functions in (62) are eigenfunctions of the Askey–Wilson second order $q$-difference operator $\mathscr{L}$ corresponding to ${}^t\lambda(n)$, is a consequence (thanks to type (37) intertwining relations) of that $\Phi_0(z;t)$ is an eigenvector of $\mathscr{L}$ corresponding to ${}^t\lambda(0)$. To be assured that the functions in (62) converge to the AWP (of course, up to a multiplication by a constant), we need essentially to remember that ${}^t\lambda(n)$ converges to $\lambda(n)$ when $t$ converges to the unit and then compare the r.h.s. of (62) and (43), which proves the propositon.

For completeness, it is perhaps worth noting that an iterative application of the 'lowering' operator $\mathscr{A}(z)\boldsymbol{E}_q +^t \mathscr{G}(z; -(i+1))$, $i = 0, 1, 2, \ldots$ to $y(z,t)$ leads to an evident 'extension' relatively to $n$ (as an index, from $Z^+$ to $Z^-$) of the functions $\Phi_n(z,t)$ in (62), generalizing thus the already evoked identically null functions (48). Morever if ${}^{\uparrow}\tilde{\Psi}_n(z)$ (or ${}^{\downarrow}\tilde{\Psi}_n(z)$) are the eigenfunctions obtained by applying $n$-times the 'raising' (or 'lowering') operator to $\tilde{y}(z,1)$, so their generalizations are obtained by performing similar operations starting now at $\tilde{y}(z,t)$. Let us note finally that the present generalizations are also difference hypergeometric functions (in the sense of [2]) reason for which they are closely related to the AWP.

## 4. Concluding remarks and outlooks

**4.1.** Using the procedure described in the sketched proof of the Proposition 2, one easily finds that the functions $\Theta_n(s)$, $n = 0, 1, 2, \ldots$ of the independent variable $s \in Z^+$ 'generated' by the IHF system (7), satisfy the second-order difference equation

$$\bar{g}(s; -1)\Theta_n(s+1) - (\bar{f}(s; n) + \bar{g}(s-1; n))\Theta_n(s) + \bar{f}(s-1; -1)\Theta_n(s-1) = 0 \tag{63}$$

and the three-term recurrence relation

$$\Theta_{n+1}(s) + (\bar{f}(s; n) - \bar{g}(s; n-1))\Theta_n(s) + (\bar{f}(-1; -1)\bar{g}(-1; -1) - \bar{f}(-1; n-1)\bar{g}(-1; n-1))\Theta_{n-1}(s) = 0. \tag{64}$$

On the other side, the functions $\tilde{\Theta}_s(n)$, $s = 0, 1, 2, \ldots$ of the independent variable $n \in Z^+$ 'generated' by the IHF system (9), satisfy the second-order difference equation

$$\bar{g}(-1; n)\tilde{\Theta}_s(n+1) + (\bar{f}(s; n) - \bar{g}(s; n-1))\tilde{\Theta}_s(n) - \bar{f}(-1; n-1)\tilde{\Theta}_s(n-1) = 0 \tag{65}$$

and the three-term recurrence relation

$$\tilde{\Theta}_{s+1}(n) + (\bar{f}(s; n) + \bar{g}(s-1; n))\tilde{\Theta}_s(n) + (\bar{f}(s-1; -1)\bar{g}(s-1; -1) - \bar{f}(-1; -1)\bar{g}(-1; -1))\tilde{\Theta}_{s-1}(n) = 0. \tag{66}$$

Comparing Eqs. (63) and (66) on the one side, Eqs. (64) and (65) on the other side, we remark that the sequences of functions $\widehat{\Theta}_n(s) := \Theta_n(s)/\varrho(n)$ and $\widehat{\tilde{\Theta}}_n(s) := \tilde{\Theta}_n(s)/\tilde{\varrho}(n)$ where $\varrho(n+1)/\varrho(n) = \bar{g}(-1; n)$ and $\tilde{\varrho}(n+1)/\tilde{\varrho}(n) = -\bar{g}(n; -1)$ are mutually-dual, i.e. $\widehat{\tilde{\Theta}}_s(n) \equiv \widehat{\Theta}_n(s)$, iff

$$\bar{f}(-1; -1)\bar{g}(-1; -1) = 0. \tag{67}$$

Remark that in the case $\bar{f}(s; n) + \bar{g}(s-1; n)$ and $\bar{f}(s; n) - \bar{g}(s; n-1)$ are linear functions respectively of say, $\lambda(n)$ (with coefficients not depending on $n$) and $y(s)$ (with coefficients not depending on $s$), $\widehat{\Theta}_n(s)$ and $\widehat{\tilde{\Theta}}_s(n)$ are (bispectral) orthogonal polynomials of the variables $y(s)$ and $\lambda(n)$, respectively. For such polynomials, the symmetry (11), constrained by Eq. (67), becomes clearly the usual duality relation between (bispectral) orthogonal polynomials (see for example [14]).

Considering (8), the two first equations in (19) and the two first ones in (50), one finds that for the AWP, the condition in Eq. (67) is not satisfied. However, we know that (see e.g. [12]) the AWP can be obtained from the $q$-Racah polynomials by a simple transformation of the parameters and the independent variable, while for the $q$-Racah polynomials, the condition in (67) is easily seen to be satisfied (see $D(0)=0$ in [12], $D(x)$ in [12] being comparable with $\bar{f}(s-1;-1)$ here). The same condition is also satisfied for the Charlier, Meixner–Kravchuk and Hahn polynomials, as well as their $q$-analogues [12]. It is on the other side a well known Leonard result [14] that any sequence of orthogonal polynomials admitting a dual sequence of orthogonal polynomials is necessarily the sequence of $q$-Racah polynomials or one of its specializations.

**4.2.** There exists surprising interconnections between the IHF technique and the Laguerre–Hahn approach to orthogonal polynomials [15]. To see this, let us combine the IHF system (7) and the condition in (53) which now reads $\bar{f}(s;n)-\bar{g}(s;n-1)=c_0(n)y(s)+c_1(n)$ (the necessity of this condition is evident for example from (64) in order that the functions $\Theta_n(s)$ be polynomials (in $y(s)$)). The result is

$$\bar{f}(s;n)=\bar{g}(s;n-1)+c_0(n)y(s)+c_1(n),$$

$$\bar{g}(s;n)=\bar{f}(s;n-1)-(c_0(n)y(s+1)+c_1(n)),$$

$$\bar{f}(s;n)\bar{g}(s;n)=\bar{f}(s;n-1)\bar{g}(s;n-1)+\hat{\mu}(n-1)-\hat{\mu}(n). \tag{68}$$

For a certain subset of Laguerre–Hahn orthogonal polynomials (including particularly all the classical (up to AWP) polynomials), there exists a correspondence between the system here in (68) and the system (3.2a)–(3.2d) in [15] (see [4,6] for details).

The main very useful consequence of that interconnection is linked on the fact that the knowledge of solutions of Eq. (68) (i.e. (5) and (53)) for a given family of orthogonal polynomials (starting from the second-order difference equation satisfied by the polynomials, as was done here for the AWP) leads (not only to the knowledge of the main characteristic of the polynomials: strict difference and three-term recurrence relations, 'Rodrigues type formula', ...) to the knowledge of the coefficients in the *fourth order difference equation* satisfied by the polynomials *r-associated* to them [5].

**4.3.** Besides the evoqued 'Laguerre–Hahn' extension of the method, there is a question of applying the technique to generate new singular solvable Hamiltonians (examples of such Hamiltonians can be obtained by 'modification' of special cases of classical polynomials, using usual Dardoux transformations [6,21]), polynomial eigenfunctions of which are not necessarily neither of Laguerre–Hahn type, nor of orthogonal type. A more natural question consists of extending the IHF method directly to the fourth order difference eigenvalue problem (for which the basic polynomial solutions are Krall type polynomials [13]). This question (as well as the 'modification' of Krall type polynomials) is attacked in [6].

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